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REINSURANCE AND DIVIDEND PROBLEMS IN INSURANCE

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par

Arian CANI

Directeur de thèse
Prof. Hansjoerg Albrecher

Co-directeur de thèse
Prof. Stefan Thonhauser

Jury

Prof. Olivier Cadot, Président
Prof. François Dufresne, expert interne
Prof. Nicole Baeuerle, experte externe
Prof. Hanspeter Schmidli, expert externe

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Bâtiment Internef
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Le doyen

Jean-Philippe Bonardi

HEC Lausanne
Le Décanat
Tél. +41 21 692 33 40 | Fax +41 21 692 33 05
www.hec.unil.ch | hecdoyen@unil.ch

Membres du Jury

Prof. **Olivier Cadot**
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Prof. **François Dufresne**
Expert interne, Université de Lausanne, Département de Sciences Actuarielles.

Prof.**Nicole Bauerle**
Expert externe, Karlsruhe Institute of Technology,
Department of Mathematics, Germany.

Prof. **Hanspeter Schmidli**
Expert externe, University of Cologne, Institute of Mathematics, Germany.

University of Lausanne
Faculty of Business and Economics

Doctorate in Actuarial Science

I hereby certify that I have examined the doctoral thesis of

Arian CANI

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made during the doctoral colloquium
have been addressed to my entire satisfaction.

Signature: Date: 21.12.2017

Prof. Hansjoerg ALBRECHER
Thesis supervisor

University of Lausanne
Faculty of Business and Economics

Doctorate in Actuarial Science

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Signature: Stefan Thonhauser Date: 12.01.2018

Prof. Stefan THONHAUSER
Thesis co-supervisor

University of Lausanne
Faculty of Business and Economics

Doctorate in Actuarial Science

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Signature: F. Dufresne Date: 2017-12-21

Prof. François DUFRESNE
Internal member of the doctoral committee

University of Lausanne
Faculty of Business and Economics

Doctorate in Actuarial Science

I hereby certify that I have examined the doctoral thesis of

Arian CANI

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made during the doctoral colloquium
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Signature: N. Bäuerle Date: 22.12.17

Prof. Nicole BAEUERLE
External member of the doctoral committee

University of Lausanne
Faculty of Business and Economics

Doctorate in Actuarial Science

I hereby certify that I have examined the doctoral thesis of

Arian CANI

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made during the doctoral colloquium
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Signature: Hanspeter Schmidli Date: 23.12.17

Prof. Hanspeter SCHMIDLI
External member of the doctoral committee

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Preface

In practice, an insurer has several possibilities to control and influence the performance of the surplus process of an insurance portfolio. The concrete nature and choice of these control variables for a certain portfolio will be determined by the objective that is pursued together with possible constraints. Typical control actions for an insurance company are the size of the capital, the choice of the premium amount, investment and reinsurance decisions, dividend payments and possibly a combination of all these.

This thesis explores the interplay between the stability of a risk business assessed by means of conservative risk measures and its profitability through various reinsurance and dividend problems. Chapter 1 reviews some basic concepts and contains some classical results that are employed in subsequent chapters. Within the framework of a compound Poisson risk model, we consider in Chapter 2 the problem of finding a dynamic reinsurance strategy that maximizes the expected discounted surplus level integrated over time. Such a performance measure is based on an economically motivated criterion as it is proportional to the expected surplus at some exogeneous exponentially distributed random life time. By combining analytical and probabilistic tools, we identify the value function as the solution to the associated Hamilton-Jacobi-Bellman equation. Chapter 3 deals with the optimality of reinsurance forms from another perspective. In a static one-year framework reinsurance model under regulatory solvency constraints and cost of capital considerations, we analyze the effects of randomizing a stop-loss treaty on the expected profit after reinsurance. While randomized treaties may be viewed as counter-intuitive and unnatural at first sight, it is illustrated that they can lead to an improved profitability. The results also underpin some of the shortfalls of using the Value-at-Risk for measuring risk. In Chapter 4, we examine another type of control action, namely, dividend payments. Prompted by the scarce literature devoted to modelling dividend strategies that yield smooth dividends over time, we propose a dividend strategy that secures a continuous dividend payment stream, the rate of which is adjusted according to the present surplus value in an affine way. Under a classical compound Poisson risk model, we derive closed-form expressions in terms of hypergeometric functions for the expected present value of dividends until ruin and the Laplace transform of the time to ruin in the case of exponentially distributed claim sizes. The results suggest that a suitably chosen affine dividend strategy can lead to an almost as large value for the expected discounted dividend payments than the one obtained under the optimal barrier strategy, while leading to considerably improved safety, measured in terms of expected ruin time of the portfolio. Finally, Chapter 5 revisits affine dividend strategies in a Brownian risk model and also studies the resulting surplus process' properties under a possibly negative interest rate.

Contents

Chapter 1

Introduction

By selling an insurance contract, the insurer promises to indemnify the policyholder in case the triggering event occurs. While this transaction guarantees safety to the policyholder, the insurer has to manage its own economic survival. Hence, for insurance companies, it is crucial to identify, measure, monitor and control their financial risks. An adequate risk management requires actuaries to assign numerical values to identified risks by implementing sound quantitative risk models and choosing suitable risk measures. Since claims represent a primary risk for an insurer, both in terms of their size and frequency, setting up a probabilistic model that adequately describes the time-evolution of the claims process is an essential first step in handling their random nature. Within the built model, the insurer can employ *control actions* such as size of the initial capital, choice of premium amount, investment and reinsurance decisions, dividend payments or opt for a combination of these to obtain the desired result on the risk reserve process. In this introductory chapter, we discuss some classical models for the surplus of an insurance company and describe the reinsurance and dividend control actions, which will be of main interest in this thesis. As they will turn out to play an important role in future chapters, a brief review of some concepts connected to the theory of differential equations and special functions is given at the end ot this chapter.

1.1 Risk models

Ruin theory is a branch of actuarial science which studies, by means of probabilistic models, an insurance company's vulnerability to insolvency due to a large number of claims within a short time period, claims of particularly high severity, or, a combination of the two. This topic is interesting not only from a theoretical perspective but also from a practical point of view as it provides insurers with an idea on how premiums shall be set to ensure the company's economic survival. The theoretical foundation of ruin theory was laid down in 1903 by Swedish actuary Filip Lundberg in his doctoral thesis and was later extended by Herald Cramér in the 1930's. Hence the Cramér-Lundberg model, also known as classical compound Poisson risk model, which has established itself as a classic over the course of time to model the evolution of the surplus of an insurance company. In the Cramér-Lundberg risk model, the time-evolution of the surplus process $X = (X_t)_{t\geq 0}$ is described by

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0. \tag{1.1}$$

Here $x \geq 0$ is the size of the initial capital. The premiums are assumed to be collected continuously over time with constant intensity c. The aggregrate claim amount at time t is given by the compound Poisson sum $\sum_{i=1}^{N_t} Y_i$, where the number of claims up to time in $[0, t]$ is a Poisson process $N = (N_t)_{t\geq 0}$ with intensity $\lambda > 0$, i.e. $N_t \sim Poi(\lambda t)$. As a consequence, the inter-occurrence times are exponentially distributed. The claim sizes $\{Y_i\}$ are a sequence of i.i.d. positive random variables independent of N.

Classical quantities of interest in this context are the time of ruin

$$\tau_x = \inf\{t \geq 0 : X_t < 0 \mid X_0 = x\},$$

and the probability of ruin

$$\psi(x) = P(\tau_x < \infty).$$

From the theory of random walks [105], it is well-known that if $c \leq \lambda\,\mathbb{E}[Y]$, ruin occurs almost surely, i.e. $\psi(x) = 1$. Therefore, the so-called *net profit condition*, namely, $c > \lambda\,\mathbb{E}[Y]$, is required to avoid almost-sure ruin.

Other alternative risk models are used in the literature. For instance, the *Sparre Andersen model*, which allows the claim inter-occurrence times to have arbitrary distribution functions. In a *diffusion approximation*, one defines a sequence of classical risk models with initial capital values $x^{(n)} = x$, claim arrival intensities $\lambda^{(n)} = \lambda n$, claim size distributions $F^{(n)}(x) = F(x\sqrt{n})$ and premium rates $c^{(n)} = c + \lambda\,\mathbb{E}[Y](\sqrt{n} - 1)$.

Then, the sequence of classical reserve processes converges weakly to a stochastic process of the form

$$x + \Lambda + \sqrt{\lambda \mathbb{E}[Y^2]} W,$$

where $\Lambda = (\Lambda_t)_{t \geq 0}$ with $\Lambda_t = (c - \lambda\, \mathbb{E}[Y])t$ and $W = (W_t)_{t \geq 0}$ is a standard Brownian motion (see [83] for a proof). Dufresne and Gerber [1] studied the so-called *perturbed classical risk process* by adding a Brownian motion in (1.1). All these basic models have been generalized into various directions, in particular by allowing for control actions such as reinsurance, dividends and economic factors such as interest rates and inflation, see [20] for a survey.

1.2 Reinsurance

1.2.1 Fundamentals of reinsurance

In a *reinsurance contract*, one party (the reinsurer) agrees to indemnify another party (called *reinsured*, *first-line insurer* or *cedent*) for a specified share of its underwritten insurance risk. In exchange of this protection, the cedent pays a *reinsurance premium*. Historically, the first formal reinsurance contract, dated July, 1370, was concluded to cover the most hazardous part of a cargo trip from Genoa, Italy to Sluis, Belgium. Since then, intermittently, the reinsurance market has developed into one of the most capitalized in the world. The nature and purpose of reinsurance is parallel to the one of insurance, namely, reduce the probability of suffering losses that are hard to cope with, in particular as a result of extremely large claims or an unusually large number of claims, whether large or not. By replacing a random component (claims) by a deterministic one (reinsurance premium), reinsurance reduces earnings variability and thus further enhances credibility in times of financial tumult and efficiency in the insurance market. It is however important to keep in mind that purchasing reinsurance means transferring a part of its insurance portfolio (core business) to the reinsurer, so that the objective is to keep the reinsured part relatively small. Reinsurance can also serve to increase an insurer's underwriting capacity by reducing its exposure from specified parts of its risks. Hence, in the presence of a reinsurance contract, insurers can underwrite more/larger policies than they would afford on their own. In this way, reinsurance creates scope for economic development. For a detailed treatment on the motivations to buy reinsurance, see

e.g. [5]. While reinsurance can be seen as a particular form of insurance, it yet differs on certain aspects. For instance the size and type of the considered risk (typically reinsured claims are heavy-tailed), the availability of data points to estimate the distribution and hence the (re)-insurance premium, the fact that reinsurance contracts are generally tailored closer to the buyer's needs; but also the regulatory framework, since reinsurance is a contract between two insurance entities.
In practice, one distinguishes between two types of reinsurance treaties: *obligatory treaties* and *facultative treaties*. In a facultative reinsurance treaty the reinsurer is able to choose which of the insurer's risks it wants to accept, whereas in an obligatory treaty, the reinsurer is obliged to accept all risks of a specified risk class. Very natural questions that arise in the reinsurance context are which reinsurance form to choose and how much reinsurance is required. A universal answer to these questions does not exist and very much depends on the nature of the involved risks as well as the specification of the objective function together with (re)insurance premium rules and further possible constraints. In the following, we describe some classical reinsurance forms of the obligatory type.

1.2.2 Reinsurance forms

Let $\{Y_i\}_{n\in\mathbb{N}}$ be the sequence of claim sizes that the first-line insurer faces and let $(N_t)_{t\geq 0}$ be a counting process (not necessarily Poissonian), where N_t represents the number of claims up to time $t > 0$. The aggregate claim amount is then given by the compound sum

$$S_t := \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0.$$

A reinsurance contract is a rule according to which the aggregate loss is sub-divided into

$$S_t = R_t + C_t,$$

where R_t is the retained loss amount (that stays with the first-line insurer) and C_t is the amount paid by the reinsurer. In many reinsurance contracts, such a rule is specified on an individual risk basis, so that $Y_i = R_i + C_i$.
We now discuss the most common forms of reinsurance contracts together with some of their properties. We start with *proportional* reinsurance forms.

Quota-share reinsurance

Quota-share reinsurance is the most straightforward reinsurance form, where

$$R_t := a\, S_t,$$

for some proportionality factor $0 < a < 1$ which also applies for premium calculation. An appealing advantage of this reinsurance form is the ease with which it can be handled and implemented. Also, due to the proportional share, moral hazard issues are alleviated. One of the major disadvantages of such a reinsurance form is that *all* claims are insured, including the small ones which is not ideal from a profitability point of view (such claims can be fully borne by the first-line insurer).

Surplus reinsurance

A reinsurance form which remedies to the non-optimality of reinsuring small claims is the so-called *surplus reinsurance*. A surplus treaty only reinsures claims Y_i whose insured sum Q_i is greater than a given retention $M > 0$, in which case the relative participation of the reinsurer is determined by a factor $1 - \frac{M}{Q_i}$. Hence,

$$R_t = \sum_{i=1}^{N_t} \left(Y_i \mathbb{1}_{\{Q_i \le M\}} + Y_i \frac{M}{Q_i} \mathbb{1}_{\{Q_i > M\}} \right), \qquad C_t = \sum_{i=1}^{N_t} \left(1 - \frac{M}{Q_i} \right) Y_i \mathbb{1}_{\{Q_i > M\}},$$

where $\mathbb{1}_A$ is the indicator function of the event A. Here again, thanks to the proportionality feature, the premium determination is simple. Surplus reinsurance is particularly popular in fire and marine insurance.

Excess-of-loss reinsurance

We now look at reinsurance forms of *non-proportional* nature. In contrast to proportional reinsurance forms, there is not a settled subdivision of claims between the first-line insurer and the reinsurer, so that the proportion of the aggregate claim amount that stays with the insurer is not known in advance, which can lead to moral hazard issues. In an *excess-of-loss treaty*, for each individual claim, the excess over some retention M is covered by the reinsurer, that is,

$$R_t = \sum_{i=1}^{N_t} \min(Y_i, M), \qquad C_t = \sum_{i=1}^{N_t} (Y_i - M)^+.$$

This intuitive form of risk sharing plays a prominent role in liability, motor and windstorm insurance, however the resulting premium calculation is not simple (due to the censored data above M) and such a treaty offers no protection against the accumulation of claims. Usually, an upper limit L determines the maximum participation of the reinsurer, leading to

$$R_t = \sum_{i=1}^{N_t} \left(\min(Y_i, M) \mathbb{1}_{\{Y_i \le M+L\}} + (Y_i - L) \mathbb{1}_{\{Y_i > M+L\}} \right), \qquad C_t = \sum_{i=1}^{N_t} \min(Y_i - M, L),$$

which is often referred to as an L xs M treaty or bounded excess-of-loss.

Stop-loss reinsurance

A *stop-loss* reinsurance contract acts on the aggregate claim amount, that is,

$$R_t = \min(S_t, M), \qquad C_t = (S_t - M)^+.$$

Here, the magnitude of individual claims and the cause of the aggregate claim amount are irrelevant for this type of cover. Again, there is typically an upper limit on C_t.

1.2.3 Reinsurance choice

Each reinsurance form has its own particularities, advantages and limitations in terms of the type of protection it offers (large claim risk, frequency risk), calculation of the reinsurance premium, administrative simplicity, practicality and moral hazard issues. Taking all the above into account, an insurer has to make a choice regarding the concrete form and amount of reinsurance among all feasible treaties. In practice, factors such as the experience in the market, scope of available treaties and reinsurance market prices will influence the reinsurance decision process. The insurer may also have a certain objective together with a specified reinsurance premium principle and possible economic constraints in which case the identification of an optimal reinsurance form becomes an optimization problem sometimes leading to mathematically tractable solutions. Basing the choice of a reinsurance form and its concrete specification solely on a mathematical result may not fully reflect the practical situation, mainly due to the involved model assumptions which are often too simplistic to accurately describe reality. Nevertheless, the theoretical results can help practicioners gain a better understanding on the implications of certain choices of reinsurance forms and help them make better decisions.

Over the last decades, there was a profileration of research on optimal reinsurance problems mostly from the cedent's perspective (it is not clear whether such an optimal form is satisfactory from a reinsurer's point of view), where researchers have been using increasingly sophisticated models to identify the corresponding optimal reinsurance forms. In the following, we discuss a non-exhaustive selection of theoretical results appearing in the literature.

1.2.4 Classical results

The study of optimal reinsurance goes as far as 1940 with the pioneer work of Bruno de Finetti [55], who considered the problem of finding the optimal quota-share proportions a_i of n independent subportfolios S_i when the objective is to minimize the variance of the total retained loss amount under the constraint that the expected profit is fixed. The optimal retention a_i^* turns out to be proportional to the reinsurance loading and inversely proportional to the variance of the corresponding subportfolio. In 1970, Bühlmann [35] solved the same problem in the presence of excess-of-loss reinsurance if the subportolios are compound distributed. In contrast to the proportional case, the optimal retention level not only depends on the first two moments but rather on the full individual claim size distribution.

Later, Karl Borch [32] used a different approach. He proved that a stop-loss reinsurance leads to the smallest variance of the cedent's retained amount for a fixed reinsurance premium calculated according to the expected value principle (see also Kahn [85] for a similar result under more general assumptions). However, this result was obtained under the fairly restricted condition that the relative safety loadings under stop-loss and quota-share reinsurance are equivalent. In this spirit, Beard [29] showed in 1977 that if the reinsurance premium is based on the variance principle (it increases linearly with the variance of the ceded risk), then the variance of the retained risk is minimized under a quota-share arrangement. In the context of risk-averse utility functions, Arrow [18] established the optimality of a stop-loss treaty when the goal is to maximize the expected utility of the resulting wealth using the expected value principle as a rule for reinsurance pricing. In the framework of the classical compound Poisson process in collective risk theory, Gerber [67] proved that, when only individual treaties are considered, an excess-of-loss treaty maximizes the adjustment coefficient, which can be seen as an approximate solution of minimizing the ruin probability (see Centeno and Simões [43] and Bowers et al. [33]).

1.2.5 Recent results and current research

The subsequent research followed the ideas outlined above, trying to take into account more general risk measures and premium principles, in which case the optimal reinsurance might not be a stop-loss (see for instance Gajek and Zagrodny [63], Kaluszka [86], Centeno and Guerra [75], Young [123] and Albrecher et al. [5, Chapter 8] for a recent survey). More recently, prompted by the regulatory developments aiming at the harmonization of risk assessment procedures among banks, insurance and other financial institutions, the Value-at-Risk (VaR) and Conditional-Tail-Expectation (CTE) became the standard risk measures to determine the appropriate solvency capital requirement. These risk measures are defined as follows:

Definition 1.2.1. ***(Value-at-Risk)*** *The VaR of a random variable* $X \sim F_X(x)$ *at level* $1-\alpha$*,* $VaR_\alpha(X)$*, is defined as the* $(1-\alpha)$*-th quantile of the distribution of* X*:*

$$VaR_\alpha(X) = \inf\{x : F_X(x) \geq 1-\alpha\}, \qquad \alpha \in (0,1).$$

In addition, we have:

Definition 1.2.2. ***(Conditional-Tail-Expectation (CTE))*** *The CTE*[1] *of a ran-*

[1]Under this definition, for absolutely continuous random variables X, the CTE turns out to be equal to what is usually named *Expected Shortfall (ES)* or *Tail-Value-at-Risk (TVaR)* in the literature, and defined by

$$\mathrm{TVaR}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_s(X)ds, \qquad \alpha \in (0,1).$$

However, this interpretation may fail to be true for discrete random variables X, typically in the case where X has a probability mass in $\mathrm{VaR}_\alpha(X)$.

dom variable $X \sim F_X(x)$ at level $1-\alpha$, is defined as

$$CTE_\alpha(X) = \mathbb{E}\left[X \mid X \geq VaR_\alpha(X)\right], \qquad \alpha \in (0,1).$$

As a result of their increasingly prevalent use, significant attention has revolved around embedding these two risk measures in the study of optimal reinsurance models. In this light, assuming that the reinsurance premium is determined by the expected value principle, Cai and Tan [36] analytically derived the optimal retention level in a stop-loss treaty which minimizes the VaR and CTE of the insurer's retained risk exposure. This result was later generalized by Cai et al. [38] and Cheung [45] who derived the optimal reinsurance forms among the class of increasing convex reinsurance treaties by convergence arguments and a geometric approach, respectively. Their results suggest that depending on the risk measure's confidence level α and the safety loading used for the reinsurance premium, the optimal reinsurance can be one of the following three types: stop-loss, quota-share or change loss (a combination of the two). Within this setting of minimizing the VaR and TVaR, Chi and Tan [47] determine the optimal reinsurance contract among a larger class of admissible reinsurance schemes, where the optimal reinsurance form turns out to be less robust for the VaR criterion. In view of these results, Guerra and Centeno [76] give further evidence against the use of quantile risk measures in optimal reinsurance problems. They show that if one relaxes the constraints of convexity and monotonicity in the corresponding ceded loss functions, discontinuous reinsurance arrangements become optimal. In addition, they argue that such treaties are not acceptable as they would lead to conflicting situations between insurer and reinsurer if the claim value is in the neighborhood of a discontinuity point.

The results summarized above all have in common that they ultimately deal with deterministic reinsurance forms. While this is a very natural and traditional way to specify the risk participation of a reinsurer, we examine in Chapter 3 whether an additional source of randomness in the specification of the retention function $r(\cdot)$ can improve the efficiency of risk sharing (measured in terms of expected profit after reinsurance) in the framework of a one-year reinsurance model under regulatory solvency constraints and cost of capital considerations. Such reinsurance contracts will be referred to as *randomized reinsurance contracts.*

1.2.6 Dynamic reinsurance

Until now we restricted our considerations to *static* reinsurance strategies, namely the concrete reinsurance form was chosen at the beginning and remained constant throughout the time period of interest, which in many cases is infinite. The concept of *dynamic* reinsurance takes a different approach on the reinsurance decision process by allowing the reinsurance form to be adjusted at some points in time according to the information available. The mathematical description on how to act optimally over time to achieve maximal rewards is known as *control theory*. In a Markovian framework, an optimal reinsurance strategy is typically given by a *feedback* strategy, i.e. it depends on the actual surplus and not on the history of the process. For discrete-time risk processes, the dynamic programming principle provides a systematic procedure to characterize the optimal sequence of reinsurance decisions. The idea behind the dynamic programming is that one tries to take the optimal decision in a first time interval and then take remaining decisions in an optimal way. Schäl [108] considers for instance the problems of minimizing the ruin probability and maximizing the expected exponential utility of the terminal surplus in a discrete-time insurance model whith reinsurance and investment possibilities. The existence of an optimal reinsurance strategy is shown using the theory of dynamic programming.

When one instead turns to classical continuous-time risk models together with continuously adjustable reinsurance forms, the analysis becomes more transparent and sometimes leads to elegant and explicit expressions for the optimal strategy. However, the gain in mathematical tractability comes at the expense of a less realistic model as it will be practically impossible to adapt the reinsurance form in a continuous fashion. Nonetheless, not only do continuously adaptable strategies allow to quantify the improvement of the objective function in comparison to the static case, they also give an insight on the long-term implications of concrete static reinsurance strategies. A potential solution to limit the adjustment possiblities is the introduction of transaction costs and/or portfolio constraints. In this direction, Højgaard and Taksar [81] examined the problem of finding a dynamic proportional reinsurance strategy in a diffusion model with transaction costs which keeps the surplus at high levels over time and not only away from zero.

In a continuous-time setting, the *value function*, which defines the best possible value the objective can take, is often determined by solving the associated *Hamilton-*

Jacobi-Bellman (HJB) equation, which can be viewed as the continuous-time equivalent of the dynamic programming principle. The concept behind the HJB equation is particulary powerful as it is obtained by letting time steps go to zero in the dynamic programming principle, hence leading to a pointwise optimization procedure, i.e. for each state of the process (and possibly time), we compute the corresponding optimal control decision. This is in contrast with the seemingly more difficult task of simultaneously finding an optimal decision for each point in time. However, the derivation of the HJB equation typically requires some regularity conditions of the value function that are difficult to verify a priori. Hence, the usual route towards a solution is to heuristically derive the HJB equation, assume that the value function is as regular as one would ask for and finally prove separately, in a so-called *verification theorem* that the candidate solution is indeed the value function of the control problem. In general, however, the HJB equation does not have a sufficiently smooth solution. A possible way to deal with such a situation is to use the concept of *viscosity solutions*, which are a generalization of the notion of classical solutions introduced by Crandall and Lions [52]. Moreover, since usually the solution of the HJB equation cannot be obtained analytically, one has to resort to numerical methods, which might be challenging in their own right (see for instance Kushner and Dupuis [89]). Classical dynamic reinsurance problems are the minimization of the ruin probability (see Schmidli [109] for dynamic proportional reinsurance in the Cramér-Lundberg model and its diffusion approximation and Hipp and Vogt [79] for dynamic excess-of-loss reinsurance in the Cramér-Lundberg model). While Schmidli showed in the diffusion case that the optimal strategy to minimize the ruin probability is to have a constant fraction of proportional reinsurance, the solution (also for excess-of-loss) in the classical risk model can only be approximated numerically. However, under some mild technical conditions, one can still retrieve significant information about the asymptotic behavior of the optimal strategy (see Schmidli [110] for more details). Several variants of this problem as well as other choices of objective functions have been studied extensively in the literature, in particular by allowing reinsurance to be simultaneously controlled with investments and dividends payments (see e.g. Schmidli [110] and Azcue and Muler [25] for a rich source of stochastic control problems in insurance).

Within this context, in Chapter 2, we take a profit-orientated approach and identify a dynamic reinsurance strategy that maximizes the surplus at an exogenous exponentially distributed time in the Cramér-Lundberg model.

1.3 Dividends

In actuarial science, there are two main paradigms to assess the performance of an insurance company. On the one hand, we have the classical ruin probability criterion which has been the focus of research under increasingly sophisticated probabilistic models since the first part of the twentieth century. Whereas such a criterion choice is concerned with the safety of an insurance company, it is often criticized as being too conservative. In particular, a trajectory of the surplus process that does not lead to ruin exceeds every finite threshold with probability one, which is typically unrealistic. This issue was first raised by the Italian mathematician Bruno de Finetti back in 1957, who asserted that no economically viable and realistic criterion could be based on such a concept.

The primary flaw being the infinite growth of the surplus, de Finetti suggested that it should be diminished from time to time according to a certain rule. These skimmings will in the sequel be called *dividends*. It certainly appears reasonable to assume that the surplus, which represents the wealth of the company, is in part redistributed to its shareholders. The rule which associates the dividends to be paid for each surplus trajectory is called a *dividend strategy*. As an alternative criterion to the ruin probability, de Finetti proposed to compare dividend strategies on the basis of their expected sum of discounted dividend payouts until ruin. The strategy which produces the maximal dividend value is said to be the *optimal dividend strategy*. Such a performance measure implies a trade-off between paying out dividends early (due to the discounting) and paying dividends later (so that due to the positive drift of the process, the time span of dividend payouts is prolongated).

Let us now state the mathematical formulation which is the basis of de Finetti's dividend problem. Suppose that in the absence of dividend payments, the surplus is described by $R = (R_t)_{t\geq 0}$. Let $u = (D_t^u)_{t\geq 0}$ with $D_0^u = 0$ be a dividend strategy which consists in a predictable, non-decreasing and càglàd (left-continuous with existing right limits) process, where D_t^u represents the total dividend payouts until time t under the strategy u. The surplus after dividends is then given by $R^u = (R_t^u)_{t\geq 0}$ with associated ruin time $\tau_x^u = \inf\{t > 0 : R_t^u < 0 | R_0^u = x\}$. A dividend strategy is said to be admissible if $D_{t+}^u - D_t^u \leq R_t^u$ for any $t < \tau^u$, that is, ruin does not occur due to dividend payments. Denoting the set of all admissible strategies by $\mathfrak{U}$, the expected value of the sum of the discounted dividend until ruin associated with the strategy $u \in \mathfrak{U}$ and initial capital $x \geq 0$ is given by

$$V^u(x) := \mathbb{E}_x \left[\int_0^{\tau_x^u} e^{-\delta t} dD_t^u \right],$$

where $\delta > 0$ is a force of interest for valuation. The associated optimal control problem consists in finding

$$V(x) := \sup_{u \in \mathfrak{U}} V^u(x), \tag{1.2}$$

and, if it exists, a strategy, u^* such that $V(x) = V^{u^*}(x)$.

1.3.1 Some frequent dividend strategies

Theoretically, the design and scope of possible dividend strategies is limited only by the bounds of the imagination. However, in the literature, some specific dividend strategies turn out to be optimal in the sense of (1.2) in certain situations. In this subsection, we briefly describe some of them.

Band strategies

A *band strategy* is characterized by the partition of the state space of the surplus process into three disjoint sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. Each set is associated with a certain dividend payment whose amount depends on the current surplus x: if $x \in \mathcal{A}$, the entire incoming premium is paid out; if $x \in \mathcal{B}$, then a lump sum payment of size $x - \{\max a : a < x, a \in \mathcal{A}\}$ brings the surplus to the largest point in $\mathcal{A}$ that is smaller than x; finally, if $x \in \mathcal{C}$ no dividends are paid. Note that the sets $\mathcal{B}$ and $\mathcal{C}$ may consist of the disjoint union of (half)-open intervals. Gerber [72] proved that if the surplus process is described by a random walk in a discrete state space, the optimal dividend strategy is of band type. He later established this result in the Cramér-Lundberg model. Employing the methodology of viscosity solutions, Azcue and Muller [24] show that such a strategy also turns out to be optimal in the presence of dynamic proportional and excess-of-loss reinsurance within the classical risk model. In this setting, Albrecher and Thonhauser [8] show that a band dividend strategy is optimal if the surplus can be invested at a positive force of interest. In a discrete-time risk model where dividends are paid out to shareholders at (random) discrete time points, Albrecher et al. [13] proved that a band strategy is optimal.

Threshold strategies

A dividend strategy is called a *threshold strategy* for a fixed threshold $b > 0$ if dividends are paid continuously at a rate a smaller than the premium rate if the surplus is above level b and no dividends are paid otherwise. A motivation for the introduction of a threshold strategy is that in contrast to the (horizontal) barrier strategy, it can lead to a positive infinite-time horizon survival probability. Such a strategy is discussed in Gerber and Shiu [70], Frostig [62] and Lin and Pavlova [92] in the classical risk model and by Gerber and Shiu [69] in its diffusion approximation. Albrecher et al. [12] compare the respective total dividend values of the threshold strategy to a linear barrier strategy in a Sparre Andersen model. An analysis of boundary crossing problems for a threshold strategy in a general Lévy set-up can be found in Kyprianou and Loeffen [90].

Barrier strategies

A *barrier strategy* is a special case of a band strategy where $\mathcal{A} = \{b\}$ consists in a single point together with $\mathcal{B} = (b, \infty)$ and $\mathcal{C} = [0, b)$. Such a strategy instantly pays out all the excess above b at $t = 0$ and subsequently all the incoming premiums that lead to a surplus above b are immediately paid as dividends. Hence, the risk process is reflected at the level b.. The accumulated dividends paid up to time t can in this case be represented as a function of the running maximum of R. If we denote

$$M_t := \max_{0 \le s \le t} R_t,$$

for $t < \tau^u$, then

$$D_t^u = (M_t - b)^+.$$

This intuitive profit distribution strategy was first proposed by de Finetti [56] in 1957 and he proved that if the risk process evolves as a simple random walk with unit step sizes, then an optimal way of paying out dividends is a barrier strategy. As a by-product of the general characterization, Gerber [72] showed that for the particular case of exponentially distributed claims in the Cramér-Lundberg model, the band strategy reduces to a barrier strategy. Albrecher and Thonhauser [8] observed that such a result still holds if there is a constant positive force of interest. In the diffusion approximation, Shreve et al. [112] proved the optimality of the barrier strategy for a certain linearly bounded drift coefficient. In a more general Lévy-insurance risk model, Loeffen and Renaud [96] give sufficient conditions (expressed in terms of the tail of the Lévy measure) under which the barrier strategy is optimal. Under the assumption that R is a stationary and skip-free Markov process (may have jumps downwards, but not upwards), the expected present value of dividends can be computed using strikingly simple probabilistic arguments, which exploit the Markovianity of the underlying risk process (see Gerber et al. [71]). In particular, denoting by $V_b(x)$ the present value of dividends for a given initial surplus and barrier $b > 0$, we have

$$V_b(x) = \frac{h(x)}{h'(b)} = \frac{h(x)}{h(b)} V_b(b), \qquad 0 \le x \le b,$$

for a suitably chosen function h. Here, the ratio $h(x)/h(b)$ can be interpreted as the expected present value of a payment of 1 payable as soon as the surplus first hits level b, conditioned on the event that ruin has not yet occured. Since ultimately the aim is to maximize the present value of dividends, b has to be chosen such that $V_b(x)$ is maximized, which here boils down to minimizing $h'(b)$, whose solution does not depend on x. A consequence of the risk process being reflected at the barrier is that the associated ruin probability is one, which in many circumstances is not desirable. A possibility to incorporate an additional safety aspect component in the process of collecting dividends is to introduce a function which penalizes early ruin of the risk process, see Albrecher and Thonhauser [119] and Hernandez and Junca [2] for studies in this direction. Another alternative is to consider time-dependent barriers (see Gerber [68], Siegl and Tichy [113] and Albrecher et al. [11] for time-linear dividend barriers and Albrecher and Kainhofer [4] for the non-linear case).

Apart from the lack of safety considerations, a commonly raised criticism is that the resulting dividend stream is far from practical acceptance. Indeed, whenever the surplus is below the barrier, no dividends are paid, which may lead to a very uneven dividend stream. This observation led to the idea of imposing restrictions on the qualitative nature of dividend payouts when considering its total expected present value. Furthermore, it is broadly accepted and backed up by empirical research (see Chapters 4 and 5 and references therein) that companies strive for a smooth dividend flow over time. In view of this aspect, Avanzi and Wong [22] introduced a mean-reverting dividend strategy which secures a continuous and smooth dividend stream over time. In part inspired by their approach, in Chapters 4 and 5, we specify a dividend strategy, the rate of which is adjusted to the current surplus through an affine function; hence the name *affine dividend strategy*. Chapter 4 examines such strategies in a classical risk model and Chapter 5 in a Brownian risk model. It turns out that affine strategies can be a competitive alternative to barrier strategies when paying dividends with the advantage of securing a smooth dividend stream. Moreover, they enable explicit expressions given in terms of hypergeometric functions for quantities of interest such as the expected present value of dividends, Laplace transform of the ruin time and the expected time to ruin. In the course of respective evaluations, differential equations of hypergeometric type will play a crucial role. In the next section, we briefly review some aspects connected to to the theory of differential equations.

1.4 Ordinary and singular points of second order linear differential equations

The classification of a point x_0 as *ordinary point*, *regular singular point* or *irregular singular point* of a linear differential equation gives a first indication on the qualitative behavior of the solution in the neighborhood of x_0 and gives rise to an appropriate method for further analysis.

In the following, we consider second order differential equations of the form

$$R(x)y'' + P(x)y' + Q(x)y = 0,$$

or, in standard form,

$$y'' + p(x)y' + q(x)y = 0, \tag{1.3}$$

where $P(x), Q(x)$ and $R(x)$ are polynomials in x and

$$p(x) = \frac{P(x)}{R(x)}, \qquad q(x) = \frac{Q(x)}{R(x)}.$$

The point $x = x_0$ is an *ordinary point* of (1.3) if both $q(x)$ and $r(x)$ are *analytic* at x_0, i.e. they have a Taylor series representation of the form

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n, \qquad q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n,$$

that both have a positive radius of convergence. If x_0 is not an ordinary point, we call it a *singular point.* Suppose, however, that the functions

$$p(x)(x-x_0), \qquad q(x)(x-x_0)^2$$

are both analytic at x_0. Then, x_0 is said to be *regular singular point.* Otherwise, it is called an *irregular singular point.* In order to classify the point $x_0 = \infty$, one can use an inverse transformation to map the point at infinity into the origin, i.e. $t = \frac{1}{x}$, together with the relations

$$\begin{aligned} \frac{dy}{dx} &= -t^2 \frac{dy}{dt}, \\ \frac{d^2y}{dx^2} &= t^4 \frac{d^2y}{dt^2} + 2t^3 \frac{dy}{dt}. \end{aligned}$$

Then, the point $x_0 = \infty$ is said to be ordinary, regular singular or irregular singular if the point $t = 0$ is accordingly classified.

1.4.1 Series solutions about ordinary points

A well-known result (see for instance [114, p. 150]) treats the case of power series solutions about ordinary points: If power series expansions of $p(x)$ and $q(x)$ are valid on an interval $|x - x_0| < R$, where $R > 0$, then the differential equation (1.3) has two linearly independent solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

whose radii of convergence are at least equal to R. Hence, the location of a singularity of a solution must coincide with the location of a singularity of the functions $p(x)$ and/or $q(x)$. An immediate consequence of such a result is that if both $p(x)$ and $q(x)$ are polynomials, hence analytic for $x \in \mathbb{R}$, the corresponding series solutions must converge for all $x \in \mathbb{R}$.

1.4.2 Series solutions about regular singular points

In case x_0 is a singular point of (1.3), the traditional power series method fails in general to deliver nontrivial solutions, so that an alternative method is necessary to study the behavior of (1.3) near x_0. This is of particular importance since a large number of differential equations arising in various fields related to mathematics have singular points and the choice of appropriate solutions is often determined by how the latter behave near those singularities. Fortunately, in most cases, the singular points are not too wild and an adequate modification of the technique of power series yields satisfactory solutions. These are the regular singular points we have previously defined. The rationale behind their definition is the following. We multiply (1.3) by $(x-x_0)^2$ to obtain

$$(x-x_0)^2 y'' + (x-x_0)\left(\sum_{n=0}^{\infty} \tilde{p}_n (x-x_0)^n\right) y' + \left(\sum_{n=0}^{\infty} \tilde{q}_n (x-x_0)^n\right) y = 0, \tag{1.4}$$

where the power series expansions

$$\sum_{n=0}^{\infty} \tilde{p}_n (x-x_0)^n = p(x)(x-x_0), \qquad \sum_{n=0}^{\infty} \tilde{q}_n (x-x_0)^n = q(x)(x-x_0)^2,$$

are valid on an interval $|x-x_0| < R$ with $R > 0$. Now, equation (1.4) resembles the well-known Euler equation

$$(x-x_0)y'' + \alpha(x-x_0)y' + \beta y = 0,$$

where α, β are constants. In particular, close to the expansion point x_0, (1.4) can be approximated by the associated Euler equation

$$(x-x_0)y'' + \tilde{p}_0(x-x_0)y' + \tilde{q}_0 y = 0.$$

Having in mind that the general solution to Euler's equation is a linear combination of powers of x (or in special cases the product of a power and a logarithm) suggests that we should look for solutions of the form

$$(x-x_0)^r \sum_{n=0} a_n (x-x_0)^n, \qquad a_0 \neq 0, \tag{1.5}$$

which is known as a *Frobenius* series. The parameter r must be chosen so that when the series (1.5) is plugged back into the ODE (1.3), the coefficient of the smallest power of $(x - x_0)$, i.e. $(x - x_0)^r$ is zero. This leads to a quadratic equation in r of the form

$$r(r-1) + \tilde{p}_0 r + \tilde{q}_0 = 0, \tag{1.6}$$

known as the *indicial equation.* Next, to obtain a solution, one equates all the other coefficients of $(x-x_0)^{r+n}$ for $n \geq 1$ in order to find a reccurence relation that depends on r. Moreover, if we denote by r_1, r_2 the roots of the indicial equation (1.6), then if $r_1 - r_2 \notin \mathbb{Z}$, two linearly independent solutions to (1.3) are given by

$$y_1(x) = (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad y_2(x) = (x - x_0)^{r_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n.$$

where the coefficients a_n and b_n are determined by respective substitution of $y_1(x)$ and $y_2(x)$ into (1.3) to obtain the corresponding reccurence relation. As for series solutions abount ordinary points, it can be shown that the radius of convergence of the linearly independent pair y_1 and y_2 is at least equal to R (see [114, p. 170-171]), i.e. the convergence extends to at least as far to the nearest other potential singularity. In the case $r_1 - r_2 \in \mathbb{Z}$, the solution may contain certain logarithmic terms (see [114, p. 171-173]).

The generalization of the power series method described in this section is named after German mathematician Georg Frobenius and is commonly referred to as the *Frobenius method.* For the sake of completeness, it is worth mentioning that the case of solutions near irregular singular points is considerably more difficult to address and lies beyond the scope of this thesis. For some techniques that can be applied in that case, see [30, Chapter 3].

1.5 The confluent hypergeometric equation and Kummer functions

The *confluent hypergeometric equation* is given by

$$zy'' + (b - z)y' - ay = 0,$$

where a, b are constants. This equation has a regular singular point at $z = 0$ and an irregular singular point at $z = \infty$. Employing the notation of (1.4), since

$$\tilde{p}(z) = b - z, \qquad \tilde{q}(z) = -az, \tag{1.7}$$

we have $\tilde{p}_0 = b$ and $\tilde{q}_0 = 0$, which yields the indicial equation

$$r(r-1) + br = 0 \Longleftrightarrow r(r-1+b) = 0.$$

Hence, the roots are $r_1 = 0$ and $r_2 = 1 - b$. This implies that for $b \notin \mathbb{Z}$, two linearly independent solutions are

$$y_1(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad y_2(z) = z^{1-b} \sum_{n=0}^{\infty} a_n z^n.$$

Substituting $y_1(z)$ in (1.7), we obtain

$$\sum_{n=0}^{\infty} \left[(n+1)(n+b)a_{n+1} - (n+a)a_n\right] z^n,$$

which leads to the reccurence relation

$$a_{n+1} = \frac{(n+a)}{(n+1)(n+b)} a_n, \qquad n = 0, 1, \ldots$$

with solution

$$a_n = \frac{(a)_n}{n!(b)_n} a_0, \qquad n = 1, 2, \ldots \tag{1.8}$$

Here, $(a)_n$ is called the *Pochhammer symbol* and can be defined as

$$(a)_0 = 1, \qquad (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad n = 1, 2, \ldots$$

where $\Gamma(z)$ is the gamma function.

Hence, setting $a_0 = 1$ in (1.8) yields the solution

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} = M(a, b, z) = {}_1F_1(a, b, z). \tag{1.9}$$

This function is known as the *Kummer function* of the first kind. A similar procedure with the second root $r_2 = 1 - b$ gives the solution

$$z^{1-b} \sum_{n=0}^{\infty} \frac{(a+1-b)_n}{(2-b)_n} \frac{z^n}{n!} = z^{1-b} M(a+1-b, 2-b, z).$$

It is worth noting, however, that $y_1(z)$ is singular when $b = 0, 1, 2, \ldots$ and that $y_2(z)$ is singular when $b = 2, 3, 4, \ldots$. Thus, it is conventional to use the *Tricomi confluent hypergeometric function* (also known as Kummer function of the second kind) $U(a, b, z)$ which is a certain linear combination of the two solutions above, defined by

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(a+1-b)} M(a,b,z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a-b+1, 2-b, z).$$

This expression is undefined for integer b, but can be extended to integer b by continuity. Note that if a is a non-positive integer and b is not a positive integer, the functions M and U fail to be linearly independent, in which case $y_2(z)$ might be used as second solution, if it exists. For special cases and adequate solutions in case of integer parameters, see [61].

1.5.1 Derivatives of hypergeometric functions w.r.t. parameters

The confluent hypergeometric function M is usually considered as a function of z; however, in some applications in physics (see for instance [65], [16]) or in risk theory as we will see later in Chapters 4 and 5, the variable of interest may be one of the parameters a or b. While the first derivative (and n-th derivative in general) of M with respect to z is known to have a simple compact expression (see [3]), the derivatives with respect to the parameters a or b have been less studied due to their more complex mathematical formulation. In the following, we shall make use of the following notation

$$\frac{d}{da} M(a,b,z) = M^{(a)}, \qquad \frac{d}{db} M(a,b,z) = M^{(b)}.$$

A first and customary approach to calculate $M^{(a)}$ and $M^{(b)}$ makes use of the derivative of the Pochhammer symbol $(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}$, which is given by

$$\frac{d}{da}(a)_n = (a)_n \left[\psi(a+n) - \psi(a)\right],$$

where $\psi(a) = \frac{d}{da}\log\Gamma(a) = \frac{\Gamma'(a)}{\Gamma(a)}$ is the logarithmic derivative of the gamma function known as the digamma function. Recalling the power series definition of M in (1.9) leads to the representations

$$M^{(a)} = \sum_{n=0}^{\infty} \left[\psi(a+n) - \psi(a)\right] \frac{(a)_n}{(b)_n} \frac{z^n}{n!},$$

and

$$M^{(b)} = \sum_{n=0}^{\infty} \left[\psi(b) - \psi(b+n)\right] \frac{(a)_n}{(b)_n} \frac{z^n}{n!}.$$

An alternative approach due to Ancarani et al. [17] is based on the solution of a certain inhomogeneous Kummer's differential equation. To begin with, recall that

$$\left(z\frac{d^2}{dz^2} + (b-z)\frac{d}{dz} - a\right) M(a,b,z) = 0. \tag{1.10}$$

Since M is analytic in a (see [60]), taking the derivative of (1.10) w.r.t. a, one gets

$$\left(z\frac{d^2}{dz^2} + (b-z)\frac{d}{dz} - a\right) M^{(a)} = M(a,b,z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}. \tag{1.11}$$

In a similar way, taking the derivative of (1.10) w.r.t. b (M is analytic in b except for poles at the non-positive integers), one finds

$$\left(z\frac{d^2}{dz^2} + (b-z)\frac{d}{dz} - a\right) M^{(b)} = -\frac{d}{dz}M(a,b,z) = -\frac{a}{b}\sum_{n=0}^{\infty} \frac{(a+1)_n}{(b+1)_n} \frac{z^n}{n!}. \tag{1.12}$$

From Babister [26], it is known that the solution to the inhomogeneous differential equation

$$\left(z\frac{d^2}{dz^2} + (b-z)\frac{d}{dz} - a\right) y = z^n,$$

is given by

$$y = \frac{z^{n+1}}{(n+1)(b+n)}\, {}_2F_2\left(1, a+n+1; 2+n, b+n+1; z\right).$$

Hence, because of the linearity of equations (1.11) and (1.12), we obtain respectively

$$M^{(a)} = \frac{z}{b}\sum_{n=0}^{\infty}\frac{(a)_n\,(1)_n}{(b+1)_n\,(2)_n}\frac{z^n}{n!}\, {}_2F_2\left(1, a+n+1; 2+n, b+n+1; z\right),$$

and

$$M^{(b)} = -\frac{a}{b}\frac{z}{b}\sum_{n=0}^{\infty}\frac{(a+1)_n\,(b)_n\,(1)_n}{(b+1)_n\,(b+1)_n\,(2)_n}\frac{z^n}{n!}\, {}_2F_2\left(1, a+n+1; 2+n, b+n+1; z\right),$$

where we used the identities

$$\frac{1}{(\xi+n)} = \frac{1}{\xi}\frac{(\xi)_n}{(\xi+1)_n}, \qquad (\xi)_{m+n} = (\xi)_m(\xi+m)_n.$$

Expanding the ${}_2F_2$ hypergeometric series together with some algebraic manipulations leads to

$$M^{(a)} = \frac{z}{b}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a+1)_{m+n}(1)_m(a)_m(1)_n}{(2)_{m+n}(b+1)_{m+n}(a+1)_m}\frac{z^{m+n}}{m!\,n!}, \tag{1.13}$$

and

$$M^{(b)} = -\frac{a}{b}\frac{z}{b}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a+1)_{m+n}(1)_m(b)_m(1)_n}{(2)_{m+n}(b+1)_{m+n}(b+1)_m}\frac{z^{m+n}}{m!\,n!}. \tag{1.14}$$

The double series (1.13) and (1.14) can be connected to the bivariate Kampé de Fériet function

$$F^{A,B,D}_{R,S,U}\left(\begin{matrix} a_1,\ldots,a_A\; b_1,\ldots,b_B\; d_1,\ldots,d_D \\ r_1,\ldots,r_R\; s_1,\ldots,s_S\; u_1,\ldots,u_U \end{matrix} x, y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m+n}\prod_{j=1}^{B}(b_j)_m\prod_{j=1}^{D}(d_j)_n}{\prod_{j=1}^{R}(r_j)_{m+n}\prod_{j=1}^{S}(s_j)_m\prod_{j=1}^{U}(u_j)_n}\frac{x^m y^n}{m!\,n!},$$

see e.g. [115, 61]. Hence, this leads to

$$M^{(a)} = \frac{z}{b}\,F^{1,2,1}_{2,1,0}\left(\begin{matrix} a+1 & 1, a & 1 \\ 2, b+1 & a+1 & - \end{matrix} z, z\right),\quad M^{(b)} = -\frac{a}{b}\frac{z}{b}\,F^{1,2,1}_{2,1,0}\left(\begin{matrix} a+1 & 1, b & 1 \\ 2, b+1 & b+1 & - \end{matrix} z, z\right),$$

where the empty product indicated by the solid horizontal line is interpreted to be unity.

Chapter 2

An optimal reinsurance problem in the classical risk model[1]

Abstract

In this chapter we consider the surplus process of an insurance company within the Cramér-Lundberg framework with the intention of controlling its performance by means of dynamic reinsurance. Our aim is to find a general dynamic reinsurance strategy that maximizes the expected discounted surplus level integrated over time. Using analytical methods we identify the value function as a particular solution to the associated Hamilton-Jacobi-Bellman equation. This approach leads to an implementable numerical method for approximating the value function and optimal reinsurance strategy. Furthermore we give some examples illustrating the applicability of this method for proportional and XL-reinsurance treaties.

2.1 Introduction

The determination of optimal insurance contracts is a classical topic in insurance mathematics. The first results are stated in a static utility theoretic framework and

[1]This chapter is based on the paper: Arian Cani and Stefan Thonhauser. An optimal reinsurance problem in the Cramér-Lundberg model. *Math. Methods Oper. Res.*, 85(2):179–205, 2017

concern the relation between a risk facing individual and the insurer. The goal is the construction of an optimal insurance arrangement for the first party with a certain constraint stemming from the second party. Classical contributions in this context are [19], [103] and [32], where one finds a collection of pioneering articles. A more recent paper by [75] studies this problem for exponential utility and provides the link to the maximization of the so-called adjustment coefficient which is the decay rate of the ruin probability for increasing initial capital. The idea of using reinsurance for maximizing the adjustment coefficient was introduced by [121], further studied by [42, 44] and [77], and can be considered as the motivation for studying optimal reinsurance.

The first paper to study dynamic optimal reinsurance in the classical risk model for the minimization of the ruin probability is [109], who dealt with the case of proportional reinsurance treaties. This approach was extended to excess of loss contracts by [79]. A general presentation on ruin probability minimization by means of reinsurance in the classical and diffusion risk model can be found in [110]. Furthermore, this reference provides some asymptotic studies of the behaviour of optimal strategies, which in certain situations coincide with the ones maximizing the adjustment coefficient. Some additional results with a focus on non-proportional reinsurance contracts are given in [78].

Using a different criterion to assess the performance of an insurance portfolio, [58] thoroughly covers a variety of capital injection minimization problems under both the classical risk model and its diffusion approximation where the insurer has the possibility to dynamically reinsure its risk. The incorporation of dynamic reinsurance to the classical problem of maximizing the dividend pay-outs of an insurance company prior to ruin in a compound Poisson framework was treated by [24] for general reinsurance schemes and by [99] for excess of loss reinsurance. In a diffusion setting, the corresponding problem was studied by [82] in the case of proportional reinsurance. Combining dividend pay-outs maximization with proportional risk exposure reduction, [107] formulated a piecewise deterministic Markov model where only jumps but not the deterministic flow can be controlled. In contrast to the aforementioned references which deal with optimal reinsurance for continuous time risk processes, [108] investigates a discrete time insurance model controlled by reinsurance and investments in a financial market with the intention to either maximize the expected exponential utility or minimize the ruin probability. An analogous problem was treated by [84], where the authors examine the purpose of maximizing the expected utility of terminal wealth by use of optimal investment and reinsurance.

Finally, we would like to mention a new approach linking ruin theoretical concepts with the framework of worst-case optimization theory explored by [87]. Embedded in a differential game setup, the authors applied a worst-case scenario approach to maximize the expected utility of the surplus of an insurance company at some given deterministic terminal time by dynamic proportional reinsurance.

In this contribution, we will study the use of dynamic reinsurance for maximizing a particular economic performance measure which for a diffusion risk model was introduced by [80, 81].
For its definition, let $X^{\mathtt{u}} = (X_t^{\mathtt{u}})_{t\geq 0}$ be a surplus process comprising a reinsurance strategy $\mathtt{u}$. The performance measure of this particular strategy is defined by

$$V^{\mathtt{u}}(x) = \mathbb{E}_x \left[\int_0^{\tau_x^{\mathtt{u}}} e^{-\delta t} X_t^{\mathtt{u}} \, dt \right], \tag{2.1}$$

where $\delta > 0$ denotes a discount or preference rate and $\tau^{\mathtt{u}}$ is the time of ruin of $X^{\mathtt{u}}$. In [117] this measure is motivated by the following arguments: the surplus of the insurance company is kept on a bank account and interest gains are immediately distributed as dividends, thus maximizing expected discounted dividend payments is equivalent to maximizing (2.1). Another way to motivate this value function in a Markovian environment is to introduce a random life time $S \sim Exp(\delta)$ which is independent of all other model ingredients. Then one observes

$$V^{\mathtt{u}}(x) = \frac{1}{\delta} \mathbb{E}_x \left[X_S^{\mathtt{u}} \mathbb{1}_{\{S < \tau_x^{\mathtt{u}}\}} \right], \tag{2.2}$$

which tells that the performance measure is proportional to the expected surplus at a random exponential time S. This means that a dynamic reinsurance strategy is used for maximizing the surplus at some exogenous point in time. Cost functions of the form (2.1), or more generally involving a running costs function $l(X_t)$, are also studied by [39] in an uncontrolled piecewise-deterministic compound Poisson environment.

The structure of the manuscript is as follows. In Section 2.2, we give a precise mathematical formulation of the problem, introducing the controlled surplus process and the value function. The analytical characterization of the value function is presented in Section 2.3. It starts with a collection of basic properties and employs the dynamic programming approach for achieving a final statement. Section 2.4 includes some comments on the numerical procedure obtained from the analytical results and two illustrative examples. Finally, a conclusion is stated in Section 2.5.

2.2 Problem statement

In the sequel, we will always work on a probability space $(\Omega, \mathcal{F}, P)$ which carries all stochastic quantities to be defined in the following. In the Cramér-Lundberg model (also known as compound Poisson model or classical risk model), the surplus process $X = (X_t)_{t\geq 0}$ of a homogeneous insurance portfolio is modeled as

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i. \tag{2.3}$$

Starting with an initial deterministic surplus $X_0 = x \geq 0$, the surplus process increases linearly due to premiums that are collected continuously over time at a constant rate $c > 0$. On the other hand, it decreases due to claims happening at the arrival times of a homogeneous Poisson process $N = (N_t)_{t\geq 0}$ with intensity $\lambda > 0$. The claims $\{Y_i\}_{i\in\mathbb{N}}$ constitute a sequence of positive independently and identically distributed random variables with a density function $f_Y(\cdot)$ and finite mean μ. Later on we will use Y as a representative random variable from this distribution. In addition, the sequence $\{Y_i\}_{i\in\mathbb{N}}$ and N are assumed to be independent. The flow of information is given by the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is generated by the surplus process X. In the remainder of the manuscript, we will use the symbol $\mathbb{E}$ for the expectation with respect to the probability measure P, for the conditional expectation $\mathbb{E}(\cdot \,|\, X_0 = x)$ we will use the expression $\mathbb{E}_x$.
Fundamental quantities in this framework are the time of ruin

$$\tau_x = \inf\{t \geq 0 \,|\, X_t < 0 \text{ and } X_0 = x\},$$

and the probability of ruin

$$\psi(x) = P\left(\tau_x < \infty\right),$$

for initial capital $x \geq 0$. In some of the proofs below we will compare pathwise, i.e., we fix an $\omega \in \Omega$, processes starting at different initial values x and y. Therefore it will be necessary to add the initial value in the definition of the time of ruin, for example $\mathbb{E}_x(X_{\tau_y})$ denotes the expected value of the surplus started at x stopped at the time of ruin as if the surplus would have started in y $(x > y)$ (thinking along the same path). Certainly, we have, using $\theta = \inf\{t \geq 0 \,|\, X_t < x - y\}$, $\mathbb{E}_x(X_{\tau_y}) = \mathbb{E}_x(X_\theta)$, but we believe that out of the context our notation will be more intuitive.

It is well known, that for avoiding almost sure ruin, it is necessary to choose a premium intensity fulfilling the *net-profit condition* $c > \lambda\mu$. Therefore, based on the expected value premium principle we set $c = (1+\eta)\lambda\mu$ with a safety loading $\eta > 0$. For further details on classical problems in risk theory and related topics we refer to Asmussen and Albrecher [20].

Assume now that in order to reduce the risk exposure of the portfolio, the insurer (cedent) has the possibility to take reinsurance in a dynamic way. Namely, at each time t, the insurer transfers a portion of the premium income to a reinsurer, who in turn commits to cover a part of the occurred claims. The dynamic reinsurance setup we are going to use follows the presentation from Schmidli [110].

Formally, a *reinsurance scheme* is given by a monotone increasing function $r : [0,\infty) \to [0,\infty)$ which fulfills $0 \le r(y) \le y$. Then r is the retention function with the meaning that for a claim of size Y, the amount $r(Y)$ is paid by the insurer and $Y - r(Y)$ is taken by the reinsurer. For introducing a control possibility a family of available schemata $\mathfrak{R}$ is parameterized by a control parameter u from a compact set $\mathcal{U}'$. This means that for $u \in \mathcal{U}'$ the chosen reinsurance contract is given by $r(\cdot,u) \in \mathfrak{R}$, where $r : [0,\infty) \times \mathcal{U}' \to \mathbb{R}^+$ with $0 \le r(y,u) \le y$. In addition we assume that $r(y,u)$ is continuous in both arguments. After fixing the family $\mathfrak{R}$, the set of available reinsurance schemes is given by

$$\mathcal{R} = \{r(\cdot,u) \in \mathfrak{R} \,|\, u \in \mathcal{U}',\, 0 \le r(y,u) \le y,\, r \text{ continuous, and increasing in } y\}.$$

For later use we denote by $\rho(y,u)$ the generalized inverse of $r(y,u)$ in the $y-$variable, which due to monotonicity exists. Naturally, when employing reinsurance there are premiums to be paid. We assume that the reinsurer uses a deterministic *premium function* $\pi : L^1(\Omega, P) \to [0,\infty)$, such that when fixing $u \in \mathcal{U}'$ the premium is based on $\pi(Y - r(Y,u))$. From an aggregated risk perspective, if the insurer chooses reinsurance $u \in \mathcal{U}'$ at time t, the premium at rate $\lambda\,\pi(Y - r(Y,u))$ is paid to the reinsurer. Consequently the premium income of the insurer reduces to $c(u) = c - \lambda\,\pi(Y - r(Y,u))$. In the sequel, we shall always assume that $c(u)$ is continuous and that full reinsurance leads to a negative premium income, i.e., $c < \lambda\pi(Y)$.

The premium function π may be based on the expected value principle,

$$\pi\,(Y - r(Y,u)) = (1+\theta)\,\mathbb{E}\,[Y - r(Y,u)]\,,$$

where $\theta > \eta$ denotes the safety loading of the reinsurer, or on the variance principle,

$$\pi\left(Y - r(Y,u)\right) = \mathbb{E}\left[Y - r(Y,u)\right] + \alpha\, Var\left[Y - r(Y,u)\right],$$

for $\alpha Var\left[Y\right] > \eta\mu$.
Possible concrete choices for $\mathfrak{R}$ and $\mathcal{U}'$ are the classical situations of proportional reinsurance and excess-of-loss reinsurance. In the first case we have $r(y,u) = uy$ and $u \in \mathcal{U}' = [0,1]$, in the second case $r(y,u) = \min(y,u)$ and $u \in \mathcal{U}' = [0,\infty]$. Notice, that in the latter case, an infinite retention level is equivalent to no reinsurance. In the following we will restrict the set of control parameters to the set $\mathcal{U} = \{u \in \mathcal{U}' \mid c(u) \geq 0\}$ for avoiding a negative premium rate. Since $\mathcal{U}'$ is supposed to be compact and $c(\cdot)$ is continuous we have that $\mathcal{U}$ is compact.

Remark 2.2.1. *The idea of a dynamic reinsurance strategy can be explained as follows. At each time instant t, the insurer chooses a control parameter $u = u_t \in \mathcal{U}$ which specifies a reinsurance scheme $r(\cdot,u)$ from an available set of schemes. The choice of u simultaneously determines the extent to which the insurer wants to reduce its risk exposure and the additional cost this protection incurs, taking the form of a reinsurance premium. Namely, if a claim occurs at time t, the insurer pays $r(Y,u_t)$ and the reinsurer pays the rest, i.e. $Y - r(Y,u_t)$. In exchange of this risk transfer, the insurer pays to the reinsurer a reinsurance premium at a rate $\lambda\pi\left(Y - r(Y,u_t)\right)$.*

Let $\mathtt{u} = (u_t)_{t\geq 0}$ be a $\mathcal{U}$-valued stochastic process which is $\{\mathcal{F}_t\}_{t\geq 0}$ previsible and called a reinsurance strategy. Then the dynamics of the controlled surplus process $X^{\mathtt{u}} = (X^{\mathtt{u}}_t)_{t\geq 0}$ are described by

$$X^{\mathtt{u}}_t = x + \int_0^t \left[c - \lambda\pi(Y - r(Y,u_s))\right] ds - \sum_{i=1}^{N_t} r(Y_i, u_{T_i}). \tag{2.4}$$

Remark 2.2.2. *From Rogers and Williams [104, p.182] we can deduce that the previsibility of $\mathtt{u}$ induces the fact that it is progressively measurable and thus also measurable as a function in time. Since the premium rate $c(\cdot)$ is assumed to be continuous and bounded by c, the integral $\int_0^t c(u_s)\, ds$ exists at least in the Lebesgue sense. Because jumps of the process $X^{\mathtt{u}}$ occur according to the fundamental Poisson process and behaves continuously between jump times, the process $X^{\mathtt{u}}$ is right continuous with existing limits from the left, i.e., cádlág. Consequently, $X^{\mathtt{u}}$ is progressively measurable as well and for fixed ω, $X^{\mathtt{u}}(\omega)$ is measurable in t. Again, integrals of the form $\int_0^t X_s\, ds$ certainly do exist in the Lebesgue sense.*

The time of ruin $\tau_x^{\mathtt{u}}$ denotes the time the controlled surplus process $X^{\mathtt{u}}$ first becomes negative,

$$\tau_x^{\mathtt{u}} = \inf\{t \geq 0 : X_t^{\mathtt{u}} < 0 \mid X_0^{\mathtt{u}} = x\}.$$

From now one we call a stochastic process $\mathtt{u} = \{u_t\}_{t\geq 0}$ admissible reinsurance strategy if it fulfills all the previously made assumptions. In this context the previsibility is crucial. That is, at claim time T_i, the reinsurance parameter is chosen based on the information up to time T_i-. The previsibility of the reinsurance strategy is a natural assumption in this setting, otherwise the insurer could change the reinsurance parameter to full reinsurance at the claim occurrence time. The reinsurer would then pay all claims while all premiums would be collected by the insurer. Let $\mathfrak{U}$ denote the set of admissible reinsurance strategies. Associated to an admissible reinsurance strategy $\mathtt{u}$ and an initial reserve $x \geq 0$, we define its *performance criterion* as the expected cumulative discounted surplus process until ruin,

$$V^{\mathtt{u}}(x) = \mathbb{E}_x\left[\int_0^{\tau_x^{\mathtt{u}}} e^{-\delta s} X_s^{\mathtt{u}}\, ds\right],$$

with $\delta > 0$ a discount or preference rate. In the sequel, we will refer to $V^{\mathtt{u}}(x)$ as the *return function*. The optimization problem then consists of finding the optimal return function, or *value function*, defined as

$$V(x) = \sup_{\mathtt{u}\in\mathfrak{U}} V^{\mathtt{u}}(x), \tag{2.5}$$

and an optimal admissible reinsurance strategy $\mathtt{u}^\star$ leading to the value function, i.e. a strategy which delivers the maximal return function (2.5).

2.3 Main results

In this section, we first derive some elementary bounds, which allow for a rough characterization of the value function. In a next step, we are able to prove the existence of a solution to an integro-differential equation which is closely related to the problem's Hamilton-Jacobi-Bellman equation. Finally, a verification argument provides the bridge between these analytical results and the stochastic optimization problem of interest.

2.3.1 Some elementary bounds

Proposition 2.3.1. *For $x \geq 0$, the value function $V(x)$ admits the following bounds:*

1. $V(x) \leq \frac{x}{\delta} + \frac{c}{\delta^2}$,
2. $V(x) \geq \frac{x}{\delta} - \frac{\lambda\pi(Y)-c}{\delta^2}\left[1 - e^{\frac{-\delta x}{\lambda\pi(Y)-c}}\right]$.

Proof. Let $\mathtt{u} = \{u_t\}_{t\geq 0}$ be an arbitrary admissible strategy. Since $c(u_s) \leq c$ for all $s \geq 0$, we get from (2.4) that

$$X_t^{\mathtt{u}} \leq x + ct,$$

holds for all $t \geq 0$. Since $\delta > 0$, this implies that,

$$V^{\mathtt{u}}(x) \leq \int_0^{\infty} e^{-\delta t}(x+ct)dt = \frac{x}{\delta} + \frac{c}{\delta^2}.$$

Taking the supremum over all admissible strategies $\mathtt{u}$ shows that the value function $V(x)$ satisfies inequality (a).
It remains now to validate inequality (b). The choice of the admissible strategy $\mathtt{u}_0$ which corresponds to buying continuously full reinsurance until the time of ruin leads to a deterministic reserve $X^{\mathtt{u}_0}$

$$X_t^{\mathtt{u}_0} = x + (c - \lambda\pi(Y))\, t,$$

with negative drift. As a consequence, the time of ruin $\tau_x^{\mathtt{u}^0}$ can be explicitly computed, that is, $\tau_x^{\mathtt{u}_0} = \frac{x}{\lambda\pi(Y)-c}$. The underlying return function $V^{\mathtt{u}_0}(x)$ is given by

$$V^{\mathtt{u}_0}(x) = \frac{x}{\delta} - \frac{\lambda\pi(Y) - c}{\delta^2}\left[1 - e^{\frac{-\delta x}{\lambda\pi(Y)-c}}\right].$$

□

The following result presents bounds on increments of the value function and also provides its continuity.

Proposition 2.3.2. *For $x > y \geq 0$, the value function satisfies:*

1. $V(x) - V(y) \leq \frac{x-y}{\delta} + C(x,y)\, V(x-y)$, *where* $C(x,y) \to 0$ *as* $|\, x-y \,| \to 0$,
2. $V(x) - V(y) \geq \frac{x-y}{\delta+\lambda}$.

Proof. For given $x > 0$ and given $\epsilon > 0$, consider an admissible ϵ-optimal strategy $\mathtt{u}$ such that

$$V(x) \leq \mathbb{E}_x \left[\int_0^{\tau_x^{\mathtt{u}}} e^{-\delta t} X_t^{\mathtt{u}} dt \right] + \epsilon.$$

Since $\mathtt{u}$ is also admissible for initial capital y with $x > y \geq 0$ (up to time $\tau_y^{\mathtt{u}}$), we have

$$V(x) - V(y) \leq \mathbb{E}_x \left[\int_0^{\tau_x^{\mathtt{u}}} e^{-\delta t} X_t^{\mathtt{u}} dt \right] - \mathbb{E}_y \left[\int_0^{\tau_y^{\mathtt{u}}} e^{-\delta t} X_t^{\mathtt{u}} dt \right] + \epsilon, \tag{2.6}$$

where $\mathbb{E}_x$, $\mathbb{E}_y$ indicate the starting value of the corresponding process. Now we are going to use a pathwise argument, let $\mathscr{E} = \{\omega \in \Omega \,|\, \tau_x^{\mathtt{u}}(\omega) = \tau_y^{\mathtt{u}}(\omega)\}$. Notice that on $\mathscr{E}$ the paths (for fixed ω) of the reserves started in x and y move parallel with a distance $x - y > 0$ and get ruined at the same point in time. Therefore, we can rewrite the above inequality in the following way,

$$\begin{aligned} V(x) - V(y) &\leq \mathbb{E}_x \left[\int_0^{\tau_y^{\mathtt{u}}} e^{-\delta t} X_t^{\mathtt{u}} dt \right] - \mathbb{E}_y \left[\int_0^{\tau_y^{\mathtt{u}}} e^{-\delta t} X_t^{\mathtt{u}} dt \right] + \mathbb{E}_x \left[\mathbb{1}_{\mathscr{E}^c} \int_{\tau_y^{\mathtt{u}}}^{\tau_x^{\mathtt{u}}} e^{-\delta t} X_t^{\mathtt{u}} dt \right] + \epsilon \\ &\leq \frac{(x-y)}{\delta} + \mathbb{E}_x \left[\mathbb{1}_{\mathscr{E}^c} \int_{\tau_y^{\mathtt{u}}}^{\tau_x^{\mathtt{u}}} e^{-\delta t} X_t^{\mathtt{u}} dt \right] + \epsilon \\ &\leq \frac{(x-y)}{\delta} + \mathbb{E}\left[\mathbb{1}_{\mathscr{E}^c}\right] V(x-y) + \epsilon. \end{aligned} \tag{2.7}$$

The first inequality is just a restatement of (2.6). It incorporates the fact that the two values, the values of the strategy $\mathtt{u}$ for surplus processes started in x and y, only differ on $\mathscr{E}^c$. This difference is given by the third expectation, in which $\mathbb{E}_x$ indicates that the surplus within the integral is started at x. The second inequality follows from the observation that

$$\mathbb{E}_x \left[\int_0^{\tau_y^{\mathtt{u}}} e^{-\delta t} X_t^{\mathtt{u}} dt \right] - \mathbb{E}_y \left[\int_0^{\tau_y^{\mathtt{u}}} e^{-\delta t} X_t^{\mathtt{u}} dt \right] \leq \mathbb{E} \left[\int_0^{\tau_y^{\mathtt{u}}} e^{-\delta t} (x-y) dt \right] \leq \int_0^{\infty} e^{-\delta t} (x-y) dt.$$

The last inequality uses $X_{\tau_y^{\mathtt{u}}}^{\mathtt{u}} \leq x - y$ for the reserve started in x and that consequently the corresponding expectation is smaller than $V(x-y)$. Define $\theta = \inf\{t \geq 0 \,|\, X_t^{\mathtt{u}} < x - y\}$ and $C(x,y) := \mathbb{E}\left[\mathbb{1}_{\mathscr{E}^c}\right] = P(\tau_y^{\mathtt{u}} < \tau_x^{\mathtt{u}}) = P_x(\theta < \tau_x^{\mathtt{u}})$. Observing that $C(x,y) \to 0$ if $|\, x - y \,| \to 0$ yields (a).

Let us now prove inequality (b). Let $y \geq 0$ and $\epsilon > 0$ be given, consider an admissible strategy $\bar{\mathtt{u}}$ such that $V^{\bar{\mathtt{u}}}(y) + \epsilon \geq V(y)$. For $x > y$, we have,

$$V(x) - V(y) \geq \mathbb{E}_x \left[\int_0^{\tau_x^{\bar{\mathtt{u}}}} e^{-\delta t} X_t^{\bar{\mathtt{u}}} dt \right] - \mathbb{E}_y \left[\int_0^{\tau_y^{\bar{\mathtt{u}}}} e^{-\delta t} X_t^{\bar{\mathtt{u}}} dt \right] - \epsilon.$$

Again, let $\mathscr{E} = \{\tau_x^{\bar{\mathtt{u}}} = \tau_y^{\bar{\mathtt{u}}}\}$ and let T_1 be the time of the first claim occurrence. We can write

$$\begin{aligned} V(x) - V(y) &\geq \mathbb{E}_x \left[\int_0^{\tau_y^{\bar{\mathtt{u}}}} e^{-\delta t} X_t^{\bar{\mathtt{u}}} dt \right] - \mathbb{E}_y \left[\int_0^{\tau_y^{\bar{\mathtt{u}}}} e^{-\delta t} X_t^{\bar{\mathtt{u}}} dt \right] + \mathbb{E}_x \left[\mathbb{1}_{\mathscr{E}^c} \int_{\tau_y^{\bar{\mathtt{u}}}}^{\tau_x^{\bar{\mathtt{u}}}} e^{-\delta t} X_t^{\bar{\mathtt{u}}} dt \right] - \epsilon \\ &\geq \mathbb{E} \left[\int_0^{T_1} e^{-\delta t} (x - y) dt \right] - \epsilon = \frac{x - y}{\delta + \lambda} - \epsilon \end{aligned}$$

From the arbitrariness of $\epsilon > 0$, we get the result. □

Additionally, we can derive the following.

Lemma 2.3.1. *The value function V is locally Lipschitz continuous.*

Proof. For given $x > 0$ and $\epsilon > 0$, consider an admissible strategy $\mathtt{u} = (u_t^x)_{t \geq 0}$ such that

$$V(x) \leq \mathbb{E}_x \left[\int_0^{\tau_x^{\mathtt{u}}} e^{-\delta t} X_t^{\mathtt{u}} dt \right] + \epsilon.$$

Let $u \in \mathcal{U}$ such that the net drift of the surplus is positive, i.e., $c(u) > \lambda \mathbb{E}(r(Y, u)) > 0$. Furthermore, we set $\theta_x = \inf\{t \geq 0 \,|\, X_t^u \geq x \text{ with } X_0^u = y\}$. Now we can define an admissible strategy $\mathtt{u}^y = (u_t^y)_t \geq 0$ for initial capital y, with $0 \leq y \leq x$, by $u_t^y = u$ for $0 \leq t < \theta_x$ and $u_t^y = u_{t-\theta_x}^x$ for $t \geq \theta_x$. Notice, if the first claim occurs at $T_1 > \frac{x-y}{c(u)}$, then level x is directly reached from level y. We have,

$$V(y) \geq \mathbb{E}_y \left(\int_0^{\tau_y^{\mathtt{u}^y}} e^{-\delta t} X_t^{\mathtt{u}^y} \, dt \right) \geq P\left(T_1 > \tfrac{x-y}{c(u)} \right) \left(\int_0^{\frac{x-y}{c(u)}} e^{-\delta t} [y + c(u)t] \, dt + e^{-\delta \frac{x-y}{c(u)}} (V(x) - \epsilon) \right). \tag{2.8}$$

Finally, after explicitly evaluating the last estimate we derive for $x > y \geq 0$,

$$0 \le V(x) - V(y) \le V(x)\left(1 - e^{-(\delta+\lambda)\frac{x-y}{c(u)}}\right) - e^{-\lambda\frac{x-y}{c(u)}}\left(\frac{c+\delta y - (c+\delta x)e^{-\delta\frac{x-y}{c(u)}}}{\delta^2}\right) + \epsilon$$
$$= V(x)\left(\frac{\delta+\lambda}{c(u)}(x-y) + \mathcal{O}(x-y)^2\right) + \frac{x}{c(u)}(x-y) + \mathcal{O}(x-y)^2 + \epsilon.$$

This implies that V is locally Lipschitz continuous. □

Finally, we can summarize the following elementary properties of the value function $V(x)$. Notice that absolute continuity follows from the local Lipschitz continuity mentioned in the previous Lemma.

Corollary 2.3.1. *The value function V is strictly positive, linearly bounded, monotone increasing and absolutely continuous.*

Remark 2.3.1. *Suppose we assume in the proof of part (a) of Proposition 2.3.2, that for all $u \in \mathcal{U}$ the random variable $r(Y,u)$ admits a bounded density f_r^u. Then, we can formally derive*

$$\begin{aligned}
P(\tau_y^{\mathbf{u}} < \tau_x^{\mathbf{u}}) &= \int_0^\infty P(\tau_y^{\mathbf{u}} < \tau_x^{\mathbf{u}} \,|\, X_{\tau_y^{\mathbf{u}}-}^{\mathbf{u}} = w) P(X_{\tau_y^{\mathbf{u}}-}^{\mathbf{u}} = w)dw \\
&= \int_0^\infty P(w < r(Y, u_{\tau_y^{\mathbf{u}}}) \le w + x - y) P(X_{\tau_y^{\mathbf{u}}-}^{\mathbf{u}} = w)dw \\
&= \int_0^\infty \int_w^{w+x-y} f_r^{u_{\tau_y^{\mathbf{u}}}}(z)dz\, P(X_{\tau_y^{\mathbf{u}}}^{\mathbf{u}} = w)dw \le (x-y)\overline{f_r},
\end{aligned}$$

where $\overline{f_r}$ denotes a bound of f_r. Since $P(\tau_y^{\mathbf{u}} < \tau_x^{\mathbf{u}}) = \mathbb{E}\left[\mathbb{1}_{\mathscr{E}^c}\right]$, we get from (2.7) that the value function is globally Lipschitz continuous. For example, this case appears when dealing with proportional reinsurance.

For further investigations, we need to improve on the lower bound from Proposition 2.3.1. When dealing with a contraction operator later on, the refined bound will allow us to describe the growth behaviour of the value function in a more precise way.

We start with showing that for

$$g(x) = \begin{cases} \frac{x}{\delta}, & x \ge 0, \\ 0, & x < 0, \end{cases}$$

$\mathcal{L}g(x) - \delta g(x) + x > 0$ holds for all $x \geq 0$, where

$$\mathcal{L}g(x) := cg'(x) + \lambda \left(\int_0^x g(x-y) dF_Y(y) - g(x) \right)$$

is the infinitesimal generator of the uncontrolled process X. For that purpose, we define

$$H(x) : = \mathcal{L}g(x) - \delta g(x) + x = x + \frac{c}{\delta} - (\delta + \lambda)\frac{x}{\delta} + \lambda \int_0^x \frac{x-y}{\delta} \, dF_Y(y),$$

which can be rewritten as

$$H(x) = \frac{c}{\delta} + \frac{\lambda}{\delta} \left(x(F_Y(x) - 1) \right) - \frac{\lambda}{\delta} \int_0^x y \, dF_Y(y).$$

From $H'(x) = \frac{\lambda}{\delta}(F_Y(x) - 1) \leq 0$ for all $x \geq 0$, we have that $H(x)$ is monotone decreasing. Determination of the boundary values, $H(0) = \frac{c}{\delta} > 0$ and $\lim_{x \to \infty} H(x) = \frac{c - \lambda\mu}{\delta} > 0$, implies that it is strictly positive as well.

Lemma 2.3.2. *The value function V is bounded from below by $\frac{x}{\delta} + \frac{c-\lambda\mu}{\delta(\delta+\lambda)}$, i.e.,*

$$V(x) \geq \frac{x}{\delta} + \frac{c - \lambda\mu}{\delta(\delta + \lambda)}. \tag{2.9}$$

Proof. Since $g(x)$ is differentiable we can apply Dynkin's formula and get

$$\mathbb{E}_x \left(e^{-\delta t \wedge \tau} g(X_{t\wedge\tau}) \right) = g(x) + \mathbb{E}_x \left(\int_0^{t\wedge\tau} e^{-\delta s} \left[\mathcal{L}g(X_s) - \delta g(X_s) \right] ds \right).$$

From above, we already know that $\mathcal{L}g(X_s) - \delta g(X_s) \geq -X_s + \frac{c-\lambda\mu}{\delta}$, using this estimate, we arrive at,

$$\begin{aligned} \mathbb{E}_x \left[e^{-\delta t\wedge\tau} g(X_{t\wedge\tau}) \right] + \mathbb{E}_x \left[\int_0^{t\wedge\tau} e^{-\delta s} X_s \, ds \right] &\geq g(x) + \mathbb{E}_x \left[\int_0^{t\wedge\tau} e^{-\delta s} \frac{c - \lambda\mu}{\delta} ds \right] \\ &\geq g(x) + \mathbb{E}_x \left[\int_0^{t\wedge T_1} e^{-\delta s} \frac{c - \lambda\mu}{\delta} ds \right], \end{aligned}$$

where T_1 denotes the time of the first claim occurrence. Using linear boundedness of $g(X_{t\wedge\tau})$ in t and monotone convergence, we arrive at

$$\mathbb{E}_x \left[\int_0^\tau e^{-\delta s} X_s \, ds \right] \geq g(x) + \frac{c - \lambda\mu}{\delta(\delta + \lambda)}.$$

From its definition, we get

$$V(x) \geq \mathbb{E}_x \left[\int_0^{\tau} e^{-\delta s} X_s \, ds \right] \geq \frac{x}{\delta} + \frac{c - \lambda\mu}{\delta(\delta+\lambda)}.$$

□

Remark 2.3.2. *By well known methods, as outlined in [20, Ch.I.4, Ch.IX.3], $f(x) := \mathbb{E}_x \left[\int_0^{\tau} e^{-\delta s} X_s \, ds \right]$ can be computed explicitly for Erlang distributed claims.*

2.3.2 Characterization of the value function

Based on the elementary properties of the value function which are collected in Corollary 2.3.1, we can work out the dynamic programming approach for solving the optimization problem.
We start with observing that V fulfills the dynamic programming principle, that is, for every $\mathcal{F}_t$-adapted stopping time $S \geq 0$ the following relation is valid:

$$V(x) = \sup_{u \in \mathfrak{U}} \mathbb{E}_x \left[\int_0^{\tau_x^u \wedge S} e^{-\delta t} X_t^u dt + e^{-\delta(\tau_x^u \wedge S)} V\left(X^u_{\tau_x^u \wedge S}\right) \right]. \tag{2.10}$$

The proof of this fact is mainly based on the continuity of V and follows standard arguments from the corresponding literature, see for instance the proof of [25, Prop.2.3].
The following Lemma shows that at least in some weak sense V fulfills the associated Hamilton-Jacobi-Bellman equation.

Lemma 2.3.3. *The value function V defined in (2.5) is a.e. a solution to:*

$$0 = \sup_{u \in \mathcal{U}} \{ x + c(u)V'(x) - (\delta+\lambda)V(x) + \lambda \int_0^{\rho(x,u)} V(x - r(y,u)) \, dF_Y(y) \}. \tag{2.11}$$

Proof. In a first step we show that (2.11) is smaller equal to zero. Fix $x > 0$, $h > 0$ and let $u \in \mathcal{U}$. Define $\tilde{\mathfrak{u}} = (u_t)_{t \geq 0}$ such that $u_t = u$ for $t \in [0,h]$ and $u_t = \tilde{u}_{t-h}$ for $t > 0$ for some $\tilde{u} \in \mathfrak{U}$. If necessary, we choose h small enough such that $x + c(u)h > 0$. Let T_1 denote the time of the first claim occurrence and set $S = \min\{T_1, h\}$. Then, (2.10) yields

$$0 \geq \mathbb{E}_x \left[\int_0^S e^{-\delta t} \left(x + c(u)t \right) \, dt + e^{-\delta S} V\left(X_S^{\tilde{\mathfrak{u}}}\right) - V(x) \right]. \tag{2.12}$$

Since u is a constant control which applies on the time horizon $[0, S]$ we can apply [105, Th.11.2.2] and get that $V \in \mathcal{D}(\mathcal{A}^u)$, i.e., V lies in the domain of the generator. In the present situation the generator $\mathcal{A}^u$ of the *constantly* controlled process X^u is given by

$$\mathcal{A}^u g(x) = c(u)g'(x) - \lambda g(x) + \lambda \int_0^{\rho(x,u)} g(x - r(x,y))dF_Y(y).$$

The particular result from [105, Th.11.2.2] applies, because the map $t \mapsto V(x+c(u)t)$ is absolutely continuous, the so-called *active boundary* is empty and the bounds from Proposition 2.3.1 and Proposition 2.3.2 guarantee the asked for integrability condition. Therefore we can apply Dynkin's formula, identifying V' with the measurable density of V, and can rewrite (2.12) to

$$0 \geq \mathbb{E}_x \Bigg[\int_0^S e^{-\delta t} \left(x + c(u)t\right) \, dt + \int_0^S e^{-\delta t} \Bigg(c(u)V'(x + c(u)t) - (\delta + \lambda)V(x + c(u)t) \\ + \lambda \int_0^{\rho(x+c(u)t,u)} V(x + c(u)t - r(y,u))dF_Y(y) \Bigg) dt \Bigg].$$

After regrouping and division by h we have

$$0 \geq \frac{1}{h} \mathbb{E}_x \Bigg[\int_0^S e^{-\delta t} \Bigg(x + c(u)t - (\delta + \lambda)V(x + c(u)t) \\ + \lambda \int_0^{\rho(x+c(u)t,u)} V(x + c(u)t - r(y,u))dF_Y(y) \Bigg) dt \Bigg] + \frac{1}{h} \mathbb{E}_x \left[\int_0^S e^{-\delta t} c(u)V'(x + c(u)t)dt \right].$$

The integral in the first expectation can be interpreted in the Riemann sense, V is continuous, such that sending $h \to 0$ leads to

$$0 \geq x - (\delta + \lambda)V(x) + \lambda \int_0^{\rho(x,u)} V(x - r(y,u))dF_Y(y) \\ + \lim_{h \searrow 0} \frac{1}{h} \mathbb{E}_x \left[\int_0^S e^{-\delta t} c(u)V'(x + c(u)t) \, dt \right].$$

The second limitation procedure needs a bit more care since the integrands as functions in t are only measurable and the respective integral is interpreted in the Lebesgue sense. For this purpose consider

$$\begin{aligned}
&\lim_{h\searrow 0}\frac{1}{h}\,\mathbb{E}_x\left[\int_0^S e^{-\delta t}c(u)V'(x+c(u)t)\,dt\right]\\
&=\lim_{h\searrow 0}e^{-\lambda h}\frac{1}{h}\int_0^h e^{-\delta t}c(u)V'(x+c(u)t)\,dt+\lim_{h\searrow 0}\frac{1}{h}\int_0^h \lambda e^{-\lambda s}\int_0^s e^{-\delta t}c(u)V'(x+c(u)t)\,dt\,ds\\
&=c(u)V'(x)\quad\text{a.e.},
\end{aligned}$$

where in the second equality we used *Lebesgue's Differentiation Theorem* from [122, Th.7.16] which applies since the measurable density V' certainly is locally integrable in the Lebesgue sense because of the bounds on the function V and its increments. One may notice that

$$\lim_{h\searrow 0}\frac{1}{h}\int_0^h \lambda e^{-\lambda s}\int_0^s e^{-\delta t}c(u)V'(x+c(u)t)\,dt\,ds=0\quad\text{a.e.},$$

since the ds integrand equals zero for $s=0$. The choice of the control parameter $u\in\mathfrak{U}$ was arbitrary, such that we have

$$0\geq\ \sup_{u\in\mathfrak{U}}\{x+c(u)V'(x)-(\delta+\lambda)V(x)+\lambda\int_0^{\rho(x,u)}V(x-r(y,u))\,dF_Y(y)\}\quad\text{a.e.}$$

We can turn to the second step, showing that (2.11) is also larger or equal to zero. Set again $S=\min\{T_1,h\}$ for some $h>0$ and let the strategy $\mathtt{u}^1=(u_t^1)_{t\geq 0}$ be h^2-optimal for the right hand side of (2.10), that is

$$\begin{aligned}
V(x)&=\sup_{u\in\mathfrak{U}}\mathbb{E}_x\left[\int_0^S e^{-\delta t}(x+\int_0^t c(u_s)ds)dt+e^{-\delta S}V(X_S^u)\right]\\
&<\ \mathbb{E}_x\left[\int_0^S e^{-\delta t}(x+\int_0^t c(u_s^1)ds)dt+e^{-\delta S}V(X_S^{\mathtt{u}^1})\right]+h^2+\epsilon\,h,
\end{aligned}$$

where we added the term $\epsilon\,h$ with some arbitrary $\varepsilon>0$ for achieving strict positivity. In the above equation we can use $T_1\sim\text{Exp}(\lambda)$ and regroup a little bit to arrive at

$$\begin{aligned}
0<&\mathbb{E}_x\left[\int_0^S e^{-\delta t}(x+\int_0^t c(u_s^1)ds)dt\right]+(e^{-(\delta+\lambda)h}-1)\mathbb{E}_x\left[V(x+\int_0^h c(u_s^1)ds)\right]\\
&+\mathbb{E}_x\left[\int_0^h\lambda e^{-\lambda t}\int_0^{\rho(x+\int_0^t c(u_s^1)ds,u_t^1)}V(x+\int_0^t c(u_s^1)ds-r(y,u_t^1)dF_Y(y)\,dt\right]\\
&+\mathbb{E}_x\left[V(x+\int_0^h c(u_s^1)ds)-V(x)\right]+h^2+\epsilon\,h\\
&=:A+B+C+D+h^2+\epsilon\,h.
\end{aligned}$$

We kept $\mathbb{E}_x$ since $\mathtt{u}^1$ is still stochastic on the time interval under consideration. In the following we divide A, B, C, D by h and study the limits as h tends to zero - for interchanging limitation and expectation we will repeatedly make use of the *dominated convergence Theorem.* We start with discussing B:

$$\lim_{h\searrow 0} \frac{e^{-(\delta+\lambda)h}-1}{h}\,\mathbb{E}_x\left[V(x+\int_0^h c(u_s^1)ds)\right] = -(\delta+\lambda)V(x),$$

which follows from continuity of V. Next we deal with C:

$$\lim_{h\searrow 0}\frac{1}{h}\,\mathbb{E}_x\left[\int_0^h \lambda e^{-\lambda t}\int_0^{\rho(x+\int_0^t c(u_s^1)ds,u_t^1)} V(x+\int_0^t c(u_s^1)ds - r(y,u_t^1)dF_Y(y)\,dt\right]$$
$$= \lambda\int_0^{\rho(x,u_0^1)} V(x-r(y,u_0^1)dF_Y(y) \quad \text{a.e.},$$

which is derived by an application of [122, Th.7.16]. For part D we exploit a similar procedure together with the absolute continuity of V,

$$\lim_{h\searrow 0}\frac{1}{h}\,\mathbb{E}_x\left[V(x+\int_0^h c(u_s^1)ds)-V(x)\right] = \lim_{h\searrow 0}\frac{1}{h}\,\mathbb{E}_x\left[\int_0^{\int_0^h c(u_s^1)ds} V'(x+y)dy\right]$$
$$= \lim_{h\searrow 0}\frac{1}{h}\,\mathbb{E}_x\left[\int_0^h c(u_t^1)V'(x+\int_0^t c(u_s^1)ds)dt\right] = c(u_0^1)V'(x) \quad \text{a.e.}$$

Part A is resolved in the same way and delivers

$$\lim_{h\searrow 0}\frac{1}{h}\,\mathbb{E}_x\left[\int_0^S e^{-\delta t}(x+\int_0^t c(u_s^1)ds)dt\right] = x.$$

Finally we arrive at

$$0 \le\; x + c(u_0^1)V'(x) - (\delta+\lambda)V(x) + \lambda\int_0^{\rho(x,u_0^1)} V(x-r(y,u_0^1)dF_Y(y)+\varepsilon \quad \text{a.e.},$$

which concludes the proof since ε was arbitrary. □

At this point, we know that the value function is in some sense a solution to the associated HJB-equation. What remains to be done for a complete analytical characterization is a complement on uniqueness. For accomplishing such a result we are going to rewrite (2.11) in a way similar as [110, p. 47] did, when transforming equation (2.14) into (2.15).

Suppose x is meaningful in the sense that $V'(x)$ exists. Since the set $\mathcal{U}$ is compact and all corresponding terms are continuous in u, a maximizer $u(x)$ exists such that the supremum equal to zero is attained. Replacing the $\sup_u$ by $u(x)$ in (2.11) we have

$$0 = x + c(u(x))V'(x) - (\delta+\lambda)V(x) + \lambda \int_0^{\rho(x,u(x))} V(x - r(y,u(x)))\, dF_Y(y), \tag{2.13}$$

from which we can observe, using the lower bound (2.9) on $V(x)$, that $c(u(x))V'(x) > 0 \;\Rightarrow\; c(u(x)) > 0$. Hence, in the supremum we can replace the set $\mathcal{U}$ by the set $\tilde{\mathcal{U}} = \{u \in \mathcal{U} \,|\, c(u) > 0\}$. Since $V(x)$ is monotone, we can rewrite (2.11) into the equivalent form:

$$V'(x) = \inf_{u\in\tilde{\mathcal{U}}} \left\{ \frac{(\delta+\lambda)V(x) - x - \lambda \int_0^{\rho(x,u)} V(x - r(y,u))dF_Y(y)}{c(u)} \right\}. \tag{2.14}$$

Formally, we know that a.e. $V(x)$ is a solution to (2.14). In addition, for x such that $V'(x)$ exists, we have the following,

$$\begin{aligned} V'(x) &= \inf_{u\in\tilde{\mathcal{U}}} \left\{ \frac{(\delta+\lambda)V(x) - x - \lambda \int_0^{\rho(x,u)} V\left(x - r(y,u)\right) dF_Y(y)}{c(u)} \right\} \\ &\le \frac{(\delta+\lambda)V(x) - x - \lambda \int_0^{x} V\left(x - y\right) dF_Y(y)}{c} \\ &\le \frac{(\delta+\lambda)\left(\frac{x}{\delta} + \frac{c}{\delta^2}\right) - x - \lambda \int_0^{x} \frac{x-y}{\delta} dF_Y(y)}{c} \\ &\le \frac{\frac{(\delta+\lambda)c}{\delta^2} + \frac{\lambda\mu}{\delta} + M(x)}{c}, \end{aligned} \tag{2.15}$$

where $M(x) := \frac{\lambda}{\delta} x\,(1 - F_Y(x))$. Clearly, $M(x) \ge 0$ for all $x \ge 0$. Moreover, we have $\frac{\lambda}{\delta}\mu \ge \frac{\lambda}{\delta}\int_x^\infty x dF_Y(z) = M(x)$, which can be used in (2.15), leading to

$$V'(x) \le \frac{\frac{(\delta+\lambda)c}{\delta^2} + \frac{2\lambda\mu}{\delta}}{c}. \tag{2.16}$$

Reinspecting (2.13) gives a positive lower bound on $c(u(x))$,

$$\begin{aligned} c(u(x))V'(x) &= (\delta+\lambda)V(x) - x - \lambda \int_0^{\rho(x,u(x))} V(x - r(y,u(x)))\, dF_Y(y) \\ &= \delta V(x) - x + \lambda \left(V(x) - \int_0^{\rho(x,u(x))} V(x - r(y,u(x)))\, dF_Y(y) \right) \\ &\ge \frac{c - \lambda\mu}{\delta+\lambda}, \end{aligned}$$

where the last inequality is due to Lemma 2.3.2. Together with (2.16) we have

$$c(u(x)) \geq \frac{c-\lambda\mu}{\delta+\lambda}\left(\frac{\frac{(\delta+\lambda)c}{\delta^2}+\frac{2\lambda\mu}{\delta}}{c}\right)^{-1} =: L > 0.$$

As a consequence, we can redefine the crucial set for taking the supremum (resp. inf) $\tilde{\mathcal{U}} = \{u \in \mathcal{U} \,|\, c(u) \geq L\}$. One may notice that in (2.14) the infimum is taken again over a compact set and that the denominator is uniformly bounded away from zero.

The first step towards a unique characterization of the value function is given in the following theorem the proof of which relies on the fixed point property of a certain operator (inspired by a similar approach used in [24, 25]).

Theorem 2.3.1. *Let $f(0) > 0$ be some given initial value, then there exists a unique a.e. differentiable solution to*

$$g'(x) = \inf_{u\in\tilde{\mathcal{U}}}\left\{\frac{(\delta+\lambda)g(x) - x - \lambda\int_0^{\rho(x,u)} g(x-r(y,u)dF_Y(y)}{c(u)}\right\},$$

with $g(0) = f(0)$.

Proof. Let $x_0 \geq 0$ and a continuous function $f : [0, x_0] \to \mathbb{R}$ be given. Fix $h > 0$ and set $\mathcal{C} = \{g : [x_0, x_0 + h] \to \mathbb{R} \,|\, g \text{ is continuous and } g(x_0) = f(x_0)\}$. The operator

$$\mathcal{T}g(x) = f(x_0) + \int_{x_0}^{x} \inf_{u\in\tilde{\mathcal{U}}}\left\{\frac{(\delta+\lambda)g(s) - s - \lambda\int_0^{\rho(s-x_0,u)} g(s-r(y,u))dF_Y(y) - \lambda\int_{\rho(s-x_0,u)}^{\rho(s,u)} f(s-r(y,u))dF_Y(y)}{c(u)}\right\} ds,$$

is defined on $\mathcal{C}$ and $x \in [x_0, x_0 + h]$ and clearly $\mathcal{T}g \in \mathcal{C}$. Since for all $s \in [x_0, x_0 + h]$ all terms involving u are continuous in it and the infimum is taken over a compact set, we know that a minimizer $u(s)$ exists.
Now let g_1, $g_2 \in \mathcal{C}$ and $u^1(s)$, $u^2(s)$ be the corresponding minimizers, we get

$$\begin{aligned}
&\mathcal{T}g_1(x) - \mathcal{T}g_2(x) \\
&\leq \int_{x_0}^{x} \frac{(\delta+\lambda)[g_1(s) - g_2(s)] - \lambda\int_0^{\rho(s-x_0,u^2(s))}[g_1(s-r(y,u^2(s))) - g_2(s-r(y,u^2(s)))]dF_Y(y)}{c(u^2(s))} ds \\
&\leq h\frac{(\delta+2\lambda)}{L}\sup_{s\in[x_0,x_0+h]} |\, g_1(s) - g_2(s)\,|.
\end{aligned}$$

Interchanging the roles of g_1 and g_2 and choosing $h = \frac{L}{2(\delta+2\lambda)}$ we get,

$$|\,\mathcal{T}g_1(x) - \mathcal{T}g_2(x)\,| \leq \frac{1}{2} \sup_{s\in[x_0,x_0+h]} |\,g_1(s) - g_2(s)\,|, \quad \forall\, x \in [x_0, x_0+h],$$

such that $\mathcal{T}$ is a contraction on $\mathcal{C}$ and that consequently an unique fixed point of it exists. Since h and the contraction factor do not depend on x_0, we can iterate this procedure on the intervals $[0, h]$, $[h, 2h], \ldots$. Finally, we observe that these fixed points, on the end points of the intervals $[k\,h, (k+1)\,h]$ continuously pasted, induce an unique solution to (2.14) with given initial value $f(0)$. By construction, this solution is absolutely continuous on $\mathbb{R}^+$, since one may alter the grid for the construction procedure. □

We are now able to finalize the analytical characterization of V.

Theorem 2.3.2. *Suppose $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = 0$ for $x < 0$ is linearly bounded by $\frac{x}{\delta} + \frac{c}{\delta^2}$ and an absolutely continuous solution to (2.14), then $g(x) = V(x)$. The optimal strategy $\mathtt{u}^* = (u_t^*)_{t\geq 0}$ is induced by the pointwise minimizer $u(x)$ of (2.14) such that $u_t^* = u(X_{t-}^{\mathtt{u}^*})$.*

Remark 2.3.3. *One can use verbatim the proof from [110, Lem.2.12] to show that the function u defining the optimal strategy is measurable. Consequently the process $(u_t^*)_{t\geq 0}$ is previsible and constitutes an admissible strategy.*

Proof. Let $t > 0$ and $\mathtt{u} = (u_t)_{t\geq 0} \in \mathfrak{U}$, since the paths of $(X_t^{\mathtt{u}})_{t\geq 0}$ are of bounded variation, we can use the Stieltjes integral to obtain

$$\begin{aligned} &e^{-\delta t\wedge\tau_x^{\mathtt{u}}} g(X_{t\wedge\tau_x^{\mathtt{u}}}^{\mathtt{u}}) - g(x) = \\ &\int_0^{t\wedge\tau_x^{\mathtt{u}}} e^{-\delta s}\left[-\delta g(X_s^{\mathtt{u}}) + c(u_s) g'(X_s^{\mathtt{u}})\right] ds + \sum_{T_i \leq t\wedge\tau_x^{\mathtt{u}}} e^{-\delta T_i}\left[g(X_{T_i}^{\mathtt{u}}) - g(X_{T_i-}^{\mathtt{u}})\right]. \end{aligned} \tag{2.17}$$

The process $M = (M_t)_{t\geq 0}$ defined by

$$\begin{aligned} M_t = &\sum_{T_i\leq t} e^{-\delta T_i}\left[g(X_{T_i}^{\mathtt{u}}) - g(X_{T_i-}^{\mathtt{u}})\right] \\ &\qquad - \lambda \int_0^t e^{-\delta s}\left[\int_0^{\rho(X_s^{\mathtt{u}}, u_s)} g(X_s^{\mathtt{u}} - r(y, u_s)) dF_Y(y) - g(X_s^{\mathtt{u}})\right] ds, \end{aligned}$$

is a zero-mean martingale, due to compensation. Therefore, taking expectations in (2.17) leads to

$$\mathbb{E}_x\left[e^{-\delta t\wedge\tau_x^{\mathbf{u}}}g(X^{\mathbf{u}}_{t\wedge\tau_x^{\mathbf{u}}})\right] = g(x)+$$
$$\mathbb{E}_x\left[\int_0^{t\wedge\tau_x^{\mathbf{u}}} e^{-\delta s}\left[-(\delta+\lambda)g(X_s^{\mathbf{u}})+c(u_s)g'(X_s^{\mathbf{u}})+\lambda\int_0^{\rho(X_s^{\mathbf{u}},u_s)} g(X_s^{\mathbf{u}}-r(y,u_s))dF_Y(y)\right]ds\right].$$

Remember that for $g'(X_s^{\mathbf{u}})$ we have (at least a.e.)

$$g'(X_s^{\mathbf{u}}) = \inf_{v\in\tilde{\mathfrak{U}}}\frac{(\delta+\lambda)g(X_s^{\mathbf{u}})-X_s^{\mathbf{u}}-\lambda\int_0^{\rho(X_s^{\mathbf{u}},v)} g(X_s^{\mathbf{u}}-r(y,v))dF_Y(y)}{c(v)},$$

which yields for the particular control parameter u_s,

$$\mathbb{E}_x\left[e^{-\delta t\wedge\tau_x^{\mathbf{u}}}g(X^{\mathbf{u}}_{t\wedge\tau_x^{\mathbf{u}}})\right] \le g(x)-\mathbb{E}_x\left[\int_0^{t\wedge\tau_x^{\mathbf{u}}} e^{-\delta s}X_s^{\mathbf{u}}ds\right]. \tag{2.18}$$

From [110, Lem.2.9], we know that either ruin occurs or the controlled surplus tends (linearly bounded) to infinity. Therefore, using bounded convergence in (2.18) results in

$$\mathbb{E}_x\left[\int_0^{t\wedge\tau_x^{\mathbf{u}}} e^{-\delta s}X_s^{\mathbf{u}}ds\right] \le g(x),$$

hence, $V(x)\le g(x)$. One observes that in (2.18) we have equality for the strategy $\mathbf{u}^*$, defined in the statement of the theorem, such that finally $V(x)=g(x)$. □

The combination of the statement of the last theorem with the uniqueness result and the properties of the value function enables us to state a complete characterization.

Corollary 2.3.2. *The value function V is the unique solution to (2.11) in the set of absolutely continuous function $g:\mathbb{R}\to\mathbb{R}$ with $g(x)=0$ for $x<0$ which are bounded by $\frac{x}{\delta}+\frac{c}{\delta^2}$. In particular just the initial value $V(0)$ for equation (2.14) allows for a solution $g(x)$ with the property $\lim_{x\to\infty}\frac{g(x)}{x}=\frac{1}{\delta}$.*

2.4 Numerical examples

In this section, we will illustrate the theoretical results and sketch a numerical solution method by means of two examples. Furthermore, for the particular case of proportional reinsurance and a reinsurer using the expected value premium principle, we can refine the analytical results and state the asymptotic behaviour of the optimal strategy as the initial capital tends to infinity. Since an explicit solution to

(2.11) is unfortunately out of reach, for deriving a solution one needs to rely on a numerical method. Luckily, the theoretical characterization stated in Theorem 2.3.2 and Corollary 2.3.2 constitutes an implementable procedure.
These results tell that an iterated application of the operator $\mathcal{T}$, defined in the proof of Theorem 2.3.1, on some linear function $g(x) = \frac{x}{\delta} + g_0$ leads to an approximation of the value function if and only if $g_0 = V(0)$ is correctly chosen, cf. Corollary 2.3.2. Consequently, the first step in the procedure asks for a good guess of g_0, which can (and needs) to be improved in later steps. For determining a meaningful approximation of g_0, we exploit the idea of policy improvement, see for instance [28].
The starting point is the value $V^{sr}(x)$ corresponding to the situation of no reinsurance, which in our parameter setting can be explicitly determined. Based on this value V^{sr}, we compute a strategy $\mathtt{u}^1 = \{u_t^1\}$ with $u_t^1 = u^1(X_t^{\mathtt{u}})$ from the HJB-equation (2.11) via

$$u^1(x) = \arg\max_{u\in\tilde{\mathfrak{U}}} \left\{ x + c(u)\frac{\partial}{\partial x}V^{sr}(x) - (\delta+\lambda)V^{sr}(x) + \lambda \int_0^{\rho(x,u)} V^{sr}(x - r(y,u))\, dF_Y(y) \right\}.$$

In a next step we determine a good approximation for $V^{\mathtt{u}^1}(0)$, which can be done by using the Monte-Carlo method with direct simulations of the controlled surplus process from (2.4).
Now we know that $V^{\mathtt{u}^1}(0)$ corresponds to an admissible strategy but does not necessarily equal $V(0)$. But with $V^{\mathtt{u}^1}(0)$ at hand we can determine $V^{\mathtt{u}^1}(x)$ for $x \geq 0$ either by an iteration of an operator, similar to $\mathcal{T}$ but without the infimum in its definition, or by a finite-difference method. We use this value $V^{\mathtt{u}^1}$ as the starting point of iterations of $\mathcal{T}$. After a number of iterations, one can improve the initial value again by using the same method as illustrated above, but with the function obtained from the iterations as basis for the policy improvement step. This newly obtained value $V^{\mathtt{u}^2}$ then serves as the basis for new iterations of $\mathcal{T}$.

Remark 2.4.1. *Alternatively, one can execute a policy iteration procedure on the basis of the original HJB-equation (2.11). Our experience showed that the obtained strategies are very close to the ones determined via the first method. Unfortunately, the quality of the simultaneously generated return functions is not always trustworthy, a fact which originates from the presence of the control parameter in front of the sensitive derivative term and inside the integral. Nevertheless, the use of these strategies allows for a considerable acceleration of the whole procedure.*

In this way we create, by the use of *policy iterations* at intermediate steps, an increasing sequence of initial values and also determine candidates for a fixed point of $\mathcal{T}$. To decide whether an initial value is significantly too small one can check the behaviour of the function obtained from the corresponding iterations of $\mathcal{T}$. If an initial value is far away from $V(0)$ we observe a violation of the lower bound from Lemma 2.3.2 for relatively small values of x. We can accept an initial value $V^*(0)$ as a good guess for $V(0)$ if the function V^* obtained from iterations stays within the theoretically given bounds. If additionally V^* matches the value of the implicitly given strategy, we can accept it as an valid approximation of the value function.

Remark 2.4.2. *Instead of starting the iteration procedure always at predetermined values V^{u}, we can also start with $g(x) = \frac{x}{\delta} + V^{\mathrm{u}}(0)$ and all previously stated arguments still apply.*
Our experience showed that this procedure leads to trustworthy results and representative illustrations of our theoretical findings. Certainly, a theoretical numerical analysis would be necessary and highly interesting but this is out of the scope of this publication.

2.4.1 Example: proportional reinsurance

In the following, we are going to use the model parameters given by: $Y_i \overset{iid}{\sim} f_Y(y)$ with $f_Y(y) = \gamma^2 y\, e^{-\gamma y}$, i.e. *Gamma*$(2, \gamma)$ distributed claim amounts. The insurer's premium rate is determined via the expected value principle and reads as $c = (1 + \eta)\lambda\mu$ with $\mu = \frac{2}{\gamma}$ and $\eta > 0$. For the reinsurer, we assume the same premium principle but with a safety loading $\theta > \eta$. The concrete numbers are given in Table 1.

γ	η	θ	λ	δ
0.2	0.1	0.11	1	0.1

Table 2.1: Set of parameters for proportional reinsurance.

The considered reinsurance schema is $r(y, u) = u\, y$ for a control parameter $u \in (\underline{u}, 1]$ with $\underline{u} = \inf\{u \in [0, 1] \mid c(u) > 0\}$, as discussed before the statement of Theorem 2.3.1.
For deriving numerical approximations to the value function and to the optimal strategy, we implemented the program we have illustrated in the introduction to this section. In contrast to the case of excess of loss reinsurance, the proportional situation turned out to be numerically demanding, requiring lots of computational

efforts for arriving at passably satisfying results.
The strategy obtained from 20 policy iterations steps, starting from V^{sr}, is depicted in Figure 2.1. In the remark following below, the shape of this strategy is discussed in some detail. Figure 2.2 contains the graphs of V^{sr} (dotted line), V^1 (dashed line) and V^{20} (full line). V^1 is computed from 30 iterations of $\mathcal{T}$ starting with g and an initial value $g_0 = 212$ corresponding to the strategy obtained from 1 policy improvement step based on V^{sr}. The function V^{20} is derived from 30 operator iterations, but using the initial value $g_0 = 226.436$ associated to the strategy from Figure 2.1. In

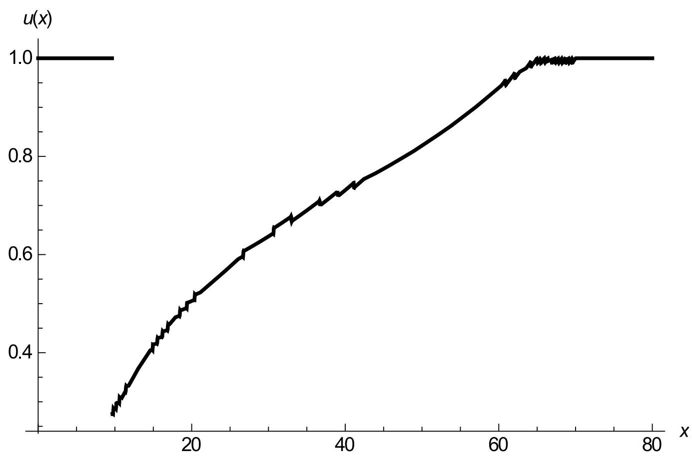

Figure 2.1: Numerically obtained proportional reinsurance strategy.

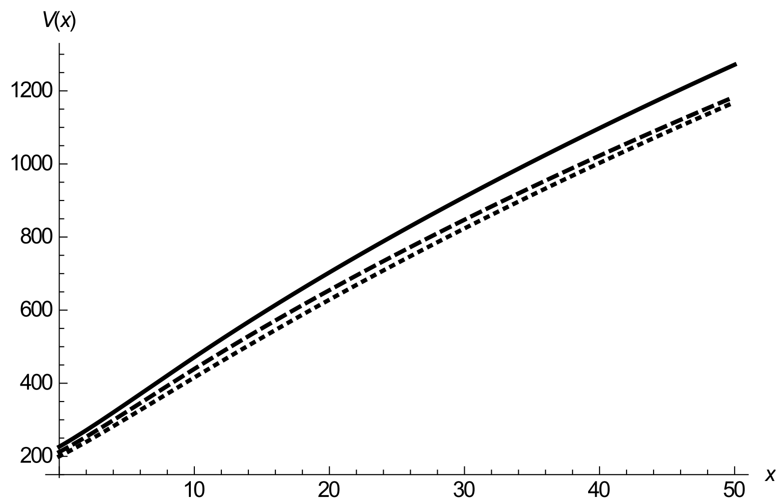

Figure 2.2: Iteration procedure in the proportional case.

Table 2, we present some exemplary function values from the iterations of $\mathcal{T}$ towards the computation of V^{20}.

	$x_0 = 10$	$x_1 = 30$	$x_2 = 50$	$x_3 = 70$
$n = 0$	326.4	526.4	726.4	926.4
$n = 5$	471.2	809.4	996.9	1153.4
$n = 10$	471.3	904.9	1215.3	1428.2
$n = 20$	471.3	909.8	1271.5	1583.9
$n = 30$	471.3	909.8	1271.6	1584.7

Table 2.2: Function values obtained from the application of $\mathfrak{T}^n$.

Remark 2.4.3. *We would like to discuss* $\lim_{x\to\infty} u^*(x)$*, which by the numerical computations is suggested to be one. Here, we exclusively deal with the case of proportional reinsurance and the expected value premium principle for both insurer and reinsurer,* $c(u) = \lambda\mu(u(1+\theta) - (\theta - \eta))$ *for safety loadings* $\theta > \eta$*. From the definition of the value function, we have*

$$\begin{aligned}
V(x) \geq & \mathbb{E}_x \left(\int_0^\tau e^{-\delta t} X_t \, dt \right) \\
= & \mathbb{E}_x \left(\frac{x}{\delta}(1 - e^{-\delta\tau}) + \lambda\mu\eta \int_0^\tau t \, e^{-\delta t} dt + \int_0^\tau e^{-\delta t} M_t dt \right). \\
= & \mathbb{E}_x \left(\frac{x}{\delta}(1 - e^{-\delta\tau}) + \frac{\eta\lambda\mu}{\delta^2} \left(1 - e^{-\delta\tau}(1 + \delta\tau)\right) + \int_0^\tau e^{-\delta t} M_t dt \right).
\end{aligned}$$

Above, we introduced the martingale $M = (M_t)_{t\geq 0}$ *which is the compensated compound Poisson process:*

$$M_t = \lambda\mu t - \sum_{k=1}^{N_t} Y_k, \quad M_0 = 0.$$

Now, we can regard $\int_0^\tau e^{-\delta t} M_t dt$ *pathwise as a Stieltjes integral and apply integration by parts, [122, Th.2.21], to arrive at*

$$\mathbb{E}_x \left(\int_0^\tau e^{-\delta t} M_t dt \right) = \mathbb{E}_x \left(-\frac{e^{-\delta\tau}}{\delta} M_{\tau-} + \int_0^\tau \frac{e^{-\delta t}}{\delta} dM_t \right) = \mathbb{E}_x \left(-\frac{e^{-\delta\tau}}{\delta} M_{\tau-} \right). \tag{2.19}$$

In (2.19), the integral with respect to the martingale is itself a martingale, leading to the second equality.

At the same time, using an $\varepsilon-$optimal strategy u^* for initial capital $x > 0$, we have

$$\begin{aligned}
V(x) - \varepsilon \leq & \frac{x}{\delta} + \mathbb{E}_x \left(\int_0^{\tau^{u^*}} e^{-\delta t} \left(\int_0^t \lambda\mu(u_s^*(1+\theta) - (\theta - \eta))ds - \sum_{k=1}^{N_t} u_{T_k}^* Y_k \right) dt \right) \\
\leq & \frac{x}{\delta} + \frac{\eta\lambda\mu}{\delta^2} + \mathbb{E}_x \left(\int_0^{\tau^{u^*}} e^{-\delta t} M_t^* dt \right) \\
= & \frac{x}{\delta} + \frac{\eta\lambda\mu}{\delta^2} - \mathbb{E}_x \left(\frac{e^{-\delta\tau^{u^*}}}{\delta} M_{\tau^{u^*}-}^* \right). \qquad (2.20)
\end{aligned}$$

If we suppose that u^* is a Markov control, then we certainly have that $M_t^* = \int_0^t \lambda\mu u_s^* ds - \sum_{k=1}^{N_t} u_{T_k}^* Y_k$ is a zero mean $\mathcal{F}_t^X$ martingale and the same integration by parts procedure as before applies. Consequently, we have for x large such that τ and τ^{u^*} (M_t is linearly bounded) are tending almost surely to infinity that:

$$V(x) \sim \frac{x}{\delta} + \frac{\eta\lambda\mu}{\delta^2} \quad \text{as } x \to \infty.$$

Now, we proceed with determining $\lim_{x\to\infty} u^*(x)$. Here, $u^*(x)$ denotes the pointwise maximizer in u of the HJB-equation (2.11), which due to continuity exists. Plugging in $c(u) = \lambda\mu(u(1+\theta) - (\theta - \eta))$ and regrouping, we see that

$$\begin{aligned}
u^*(x) = & \frac{(\delta+\lambda)V(x) - x + \lambda\mu(\theta-\eta)V'(x) - \lambda \int_0^{\frac{x}{u^*(x)}} V(x - u^*(x)y) dF_Y(y)}{\lambda\mu(1+\theta)V'(x)} \\
\approx (\geq) & \frac{(\delta+\lambda)(\frac{x}{\delta} + \frac{\eta\lambda\mu}{\delta^2}) - x + \lambda\mu(\theta-\eta)\frac{1}{\delta} - \lambda \int_0^{\frac{x}{u^*(x)}} \left[\frac{x - u^*(x)y}{\delta} + \frac{\eta\lambda\mu}{\delta^2} \right] dF_Y(y)}{\lambda\mu(1+\theta)\frac{1}{\delta}} \\
\approx (\geq) & \frac{\frac{\lambda x}{\delta}(1 - F_Y(x/u^*(x))) + \frac{\lambda\mu}{\delta}(\theta-\eta) + \frac{\lambda u^*(x)}{\delta} \int_0^{\frac{x}{u^*(x)}} y dF_Y(y) + (\delta+\lambda)\frac{\eta\lambda\mu}{\delta^2} - \frac{\lambda^2\mu\eta}{\delta^2} F(x/u^*(x))}{\frac{\lambda\mu}{\delta}(1+\theta)}.
\end{aligned}$$

We wrote "$\approx$ ($\geq$)" because in the integral, $V(x - u^*(x)y) \leq \frac{x - u^*(x)y}{\delta} + \frac{\eta\lambda\mu}{\delta^2}$, compare with (2.20). But since we have more or less a similar lower bound if $x \to \infty$, it becomes "$\approx$".
If we now assume that $\lim_{x\to\infty} u^*(x) = u^*$ exists, it should fulfill

$$u^* \approx \frac{\theta + u^*}{1+\theta},$$

which can be fulfilled only if $u^* = 1$. The two plots in Figures 2.3 and 2.4 illustrate the sharp linear upper bound together with $V(x)$ and $f(x) = \mathbb{E}_x \left(\int_0^\tau e^{-\delta t} X_t \, dt \right)$ for exponentially ν distributed claims and the following set of parameters given in Table 3.

ν	η	θ	λ	δ
1	0.6	0.61	1	0.01

Table 2.3: Set of parameters

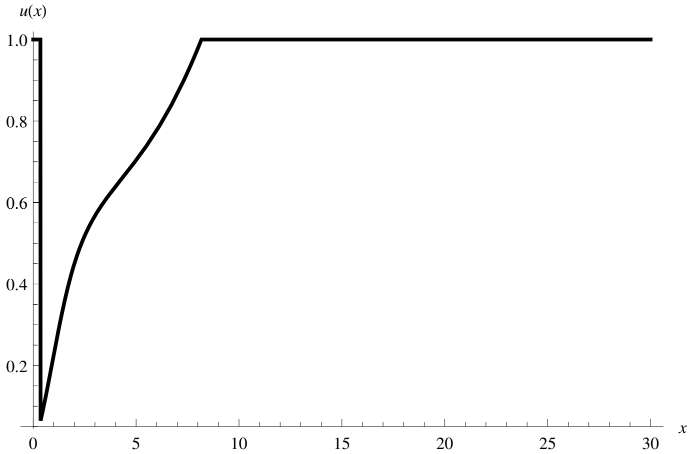

Figure 2.3: Illustrative optimal strategy for exponentially distribted claims

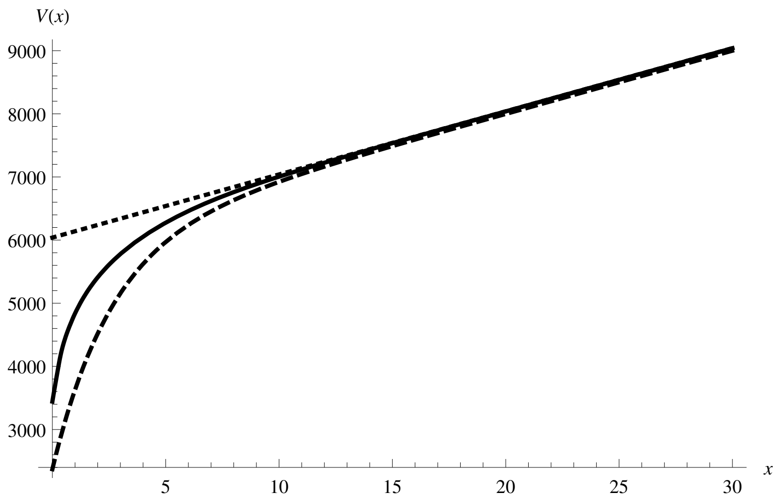

Figure 2.4: The sharp upper bound for proportional reinsurance and expected value principle for premiums

2.4.2 Example: XL-reinsurance

As a second example, we consider the case of dynamic XL-reinsurance with $Exp(\nu)$ distributed claim amounts. The particular numbers chosen are close to the ones chosen by [79] and can be found in Table 4. The numerically determined approximative

ν	η	θ	λ	δ
1	0.5	0.65	1	0.01

Table 2.4: Set of parameters

optimal strategy is displayed in Figure 2.5. The corresponding value function's numerical approximation (full line) is shown in Figure 2.6 together with V^1 (dashed line). It is remarkable to observe that this strategy consists of $u(x) = \infty$, i.e. buying no reinsurance, followed by taking exactly $u(x) = x$ and finally $u(x) \approx const.$ as the maximizing retention level for large initial capital x.

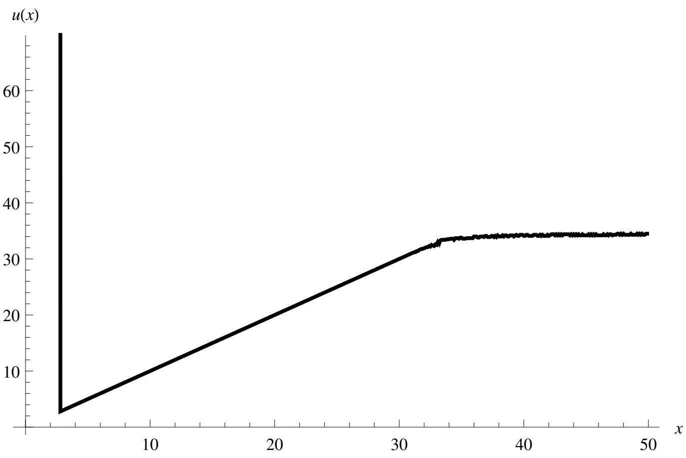

Figure 2.5: Numerical optimal XL strategy

Remark 2.4.4 (Comparison with ruin probability minimization)**.** *When numerically determining the approximative optimal strategies, one observes some similarities but also differences to the situation of optimal dynamic reinsurance strategies for minimizing ruin probabilities, see [110, Ch. 2.3.1] and [79]. In both situations, proportional and XL, the behaviour for small initial capital is similar, one finds that for some $x_0 > 0$ on $[0, x_0]$, it is optimal to take no reinsurance. From that point on, a certain amount of reinsurance is bought. For larger x, the reinsurance choice is either returning to the no reinsurance case (proportional) or converging towards a constant level (XL).*
Here, the proportional case is in contrast to the situation when minimizing the ruin probability. There, for small claims *the optimal reinsurance choice converges to a finite value as x tends to infinity. This different behaviour may be explained by the underlying performance measure which in the present framework is profit orientated. Because of discounting, a ruin event late in time does not bother the insurer which*

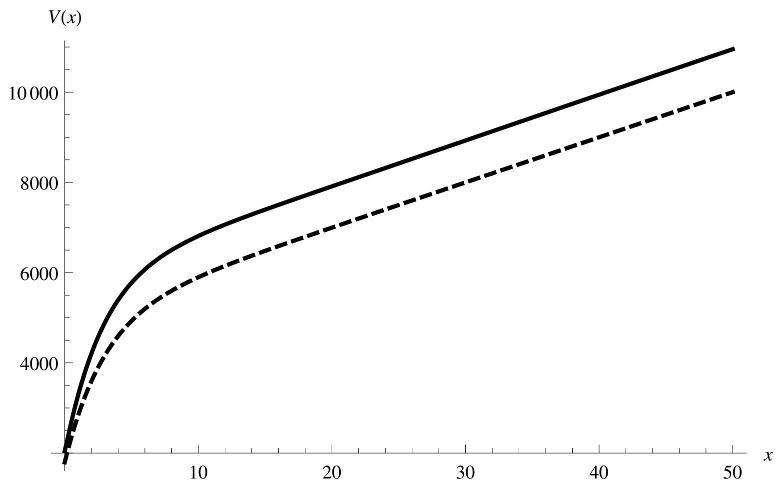

Figure 2.6: Value function for XL-reinsurance

implies that above a certain surplus level (large enough for having early ruin just with a low probability) one is focusing on the maximal drift and not buying reinsurance. The question: "why does the numerically optimal XL strategy behave differently?" is interesting as a future research project on its own. The answer to this question may be based on the comparison of solutions to integro-differential equations.

2.5 Conclusion

In this chapter, we studied a dynamic optimal reinsurance problem which is derived from an economical valuation criterion in risk theory. An interplay between analytical and probabilistic arguments allowed us to characterize the associated value function and finally the theoretical results were complemented by numerical examples. Based on the alternative interpretation of the studied value function, which is given in (2.2), we can state, that our results suggest that reinsurance can accelerate the process of building up a free reserve and that the use of reinsurance is beneficial in the economical context.

Chapter 3

Randomized reinsurance contracts[1]

Abstract

In this chapter we discuss the potential of randomizing reinsurance treaties for efficient risk management. While it may be considered counter-intuitive to introduce additional external randomness in the determination of the retention function for a given occurred loss, we indicate why and to what extent randomizing a treaty can be interesting for the insurer. We illustrate the approach with a detailed analysis of the effects of randomizing a stop-loss treaty on the expected profit after reinsurance in the framework of a one-year reinsurance model under regulatory solvency constraints and cost of capital considerations.

3.1 Introduction and Motivation

Reinsurance is a classical tool for the risk management of an insurance company. Among the many motivations for entering a reinsurance treaty, one that is of particular importance from an actuarial point of view is its function as a risk transfer, as it helps to reduce the risk exposure of the insurer and hence to stabilize the business (see e.g. Albrecher et al. [5] for a recent overview). Passing on some part

[1]This chapter is based on the paper: Hansjörg Albrecher and Arian Cani. On randomized reinsurance contracts. Submitted

of the insurance risk to a reinsurance company comes at the expense of paying a respective reinsurance premium, which reduces the potential profits, so that there is a tradeoff as to how much reinsurance is desirable for the insurance company. The solution naturally depends on the criteria that are used to quantify the performance of the retained portfolio as well as the pricing rule that is applied by the reinsurer for accepting the ceded part of the risk. Historically, the study of optimal reinsurance treaties can be traced back to the seminal papers of Borch [32] and Arrow [18] and has been an active research field both for academics and practicioners since then. Borch [32] showed that a stop-loss treaty minimizes the variance of the insurer's retained loss when the reinsurance premium is prespecified and determined according to an expected value premium principle. In the framework of risk-averse utility functions, Arrow [18] established that such a stop-loss contract more generally maximizes the expected utility of the terminal wealth of the insurer. Over the following decades, there were many contributions in the field, generalizing these classical results for more intricate optimality criteria and/or more general premium principles (see for instance Gajek & Zagrodny [63], Kaluszka [86], Centeno & Guerra [74] as well as Tan et al. [118], Malamud et al. [97] and Chi et al. [49] for some recent contributions, and [5, Ch.8] for a survey).

Prompted by the recent insurance regulatory developments aiming at the harmonization of risk assessment procedures, the Value-at-Risk (VaR) and Conditional-Tail-Expectation (CTE) became benchmark risk measures to reflect risk and subsequently determine capital requirements of an insurance company. Consequently, considerable attention has turned to embedding these two risk measures in the study of optimal reinsurance models. Cai & Tan [36] derive analytically the optimal retention of a stop-loss reinsurance treaty which minimizes the VaR and CTE of the insurer's remaining risk exposure under the expected value premium principle. These results were later generalized by Cai et al. [38] who examine optimal reinsurance schemes within the class of increasing convex functions. Using a geometric approach, Cheung [45] simplifies the arguments in Cai et al. [38] and identifies the stop-loss treaty as optimal also when the expected value premium principle is replaced by Wang's premium principle in the VaR-minimization problem. Within the setting of minimizing the VaR and CTE of the total retained loss of the insurer, Chi & Tan [47] determine the optimal reinsurance contract among a larger class of admissible reinsurance schemes, see also Chi [46] and Chi & Tan [48] for further extensions.

All the reinsurance forms considered above are of a deterministic form, i.e. for a risk X there is a fixed pre-defined function $r(X)$ that determines how much of the risk X is retained by the first-line insurer. While this is a traditional and intuitive way to specify the risk participation of the reinsurer, the question arises whether there could not exist situations in which additional randomness in the specification of $r(\cdot)$ could be advantageous. For instance, consider a reinsurance treaty that provides stop-loss coverage of the following form: at the end of the year a coin is flipped, and if the outcome is "Heads", then the reinsurer participates in the claim payment according to a stop-loss treaty with some pre-defined retention d, otherwise no reinsurance is provided. An immediate generalization of such a mechanism is to draw the realized retention level independently from a more general distribution (it will, however, turn out that a two-point distribution can not be outperformed for the optimization criteria considered below).
Guerra & Centeno [76] in fact used randomized treaties as a mathematical tool to identify optimal reinsurance forms under a general class of risk measures and premium principles, when the criterion is to minimize the risk measure of the retained risk exposure. As in other mathematical contexts (like the identification of Nash equilibria in game theory), this (in a certain sense) implicit 'convexification' allows to show the existence of an optimal strategy among such an enlarged set of admissible reinsurance forms. One can then (for the same premium) achieve an identical resulting cumulative distribution function of the retained loss through a deterministic treaty, which finally is the optimal reinsurance form (see [76] for details). While the latter argument at first glance seems to render the practical implementation of randomized treaties unnecessary, the 'equivalent' deterministic treaty can have unfavourable properties (like non-monotonicities or even discontinuities). Also, as will be discussed later, randomization of treaties may be simpler and may have some particular advantages to avoid moral hazard problems. We therefore in this chapter would like to take up the discussion of randomized reinsurance treaties from a more practical perspective, namely to study how randomization of classical treaties possibly increases the efficiency of risk sharing, and how it affects the resulting loss distribution. Eventually, randomization can be seen as an alternative method to reshape the loss distribution of the insurer.

We would like to point out that Gajek and Zagrodny [64] also discovered randomized reinsurance treaties as 'curious' possible solutions in the presence of discrete loss variables when the goal is to minimize the ruin probability of an insurer and there is a constraint on the available reinsurance premium, a problem which they nicely linked to the Neyman-Pearson lemma in statistical hypothesis testing (and in that case the performance of these randomized treaties can not be matched by a deterministic treaty). This connection between optimal reinsurance and the design of most powerful tests in statistics was recently studied in more detail in Lo [95].

In order to maintain transparency of the ideas involved, we prefer in this chapter to restrict our analysis to a simple stop-loss treaty on the aggregate loss of an insurance portfolio, and randomize it according to an independent mechanism (a lottery) that – after the aggregate claim has been settled – determines the retention of the stop-loss cover. Should such a randomized reinsurance cover be realized in practice, one could for instance think of a random experiment that both parties agree upon, possibly in the presence of a notary. At a first glance, such a random mechanism to determine the final participation of the reinsurer may seem unnatural, not the least because a reinsurer intends to help the insurer in adverse cases of large claims. However, reinsurance as well as direct insurance in the first place, is about efficiently dealing with risks, and if a non-standard reinsurance form is useful to reshape the loss distribution of the insurer in a cost-efficient and simple way, it may be worthwhile to be considered. From an insurer's viewpoint, such an uncertainty in the reinsurance cover could be compared with hearing about an event (like a natural catastrophe), but not yet knowing what the implications for the actual claim payments to policyholders will be, or also with the uncertainty until the full development of some claim. In the randomization case one knows the original claim size but does not yet know how much of it will finally remain with the insurer, so the main difference being that in the latter case the randomization is introduced artificially (but for efficiency reasons). Such additional introduced randomness can in fact be observed in some reinsurance treaties already implemented in practice, where the coverage is made dependent on a financial index or the financial performance of the insurance company itself (like in certain finite-risk reinsurance setups, see e.g. Culp [53]). For the 'marginal' analysis of the insurance liabilities, this introduced randomness can be interpreted as independent of the insurance risks.

The criterion for studying the effectiveness of reinsurance contracts in this chapter will be the one of maximizing expected profit after reinsurance, taking into account capital costs from the resulting solvency constraint for some fixed cost-of-capital rate, which goes back to Kull [88]. For a comparison to other criteria recently popular in the literature on optimal reinsurance forms, we refer to Remark 3.2.1 or [5, Ch.8]. For the sake of simplicity, we focus here solely on the insurance risk (no market risk, counterparty risk etc.) in a one-period framework and assume that there is no settlement delay of claims. As a risk measure for the determination of the required solvency capital, we restrict the analysis to the VaR. As amply emphasized in the literature, the choice of VaR in practice is questionable for several reasons, in the present context notably because it encourages excessive protection of medium-sized claims rather than large ones (see also Basak & Shapiro [27], Bernard & Tian [31] and Guerra & Centeno [76]). Yet this risk measure is currently implemented by many regulators and it seems that this will continue to be the case in the near future. The results below may also reinforce from a methodological point of view the doubtfulness of the use of VaR in practice for measuring risk in this context.

The rest of the chapter is organized as follows. In Section 3.2, we introduce the particular randomized stop-loss reinsurance treaty, the model and the objective function. Section 3.3 derives the optimal randomized treaty and discusses some concrete cases in more detail. In Section 3.4, it is then studied which retention level of a stop-loss contract is optimal for any given probability level of the randomization procedure, which gives some additional insight in the structure of the problem. Section 3.5 gives some numerical illustrations of the potential of randomizing classical contracts. Moreover, in Section 3.6 we compare the randomized stop-loss treaties with (deterministic) bounded stop-loss treaties, as the two share certain similarities. Finally, Section 3.7 contains some further practical considerations and conclusions.

3.2 The model

In this chapter we will study the effects of randomizing a simple stop-loss treaty. Let the random variable X denote the agregate loss that the insurer faces over one year. For convenience, let us assume here that X is continuous. Let further Y be a Bernoulli random variable, independent of X, with $\mathbb{P}(Y = 1) = p$ and $\mathbb{P}(Y = 0) = 1 - p$ for some fixed p $(0 \leq p \leq 1)$. Consider now a randomized reinsurance contract of the form

$$r(X) = r(X, d) = \begin{cases} \min(X, d), & \text{if } Y = 1, \\ X, & \text{if } Y = 0, \end{cases} \tag{3.1}$$

where $r(X)$ denotes the retained loss of the insurer after reinsurance. That is, after the realization of X there is a random experiment (which is independent of the outcome of X) that decides whether the reinsurance coverage of X is according to a SL treaty with retention d or whether no reinsurance takes place. While we later will allow for a random retention following a more general distribution than only the two-point distribution on $\{d, \infty\}$, the latter in fact will turn out to be optimal, so we focus the analysis first on this case. The resulting cumulative distribution function (c.d.f.) for the insurer then is

$$F_{r(X)}(x) = \begin{cases} F_X(x), & 0 \leq x < d, \\ p + (1-p)F_X(x), & x \geq d, \end{cases}$$

cf. Figure 3.1. For the survival function $\overline{F}_{r(X)}(x) = 1 - F_{r(X)}(x)$, we equivalently

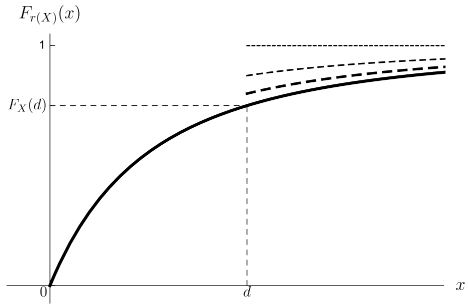

Figure 3.1: $F_{r(X)}$ for various values of p (for $x < d$ all c.d.f.'s coincide)

have

$$\overline{F}_{r(X)}(x) = \begin{cases} \overline{F}_X(x), & 0 \leq x < d, \\ (1-p)\overline{F}_X(x), & x \geq d. \end{cases} \tag{3.2}$$

From the latter expression, one easily deduces the expected retained claim amount

$$\mathbb{E}\left[r(X,d)\right] = \mathbb{E}\left[X\right] - p\int_d^{\infty} \overline{F}_X(x)dx.$$

Let $\pi(X)$ denote the total premium that the first-line insurer received from policyholders for accepting the aggregate risk X. Following a suggestion of Kull [88], one can consider the annual loss

$$\text{Loss} = X - \pi(X) + r_{CoC} \cdot \rho(\text{Loss}),$$

where $r_{CoC} \cdot \rho(\text{Loss})$ reflects capital costs, with r_{CoC} denoting a cost-of-capital rate and ρ a solvency risk measure. For a positively homogeneous and translationinvariant risk measure ρ, this leads to

$$\rho(\text{Loss}) = \frac{\rho(X) - \pi(X)}{1 - r_{CoC}},$$

and consequently the annual profit (i.e. negative loss) is given by

$$\frac{\pi(X)}{1 - r_{CoC}} - X - \frac{r_{CoC}}{1 - r_{CoC}} \cdot \rho(X) \tag{3.3}$$

(note that this approach for incorporating solvency capital requirements focuses on the current-year *insurance* risk only, which could then be complemented by market risk, counterparty risk, multi-year loss development patterns etc., see [5, Ch.8] for further details). If a reinsurance treaty of the form (3.1) is entered for a premium $\pi_R(d)$, then (3.3) changes into

$$Z(d) = \frac{\pi(X) - \pi_R(d)}{1 - r_{CoC}} - r(X, d) - \frac{r_{CoC}}{1 - r_{CoC}} \cdot \rho(r(X, d)). \tag{3.4}$$

As a performance measure of a reinsurance treaty, we will in this chapter choose the resulting expected annual profit $\mathbb{E}(Z(d))$, since it combines the solvency aspect with the profitability considerations in an intuitive way. Furthermore, to simplify calculations we will assume an expected value principle for the reinsurance premium (with relative safety loading $\theta > 0$):

$$\pi_R(d) = (1 + \theta)\mathbb{E}\left[X - r(X, d)\right] = (1 + \theta)p \int_d^\infty \overline{F}_X(x)dx.$$

For the risk measure ρ, we choose the Value-at-Risk (VaR) at level $1 - \alpha$ and use the notation

$$\rho(X) = \text{VaR}_\alpha(X) = \inf\{x : \overline{F}_X(x) \leq \alpha\}, \quad \alpha \in (0, 1).$$

This leads to the optimization problem

$$\max_{0 \leq p \leq 1,\, d \geq 0} \mathbb{E}\left[Z(d)\right] \tag{3.5}$$

with

$$\mathbb{E}\left[Z(d)\right] = \frac{\pi(X)}{1 - r_{CoC}} - \mathbb{E}\left[X\right] - \frac{r_{CoC}}{1 - r_{CoC}} \left(\left(1 + \frac{\theta}{r_{CoC}}\right) p \int_d^\infty \overline{F}_X(x)dx + \mathrm{VaR}_\alpha(r(X,d)) \right). \tag{3.6}$$

In view of (3.6), the optimization problem (3.5) can be reformulated as

$$g(d^*, p^*) := \min_{d \geq 0,\, 0 \leq p \leq 1} g(d, p), \tag{3.7}$$

where

$$g(d,p) := \left(1 + \frac{\theta}{r_{CoC}}\right) p \int_d^\infty \overline{F}_X(x)dx + \mathrm{VaR}_\alpha(r(X,d)). \tag{3.8}$$

Clearly, the trade-off to consider is to reduce the capital costs with a not too expensive reinsurance premium, and we will see in the sequel that under the present assumptions this trade-off can be more efficiently resolved introducing randomized reinsurance forms, i.e. $0 < p < 1$.

Remark 3.2.1. For general reinsurance treaties $r(X)$, a general premium principle π_R and risk measure ρ the above optimization criterion leads to minimizing

$$\pi_R(X - r(X)) - (1 - r_{CoC})\mathbb{E}\left[X - r(X)\right] + r_{CoC} \cdot \rho(r(X))$$

over all admissible $r(X)$. Note that this in general differs from the purely risk-averse objective function $\rho(\pi_R(X - r(X)) + r(X))$ used by Cai & Tan [36] and several subsequent papers in the literature, but in case of the expected value premium principle and translation invariance of ρ the two can be identified for a modified value of the safety loading coefficient (and hence different weighting), cf. [5, Sec.8.4] for details.

3.3 The optimization problem

In view of (3.8), it is clear that in our setting only retention values $d < \overline{F}_X^{-1}(\alpha)$ are of interest, as otherwise the reinsurance treaty does not improve $\mathrm{VaR}_\alpha(X)$ and therefore it is better not to take reinsurance at all (and keep the saved reinsurance premium for profit). Whenever $p > 0$ is optimal, for each potentially optimal candidate retention $d < \overline{F}_X^{-1}(\alpha)$, the optimal value of p has to be the one such that $F_{r(X,d)}(d) = p + (1-p)F_X(d) = 1-\alpha$, i.e.

$$p(d) = 1 - \frac{\alpha}{\overline{F}_X(d)}, \tag{3.9}$$

cf. Figure 3.2.

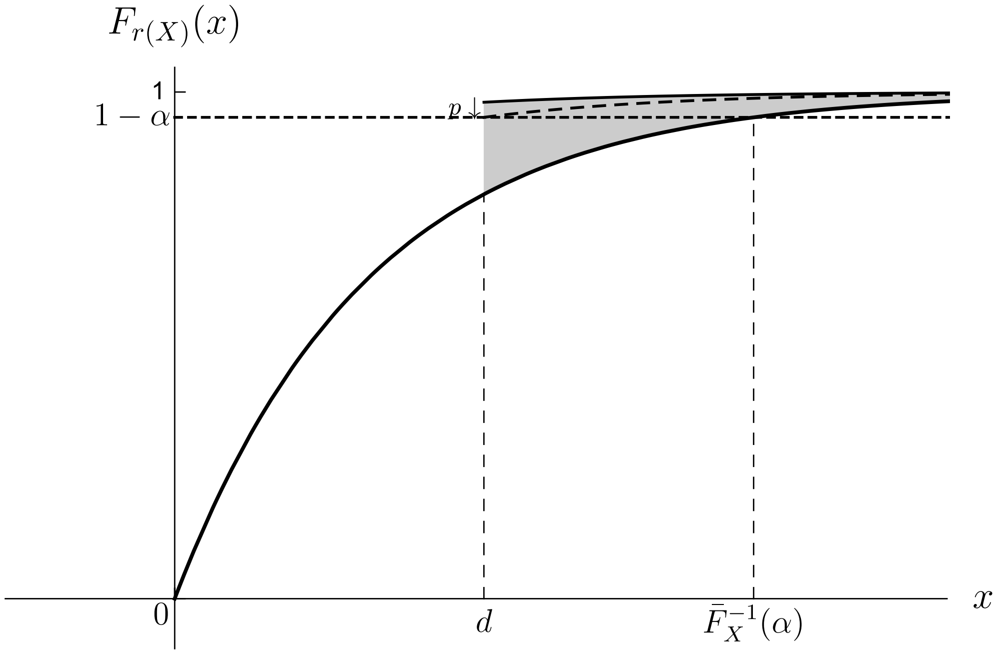

Figure 3.2: Optimal value of p in a randomized stop-loss treaty

Indeed, if $p + (1-p)F_X(d) > 1-\alpha$, then the solvency capital requirement is over-fulfilled in the sense that the same level of $\mathrm{VaR}_\alpha(r(X,d))$ could be achieved for a lower reinsurance premium simply by decreasing p to (3.9). On the other hand, for any p with $p + (1-p)F_X(d) < 1-\alpha$, one could attain the same level of $\mathrm{VaR}_\alpha(r(X,d))$ by increasing d (i.e. transferring the location of the jump of $F_{r(X,d)}$) to the level $\widetilde{d} > d$ such that $p + (1-p)F_X(\widetilde{d}) = 1-\alpha$ holds, so that the same value $\mathrm{VaR}_\alpha(r(X,d)) = \mathrm{VaR}_\alpha(r(X,\widetilde{d})) = \overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right)$ is achieved by a smaller reinsurance premium. Consequently, the original d could not have been optimal for the overall optimization problem. One can hence fix $p(d)$ according to (3.9) and the optimization problem (3.7) reduces to the one-dimensional problem

$$\min_{0 \le d \le \overline{F}_X^{-1}(\alpha)} g(d, 1-\alpha/\overline{F}_X(d)). \tag{3.10}$$

Note that if the optimal retention is the right-end point of this interval, i.e. $d^* = \overline{F}_X^{-1}(\alpha)$, the corresponding probability is $p^* = p(d^*) = 0$, which means no reinsurance (this also corresponds to $d = \infty$ for any p) and the resulting objective function then is $\mathrm{VaR}_\alpha(X) = \overline{F}_X^{-1}(\alpha)$.

It is clear from (3.9), but useful to note for later purposes, that we always have $p(d) \leq 1 - \alpha$, with equality for $d = 0$.

Problem (3.10) translates into

$$\min_{0 \leq d \leq \overline{F}_X^{-1}(\alpha)} (1 + \theta/r_{CoC}) \; (1 - \alpha/\overline{F}_X(d)) \int_d^\infty \overline{F}_X(x)dx + d.$$

This can also be expressed in terms of the mean-excess function $e_X(u) = \mathbb{E}(X - u|X > u)$ and the pure reinsurance premium $\pi_{SL}(d) = \int_d^\infty \overline{F}_X(x)dx$ of a classical unbounded stop-loss contract (i.e. $p = 1$):

$$\min_{0 \leq d \leq \overline{F}_X^{-1}(\alpha)} (1 + \theta/r_{CoC}) \; (\pi_{SL}(d) - \alpha \cdot e_X(d)) + d. \tag{3.11}$$

One observes that the shape of this function strongly depends on the distribution of the loss variable X and a general analysis is difficult. In any case, a particular candidate for an optimal retention d is the solution of the equation

$$(1 + \theta/r_{CoC}) \; (\overline{F}_X(d) + \alpha \cdot e'_X(d)) = 1. \tag{3.12}$$

Example 3.3.1. *If X is exponentially distributed with parameter ν, then $e_X(d) = 1/\nu$ and the solution of* (3.12) *is indeed*

$$d = \frac{1}{\nu} \log\left(1 + \theta/r_{CoC}\right). \tag{3.13}$$

Since in this case $\overline{F}_X^{-1}(\alpha) = \frac{1}{\nu}\log(1/\alpha)$, the solution of the overall optimization problem (3.5) *is the following: If $\frac{1}{\alpha} > 1 + \theta/r_{CoC}$, then the optimal retention d^* is given by* (3.13) *together with the corresponding $p^* = 1 - \alpha\,(1 + \theta/r_{CoC})$ (cf.* (3.9)*). If $\frac{1}{\alpha} \leq 1 + \theta/r_{CoC}$, then $d^* = \infty$, i.e. no reinsurance of the form* (3.1) *should be taken (in this case the reinsurance premium, through the loading θ, is too expensive or the cost-of-capital rate is too small relative to the solvency quantile α, so that reinsurance is not efficient).*

Example 3.3.2. *If X follows a shifted Pareto distribution, i.e.*

$$F_X(x) = 1 - \left(\frac{\xi}{x + \xi}\right)^{1/\gamma}, \qquad \xi > 0\,;\, \gamma < 1,$$

then $e_X(d) = \frac{d+\xi}{\frac{1}{\gamma}-1}$*, and the solution of* (3.12) *is given by*

$$d = \xi \left(\left(\frac{1}{1+\frac{\theta}{r_{CoC}}} - \frac{\alpha}{\frac{1}{\gamma}-1} \right)^{-\gamma} - 1 \right). \tag{3.14}$$

Since $\bar{F}_X^{-1}(\alpha) = \xi\,(\alpha^{-\gamma} - 1)$*, the solution of the overall optimization problem* (3.5) *is as follows: If* $\frac{1}{\alpha}(1-\gamma) > 1 + \theta/r_{CoC}$*, then* d^* *is given by* (3.14) *together with*

$$p^* = \frac{\frac{1}{\gamma}\left(1 - \alpha\left(1 + \frac{\theta}{r_{CoC}}\right)\right) - 1}{\frac{1}{\gamma} - \alpha\left(1 + \frac{\theta}{r_{CoC}}\right) - 1}.$$

If $\frac{1}{\alpha}(1-\gamma) \leq 1 + \theta/r_{CoC}$*, then it is optimal to take no reinsurance.*

Example 3.3.3. *If* X *is uniformly distributed in* $[0, b]$*, one has* $e_X(d) = 1/2(b-d)$ *for* $d < b$*, and*

$$d = b\left(1 - \left(\frac{1}{1+\frac{\theta}{r_{CoC}}} + \frac{\alpha}{2}\right)\right), \tag{3.15}$$

solves (3.12)*. Because* $\bar{F}_X^{-1}(\alpha) = b(1-\alpha)$*, the solution of the overall optimization problem* (3.5) *then reads the following: If* $\frac{2}{\alpha} > 1 + \theta/r_{CoC}$*, then* d^* *is given by* (3.15) *together with*

$$p^* = \frac{\frac{1}{1+\frac{\theta}{r_{CoC}}} - \frac{\alpha}{2}}{\frac{1}{1+\frac{\theta}{r_{CoC}}} + \frac{\alpha}{2}}.$$

If $\frac{2}{\alpha} \leq 1 + \theta/r_{CoC}$*, then* $d^* = \infty$*, i.e. the expected profit is maximized when no reinsurance is purchased.*

Remark 3.3.1. One could equivalently have started the analysis from the viewpoint of choosing candidate values p first. Clearly, only $p \leq 1-\alpha$ can be optimal, since the resulting $\mathrm{VaR}_\alpha(r(X,d))$ would not be improved by choosing $p > 1-\alpha$, which is more expensive (note that in particular a classical unbounded stop-loss contract ($p = 1$) can not be optimal for (3.5), since reinsurance beyond the solvency quantile is not efficient). In much the same way as above, one can now argue that the optimal choice of d for a given $p \leq 1-\alpha$ again has to fulfill $p + (1-p)F_X(d) = 1-\alpha$, i.e.

$$d(p) = \overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right). \tag{3.16}$$

For any smaller (larger) d one can achieve the same VaR with a larger retention (smaller p, respectively) and hence a smaller premium in each case. Fixing (3.16), the optimization problem (3.7) then reduces to the one-dimensional problem

$$\min_{0\le p\le 1-\alpha}\left(1+\frac{\theta}{r_{CoC}}\right)p\int_{\overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right)}^{\infty}\overline{F}_X(x)dx+\overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right), \tag{3.17}$$

which has a less intuitive form than (3.11). □

In fact, the randomized treaty studied in this section (based on a two-point distribution on $\{d,\infty\}$) is the optimal treaty among all randomized stop-loss treaties with arbitrary distribution for the random retention:

Theorem 3.3.1. *Let $\mathfrak{R}$ be the set of all stop-loss treaties with a random retention level D with c.d.f. F_D, where D is independent of X. Assume that the reinsurance premium is determined by $\pi_R(\mathcal{R})=(1+\theta)\mathbb{E}(\mathcal{R})$ for every $\mathcal{R}\in\mathfrak{R}$. Then the expected value of*

$$\mathbb{E}\left(\frac{\pi(X)-\pi_R(X-\mathcal{R})}{1-r_{CoC}}-(X-\mathcal{R})-\frac{r_{CoC}}{1-r_{CoC}}\cdot VaR_\alpha(X-\mathcal{R})\right)$$

is maximized for

$$D=\begin{cases} d^*, & \textit{with prob. } p^*,\\ \infty, & \textit{with prob. } 1-p^*. \end{cases} \tag{3.18}$$

Proof. Consider the optimal two-point solution (3.18). Since the reinsurance premium follows an expected value principle, it is proportional to the grey area in Figure 3.2. Whenever another random variable D leads to a different value of $\mathrm{VaR}_\alpha(X-\mathcal{R})$, the two-point distribution on $\{\mathrm{VaR}_\alpha(X-\mathcal{R}),\infty\}$ with the respective value $p(\mathrm{VaR}_\alpha(X-\mathcal{R}))$ according to (3.9) can generate the same VaR value, but for a cheaper reinsurance premium. Hence the optimal choice of d^* (together with p^*) can not be outperformed by any other random variable D that is independent of X. □

3.4 Optimizing the retention for fixed p

While the determination of the optimal pair (d^*,p^*) is already studied in Section 3.3, we now identify the optimal retention level for an arbitrary (possibly non-optimal) given probability level p. This will give some additional insight into the nature and consequences of the randomization procedure. We will now also allow $F_X(0)>0$

(which we refrained from in the previous sections for the sake of clarity of exposition, but which may for instance be relevant in some catastrophe insurance portfolios). Let us start with some general observations.

3.4.1 Preliminary properties

First, observe that if $1-\alpha \le F_X(0)$, then clearly $\mathrm{VaR}_\alpha(r(X,d)) = 0$ for all $d \ge 0$ in which case the expected profit in (3.5) is trivially maximized for $p = 0$, i.e. no reinsurance. We hence assume $\alpha < \overline{F}_X(0)$ in the following.

If $1-\alpha < F_X(d)$, i.e. $d > \overline{F}_X^{-1}(\alpha)$, then $\mathrm{VaR}_\alpha(r(X,d)) = \mathrm{VaR}_\alpha(X) = \overline{F}_X^{-1}(\alpha)$. In this case the retention d exceeds the VaR of the original (and also the retained) risk, so reinsurance is again not of interest, as the reinsurance premium reduces the expected profit, but does not lower the capital costs.

Next, for $F_X(d) \le 1-\alpha \le p + (1-p)F_X(d)$, i.e. $(1-p)\overline{F}_X(d) \le \alpha \le \overline{F}_X(d)$, we have $\mathrm{VaR}_\alpha(r(X,d)) = d$.

Finally, in case $1-\alpha > p+(1-p)F_X(d)$, i.e. $d < \overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right)$, we have $\mathrm{VaR}_\alpha(r(X,d)) = \overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right)$.

Summarizing this differently, for each fixed retention $d \ge 0$, the Value-at-Risk of the retained loss amount reads

$$\mathrm{VaR}_\alpha(r(X,d)) = \overline{F}_X^{-1}(\alpha) \tag{3.19}$$

for $p = 0$,

$$\mathrm{VaR}_\alpha(r(X,d)) = \begin{cases} \overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right), & 0 \le d < \overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right), \\ d, & \overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right) \le d \le \overline{F}_X^{-1}(\alpha), \\ \overline{F}_X^{-1}(\alpha), & d > \overline{F}_X^{-1}(\alpha) \end{cases} \tag{3.20}$$

for $p \in \left(0, 1-\frac{\alpha}{\overline{F}_X(0)}\right)$, and

$$\mathrm{VaR}_\alpha(r(X,d)) = \begin{cases} d, & 0 \le d \le \overline{F}_X^{-1}(\alpha), \\ \overline{F}_X^{-1}(\alpha), & d > \overline{F}_X^{-1}(\alpha) \end{cases} \tag{3.21}$$

for $p \in \left[1 - \frac{\alpha}{\overline{F}_X(0)}, 1\right]$. Note that in the latter case the Value-at-Risk is bounded by d for any $d \ge 0$ (cf. Figure 3.3 (right)), whereas it exceeds d in the range $0 \le d < \overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right)$ in the case $p < 1 - \frac{\alpha}{\overline{F}_X(0)}$ (cf. Figure 3.3 (left)). One observes that the domain of the Value-at-Risk as a function of d is enlarged for increasing p, reaching the situation on the right-hand picture when p tends to $1 - \alpha/\overline{F}_X(0)$ (and, conversely, for $p \to 0$ the constant $\overline{F}_X^{-1}(\alpha)$ is reached for all values of d).

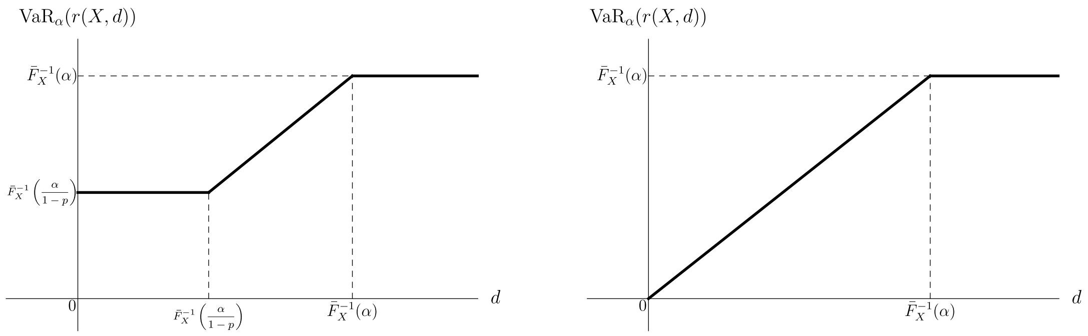

Figure 3.3: $\mathrm{VaR}_\alpha(r(X,d))$ as a function of d for $p < 1 - \alpha/\overline{F}_X(0)$ (left) and $p \ge 1 - \alpha/\overline{F}_X(0)$ (right).

The grey area depicted in Figure 3.4 represents all additional pairs $(d, \mathrm{VaR}_\alpha(r(X,d)))$ that can be obtained by varying p in the range $p \in \left(0, 1 - \alpha/\overline{F}_X(0)\right)$.

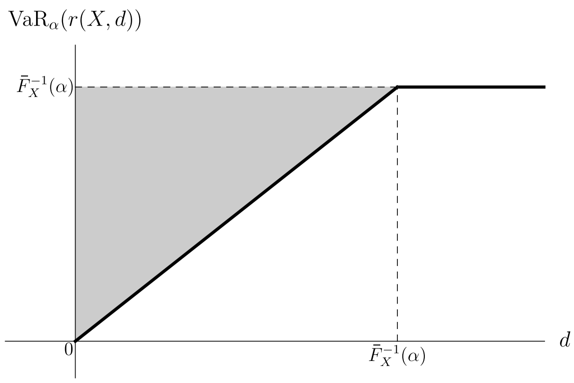

Figure 3.4: Effect of randomizing on the Value-at-Risk as a function of d

3.4.2 Optimization w.r.t. d for fixed p

Proposition 3.4.1. *Fix the value of p and let $\kappa := \frac{1}{p(1+\theta/r_{CoC})}$.*
(i) Consider first the case $p \geq 1 - \alpha/\overline{F}_X(0)$. If

$$\alpha < \kappa < \overline{F}_X(0) \tag{3.22}$$

and

$$g\left(\overline{F}_X^{-1}(\kappa), p\right) \leq \overline{F}_X^{-1}(\alpha), \tag{3.23}$$

then a finite optimal retention d^* exists and is given by

$$d^* = \overline{F}_X^{-1}(\kappa).$$

If

$$\alpha < \overline{F}_X(0) \leq \kappa$$

and

$$\mathbb{E}[X] \leq \kappa \overline{F}_X^{-1}(\alpha)$$

hold, then the finite optimal retention is $d^* = 0$.
(ii) For $p \in \left(0, 1 - \alpha/\overline{F}_X(0)\right)$, a finite optimal retention d^* exists if

$$\alpha < \kappa < \frac{\alpha}{1-p} \tag{3.24}$$

and

$$g\left(\overline{F}_X^{-1}(\kappa), p\right) \leq \overline{F}_X^{-1}(\alpha), \tag{3.25}$$

and then its value is also

$$d^* = \overline{F}_X^{-1}(\kappa).$$

Alternatively, if

$$\kappa \geq \frac{\alpha}{1-p} \tag{3.26}$$

and

$$g\left(\overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right),p\right) \leq \overline{F}_X^{-1}(\alpha), \tag{3.27}$$

the optimal retention is given by

$$d^* = \overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right).$$

If none of the above conditions hold, $d^* = \infty$ (i.e. no reinsurance).

Proof. Let us first consider the case $p \in \left[1-\alpha/\overline{F}_X(0), 1\right]$. Then,

$$g(d) := g(d,p) = \begin{cases} g_L(d), & 0 \leq d \leq \overline{F}_X^{-1}(\alpha), \\ g_U(d), & d > \overline{F}_X^{-1}(\alpha) \end{cases} \tag{3.28}$$

with

$$\begin{aligned} g_L(d) &:= \left(1+\frac{\theta}{r_{CoC}}\right) p \int_d^\infty \overline{F}_X(x)dx + d, \\ g_U(d) &:= \left(1+\frac{\theta}{r_{CoC}}\right) p \int_d^\infty \overline{F}_X(x)dx + \overline{F}_X^{-1}(\alpha). \end{aligned}$$

Clearly, from (3.28), $g(d)$ is continuous on $d \in [0,\infty)$ and tends to $\overline{F}_X^{-1}(\alpha)$ as $d \to \infty$. In addition, observe that for $\kappa < \overline{F}_X(0)$, $g_L(d)$ is decreasing on $\left[0, \overline{F}_X^{-1}(\kappa)\right)$, increasing on $\left(\overline{F}_X^{-1}(\kappa), \infty\right)$ and attains a minimum at $\overline{F}_X^{-1}(\kappa)$. Therefore, since $g_U(d)$ is decreasing on $d \in [0,\infty)$, $g(d)$ attains a global minimum at $\overline{F}_X^{-1}(\kappa)$ if $\alpha < \kappa < \overline{F}_X(0)$ and $g\left(\overline{F}_X^{-1}(\kappa)\right) \leq \overline{F}_X^{-1}(\alpha)$. Here, the latter condition ensures that a finite global minimum of $g(d)$ exists, namely that the expected profit $\mathbb{E}\left[Z(d)\right]$ can be increased through reinsurance. In this case, the optimal retention is $d^* = \overline{F}_X^{-1}(\kappa)$. The condition $\alpha < \kappa$ is necessary, otherwise $\overline{F}_X^{-1}(\kappa) \geq \overline{F}_X^{-1}(\alpha)$ and $g(d)$ is then decreasing on $d \in [0,\infty)$ in which case a finite optimal retention d^* does not exist (i.e. it would be preferable not to buy reinsurance from a profitability aspect). Note that if $\kappa \geq \overline{F}_X(0)$, $g_L(d)$ attains its minimum at $d = 0$ and is increasing on $d \in (0,\infty)$. Consequently, because $g_U(d)$ is decreasing on $d \in [0,\infty)$, a finite optimal retention exists if and only if $g(0) = \left(1+\frac{\theta}{r_{CoC}}\right) p\,\mathbb{E}[X] \leq \overline{F}_X^{-1}(\alpha)$ which can be rewritten as $\mathbb{E}[X] \leq \kappa \overline{F}_X^{-1}(\alpha)$. Hence, in this case, the optimal retention is

$d^* = 0$ meaning that the expected profit is maximized by passing the entire risk to the reinsurer.
Let us now examine the case $p \in \left(0, 1 - \frac{\alpha}{\overline{F}_X(0)}\right)$, where

$$g(d) = \begin{cases} \left(1 + \frac{\theta}{r_{CoC}}\right) p \int_d^\infty \overline{F}_X(x)dx + \overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right), & 0 \le d < \overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right), \\ \left(1 + \frac{\theta}{r_{CoC}}\right) p \int_d^\infty \overline{F}_X(x)dx + d, & \overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right) \le d \le \overline{F}_X^{-1}(\alpha), \\ \left(1 + \frac{\theta}{r_{CoC}}\right) p \int_d^\infty \overline{F}_X(x)dx + \overline{F}_X^{-1}(\alpha), & d > \overline{F}_X^{-1}(\alpha). \end{cases} \tag{3.29}$$

Again, $g(d)$ is continuous on $d \in [0, \infty)$ with the limiting value $\overline{F}_X^{-1}(\alpha)$ as $d \to \infty$. Furthermore, observe from (3.29) that $g(d)$ is decreasing on $d \in \left[0, \overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right)\right)$. The subsequent behavior of $g(d)$ is determined by the relations between κ, α and p. More precisely, if $\kappa \ge \frac{\alpha}{1-p}$, $g_L(d)$ is increasing on $d \in \left(\overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right), \infty\right)$, so is $g(d)$ on $d \in \left(\overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right), \overline{F}_X^{-1}(\alpha)\right]$. Hence, because $g_U(d)$ is decreasing on $d \in [0, \infty)$, a finite optimal retention d^* exists only if $g\left(\overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right)\right) \le \overline{F}_X^{-1}(\alpha)$. In this case, $g(d)$ attains a global minimum at $\overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right)$, which is the optimal retention. In the case $\alpha < \kappa < \frac{\alpha}{1-p}$, $g(d)$ is decreasing on $d \in \left[0, \overline{F}_X^{-1}(\kappa)\right)$, increasing on $\left(\overline{F}_X^{-1}(\kappa), \overline{F}_X^{-1}(\alpha)\right]$ and then decreasing again towards $\overline{F}_X^{-1}(\alpha)$ as $d \to \infty$. The function $g(d)$ then attains a global minimum value at $\overline{F}_X^{-1}(\kappa)$ if $g\left(\overline{F}_X^{-1}(\kappa)\right) \le \overline{F}_X^{-1}(\alpha)$. Hence, $d^* = \overline{F}_X^{-1}(\kappa)$ is the optimal retention. Finally, if $\kappa \le \alpha$, $f(d)$ is decreasing on $d \in [0, \infty)$. Consequently, a finite optimal retention d^* does not exist. □

3.5 Numerical illustrations

In this section, we illustrate the effects of the proposed randomized stop-loss treaty on the expected profit and discuss some quantitative properties of the resulting optimal retention level d^*. Assume that the distribution function of the aggregate loss of the insurer is given by

$$F_X(x) = \begin{cases} 0.05, & x = 0, \\ 1 - 0.95\left(\frac{1000}{1000+x}\right)^3, & x > 0, \end{cases}$$

i.e. a shifted Pareto distribution with an atom at 0. Furthermore, assume that the first-line insurance premium is determined by $\pi(X) = (1 + 0.1) \cdot \mathbb{E}[X] = 522.5$.

3.5.1 Optimal retention level d^* as a function of p

Figure 3.5 depicts the optimal retention d^* as a function of p for $\theta = 0.2$, $r_{CoC} = 0.07$ and $\alpha = 0.05$. Recall from Section 3.4.1 that one needs to distinguish the regions $p \in (0, a)$ and $p \in [a, 1]$ with $a := 1 - \alpha/\overline{F}_X(0) \approx 0.947$ for the analysis (indicated by the right vertical dashed line). In both cases, the existence and representation of the optimal retention d^* is contingent on the value of $\kappa = \kappa(p) \approx \frac{0.259}{p}$. For $p \in (0, a)$, the two subcases $\kappa \geq \frac{\alpha}{1-p}$ and $\alpha < \kappa < \frac{\alpha}{1-p}$ have to be treated separately (see the left vertical dashed line at $p \approx 0.838$ for which $\kappa = \frac{\alpha}{1-p}$). In all these cases, the conditions of Proposition 3.4.1 are verified for the considered parameter set and any $p \in (0, 1]$, so that a finite optimal retention level d^* is known to exist. For $p \in (0, 0.838]$ (i.e., $\kappa \geq \frac{\alpha}{1-p}$), the optimal retention d^* is given by $\overline{F}_X^{-1}\left(\frac{\alpha}{1-p}\right)$ and is decreasing in p. In this region, in order to maximize the expected profit, it is optimal to choose d such that the VaR is minimized; the gains from a cheaper reinsurance premium with a larger retention would not offset the additional costs arising from a larger VaR. As p increases within this region, smaller VaR values can be attained (cf. Figure 3.4), explaining the decrease in the optimal retention d^* up to $p = 0.838$. The rate of this decrease corresponds to the rate at which the VaR domain is enlarged (as a function of d) when p increases. At $p = 0.838$, the savings on the reinsurance premium from choosing larger values of d start to dominate the capital costs for resulting higher VaR values. As a result, the optimal retention given by $d^* = \overline{F}_X^{-1}(\kappa)$ increases on $p \in (0.838, 1]$ with a smooth transition through the right vertical line at $p = 0.947$.

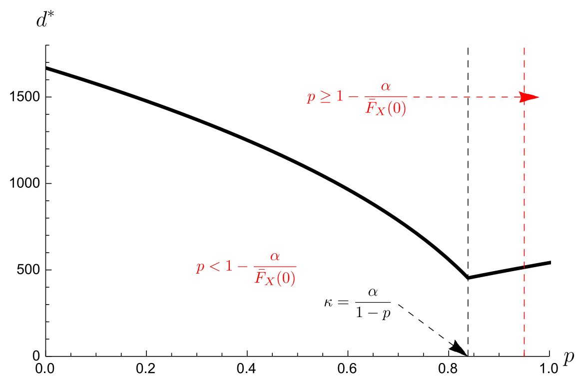

Figure 3.5: Optimal retention d^* as a function of p for $\theta = 0.2$ (solid) with $\alpha = 0.05$ and $r_{CoC} = 0.07$.

Let us now examine the effects of the reinsurance loading θ on the optimal retention level d^*. When reinsurance becomes cheaper, i.e. θ decreases, reinsurance premium

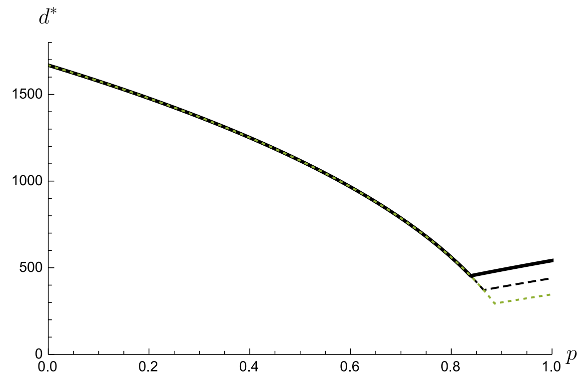

Figure 3.6: Optimal retention d^* as a function of p for $\theta = 0.11$ (dotted), $\theta = 0.15$ (dashed) and $\theta = 0.2$ (solid) with $\alpha = 0.05$ and $r_{CoC} = 0.07$.

savings are reduced. This has the effect of shifting the solution p of $\kappa(p) = \frac{\alpha}{1-p}$ towards higher p-values, in the present case to 0.864 for $\theta = 0.15$ and 0.886 for $\theta = 0.11$. Since reinsurance premium savings become worth considering only for higher p-values, minimizing the VaR is of interest in an extended region, resulting in an extended decrease of d^* for smaller reinsurance loadings (cf. Figure 3.6).

Note that θ and r_{CoC} enter in κ as a ratio, so if both these parameters increase or decrease to the same relative extent, the resulting shape of the optimal retention d^* as a function of p will remain unchanged. Another observation is that the optimal retention d^* is not affected by a change in the reinsurance premium loading up to $p = 0.838$ for the considered θ-values (here, θ almost doubles form 0.11 to 0.2). The reason is again the trade-off between VaR and reinsurance premium (and the fact the reinsurance premium is based on the expected value principle). In other words, if p is fixed at such (not too large) values, an increased reinsurance premium will still lead to the same insurer's preference choice of the retention. In order to further illustrate this point, Figure 3.7 depicts d^* as a function of θ for a fixed value of $p = 0.8$. For all values of θ up to $\theta = 0.28$ (which signifies the value for which $\kappa(\theta) = \frac{\alpha}{1-p}$), d^* remains unchanged. Beyond that value, d^* increases. Finally, for $\theta \geq 0.449$ condition (3.25) is not fulfilled any more, and reinsurance becomes too expensive for the insurer to enter a reinsurance agreement of this type at all.

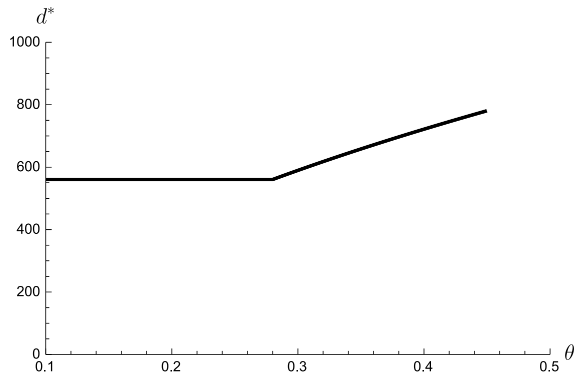

Figure 3.7: Optimal retention d^* as a function of θ for $p = 0.8$.

3.5.2 Optimal p^* as a function of the retention d

For each given retention level $d \geq 0$, one can also look for the optimal $p \in [0, 1]$ that maximizes the expected profit. Figure 3.6 depicts p^* as a function of the retention level d for different reinsurance loadings. Note that for small values of d and high reinsurance loading, it is preferable to have no reinsurance at all.

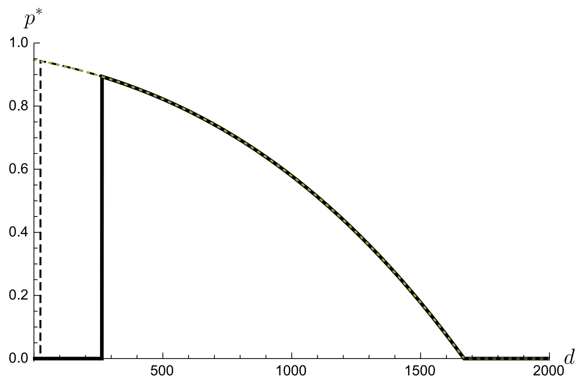

Figure 3.8: Optimal p^* as a function of d for $\theta = 0.11$ (dotted), $\theta = 0.2$ (dashed) and $\theta = 0.3$ (solid).

3.5.3 Maximal expected profit as a function of p

Let us now analyze the impact of introducing randomness in the reinsurance treaty on the expected profit. Figure 3.9 depicts the maximal expected profit (under the choice of the respective best $d^*(p)$) as a function of p for various reinsurance loadings. It is interesting to observe that although d^* is first decreasing in p on $\kappa \geq \frac{\alpha}{1-p}$, the expected profit is first increasing in p. Thus, having the possibility to choose a smaller VaR (by decreasing d^*) in response to an increase in p outbalances the increase in the reinsurance premium (through both an increase of p and decrease of d^*) in an increasing fashion. The maximal expected profit is attained when the optimal pair (d^*, p^*) is chosen. In the present illustration p^* is 0.863 for $\theta = 0.11$, 0.829 for $\theta = 0.15$ and 0.804 for $\theta = 0.18$. As p approaches 1, the gains diminish again. Note that the randomized strategy outperforms the classical deterministic stop-loss ($p = 1$) for a variety of p-values. One also sees that for higher reinsurance premiums (here $\theta = 0.15$ and $\theta = 0.18$), an over-all positive expected profit can only be achieved through randomization, not with a determinstic stop-loss contract (even when using the optimal retention). Figure 3.10 depicts the expected profit for arbitrary combinations of retentions d and probabilities p.

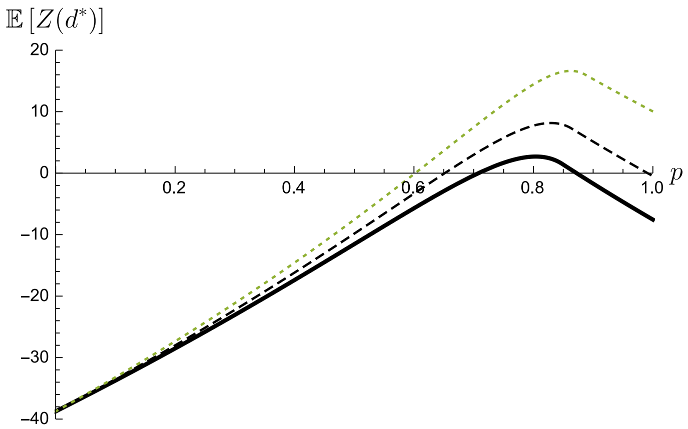

Figure 3.9: Maximal expected profit $\mathbb{E}[Z(d^*)]$ as a function of p for $\theta = 0.11$ (dotted), $\theta = 0.15$ (dashed) and $\theta = 0.18$ (black) with $\alpha = 0.05$ and $r_{CoC} = 0.07$.

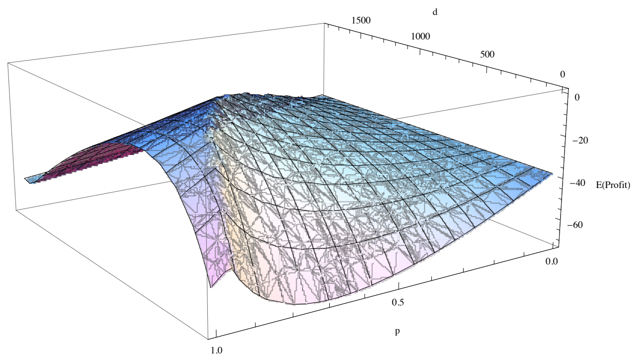

Figure 3.10: Expected profit as a function of d and p.

Another interpretation is to see this as a stop-loss contract, the retention of which is either ∞

3.6 Comparison with bounded stop-loss contracts

Randomization adds a degree of freedom to the classical stop-loss treaty, and so one may argue that this naturally leads to an improved solution. From Figure 3.1 it becomes clear that the resulting shape of the retained loss distribution of a randomized stop-loss treaty resembles a deterministic bounded stop-loss treaty

$$r_B(X) = x - \min\{(X - d_B)_+, l_B\} \tag{3.30}$$

with retention d_B and upper limit l_B. It is hence particularly instructive to compare the two. Note that beyond the retention the former takes a convex combination of the original loss c.d.f. F_X and the constant 1, whereas the latter shifts the part of F_X to the right of $d_B + l_B$ by l_B units to the left, cf. Figure 3.11. Hence, even for $d = d_B$, the resulting contracts will in general be different.

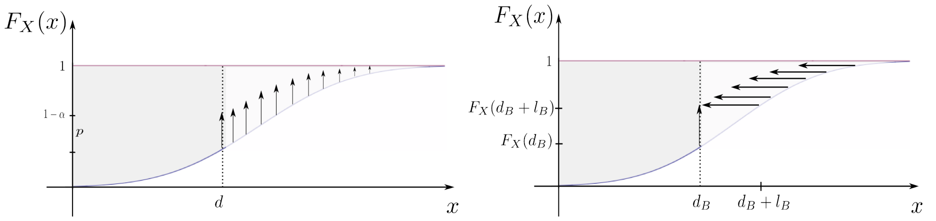

Figure 3.11: Original and retained loss distribution under randomized reinsurance (3.1) (left) and the bounded stop-loss contract (3.30) (right)

In [47], it has been shown that a bounded stop-loss treaty minimizes the total retained risk exposure of an insurer within the class of deterministic reinsurance forms where both the ceded and retained loss functions are non-decreasing. As outlined in Remark 3.2.1, under the expected value principle for the reinsurance premium, this then also applies to the objective function used in the present chapter, but under different weights for the sum of the competing terms. We now want to compare the optimal randomized strategy (d^*, p^*) with the optimal bounded stop-loss treaty $(d_B^*, l_B^*(d_B^*))$. By similar arguments as in Section 3.3 (or also following the reasoning in [47]), it is clear that it is better not to take any reinsurance if $d_B^* \geq \overline{F}_X^{-1}(\alpha)$, and in the other case necessarily $l_B^*(d_B^*) = \overline{F}_X^{-1}(\alpha) - d_B^*$. That is,

$$r_B^*(x) = \begin{cases} x - \min\left((x - d_B^*)_+, \overline{F}_X^{-1}(\alpha) - d_B^*\right), & \text{if } d_B^* < \overline{F}_X^{-1}(\alpha), \\ x, & \text{if } d_B^* \geq \overline{F}_X^{-1}(\alpha), \end{cases} \tag{3.31}$$

with $d_B^* = \overline{F}_X^{-1}\left(\frac{1}{1+\theta/r_{CoC}}\right)$. The analogue of (3.8) for the optimal bounded stop-loss then is

$$g_B(d_B^*) := \begin{cases} \left(1+\frac{\theta}{r_{CoC}}\right)\int_{d_B^*}^{\overline{F}_X^{-1}(\alpha)} \overline{F}_X(x)dx + d_B^*, & \text{if } d_B^* < \overline{F}_X^{-1}(\alpha), \\ \overline{F}_X^{-1}(\alpha), & \text{if } d_B^* \geq \overline{F}_X^{-1}(\alpha). \end{cases} \tag{3.32}$$

For the best randomized stop-loss treaty, the respective amount reads

$$g\left(d^*, 1-\frac{\alpha}{\overline{F}_X(d^*)}\right) = \begin{cases} \left(1+\frac{\theta}{r_{CoC}}\right)\left(1-\frac{\alpha}{\overline{F}_X(d^*)}\right)\int_{d^*}^{\infty} \overline{F}_X(x)dx + d^*, & \text{if } d^* < \overline{F}_X^{-1}(\alpha), \\ \overline{F}_X^{-1}(\alpha), & \text{if } d^* \geq \overline{F}_X^{-1}(\alpha), \end{cases}$$

where d^* is determined according to Section 3.3.

Let us consider any candidate retention $0 \leq d < \overline{F}_X^{-1}(\alpha)$ and $d = d_B$. Then for both the randomized stop-loss and the bounded stop-loss, the choice $p^*(d)$ and $l_B^*(d)$ will be such that the resulting $\text{VaR}_\alpha(r(X))$ is equal to d. To quantify the performance difference of the two treaties one is thus left with comparing the pure reinsurance premiums:

$$\begin{aligned} h(d) :&= \left(1-\frac{\alpha}{\overline{F}_X(d)}\right)\int_d^\infty \overline{F}_X(x)dx - \int_d^{\overline{F}_X^{-1}(\alpha)} \overline{F}_X(x)dx, \\ &= \int_{\overline{F}_X^{-1}(\alpha)}^\infty \overline{F}_X(x)dx - \alpha \cdot e_X(d), \\ &= \alpha\left(e_X\left(\overline{F}_X^{-1}(\alpha)\right) - e_X(d)\right). \end{aligned}$$

Correspondingly, if the mean-excess function is increasing, which is a property typically shared by the class of heavy-tailed distributions (see e.g. Embrechts et al. [59, Ch.6]), it follows that a bounded stop-loss treaty is preferable to a randomized stop-loss treaty for each fixed retention level $d = d_B$. In other words, shifting the distribution by $l_B^*(d)$ to obtain $\text{VaR}_\alpha(r(X)) = d$ then leads to a cheaper premium than reshaping the c.d.f. by randomization towards $\text{VaR}_\alpha(r(X)) = d$. Since this is true for all d, the best bounded stop-loss treaty then also outperforms the best randomized stop-loss treaty.
On the other hand, for distributions with decreasing mean-excess function (like the uniform distribution, certain Gamma distributions or the light-tailed Weibull distribution), randomization outperforms bounded stop-loss for each retention level and correspondingly also for the respective optimal retention levels.
When the mean-excess function is not monotone, the performance comparison can be more intricate, cf. Example 3.6.4.

In the following, we consider some concrete examples.

Example 3.6.1. *If X is exponentially distributed, the mean-excess function $e_X(d)$ is constant, so that $h(d) = 0$ for all $d < \overline{F}_X^{-1}(\alpha)$. This means that in this case the best randomized stop-loss and the best bounded stop-loss treaty lead to the same resulting loss distribution, and correspondingly the optimal values d^* and d_B^* must coincide. This is of course due to the lack-of-memory property of the exponential distribution: for the region to the right of the retention level d, shifting the distribution function from the right by $l_B^*(d)$ into the point $(d, 1-\alpha)$ is equivalent to rescaling it up into that same attachment point. This can also easily be verified analytically by realizing that $\mathbb{P}(r(X,d) > d + y) = \alpha\mathbb{P}(X > y)$ as well as $\mathbb{P}(r_B(X) > d_B + y) = \alpha\mathbb{P}(X > y)$ for the respective optimal values $p^*(d)$ and $l_B^*(d_B)$ and all $y > 0$.*

Example 3.6.2. *Let X be uniformly distributed in $[0, b]$, in which case $\overline{F}_X^{-1}(\alpha) = b(1-\alpha)$. Here $e_X(d)$ is decreasing in d, so a randomized stop-loss will lead to a better profitability. The optimal bounded stop-loss is the following: If $1/\alpha > 1 + \theta/r_{CoC}$, then the retention $d_B^* = b\left(1 - \frac{1}{1+\frac{\theta}{r_{CoC}}}\right)$ is chosen together with the layer $l_B^*(d_B^*) = b\left(\frac{1}{1+\frac{\theta}{r_{CoC}}} - \alpha\right)$, otherwise it is preferable not to buy reinsurance. After some calculations, one gets*

$$g_B(d_B^*) = \begin{cases} \frac{b}{2}\left(-\alpha^2\left(1 + \frac{\theta}{r_{CoC}}\right) + \frac{1+\frac{2\theta}{r_{CoC}}}{1+\frac{\theta}{r_{CoC}}}\right), & \text{if } \frac{1}{\alpha} > 1 + \frac{\theta}{r_{CoC}}, \\ b(1-\alpha), & \text{if } \frac{1}{\alpha} \le 1 + \frac{\theta}{r_{CoC}}. \end{cases}$$

At the same time, under the optimal randomized stop-loss, we have in view of (3.15)

$$g\left(d^*, p^*\right) = \begin{cases} b\left(1 - \frac{1}{2}\left(\alpha + \frac{1}{1+\frac{\theta}{r_{CoC}}}\right) - \frac{1}{8}\alpha^2\left(1 + \frac{\theta}{r_{CoC}}\right)\right), & \text{if } \frac{2}{\alpha} > 1 + \frac{\theta}{r_{CoC}}, \\ b(1-\alpha), & \text{if } \frac{2}{\alpha} \le 1 + \frac{\theta}{r_{CoC}}. \end{cases}$$

The difference $g_B(d_B^*) - g(d^*, p^*) := D$ reads

$$D = \begin{cases} 0, & \textit{if } \frac{1}{\alpha} \leq \frac{1}{2}\left(1+\frac{\theta}{r_{CoC}}\right), \\ \frac{b}{2}\left(\frac{1}{1+\frac{\theta}{r_{CoC}}} + \alpha\left(\frac{\alpha}{2}\left(1+\frac{\theta}{r_{CoC}}\right)-1\right)\right) > 0, & \textit{if } \frac{1}{2}\left(1+\frac{\theta}{r_{CoC}}\right) < \frac{1}{\alpha} \leq \left(1+\frac{\theta}{r_{CoC}}\right), \\ \frac{b\alpha}{2}\left(1-\frac{3}{4}\alpha\left(1+\frac{\theta}{r_{CoC}}\right)\right) > 0, & \textit{if } \frac{1}{\alpha} > 1+\frac{\theta}{r_{CoC}}. \end{cases} \tag{3.33}$$

Correspondingly, the best randomized stop-loss treaty is always at least as good as the best bounded stop-loss contract, and typically better. Note that the performance difference increases in b. *It is also worth mentioning that for a uniformly distributed risk with a bounded stop-loss, the c.d.f. of the retained amount attains* 1 *at* $b - l_B^* < b$, *which is sub-optimal in view of minimizing the reinsurance premium. Conversely, by construction, the resulting c.d.f. of randomized stop-loss attains* 1 *only at* b. *Figure 3.12 illustrates the expected profit under the optimal bounded stop-loss (dashed) and the optimal randomized stop-loss (solid) for* $\alpha = 0.05$, $r_{CoC} = 0.07$ *and* $b = 5$ *as a function of the premium loading* θ.

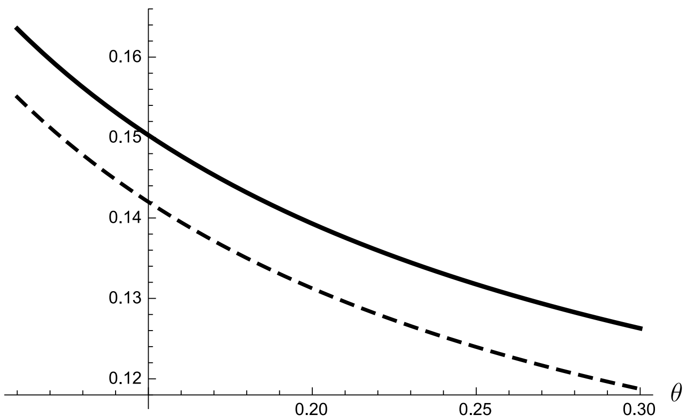

Figure 3.12: Expected profit with the optimal bounded stop-loss (dashed) vs. optimal randomized stop-loss (solid)

Example 3.6.3. *Let* X *be a shifted Pareto random variable with*

$$F_X(x) = 1 - \left(\frac{\xi}{x+\xi}\right)^{1/\gamma}, \qquad \xi > 0\,;\, \gamma < 1.$$

In this case the mean-excess function $e_X(d)$ *is increasing, so a randomized stop-loss treaty can not outperform the best bounded stop-loss. The optimal bounded stop-loss strategy is as follows: If* $1/\alpha > 1 + \theta/r_{CoC}$, *then one chooses the retention* $d_B^* =$

$\xi\left(\left(\frac{1}{1+\frac{\theta}{r_{CoC}}}\right)^{-\gamma}-1\right)$ *together with the layer* $l_B^*(d_B^*) = \xi\left(\alpha^{-\gamma}-\left(\frac{1}{1+\frac{\theta}{r_{CoC}}}\right)^{-\gamma}\right)$, *otherwise no reinsurance is taken. In view of (3.32), this translates into*

$$g_B(d_B^*) := \begin{cases} \xi\left(\left(\frac{1}{1+\frac{\theta}{r_{CoC}}}\right)^{-\gamma} - 1 + \frac{\left(1+\frac{\theta}{r_{CoC}}\right)}{\frac{1}{\gamma}-1}\left(\left(\frac{1}{1+\frac{\theta}{r_{CoC}}}\right)^{1-\gamma} - \alpha^{1-\gamma}\right)\right), & \text{if } \frac{1}{\alpha} > 1+\frac{\theta}{r_{CoC}}, \\ \xi\left(\alpha^{-\gamma}-1\right), & \text{if } \frac{1}{\alpha} \le 1+\frac{\theta}{r_{CoC}}. \end{cases}$$

On the other hand, in view of (3.14), *the optimal randomized stop-loss strategy is given by*

$$g\left(d^*, p^*\right) = \begin{cases} \xi\left(\left(\frac{1}{a} - \frac{\alpha}{\frac{1}{\gamma}-1}\right)^{-\gamma}\left(1+\frac{a}{\frac{1}{\gamma}-1}\left(\frac{1}{a}-\frac{\alpha}{1-\gamma}\right)\right) - 1\right), & \text{if } \frac{1}{\alpha}(1-\gamma) > a, \\ \xi\left(\alpha^{-\gamma}-1\right), & \text{if } \frac{1}{\alpha}(1-\gamma) \le a, \end{cases}$$

where $a := 1+\frac{\theta}{r_{CoC}}$. *The difference* $D := g_B(d_B^*) - g\left(d^*, 1-\frac{\alpha}{F_X(d^*)}\right)$ *then takes the form*

$$D = \begin{cases} 0, & \text{if } \frac{1}{\alpha} \le a, \\ \xi\left(a^{\gamma} - \alpha^{-\gamma} + \frac{a}{\frac{1}{\gamma}-1}\left(a^{\gamma-1} - \alpha^{1-\gamma}\right)\right) < 0, & \text{if } a < \frac{1}{\alpha} \le \frac{a}{1-\gamma}, \\ \frac{\xi\left(\left(\frac{1}{\gamma}-1\right)\left(\frac{a^{\gamma}}{\gamma} - a\alpha^{1-\gamma}\right) + \frac{1}{\gamma}\left(1+a\alpha-\frac{1}{\gamma}\right)\left(\frac{1}{a}-\frac{\alpha}{\frac{1}{\gamma}-1}\right)^{-\gamma}\right)}{\left(\frac{1}{\gamma}-1\right)^2} < 0, & \text{if } \frac{1}{\alpha} > \frac{a}{1-\gamma}, \end{cases} \tag{3.34}$$

so that indeed here a bounded stop-loss contract is always preferable.

Example 3.6.4. *Let us now consider an example of a distribution with non-monotone mean-excess function. Concretely, let us introduce an upper truncation point* $T > 0$ *to the shifted Pareto distribution considered in Example 3.6.3, i.e.*

$$F_X(x) = \frac{1-\left(\frac{\xi}{x+\xi}\right)^{\frac{1}{\gamma}}}{1-\left(\frac{\xi}{T+\xi}\right)^{\frac{1}{\gamma}}}, \qquad 0 \le x \le T\ ; \xi > 0\ ; \gamma < 1.$$

Such distributions recently gained some popularity in insurance claims modelling (see *e.g. [5, Ch.4]). The corresponding mean-excess function is*

$$e_X(d) = \frac{\left(\frac{\xi}{T+\xi}\right)^{\frac{1}{\gamma}}\left(d+\xi+\frac{(T-d)}{\gamma}\right) - \left(\frac{\xi}{d+\xi}\right)^{\frac{1}{\gamma}}(d+\xi)}{\left(\frac{1}{\gamma}-1\right)\left(\left(\frac{\xi}{T+\xi}\right)^{\frac{1}{\gamma}} - \left(\frac{\xi}{d+\xi}\right)^{\frac{1}{\gamma}}\right)}, \qquad 0 \le d < T,$$

which is non-monotone (first increasing and then decreasing to 0*) for triples* (ξ, γ, T) *with* $e'_X(0) > 0$ *(cf. Figure 3.13).*

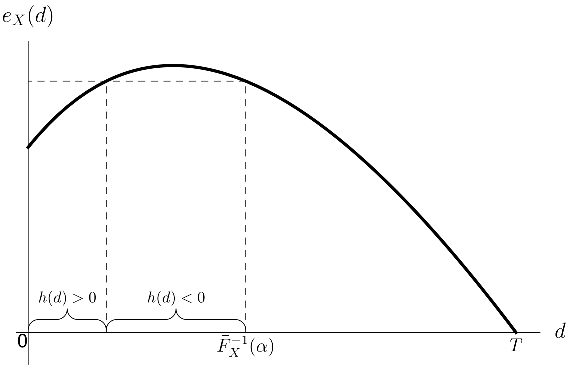

Figure 3.13: Possible shape of the mean-excess function for a shifted truncated Pareto random variable

Figure 3.14 depicts the expected profit under both treaties as a function of the truncation point T *for* $\xi = 20, \gamma = 0.5, \alpha = 0.05, \theta = 0.2$ *and* $r_{CoC} = 0.07$*. One sees that there is a threshold value for* T *above which the heavy-tailed feature of the risk* X *starts to dominate, making bounded stop-loss more attractive. However, for smaller values of* T *the randomized stop-loss is preferable.*

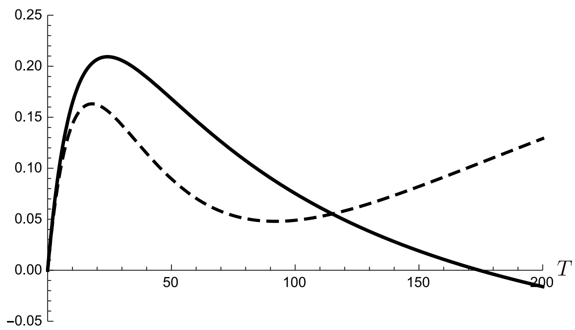

Figure 3.14: Expected profit with optimal bounded stop-loss (dashed) vs. optimal randomized stop-loss (solid) as a function of T

3.7 Conclusion

In this chapter, we showed that randomizing classical reinsurance treaties can be beneficial for the insurer. While randomization is a well-known mathematical tool to help identifying optimal deterministic solutions, the purpose here was to initiate a discussion and ponder the possibilities of actually implementing additional randomness in the settlement of the risk sharing arrangement between insurer and reinsurer. In this context, one should keep in mind that randomization has potential advantages with respect to moral hazard problems, as it is unclear during the settlement procedure who will finally have to pay the claim. Also, when comparing the randomized stop-loss treaty with deterministic bounded stop-loss, one may argue that in the former case the resulting retained loss distribution beyond the retention is determined by a part closer to the center for which one may have more confidence in the chosen model (as in the latter the respective part is further in the tail of the original loss distribution).

We deliberately chose a simple form of randomization as well as simple model assumptions here, in order to make the reasoning transparent, and clearly many variants and generalizations are possible. This includes considering more general reinsurance premium principles, but also randomization of individual claims (like in excess-of-loss treaties). For instance, rather than participating with a fraction p in all claims like in a quota-share arrangement, the reinsurer could achieve a similar result by paying each claim fully, but only with a probability p, independently for each claim (which can be preferable in terms of administrative expenses). On the aggregate level, one can view the introduced randomization also as a simple alternative way to reshape the loss distribution (for instance when 'picking' any target point above the original loss distribution function for the retained loss distribution function, one can realize the resulting risk transfer through simple randomization. This can in general be a simple means to taylor the needs of clients for reinsurance companies (in terms of target shapes of the retained loss), and more intricate randomization mechanisms can further increase the possible variations. While the concept can seem non-intuitive in the first place, it may provide a thought-provoking additional perspective on the nature of the problem (as well as on the choice of objective functions and constraints).

In this chapter, we focused on the Value-at-Risk for measuring risk, and the results depend crucially on this choice. In a subsequent study, we will consider the effects of randomization for other choices of risk measures. However, the Value-at-Risk is the risk measure implemented in many regulatory systems nowadays, and the arguments in this chapter may underline some of the shortfalls of this risk measure (particularly the encouragement to 'only' optimize the retained situation up to the point of the solvency requirement). Clearly, in practical situations the target solvency ratio will often be considerably larger than 1, and corresponding adaptations of the arguments can then be made.

Finally, in this chapter we considered the reinsurer's preferences solely through the reinsurance premium rule. It will be interesting future work to include the reinsurer's viewpoint on the suitability of randomized contracts by considering joint optimization criteria.

Chapter 4

Affine dividend strategies in a classical risk model[1]

Abstract

We consider a classical compound Poisson risk model with affine dividend payments. We illustrate how both by analytical and probabilistic techniques closed-form expressions for the expected discounted dividends until ruin and the Laplace transform of the time to ruin can be derived for exponentially distributed claim amounts. Moreover, numerical examples are given which compare the performance of the proposed strategy to classical barrier strategies and illustrate that such affine strategies can be a noteworthy compromise between profitability and safety in collective risk theory.

4.1 Introduction

The question of how to pay dividends from a surplus process of an insurance portfolio has a long tradition in collective risk theory. The classical criterion to measure the performance of such a dividend strategy is the expected sum of discounted div-

[1]This chapter is based on the paper: Hansjörg Albrecher and Arian Cani. Risk theory with affine dividend payment strategies. In *Number theory—Diophantine problems, uniform distribution and applications*, pages 25–60. Springer, Cham, 2017

idend payments over the lifetime of the process, where typically the discount rate is assumed to be positive and constant over time. In this case the optimal strategy is a balance between paying out dividends early (in view of the discounting) and paying dividends later (so that due to the typically positive drift of the process the lifetime (and hence the time span of dividend payments) is prolongated). This criterion was first proposed by de Finetti [56], who proved for a simple random walk model that the optimal strategy is a barrier strategy, that is, dividends are paid out whenever the surplus process exceeds a threshold value (the horizontal dividend barrier), and no dividends are paid out below that level (i.e. the process is reflected at this barrier). Later, Gerber [72] proved that for a Cramér-Lundberg risk process, a so-called band strategy is optimal, which simplifies to a barrier strategy in some particular cases (including the one with exponential claim size distribution). More recently, this stochastic control problem was embedded in modern control theory, which led to surprisingly challenging mathematical problems (see e.g. Schmidli [110] and Azcue and Muler [25]). The optimal dividend problem was also studied intensively in many different variants, including model variations, transaction costs, as well as other objective functions and constraints, see [9] and [21] for an overview.

One disadvantage of the classical criterion of maximizing the expected sum of discounted dividend payments until ruin is that it focuses on profitability only, and does not consider the lifetime of the controlled process (in particular, under the optimal band strategy, the process will be ruined with probability 1, and if the barrier is at level 0, then it is even optimal to pay out all the surplus immediately and get ruined at the occurrence of the first claim payment; we refer to [20] for an overview of the ruin concept and its many mathematical implications). In [119], a variant of the dividend problem was studied, where the objective function is a weighted sum of expected discounted dividend payments until ruin and expected ruin time. It turns out that in such a setting, again a band strategy (respectively, barrier strategy) is optimal, albeit with modified parameters. This approach was then extended to more general models in [96].

The criterion of maximizing the expected sum of discounted dividend payments until ruin may be considered as a somewhat natural target, which also has economic motivation in terms of valuating a company on the basis of this quantity (starting with [73] and later variants within the corporate finance literature). However, if a barrier strategy is optimal, in addition to the solvency aspect mentioned above, this strategy does not pay any dividends whenever the surplus is below the barrier and it pays the maximal feasible amount above the barrier, so that the dividend stream may be very uneven over time. At the same time, empirical research suggests that companies typically strive for a smooth dividend distribution over time with the incentive to gradually move towards a long-term payout ratio (see e.g. Lintner [94] for a pioneering study on this topic). This goes in line with the observation that dividend payments in practice often adjust to changes in earnings only slowly (indicating that the management exhibits some reluctance to either increase or decrease established dividend levels unless there is sufficient confidence that the new levels are justified for the future, not the least to avoid psychological effects entailed by dividend reductions), see also Brav et al. [34].

In view of these aspects, in this chapter we propose a dividend strategy that secures a continuous dividend payment stream, the rate of which is adjusted according to the present surplus value in an affine way. We will study such a strategy for a compound Poisson surplus model. Our approach is in part inspired by Avanzi and Wong [22] who studied a related strategy for a diffusion process and also gave an extensive numerical study of its performance. Mathematically, our model in the Cramér-Lundberg framework will lead to an Ornstein-Uhlenbeck process driven by the compound Poisson subordinator. For such a setup we will derive equations for the expected discounted dividend payments until ruin as well as for the Laplace transform of the time of ruin. These equations turn out to be challenging in their own right, and various different approaches to solve them will lead to interesting relations between special functions of hypergeometric type.[2]

An interesting consequence of the numerical results at the end of the chapter is that utilizing such an affine dividend strategy leads to almost the same performance as the barrier strategy in terms of expected sum of discounted dividend payments, but has – in many different parameter settings – a considerably longer lifetime. Consequently, in view of a compromise between profitability and safety, such an affine strategy is certainly an interesting alternative. In fact, such a strategy is known to be optimal in a somewhat different context of linear quadratic optimal control problems, where quadratic deviations of a target 'dividend' rate are punished in the

[2] In this way, a practically motivated question of insurance risk theory leads to non-trivial mathematical problems and relations, a connection which is also in the tradition of Robert Tichy's work, to whom this paper is dedicated. For the application of Quasi-Monte Carlo results to risk theory by Robert Tichy, see e.g. [120, 10].

objective function, see Steffensen [116] for an application in the control of pension funds and Parlar [101] for a model in forest management systems.

The rest of the chapter is organized as follows. In Section 4.2, we introduce the model and discuss some basic properties. Section 4.3 then derives the integro-differential equation for the expected discounted dividend payments and studies its solution for the case of exponentially distributed claim amounts. In Section 4.4 we pursue an alternative approach for the solution of the latter equation via Laplace transforms, leading to a rather intricate study of certain special functions and suggesting an identity that seems to be new and non-obvious. In Section 4.5 we adapt the calculations of Section 4.3 to study the Laplace transform of the time to ruin. In order to retrieve a concrete formula for the expected ruin time from the Laplace transform, we then employ an approach based on digamma functions and another one based on Kampé de Fériet functions. Section 4.6 gives a simple and intuitive probabilistic view to connect the quantities of Sections 4.3 and 4.5. Finally, Section 4.7 provides detailed numerical illustrations to test the proposed strategy and determines optimal parameters. The results are then compared to the optimal barrier strategies showing that affine strategies can be a competitive alternative to barrier strategies when paying dividends.

4.2 The model

In the classical Cramér-Lundberg risk model, the surplus process of an insurance company $(R_t)_{t\geq 0}$ is described by

$$R_t = x + ct - \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0, \tag{4.1}$$

where $x = R_0$ is the initial capital, $c > 0$ is the constant premium rate and the claims $\{Y_i\}_{i\in\mathbb{N}}$ are a sequence of independent and identically distributed positive random variables with distribution function F_Y, bounded density f_Y and finite mean μ. The number of claims up to time $t \geq 0$ is assumed to be a homogeneous Poisson process N_t with intensity $\lambda > 0$, independent of $\{Y_i\}_{i\in\mathbb{N}}$.

Let D_t denote the accumulated dividends paid up to time t, so $X_t := R_t - D_t$ is the surplus process after dividend payments. Assume now that dividends are paid according to an affine strategy, i.e.

$$dD_t = (qX_t + \beta)dt, \tag{4.2}$$

where $q > 0$ is a fixed proportionality constant and $0 \le \beta \le c$ is a constant rate. Then

$$dX_t = (c - (qX_t + \beta))\, dt - dS_t, \tag{4.3}$$

which identifies X_t as a Lévy-driven Ornstein-Uhlenbeck process (in the present case, the driving Lévy process is the compound Poisson process $S_t = \sum_{i=1}^{N_t} Y_i$). The unique solution to (4.3) is given in terms of the stochastic integral

$$X_t = \frac{c-\beta}{q} + \left(x - \frac{c-\beta}{q}\right) e^{-qt} - \int_0^t e^{-q(t-u)} dS_u, \tag{4.4}$$

i.e.

$$X_t = \frac{c-\beta}{q} + \left(x - \frac{c-\beta}{q}\right) e^{-qt} - \sum_{i=1}^{N_t} e^{-q(t-T_i)} Y_i,$$

which embeds X_t into the class of shot-noise processes. One sees that the process X_t behaves like an exponentially decaying function between the claim occurrences, and the influence of past claims on the value of X_t also decays exponentially in time (see Figure 4.1 for a sample path of X_t).

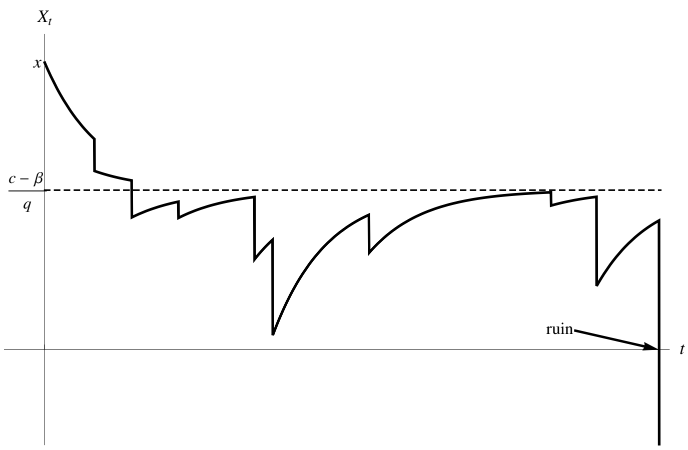

Figure 4.1: Sample path of X_t

Let

$$\tau_x := \inf\{t \geq 0 : X_t < 0 \mid X_0 = x\}$$

denote the time of ruin of X_t and note that $P(\tau_x < \infty) = 1$ for all $x \geq 0$ (i.e. ruin is certain). The latter holds true, since the process X_t is upper-bounded by $\max\{x, (c-\beta)/q\}$ (above $(c-\beta)/q$ there is a negative drift down to this level and below it is bounded by this level).

If X_t is not stopped at ruin, then as $t \to \infty$

$$X_t \xrightarrow{d} X_\infty := \frac{c-\beta}{q} - \int_0^\infty e^{-qu} dS_u, \tag{4.5}$$

see e.g. [106]. If the claim sizes Y_i are Exp(α)-distributed, then the self-decomposable limit random variable X_∞ simplifies further to a shifted Gamma random variable $X_\infty = (c-\beta)/q - Z$ with $Z \sim \Gamma(\lambda/q, \alpha)$, see also [50, 111].

4.3 Expected discounted dividend payments

We are now interested in the expected value of the sum of the discounted dividend payments up to the time of ruin

$$V(x) := \mathbb{E}_x \left[\int_0^{\tau_x} e^{-\delta t} (qX_t + \beta)\, dt \right], \tag{4.6}$$

where $\delta \geq 0$ is a force of interest for valuation. Let us first consider some elementary, but general properties of the function $V(x)$ regarding bounds and growth rate.

Proposition 4.3.1. *For $x \geq 0$, the function $V(x)$ satisfies the following bounds:*

$$\underline{l} + \frac{qx}{q+\delta} \leq V(x) \leq \bar{l} + \frac{qx}{q+\delta}, \tag{4.7}$$

where $\underline{l} = \frac{\lambda(q(c-\lambda\mu)+\delta\beta)}{(q+\delta)(\delta+\lambda)}$ and $\bar{l} = \frac{cq+\delta\beta}{\delta(q+\delta)}$.

Proof. For any $t \geq 0$, the process X_t in (4.3) satisfies $X_t \leq \left(x - \frac{c-\beta}{q}\right) e^{-qt} + \frac{c-\beta}{q} := \tilde{X}_t$. Then, clearly,

$$V(x) \leq \mathbb{E}_x \left[\int_0^\infty e^{-\delta t} \left(q\tilde{X}_t + \beta\right) dt \right] = \frac{qx - c + \beta}{q+\delta} + \frac{c}{\delta},$$

which yields the upper bound.
For the lower bound, define $h(x) := \frac{qx}{q+\delta}\mathbb{1}_{\{x\geq 0\}}$ and let M be an operator acting on h defined as

$$(Mh)(x) := \mathcal{L}h(x) - \delta h(x) + qx + \beta, \tag{4.8}$$

for $x \geq 0$, where $\mathcal{L}h(x) := (c - (qx+\beta))\, h'(x) + \lambda \left(\int_0^x h(x-y)dF_Y(y) - h(x)\right)$ is the infinitesimal generator of the process (4.3). More concretely, (4.8) can be rewritten as

$$\begin{aligned}(Mh)(x) &= (c-(qx+\beta))\,\frac{q}{q+\delta} + \lambda\left(\int_0^x \frac{q(x-y)}{q+\delta}dF_Y(y) - \frac{qx}{q+\delta}\right) - \frac{\delta qx}{q+\delta} + qx + \beta,\\ &= \frac{cq+\delta\beta}{q+\delta} + \frac{\lambda qx}{q+\delta}(F_Y(x)-1) - \frac{\lambda q}{q+\delta}\int_0^x y\, dF_Y(y).\end{aligned}$$

Observe that $(Mh)'(x) = \frac{\lambda q}{q+\delta}(F_Y(x)-1) \leq 0$ with boundary values $(Mh)(0) = \frac{cq+\delta\beta}{q+\delta} > 0$ and $\lim_{x\to\infty}(Mh)(x) = \frac{q(c-\lambda\mu)+\delta\beta}{q+\delta} > 0$. Thus, $(Mh)(x)$ is strictly positive and monotone decreasing, bounded from below by $\frac{q(c-\lambda\mu)+\delta\beta}{q+\delta}$.

In view of the Dynkin formula applied to the function $e^{-\delta t}h(X_t)$, the process

$$e^{-\delta t}h(X_t) - h(x) - \int_0^t e^{-\delta s}\left[\mathcal{L}h(X_s) - \delta h(X_s)\right] ds$$

is a zero-expectation martingale. Bearing in mind that the stopped process $X_{t\wedge\tau}$ is also a martingale, we obtain

$$\mathbb{E}_x\left(e^{-\delta t\wedge\tau}h(X_{t\wedge\tau})\right) = h(x) + \mathbb{E}_x\left(\int_0^{t\wedge\tau} e^{-\delta s}\left[\mathcal{L}h(X_s) - \delta h(X_s)\right] ds\right).$$

From the properties of M, we get that the integrand on the right-hand side is bounded from below by $-(qX_s+\beta) + \frac{q(c-\lambda\mu)+\delta\beta}{q+\delta}$. Furthermore, since $h(X_{t\wedge\tau})$ is linearly bounded in t, an application of the monotone convergence theorem implies that as $t\to\infty$, the right-hand side converges to 0. Combining the above and rearranging terms yields

$$\begin{aligned}\mathbb{E}_x\left[\int_0^\tau e^{-\delta s}\left(qX_s+\beta\right)\, ds\right] &\geq h(x) + \mathbb{E}_x\left[\int_0^\tau e^{-\delta s}\frac{q(c-\lambda\mu)+\delta\beta}{q+\delta}ds\right],\\ &\geq h(x) + \mathbb{E}_x\left[\int_0^{T_1} e^{-\delta s}\frac{q(c-\lambda\mu)+\delta\beta}{q+\delta}ds\right],\\ &= h(x) + \frac{\lambda\left(q(c-\lambda\mu)+\delta\beta\right)}{(q+\delta)(\delta+\lambda)},\end{aligned}$$

which gives the result. □

Proposition 4.3.2. *For $0 \leq y < x$ and $\overline{f_Y} := \max_x f_Y(x) < \infty$, the following inequality holds*

$$\frac{\lambda q(x-y)}{q+\lambda+\delta} \leq V(x) - V(y) \leq \frac{q(x-y)}{q+\delta}\left(1 + \left(x - y + \frac{c}{\delta} + \frac{\beta}{q}\right)\overline{f_Y}\right).$$

Proof. Let $0 \leq y < x$ and let X_t^y and X_t^x be the processes in (4.4) started in y and x with respective times of ruin τ_y and τ_x.
Additionally, define $\mathcal{M} = \{\omega \in \Omega \mid \tau_x(\omega) = \tau_y(\omega)\}$ and denote by $\mathcal{M}^c$ its complementary set. A pathwise comparison of both processes on $\mathcal{M}$ gives $X_t^x(\omega) - X_t^y(\omega) = (x-y)e^{-qt}$. We have

$$\begin{aligned} V(x) - V(y) &= \mathbb{E}\left[\int_0^{\tau_y} e^{-\delta t} q X_t^x dt\right] - \mathbb{E}\left[\int_0^{\tau_y} e^{-\delta t} q X_t^y dt\right] + \mathbb{E}\left[\mathbb{1}_{\mathcal{M}^c}\int_{\tau_y}^{\tau_x} e^{-\delta t}\left(qX_t^x + \beta\right) dt\right], \\ &= \mathbb{E}\left[\int_0^{\tau_y} e^{-(q+\delta)t} q(x-y) dt\right] + \mathbb{E}\left[\mathbb{1}_{\mathcal{M}^c}\int_{\tau_y}^{\tau_x} e^{-\delta t}\left(qX_t^x + \beta\right) dt\right], \\ &\geq \mathbb{E}\left[\int_0^{T_1} e^{-(q+\delta)t} q(x-y) dt\right], \\ &= \frac{\lambda q(x-y)}{q+\lambda+\delta}. \end{aligned}$$

For the reverse direction, we can write

$$\begin{aligned} V(x) - V(y) &= \mathbb{E}\left[\int_0^{\tau_y} e^{-(q+\delta)t} q(x-y) dt\right] + \mathbb{E}\left[\mathbb{1}_{\mathcal{M}^c}\int_{\tau_y}^{\tau_x} e^{-\delta t} X_t^x dt\right], \\ &\leq \int_0^{\infty} e^{-(q+\delta)t} q(x-y) dt + V(x-y)\mathbb{E}\left[\mathbb{1}_{\mathcal{M}^c}\right]. \end{aligned} \tag{4.9}$$

The last inequality follows from the a.s. finiteness of τ_y in the first integral combined with the strong Markov property of the process X^x and observing that on $\mathcal{M}^c$, $X_{\tau_y}^x(\omega) \leq (x-y)e^{-q\tau_y(\omega)} \leq x - y$ in the second integral. By definition, $\mathcal{M}^c$ comprises all paths ω such that $\tau_x(\omega) > \tau_y(\omega)$, therefore $\mathbb{E}[\mathbb{1}_{\mathcal{M}^c}] = P(\tau_x > \tau_y)$. Writing $X_{\tau_y -}^y$ for the surplus immediately prior to ruin of the surplus started in y and conditioning on the latter leads to

$$\begin{aligned} P(\tau_x > \tau_y) &= \int_0^{\max(y,\frac{c-\beta}{q})} P(\tau_x > \tau_y \mid X^y_{\tau_{y-}} = z) P(X^y_{\tau_{y-}} \in dz) \\ &= \int_0^{\max(y,\frac{c-\beta}{q})} P(z < Y \leq z + x - y)\, P(X^y_{\tau_{y-}} \in dz), \\ &= \int_0^{\max(y,\frac{c-\beta}{q})} \int_z^{z+x-y} f_Y(w)\, dw\, P(X^y_{\tau_{y-}} \in dz) \leq (x-y)\overline{f_Y}. \end{aligned}$$

Substituting the last result in (4.9) and explicitly evaluating the first integral in the aforementioned expression gives

$$V(x) - V(y) \leq \frac{q(x-y)}{q+\delta} + V(x-y)(x-y)\overline{f_Y}.$$

Combining this with the upper bound obtained in Proposition 4.3.1 establishes the result. □

Hence $V(x)$ is increasing. If the derivative exists, then using the typical infinitesimal generator arguments for X_t and in view of (4.7), one gets that $V(x)$ is characterized as a solution to the integro-differential equation (IDE)

$$(c - (qx+\beta))\, V'(x) - (\lambda+\delta)V(x) + \lambda \int_0^x V(x-y) dF_Y(y) = -(qx+\beta), \qquad x \geq 0. \tag{4.10}$$

4.3.1 Constructing an exact solution for exponential claims

We now assume that the claims are exponentially distributed with rate $\alpha > 0$. Then, applying the operator $(\frac{d}{dx} + \alpha)$ to both sides of (4.10) leads to the second-order differential equation

$$\begin{aligned} (c - (qx+\beta))\, V''(x) + [\alpha\,(c - (qx+\beta)) - (q+\lambda+\delta)]\, V'(x) - \alpha\delta V(x) \\ = -q(1+\alpha x) - \alpha\beta. \end{aligned} \tag{4.11}$$

Let V_h be the solution to the related homogeneous differential equation of (4.11). Choosing $f(z) := V_h(x)$ associated to the change of variable $z := z(x) = \frac{\alpha(c-(qx+\beta))}{q}$

produces *Kummer's* confluent hypergeometric equation

$$z f''(z) + (b - z)\, f'(z) - a f(z) = 0, \qquad z \leq \frac{\alpha(c-\beta)}{q}, \tag{4.12}$$

with parameters

$$a = \frac{\delta}{q}, \qquad b = 1 + \frac{\lambda + \delta}{q},$$

which has a regular singular point at $z = 0$ and an irregular singular point at $z = -\infty$ (which in the original coordinates correspond to $x = (c-\beta)/q \geq 0$ and $x = \infty$, respectively). This gives

$$V_h(x) = f(z) = \begin{cases} A_1\, M\left(\frac{\delta}{q}, 1 + \frac{\lambda+\delta}{q}, z(x)\right) + A_2\, U\left(\frac{\delta}{q}, 1 + \frac{\lambda+\delta}{q}, z(x)\right), & 0 \leq x \leq \frac{c-\beta}{q}, \\ A_3\, M\left(\frac{\delta}{q}, 1 + \frac{\lambda+\delta}{q}, z(x)\right) + A_4\, e^{z(x)} U\left(1 + \frac{\lambda}{q}, 1 + \frac{\lambda+\delta}{q}, -z(x)\right), & x > \frac{c-\beta}{q}, \end{cases} \tag{4.13}$$

for arbitrary constants $A_i, i = 1, \ldots 4$. Here

$$M(a, b, z) = {}_1F_1(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \tag{4.14}$$

denotes the Kummer confluent hypergeometric function with the Pochhammer symbol $(a)_n = \Gamma(a+n)/\Gamma(n)$, and

$$U(a, b, z) = \begin{cases} \frac{\Gamma(1-b)}{\Gamma(1+a-b)} M(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(1+a-b, 2-b, z) & b \notin \mathbb{Z}, \\ \lim\limits_{\theta \to b} U(a, \theta, z) & b \in \mathbb{Z}, \end{cases} \tag{4.15}$$

is Tricomi's confluent hypergeometric function. The piecewise construction of V_h originates from the fact that Tricomi's function $U(a, b, z)$ is in general complex-valued when its argument z is negative, that is, when $x > (c-\beta)/q$. Since we are looking for a real-valued solution V over the entire domain $x \geq 0$, another independent pair of solutions to (4.12), here, $M(a, b, z)$ and $e^z U(b-a, b, -z)$ needs to be chosen for $z < 0$, namely, $x > (c-\beta)/q$.

The general solution to (4.11) can then be written as

$$V(x) = V_h(x) + V_p(x),$$

where $V_p(x)$ is a particular solution to (4.11). Looking for a form $V_p(x) = Ax + B$, one finds

$$V_p(x) = \frac{1}{q+\delta}\left(qx + \beta + \frac{q}{\delta}\left(c - \frac{\lambda}{\alpha}\right)\right), \qquad x \geq 0.$$

To determine the constant coefficients $A_i, i = 1, \ldots 4$, we first investigate the components of V_h involving the Tricomi function U. For $a = \delta/q$ and $b = 1 + \frac{\lambda+\delta}{q} > 1$, $U(a,b,z)$ is singular at $z = 0$. Linear boundedness of V established in Proposition 4.3.1 then leads to the requirement $A_2 = 0$. Next, we focus on A_4: one has (cf. Olver [100])

$$\lim_{z\to 0} U(a,b,z) = \begin{cases} \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + \mathcal{O}\left(z^{2-\Re(b)}\right) & \text{if } 1 \leq \Re(b) < 2, \\ \frac{1}{\Gamma(a)} z^{-1} + \mathcal{O}(\log z) & \text{if } b = 2, \\ \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + \mathcal{O}\left(z^{2-\Re(b)}\right) & \text{if } \Re(b) \geq 2, b \neq 2. \end{cases}$$

In the original coordinates, this translates to

$$\lim_{x \to \frac{c-\beta}{q}+} e^{z(x)} U\left(1 + \frac{\lambda}{q}, 1 + \frac{\lambda+\delta}{q}, -z(x)\right) =$$

$$\begin{cases} \frac{\Gamma\left(\frac{\lambda+\delta}{q}\right)}{\Gamma\left(1+\frac{\lambda}{q}\right)} (-z(x))^{-\frac{\lambda+\delta}{q}} + \frac{\Gamma\left(-\frac{\lambda+\delta}{q}\right)}{\Gamma\left(1-\frac{\delta}{q}\right)} + +\mathcal{O}\left(x - \frac{c-\beta}{q}\right)^{1-\frac{\lambda+\delta}{q}} & \text{if } \frac{\lambda+\delta}{q} < 1, \\ \frac{1}{\Gamma\left(1+\frac{\lambda}{q}\right)} (-z(x))^{-1} + \mathcal{O}\left(\log\left(x - \frac{c-\beta}{q}\right)\right) & \text{if } \frac{\lambda+\delta}{q} = 1, \\ \frac{\Gamma\left(\frac{\lambda+\delta}{q}\right)}{\Gamma\left(1+\frac{\lambda}{q}\right)} (-z(x))^{-\frac{\lambda+\delta}{q}} + \frac{\Gamma\left(-\frac{\lambda+\delta}{q}\right)}{\Gamma\left(1-\frac{\delta}{q}\right)} + \mathcal{O}\left(x - \frac{c-\beta}{q}\right)^{1-\frac{\lambda+\delta}{q}} & \text{if } \frac{\lambda+\delta}{q} > 1. \end{cases}$$

The latter expression is unbounded for all choices of $(\lambda+\delta)/q$, so that by the linear boundedness of V we can also conclude $A_4 = 0$. On the other hand, the Kummer function M is analytic over the entire domain $x \geq 0$.

Next, the constant A_1 is determined by setting $x = 0$ in (4.10) which yields $(c - \beta)V'(0) = (\lambda+\delta)V(0) = -\beta$. Using the differentiation property

$$\frac{d}{dz} M(a,b,z) = \frac{a}{b} M(a+1, b+1, z) \tag{4.16}$$

(see [3]), this translates into

$$
\begin{aligned}
&(c-\beta)\left[\frac{-\alpha\delta}{q+\lambda+\delta}A_1 M\left(1+\frac{\delta}{q},2+\frac{\lambda+\delta}{q},z(0)\right)+\frac{q}{q+\delta}\right]+(\lambda+\delta)A_1 M\left(\frac{\delta}{q},1+\frac{\lambda+\delta}{q},z(0)\right)\\
&+(\lambda+\delta)\left[\frac{q}{\alpha\delta(q+\delta)}\left(\alpha(c-\beta)-(q+\lambda+\delta)\right)+\frac{q+\alpha\beta}{\alpha\delta}\right]\\
&=-\beta.
\end{aligned}
$$

Solving for A_1 gives

$$
A_1=\frac{\beta+\frac{q(c-\beta)}{q+\delta}-(\lambda+\delta)\left[\frac{q}{\alpha\delta(q+\delta)}\left(\alpha(c-\beta)-(q+\lambda+\delta)\right)+\frac{q+\alpha\beta}{\alpha\delta}\right]}{\frac{\alpha\delta(c-\beta)}{q+\lambda+\delta}M\left(1+\frac{\delta}{q},2+\frac{\lambda+\delta}{q},z(0)\right)+(\lambda+\delta)M\left(\frac{\delta}{q},1+\frac{\lambda+\delta}{q},z(0)\right)}.
$$

Finally, by the continuity of V at $x=(c-\beta)/q$ (which follows from Proposition 4.3.2), we get $A_3=A_1$, so that we arrive at the following result.

Proposition 4.3.3. *For any $x\geq 0$, the sum of the expected discounted dividend payments up to the time of ruin in a Cramér-Lundberg model with affine dividend strategy* (4.2) *and Exp(α)-distributed claims is given by*

$$
\begin{aligned}
V(x)=&\frac{\beta+\frac{q(c-\beta)}{q+\delta}-\frac{\lambda+\delta}{q+\delta}\left(\beta+\frac{q}{\delta}\left(c-\frac{\lambda}{\alpha}\right)\right)}{\frac{\alpha\delta(c-\beta)}{q+\lambda+\delta}M\left(1+\frac{\delta}{q},2+\frac{\lambda+\delta}{q},z(0)\right)+(\lambda+\delta)\,M\left(\frac{\delta}{q},1+\frac{\lambda+\delta}{q},z(0)\right)}M\left(\frac{\delta}{q},1+\frac{\lambda+\delta}{q},z(x)\right)\\
&+\frac{1}{q+\delta}\left(qx+\beta+\frac{q}{\delta}\left(c-\frac{\lambda}{\alpha}\right)\right),
\end{aligned}
\tag{4.17}
$$

where $z(x)=\frac{\alpha(c-(qx+\beta))}{q}$.

Remark 4.3.1. For $q\to\infty$ (i.e. infinite dividend rate), $V(x)$ in (4.17) tends to $x+\frac{c}{\lambda+\delta}$. Note that an infinite rate q instantaneously drives the process X_t to 0 implying an immediate lump sum dividend payment of size x. From then on, all incoming premium at rate c is immediately paid out as dividends (of which a magnitude $c-\beta$ is due to the proportional factor and β is due to the constant part), and the process X_t is continuously pushed back towards 0. The first claim will then lead to ruin and stops the dividend payments. That is, $q\to\infty$ corresponds to a horizontal dividend barrier strategy with barrier $b=0$.

4.4 A Laplace transform approach

The structure of equation (4.10) suggests that a Laplace transform approach could in general also be a feasible tool to determine V. Indeed, denote by

$$\widetilde{V}(s) := \int_0^\infty e^{-sx} V(x)\, dx, \quad \widetilde{f_Y}(s) := \int_0^\infty e^{-sx} f_Y(x)\, dx$$

the corresponding Laplace transforms. Then (4.10) turns into a first-order differential equation for $\widetilde{V}(s)$:

$$\widetilde{V}'(s) = \widetilde{V}(s) \frac{(\beta - c)s - q + \lambda + \delta - \lambda \widetilde{f_Y}(s)}{qs} + \frac{(c-\beta)V(0) - \frac{q}{s^2} - \frac{\beta}{s}}{qs}.$$

It has the solution

$$\widetilde{V}(s) = e^{-\frac{(c-\beta)}{q}s} s^{\frac{\lambda+\delta}{q}-1} e^{-\frac{\lambda}{q}\int \frac{\widetilde{f_Y}(s)}{s} ds} \cdot \left(\int \frac{(c-\beta)V(0) - \frac{q}{s^2} - \frac{\beta}{s}}{qs} e^{\frac{(c-\beta)}{q}s} s^{1-\frac{\lambda+\delta}{q}} e^{\frac{\lambda}{q}\int \frac{\widetilde{f_Y}(s)}{s} ds} ds + C \right)$$

for some constant C. In addition to the algebraic manipulations required in the Laplace transform domain, the inversion of $\widetilde{V}(s)$ is another intricate problem, see Section 4.4.1.

4.4.1 Exponential claims

It is instructive to see how for exponential claims with rate α the above expression simplifies to the explicit solution derived in the previous section. While it will become clear that for this case the approach of Section 4.3.1 leads to the result with considerably less effort, a comparison of the two approaches gives rise to identities between special functions which are interesting in their own right.
From $\widetilde{f_Y}(s) = \alpha/(s+\alpha)$ one gets after standard algebraic manipulations

$$\widetilde{V}(s) = e^{-\frac{(c-\beta)}{q}s} s^{\frac{\delta}{q}-1} (s+\alpha)^{\frac{\lambda}{q}} \left(C + \int \left(\frac{(c-\beta)V(0) - \frac{q}{s^2} - \frac{\beta}{s}}{q} \right) e^{\frac{(c-\beta)}{q}s} s^{-\frac{\delta}{q}} (s+\alpha)^{-\frac{\lambda}{q}} ds \right).$$

Expanding the exponential term inside the integral gives

$$\widetilde{V}(s) = e^{-\frac{(c-\beta)}{q}s} s^{\frac{\delta}{q}-1}(s+\alpha)^{\frac{\lambda}{q}} \left(C + \frac{(c-\beta)}{q} V(0) \sum_{n=0}^{\infty} \frac{\left(\frac{c-\beta}{q}\right)^n}{n!} \int s^{n-\frac{\delta}{q}}(s+\alpha)^{-\frac{\lambda}{q}}\, ds \right.$$
$$\left. - \sum_{n=0}^{\infty} \frac{\left(\frac{c-\beta}{q}\right)^n}{n!} \int s^{n-\frac{\delta}{q}-2}(s+\alpha)^{-\frac{\lambda}{q}}\, ds - \frac{\beta}{q} \sum_{n=0}^{\infty} \frac{\left(\frac{c-\beta}{q}\right)^n}{n!} \int s^{n-\frac{\delta}{q}-1}(s+\alpha)^{-\frac{\lambda}{q}}\, ds \right). \tag{4.18}$$

The following Lemma establishes a connection between the integrals in (4.18) and the Gauss hypergeometric function ${}_2F_1$. Recall (also for later use) that the generalized hypergeometric function ${}_pF_q$ is defined through

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

Lemma 4.4.1. *For $(n,k) \in \mathbb{N}_0 \times \mathbb{N}_0$ and $n - \frac{\delta}{q} - k \notin \mathbb{Z}^-$, one has*

$$\int s^{n-\frac{\delta}{q}-k}(s+\alpha)^{-\frac{\lambda}{q}}\, ds = \frac{\alpha^{-\frac{\lambda}{q}}}{\left(n-\frac{\delta}{q}-k+1\right)} s^{n-\frac{\delta}{q}-k+1} {}_2F_1\left(\frac{\lambda}{q}; n-\frac{\delta}{q}-k+1, n-\frac{\delta}{q}-k+2; -\frac{s}{\alpha}\right),$$

for $s \in \mathfrak{S} = \left\{s : |\frac{s}{\alpha}| < 1, s \neq 0\right\}$.

Proof. Let $(n,k) \in \mathbb{N}_0 \times \mathbb{N}_0$ and define the new variable $\xi = s/\alpha$, i.e.

$$\begin{aligned} \int s^{n-\frac{\delta}{q}-k}(s+\alpha)^{-\frac{\lambda}{q}}\, ds &= \alpha^{n-\frac{\delta+\lambda}{q}-k+1} \int \xi^{n-\frac{\delta}{q}-k}(1+\xi)^{-\frac{\lambda}{q}}\, d\xi \\ &= \alpha^{-\frac{\lambda}{q}}(\alpha\xi)^{n-\frac{\delta}{q}-k+1} \sum_{j=0}^{\infty} \frac{\left(\frac{\lambda}{q}\right)_j}{\left(n-\frac{\delta}{q}-k+1+j\right)} \frac{(-\xi)^j}{j!}. \end{aligned}$$

In terms of the original variable s, this can be recast into the form

$$\frac{\alpha^{-\frac{\lambda}{q}}}{\left(n-\frac{\delta}{q}-k+1\right)} s^{n-\frac{\delta}{q}-k+1} \sum_{j=0}^{\infty} \frac{\left(\frac{\lambda}{q}\right)_j \left(n-\frac{\delta}{q}-k+1\right)_j}{\left(n-\frac{\delta}{q}-k+2\right)_j} \frac{\left(-\frac{s}{\alpha}\right)^j}{j!}.$$

□

Using Lemma 4.4.1, we can rewrite (4.18) as

$$\widetilde{V}(s) = e^{-\frac{(c-\beta)}{q}s}\left(C\, s^{\frac{\delta}{q}-1}(s+\alpha)^{\frac{\lambda}{q}} - \left(1+\frac{s}{\alpha}\right)^{\frac{\lambda}{q}}\frac{1}{s^2}\sum_{n=0}^{\infty}\frac{\left(\frac{c-\beta}{q}\right)^n}{n!}\frac{s^n}{(n-\frac{\delta}{q}-1)}A(n-1,s)\right.$$
$$-\left(1+\frac{s}{\alpha}\right)^{\frac{\lambda}{q}}\frac{\beta}{qs}\sum_{n=0}^{\infty}\frac{\left(\frac{c-\beta}{q}\right)^n}{n!}\frac{s^n}{(n-\frac{\delta}{q})}A(n,s)$$
$$\left.+\frac{(c-\beta)}{q}V(0)\left(1+\frac{s}{\alpha}\right)^{\frac{\lambda}{q}}\sum_{n=0}^{\infty}\frac{\left(\frac{c-\beta}{q}\right)^n}{n!}\frac{s^n}{(n-\frac{\delta}{q}+1)}A(n+1,s)\right). \tag{4.19}$$

where

$$A(n,s) := {}_2F_1\left(\frac{\lambda}{q}, n-\frac{\delta}{q}; n-\frac{\delta}{q}+1; -\frac{s}{\alpha}\right)$$

Denote

$$\mathbf{{}_2F_1}\left(a,b;\nu;z\right) := \frac{{}_2F_1\left(\nu-a,\nu-b;\nu;z\right)}{\Gamma(\nu)}.$$

Successively using the transformation formulas

$$\mathbf{{}_2F_1}\left(a,b;\nu;z\right) = (1-z)^{\nu-a-b}\,\mathbf{{}_2F_1}\left(\nu-a,\nu-b;\nu;z\right),$$

and

$$\mathbf{{}_2F_1}\left(a,b;\nu;z\right) = \frac{\pi}{\sin\left(\pi(b-a)\right)}\left[\frac{(-z)^{-a}}{\Gamma(b)\Gamma(\nu-a)}\mathbf{{}_2F_1}\left(a,a-\nu+1;a-b+1;\frac{1}{z}\right)\right.$$
$$\left.-\frac{(-z)^{-a}}{\Gamma(b)\Gamma(\nu-a)}\mathbf{{}_2F_1}\left(a,a-\nu+1;a-b+1;\frac{1}{z}\right)\right],$$

(cf. [100]), we can then rewrite (4.19) as

$$
\begin{aligned}
\widetilde{V}(s) = e^{\frac{(c-\beta)}{q}s} \Bigg[\Bigg(& C - \alpha^{-\frac{\lambda+\delta}{q}-1} \sum_{n=0}^{\infty} \frac{(z(0))^n}{n!} \varphi(n-1) \\
& -\alpha^{-\frac{\lambda+\delta}{q}-1} \sum_{n=0}^{\infty} \frac{(z(0))^n}{n!} \varphi(n-1) - \frac{\beta}{q} \alpha^{-\frac{\lambda+\delta}{q}} \sum_{n=0}^{\infty} \frac{(z(0))^n}{n!} \varphi(n) \\
& + \frac{(c-\beta)}{q} V(0)\, \alpha^{-\frac{\lambda+\delta}{q}+1} \sum_{n=0}^{\infty} \frac{(z(0))^n}{n!} \varphi(n+1) \Bigg) s^{\frac{\delta}{q}-1} (s+\alpha)^{\frac{\lambda}{q}} \\
& + \frac{\left(1+\frac{\alpha}{s}\right)}{s^2} \sum_{n=0}^{\infty} \frac{\left(\frac{(c-\beta)}{q}\right)^n}{n!} s^n \overline{A}(n-2,s) + \frac{\beta}{q} \frac{\left(1+\frac{\alpha}{s}\right)}{s} \sum_{n=0}^{\infty} \frac{\left(\frac{(c-\beta)}{q}\right)^n}{n!} s^n \overline{A}(n-1,s) \\
& - \frac{(c-\beta)}{q} V(0) \left(1+\frac{\alpha}{s}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{(c-\beta)}{q}\right)^n}{n!} s^n\, \overline{A}(n,s) \Bigg],
\end{aligned}
\tag{4.20}
$$

where

$$
\varphi(n) := \frac{\Gamma\left(-n+\frac{\lambda+\delta}{q}\right) \Gamma\left(n-\frac{\delta}{q}\right)}{\Gamma\left(\frac{\lambda}{q}\right)},
$$

and,

$$
\overline{A}(n,s) := \frac{{}_2F_1\left(1, \frac{\delta}{q}-n; \frac{\lambda+\delta}{q}-n; -\frac{\alpha}{s}\right)}{\left(-(n+1)+\frac{\lambda+\delta}{q}\right)}.
$$

Since

$$
\sum_{n=0}^{\infty} \frac{z^n}{n!} \varphi(n-\kappa) = \varphi(-\kappa)\, M\left(-\frac{\delta}{q}-\kappa, 1-\kappa-\frac{\lambda+\delta}{q}, -z\right),
$$

equation (4.20) simplifies to

$$
\begin{aligned}
\widetilde{V}(s) = e^{-\frac{(c-\beta)}{q}s} \Bigg[& s^{\frac{\delta}{q}-1} (s+\alpha)^{\frac{\lambda}{q}} (C+D) \\
& + \frac{\left(1+\frac{\alpha}{s}\right)}{s^2} \sum_{n=0}^{\infty} \frac{\left(\frac{(c-\beta)}{q}\right)^n}{n!} s^n \overline{A}(n-2,s) + \frac{\beta}{q} \frac{\left(1+\frac{\alpha}{s}\right)}{s} \sum_{n=0}^{\infty} \frac{\left(\frac{(c-\beta)}{q}\right)^n}{n!} s^n \overline{A}(n-1,s) \\
& - \frac{(c-\beta)}{q} V(0) \left(1+\frac{\alpha}{s}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{(c-\beta)}{q}\right)^n}{n!} s^n \overline{A}(n,s) \Bigg]
\end{aligned}
\tag{4.21}
$$

where

$$D = \frac{(c-\beta)}{q} V(0)\,\alpha^{-\frac{\lambda+\delta}{q}+1} \varphi(1)\, M\left(1-\frac{\delta}{q}, 2-\frac{\lambda+\delta}{q}, -\frac{\alpha(c-\beta)}{q}\right)$$
$$-\alpha^{-\frac{\lambda+\delta}{q}-1} \varphi(-1)\, M\left(-1-\frac{\delta}{q}, -\frac{\lambda+\delta}{q}, -\frac{\alpha(c-\beta)}{q}\right)$$
$$-\frac{\beta}{q}\,\alpha^{-\frac{\lambda+\delta}{q}} \varphi(0)\, M\left(-\frac{\delta}{q}, 1-\frac{\lambda+\delta}{q}, -\frac{\alpha(c-\beta)}{q}\right).$$

From

$$\frac{1}{\Gamma(b)} \int_0^\infty e^{-sx} x^{b-1} M(a,b,x)\, dx = s^{a-b}(s-1)^{-a}, \qquad \Re(b) > 0, \Re(s) > 1,$$

one deduces that for $\lambda + \delta < q$ and $\Re(s) > 0$, the inverse Laplace transform of $s^{\frac{\delta}{q}-1}(s+\alpha)^{\frac{\lambda}{q}}$ is given by

$$\Re\left\{\left(\frac{\Gamma\left(\frac{\delta}{q}\right)}{\Gamma\left(\frac{\lambda+\delta}{q}\right)} U\left(\frac{\delta}{q}, 1+\frac{\lambda+\delta}{q}, -\alpha x\right)\right.\right.$$
$$\left.\left. -\frac{\Gamma\left(\frac{\delta}{q}\right)}{\Gamma\left(\frac{\lambda+\delta}{q}\right)} \frac{\Gamma\left(-\frac{\lambda+\delta}{q}\right)}{\Gamma\left(-\frac{\lambda}{q}\right)} M\left(\frac{\delta}{q}, 1+\frac{\lambda+\delta}{q}, -\alpha x\right)\right) \frac{(-\alpha)^{\frac{\lambda+\delta}{q}}}{\Gamma\left(1-\frac{\lambda+\delta}{q}\right)}\right\}.$$

For the sake of brevity, define $g(x) := x^{-\frac{\lambda+\delta}{q}} M\left(-\frac{\lambda}{q}, 1-\frac{\lambda+\delta}{q}, -\alpha x\right)/\Gamma(1-\frac{\lambda+\delta}{q})$. The first term of (4.21) can be interpreted as

$$e^{-\frac{(c-\beta)}{q}s}\,\tilde{g}(s) = \mathcal{L}\left\{u\left(x-\frac{c-\beta}{q}\right) g\left(x-\frac{c-\beta}{q}\right)\right\}.$$

where u is the Heaviside function. Since $u\left(x-\frac{c-\beta}{q}\right) g\left(x-\frac{c-\beta}{q}\right)$ becomes unbounded as x approaches $(c-\beta)/q$ from the right, the linear boundedness of $V(x)$ imposes $C = -D$.

The three hypergeometric functions ${}_2F_1$ in (4.21) have parameters differing by an integer. A connection between those terms is given by the identity

$$
{}_2F_1\left(1, \frac{\delta}{q} - n; \frac{\lambda+\delta}{q} - n; -\frac{\alpha}{s}\right) = \sum_{k=0}^{m-1} \frac{\left(n+1-k-\frac{\delta}{q}\right)_k}{\left(n+1-k-\frac{\lambda+\delta}{q}\right)_k} \left(-\frac{\alpha}{s}\right)^k
$$

$$
+ \left(-\frac{\alpha}{s}\right)^m \frac{\left(1-\frac{\delta}{q}+n-m\right)_m}{\left(1-\frac{\lambda+\delta}{q}+n-m\right)_m} \, {}_2F_1\left(1, \frac{\delta}{q} - n + m; \frac{\lambda+\delta}{q} - n + m; -\frac{\alpha}{s}\right)
$$

for $(n, m) \in \mathbb{N}_0 \times \mathbb{N}$. Respective substitution in (4.21) gives

$$
\widetilde{V}(s) = e^{-\frac{(c-\beta)}{q}s} \left[-\frac{(c-\beta)}{q} V(0) \left(1+\frac{\alpha}{s}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{(c-\beta)}{q}\right)^n}{n!} \frac{s^n}{\left(-(n+1)+\frac{\lambda+\delta}{q}\right)} \cdot \right.
$$

$$
\cdot \, {}_2F_1\left(1, \frac{\delta}{q} - n; \frac{\lambda+\delta}{q} - n; -\frac{\alpha}{s}\right)
$$

$$
+ \frac{\left(1+\frac{\alpha}{s}\right)}{\alpha^2} \sum_{n=0}^{\infty} \frac{\left(\frac{(c-\beta)}{q}\right)^n}{n!} \frac{s^n \left(1 - {}_2F_1\left(1, \frac{\delta}{q} - n; \frac{\lambda+\delta}{q} - n; -\frac{\alpha}{s}\right)\right)\left(n - \frac{\lambda+\delta}{q}\right)}{\left(n-\frac{\delta}{q}-1\right)\left(n-\frac{\delta}{q}\right)}
$$

$$
- \frac{\beta}{q} \frac{\left(1+\frac{\alpha}{s}\right)}{\alpha} \sum_{n=0}^{\infty} \frac{\left(\frac{(c-\beta)}{q}\right)^n}{n!} \frac{s^n \left(1 - {}_2F_1\left(1, \frac{\delta}{q} - n; \frac{\lambda+\delta}{q} - n; -\frac{\alpha}{s}\right)\right)}{\left(n-\frac{\delta}{q}\right)}
$$

$$
\left. - \frac{\left(1+\frac{\alpha}{s}\right)}{\alpha s} \sum_{n=0}^{\infty} \frac{\left(\frac{(c-\beta)}{q}\right)^n}{n!} \frac{s^n}{\left(n-\frac{\delta}{q}-1\right)} \right].
$$

With considerable effort, the latter expression can be represented as

$$
\widetilde{V}(s) = e^{-\frac{(c-\beta)}{q}s} \left(1+\frac{\alpha}{s}\right) \left[B \cdot {}_2F_1\left(1, \frac{\delta}{q}; \frac{\lambda+\delta}{q}; -\frac{\alpha}{s}\right) \right.
$$

$$
+ \frac{(c-\beta)}{q} V(0) \sum_{n=1}^{\infty} \left(\frac{(c-\beta)}{q}\right)^n s^n \, Z(n) - \frac{\beta}{q} \frac{(c-\beta)}{q} \sum_{n=1}^{\infty} \left(\frac{(c-\beta)}{q}\right)^n s^n \, Z(n+1)
$$

$$
\left. - \left(\frac{c-\beta}{q}\right)^2 \sum_{n=1}^{\infty} \left(\frac{(c-\beta)}{q}\right)^n s^n \, Z(n+2) + \frac{1}{\alpha s \left(1+\frac{\delta}{q}\right)} - \frac{q(\lambda+\delta)}{\alpha^2 \delta (q+\delta)} + \frac{c}{\alpha \delta} \right] \tag{4.22}
$$

where

$$Z(j) := \frac{{}_2F_2\left(1, n+1-\frac{\delta}{q}; j+1, n+2-\frac{\lambda+\delta}{q}; -\frac{\alpha(c-\beta)}{q}\right)}{j!\left(n+1-\frac{\lambda+\delta}{q}\right)}, \qquad j \in \mathbb{N},$$

and

$$B = \frac{(c-\beta)}{q}V(0)\frac{M\left(1-\frac{\delta}{q}, 2-\frac{\lambda+\delta}{q}, -\frac{\alpha(c-\beta)}{q}\right)}{\left(1-\frac{\lambda+\delta}{q}\right)} + \frac{q(\lambda+\delta)}{\alpha^2\delta(q+\delta)}M\left(-1-\frac{\delta}{q}, -\frac{\lambda+\delta}{q}, -\frac{\alpha(c-\beta)}{q}\right)$$
$$-\frac{\beta}{\alpha\delta}M\left(-\frac{\delta}{q}, 1-\frac{\lambda+\delta}{q}, -\frac{\alpha(c-\beta)}{q}\right).$$

Using the contiguous relation

$$\nu(1-z)\,{}_2F_1\left(a, b; \nu; z\right) = \nu\,{}_2F_1\left(a-1, b; \nu; z\right) - (b-\nu)z\,{}_2F_1\left(a, b; \nu+1; z\right)$$

for our context, we obtain

$${}_2F_1\left(1, \frac{\delta}{q}; \frac{\lambda+\delta}{q}; -\frac{\alpha}{s}\right) = \frac{1}{1+\frac{\alpha}{s}}\left(1+\frac{\alpha\lambda}{\lambda+\delta}\frac{{}_2F_1\left(1, \frac{\delta}{q}; 1+\frac{\lambda+\delta}{q}; -\frac{\alpha}{s}\right)}{s}\right),$$

which transforms (4.22) into

$$\begin{aligned}\widetilde{V}(s) = {}& e^{-\frac{(c-\beta)}{q}s}\frac{B\alpha\lambda}{(\lambda+\delta)}\frac{{}_2F_1\left(1, \frac{\delta}{q}; 1+\frac{\lambda+\delta}{q}; -\frac{\alpha}{s}\right)}{s} \\ & + e^{-\frac{(c-\beta)}{q}s}\left\{B + \left(1+\frac{\alpha}{s}\right)\left[\frac{(c-\beta)}{q}V(0)\sum_{n=1}^{\infty}\left(\frac{(c-\beta)}{q}\right)^n s^n Z(n)\right.\right. \\ & -\frac{\beta}{q}\frac{(c-\beta)}{q}\sum_{n=1}^{\infty}\left(\frac{(c-\beta)}{q}\right)^n s^n Z(n+1) - \left(\frac{c-\beta}{q}\right)^2\sum_{n=1}^{\infty}\left(\frac{(c-\beta)}{q}\right)^n s^n Z(n+2) \\ & \left.\left.+\frac{1}{\alpha s\left(1+\frac{\delta}{q}\right)} - \frac{q(\lambda+\delta)}{\alpha^2\delta(q+\delta)} + \frac{c}{\alpha\delta}\right]\right\}.\end{aligned} \tag{4.23}$$

Utilizing the relationship (cf. Olver [100])

$$\int_0^{\infty} e^{-sx}x^{b-1}\mathbf{M}\left(a, \nu, kx\right)\,dx = \frac{\Gamma(b)}{s^b}\,\mathbf{{}_2F_1}\left(a, b; \nu; \frac{k}{s}\right), \qquad \Re(b) > 0, \Re(s) > \max\left(\Re(k), 0\right),$$

gives the inverse Laplace transform of the first term of (4.23):

$$
\begin{aligned}
e^{-\frac{(c-\beta)}{q}s}\frac{B\alpha\lambda}{(\lambda+\delta)}\frac{{}_2F_1\left(1,\frac{\delta}{q};1+\frac{\lambda+\delta}{q};-\frac{\alpha}{s}\right)}{s}\\
&= \frac{B\alpha\lambda}{(\lambda+\delta)}\mathcal{L}\left\{u\left(x-\frac{c-\beta}{q}\right)M\left(\frac{\delta}{q},1+\frac{\lambda+\delta}{q},-\alpha\left(x-\frac{c-\beta}{q}\right)\right)\right\}.
\end{aligned}
$$

Since we know by different means from Section 4.3 the expression for $V(x)$, we can take the (direct) Laplace transform of (4.17) in order to compare it with the above expression. After some efforts, one obtains from (4.17)

$$
\tilde{V}(s) = e^{-\frac{(c-\beta)}{q}s}\tilde{C}\left[H(s)+\frac{{}_2F_1\left(\frac{\delta}{q},1;1+\frac{\lambda+\delta}{q};-\frac{\alpha}{s}\right)}{s}\right]+\frac{1}{q+\delta}\left(\frac{q}{s^2}+\frac{\beta}{s}+\frac{q\left(c-\frac{\lambda}{\alpha}\right)}{\delta s}\right), \tag{4.24}
$$

where

$$
H(s) := \int_{-\frac{(c-\beta)}{q}}^{0} e^{-sy}M\left(\frac{\delta}{q},1+\frac{\lambda+\delta}{q},-\alpha y\right)dy.
$$

One can show that $\frac{B\alpha\lambda}{(\lambda+\delta)} = \tilde{C}$ and

$$
H(s) = \sum_{n=0}^{\infty}\frac{\left(\frac{c-\beta}{q}\right)^{n+1}s^n}{(n+1)!}\,{}_2F_2\left(n+1,\frac{\delta}{q};2+n,1+\frac{\lambda+\delta}{q};\frac{\alpha(c-\beta)}{q}\right);
$$

so the first term coincides with the one in (4.23). Since Expressions (4.24) and (4.23) have to coincide altogether, this leads to the identity

$$
e^{-\frac{(c-\beta)}{q}s}\frac{B\alpha\lambda}{(\lambda+\delta)}H(s)+\frac{1}{q+\delta}\left(\frac{q}{s^2}+\frac{\beta}{s}+\frac{q\left(c-\frac{\lambda}{\alpha}\right)}{\delta s}\right)=e^{-\frac{(c-\beta)}{q}s}\left\{B+\left(1+\frac{\alpha}{s}\right)\Bigg[\right.
$$

$$
\frac{(c-\beta)}{q}V(0)\sum_{n=1}^{\infty}\frac{\left(\frac{(c-\beta)}{q}\right)^n s^n}{n!\left(n+1-\frac{\lambda+\delta}{q}\right)}\,{}_2F_2\left(1,n+1-\frac{\delta}{q};n+1,n+2-\frac{\lambda+\delta}{q};-\frac{\alpha(c-\beta)}{q}\right)
$$

$$
-\left(\frac{c-\beta}{q}\right)^2\sum_{n=1}^{\infty}\frac{\left(\frac{(c-\beta)}{q}\right)^n s^n}{(n+2)!\left(n+1-\frac{\lambda+\delta}{q}\right)}\,{}_2F_2\left(1,n+1-\frac{\delta}{q};n+3,n+2-\frac{\lambda+\delta}{q};-\frac{\alpha(c-\beta)}{q}\right)
$$

$$
-\frac{\beta}{q}\frac{(c-\beta)}{q}\sum_{n=1}^{\infty}\frac{\left(\frac{(c-\beta)}{q}\right)^n s^n}{(n+1)!\left(n+1-\frac{\lambda+\delta}{q}\right)}\,{}_2F_2\left(1,n+1-\frac{\delta}{q};n+2,n+2-\frac{\lambda+\delta}{q};-\frac{\alpha(c-\beta)}{q}\right)
$$

$$
\left.+\frac{1}{\alpha s\left(1+\frac{\delta}{q}\right)}-\frac{q(\lambda+\delta)}{\alpha^2\delta(q+\delta)}+\frac{c}{\alpha\delta}\Bigg]\right\}. \tag{4.25}
$$

While it is far from obvious to show analytically that (4.25) holds true, it is indeed the case, as numerical verifications show. In fact, the two alternative approaches of Sections 4.3 and 4.4 – through identity (4.25) – suggest new relations between ${}_2F_2$-hypergeometric functions of argument $\pm z$. A detailed study of such relations is, however, beyond the scope of this chapter.

4.5 The time of ruin

Let us now study the effect of the proposed dividend strategy on the distribution of the ruin time τ_x. For this purpose, consider the expected discounted penalty at ruin

$$
m_\delta(x) := \mathbb{E}\left[e^{-\delta\tau_x}w\left(|X_{\tau_x}|\right)\right],
$$

where w is a non-negative penalty function of the deficit at ruin. Given differentiability of $m_\delta(x)$, the standard arguments based on the infinitesimal generator then lead to the integro-differential equation

$$
\left(c-(qx+\beta)\right)m_\delta'(x)-(\lambda+\delta)m_\delta(x)+\lambda\int_0^x m_\delta(x-y)dF_Y(y)=-\lambda A(x),\ \ x\geq 0, \tag{4.26}
$$

where

$$A(x) := \int_x^\infty w(y-x)\, dF_Y(y).$$

We will again restrict our considerations to exponentially distributed claims with parameter $\alpha > 0$. In this case, $|X_{\tau_x}| \sim$Exp(α) due to lack of memory, so that we can focus on the (Laplace transform of the) time of ruin, i.e. $w(x) = 1$.
Similarly to Section 4.3.1, this leads to the second-order homogeneous differential equation

$$(c-(qx+\beta))\, m_\delta''(x) + [\alpha\,(c-(qx+\beta)) - (q+\lambda+\delta)]\, m_\delta'(x) - \alpha\delta m_\delta(x) = 0, \quad x \geq 0. \tag{4.27}$$

Because of the linear boundedness of $m_\delta(x)$ in x, the solution to (4.27) matches the homogeneous solution $V_h(x) := V(x) - V_p(x)$ to (4.11) up to a constant factor. That is, for $x \geq 0$, we can write $m_\delta(x) = B\, V_h(x)$ for some constant B. Letting $x = 0$ in (4.26) yields

$$(c-\beta)m_\delta'(0) - (\lambda+\delta)m_\delta(0) = -\lambda,$$

that is

$$(c-\beta)B\left(V'(0) - V_p'(0)\right) - (\lambda+\delta)B\left(V(0) - V_p(0)\right) = -\lambda$$

and hence

$$B = \frac{\lambda}{\beta + \frac{q(c-\beta)}{q+\delta} - \frac{\lambda+\delta}{q+\delta}\left(\beta + \frac{q}{\delta}\left(c - \frac{\lambda}{\alpha}\right)\right)}.$$

Proposition 4.5.1. *For any $x \geq 0$, the Laplace transform of the ruin time in a Cramér-Lundberg model with affine dividend strategy* (4.2) *and Exp(α)-distributed claims is given by*

$$m_\delta(x) = \frac{\lambda M\left(\frac{\delta}{q}, 1 + \frac{\lambda+\delta}{q}, z(x)\right)}{\frac{\alpha\delta(c-\beta)}{q+\lambda+\delta} M\left(1+\frac{\delta}{q}, 2 + \frac{\lambda+\delta}{q}, z(0)\right) + (\lambda+\delta) M\left(\frac{\delta}{q}, 1 + \frac{\lambda+\delta}{q}, z(0)\right)}, \qquad x \geq 0, \tag{4.28}$$

where $z(x) = \frac{\alpha(c-(qx+\beta))}{q}$.

One particular quantity of interest is the expected ruin time. While it can be simply obtained by taking the derivative

$$\mathbb{E}[\tau_x] = -\frac{d}{d\delta} m_\delta(x) \Big|_{\delta=0},$$

the concrete calculation is considerably involved. It requires differentiating M with respect to its first two parameters:

$$M^{(a)} = \frac{d}{da} M(a,b,z), \qquad M^{(b)} = \frac{d}{db} M(a,b,z).$$

The so-called Kummer transformation $M(a,b,z) = e^z M(b-a, b, -z)$ will facilitate the mathematical tractability, under which (4.28) reads

$$m_\delta(x) = \frac{\lambda e^{-\alpha x} M\left(a, a+\frac{\delta}{q}, -z(x)\right)}{\frac{\alpha\delta(c-\beta)}{q+\lambda+\delta} M\left(a, a+1+\frac{\delta}{q}, -z(0)\right) + (\lambda+\delta) M\left(a, a+\frac{\delta}{q}, -z(0)\right)}, \qquad x \geq 0, \tag{4.29}$$

where $a = 1 + \frac{\lambda}{q}$.
We now discuss two possible representations for the derivative of M:

4.5.1 Digamma functions

From Section 1.5.1, recall that

$$M^{(b)} = \sum_{n=0}^{\infty} [\psi(b) - \psi(b+n)] \frac{(a)_n}{(b)_n} \frac{z^n}{n!}. \tag{4.30}$$

With that in mind, the derivative of (4.29) with respect to δ (together with the property $M(a,a,z) = e^z$) yields

$$\begin{aligned} \frac{d}{d\delta} m_\delta(x) \Big|_{\delta=0} &= \frac{e^{z(x)}}{q} \sum_{n=0}^{\infty} [\psi(a) - \psi(a+n)] \frac{(-z(x))^n}{n!} - \frac{\alpha(c-\beta)}{\lambda(q+\lambda)} e^{z(0)} M(a, a+1, -z(0)) \\ &\quad - \frac{1}{\lambda} - \frac{e^{z(0)}}{q} \sum_{n=0}^{\infty} [\psi(a) - \psi(a+n)] \frac{(-z(0))^n}{n!}. \end{aligned}$$

But for series of the above type there are expressions in terms of the generalized hypergeometric function ${}_2F_2$ available (cf. [54]):

$$\sum_{n=0}^{\infty}[\psi(a+n)-\psi(a)]\frac{(z(x))^n}{n!}=\frac{z(x)}{a}e^{z(x)}\,{}_2F_2\left(1,1;2,a+1;-z(x)\right).$$

Using this last result together with the relation $M(a,a+1,-z)=az^{-a}\gamma(a,z)$ (where $\gamma(a,z)$ denotes the lower incomplete gamma function) leads to the following formula:

Proposition 4.5.2. *For any $x\geq 0$, the expected time of ruin under the the proposed dividend strategy in model (4.1) with exponentially distributed claims with parameter α is given by*

$$\mathbb{E}[\tau_x]=\frac{1+e^{z(0)}\gamma\left(1+\frac{\lambda}{q},z(0)\right)z(0)^{-\frac{\lambda}{q}}}{\lambda}\\+\frac{{}_2F_2\left(1,1;2,2+\frac{\lambda}{q};z(0)\right)z(0)-{}_2F_2\left(1,1;2,2+\frac{\lambda}{q};z(x)\right)z(x)}{q+\lambda},\quad(4.31)$$

where $z(x)=\frac{\alpha(c-(qx+\beta)}{q}$.

4.5.2 Kampé de Fériet functions

As an alternative, one can express the derivatives of the Kummer function also in terms of the bivariate Kampé de Fériet function

$$F^{A,B,D}_{R,S,U}\left(\begin{matrix}a_1,\dots,a_A\;b_1,\dots,b_B\;d_1,\dots,d_D\\r_1,\dots,r_R\;s_1,\dots,s_S\;u_1,\dots,u_U\end{matrix}x,y\right)\\=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m+n}\prod_{j=1}^{B}(b_j)_m\prod_{j=1}^{D}(d_j)_n}{\prod_{j=1}^{R}(r_j)_{m+n}\prod_{j=1}^{S}(s_j)_m\prod_{j=1}^{U}(u_j)_n}\frac{x^m y^n}{m!\,n!},$$

see e.g. [115, 61]. Recall from Section 1.5.1 that the concrete connection is given by

$$M^{(a)}=\frac{z}{b}F^{1,2,1}_{2,1,0}\left(\begin{matrix}a+1\quad 1,a\quad 1\\2,b+1\;a+1\;-\end{matrix}z,z\right),\quad M^{(b)}=-\frac{a}{b}\frac{z}{b}F^{1,2,1}_{2,1,0}\left(\begin{matrix}a+1\quad 1,b\quad 1\\2,b+1\;b+1\;-\end{matrix}z,z\right),$$

where the empty product indicated by the solid horizontal line is interpreted to be unity. Employing this formula and proceeding similarly as in Section 4.5.1, one then obtains

$$\mathbb{E}[\tau_x] = -\frac{d}{d\delta} m_\delta(x)\bigg|_{\delta=0} = \frac{1 + e^{z(0)}\gamma\left(1+\frac{\lambda}{q}, z(0)\right) z(0)^{-\frac{\lambda}{q}}}{\lambda}$$
$$+ \frac{e^{z(0)} z(0) F^{0,2,1}_{1,1,0}\left(\begin{smallmatrix} - & 1,1+\frac{\lambda}{q} & 1 \\ 2, & 2+\frac{\lambda}{q} & - \end{smallmatrix} - z(0), -z(0)\right) - e^{z(x)} z(x) F^{0,2,1}_{1,1,0}\left(\begin{smallmatrix} - & 1,1+\frac{\lambda}{q} & 1 \\ 2, & 2+\frac{\lambda}{q} & - \end{smallmatrix} - z(x), -z(x)\right)}{q+\lambda}$$

The equivalency of this expression with (4.31) follows from the reduction formula (cf. [98])

$$F^{0,2,1}_{1,1,0}\left(\begin{matrix} - & a, b & d - a \\ d & f & - \end{matrix}\; z, z\right) = e^z\, {}_2F_2\left(a, f-b; d, f; -z\right),$$

with $a = 1, b = 1 + \frac{\lambda}{q}, d = 2$ and $f = 2 + \frac{\lambda}{q}$.

4.6 A probabilistic argument

Inspired by Avanzi & Wong [22], we also present here a probabilistic argument to connect the function $V(x)$ of Section 4.3 with the Laplace transform of the time to ruin in the previous section. To that end, consider first a surplus process of the form (4.4), but let it continue after ruin (i.e. whenever $X_t < -\frac{\beta}{q}$, negative dividends are paid, which could be interpreted as capital injections). Denote with

$$\overline{V}(x) := \mathbb{E}_x\left[\int_0^\infty e^{-\delta t}(qX_t + \beta)\, dt\right],$$

the respective expected discounted dividend payments (or, more precisely, the difference between expected discounted dividend payments and expected discounted amount of such capital injections).

Proposition 4.6.1. *For $x \geq 0$,*

$$\overline{V}(x) = \frac{c - \lambda\mu}{\delta} + \frac{x - \frac{c-\beta-\lambda\mu}{q}}{1 + \delta/q}.$$

Proof.

$$\begin{aligned}\overline{V}(x) &= \mathbb{E}_x\left[\int_0^\infty e^{-\delta t}\left[q\left(\left(x-\frac{c-\beta}{q}\right)e^{-qt}+\frac{c-\beta}{q}-\int_0^t e^{-q(t-u)}dS_u\right)+\beta\right]dt\right],\\ &= \frac{qx}{q+\delta}+\frac{(c-\beta)q}{\delta(q+\delta)}+\frac{\beta}{\delta}-\mathbb{E}\left[\int_0^\infty qe^{qu}\int_u^\infty e^{-(q+\delta)t}dt\,dS_u\right],\\ &= \frac{qx}{q+\delta}+\frac{(c-\beta)q}{\delta(q+\delta)}+\frac{\beta}{\delta}-\frac{q}{q+\delta}\,\mathbb{E}\left[\sum_{i=1}^\infty e^{-\delta T_i}Y_i\right],\end{aligned}$$

where the last equality follows from $\lim_{t\to\infty} N_t = \infty$ a.s. Since the i-th claim arrival time T_i in the Poisson model is independent of Y_i and $\Gamma(i,\lambda)$-distributed, we then have

$$\overline{V}(x) = \frac{qx}{q+\delta}+\frac{(c-\beta-\lambda\mu)q}{\delta(q+\delta)}+\frac{\beta}{\delta}, \qquad x \geq 0.$$

□

Due to the strong Markov property of X_t, we can now deduce

$$V(x) = \overline{V}(x) - \mathbb{E}\left[e^{-\delta\tau_x}\overline{V}\left(X_{\tau_x}\right)\right], \qquad x \geq 0.$$

For exponentially distributed claims with mean $\mu = 1/\alpha$, the lack-of-memory property implies that the deficit at $t = \tau_x$ is again exponentially distributed and independent of the time of ruin. This leads to

$$V(x) = \overline{V}(x) - \mathbb{E}\left[e^{-\delta\tau_x}\right]\overline{V}\left(-\frac{1}{\alpha}\right).$$

Combining Proposition 4.6.1 with the expression for the Laplace transform of the time of ruin derived in (4.28) then again leads to the formula given in Proposition 4.3.3. In particular, this approach gives a complementary probabilistic interpretation of the particular solution $V_p(X) = \overline{V}(x)$ in Section 4.3 as well as the relation between the time of ruin and amount of dividend payments, in a certain sense akin to the dividends-penalty identity of Gerber et al. [66] in the model with horizontal dividend barrier.

4.7 Numerical Illustrations

4.7.1 General considerations

In this section we analyze the effects of the affine dividend strategy numerically. In particular, we are interested in the influence of the parameters q and β on the expected discounted dividend payments. Assume that $\alpha = 1/3$, $\lambda = 1$, $c = 3.5$ and $\delta = 0.05$. With each pair $(q, \beta) \in \mathbb{R}_+ \times [0, c]$, we associate $V(x; q, \beta) := V(x)$. Table 4.1 shows the influence of q on $V(x)$ for $\beta = 1.5$. We observe that $V(x; q, 1.5)$ increases in q up to a certain value and decreases thereafter. This demonstrates the compromise between paying larger amounts early (which is preferable due to discounting) and maintaining a longer survival in order to receive more payments later, i.e. too large proportions q (in addition to the constant rate β) reduce the lifetime of the process too much. One observes that this turning point appears for larger values of q the larger the initial surplus value x is (and for $x = 20$, this turning point is not yet visible for the depicted range of q). Incidentally, for $x = 0$ one sees that the choices $q = 0.2$ and $q = 0.5$ lead to roughly the same total expected dividend payouts (where for the larger q, more dividends are collected earlier and over a shorter portfolio's lifetime compared to the case with the smaller q, i.e. different time patterns of dividend payments here lead to the same aggregate payout in expectation).

$V(x; q, 1.5)$						
x	$q = 0.1$	$q = 0.2$	$q = 0.3$	$q = 0.5$	$q = 1$	$q = 10$
0	3.385	3.403	3.406	3.403	3.389	3.344
0.5	3.896	3.919	3.923	3.920	3.903	3.846
1	4.401	4.430	4.436	4.433	4.414	4.349
2	5.396	5.440	5.452	5.451	5.430	5.352
3	6.371	6.435	6.454	6.459	6.440	6.354
4	7.327	7.415	7.445	7.458	7.443	7.356
5	8.268	8.384	8.426	8.450	8.442	8.356
10	12.763	13.079	13.213	13.321	13.381	13.352
20	21.052	22.007	22.433	22.818	23.117	23.324

Table 4.1: Expected present value of dividends for different rates q with $\alpha = 1/3$, $\lambda = 1$, $c = 3.5$, $\beta = 1.5$ and $\delta = 0.05$.

Next, let us consider the effect of β on $V(x)$ for a given level of q. Table 4.2 illustrates a qualitatively similar pattern: the larger x is, the higher constant rate β can be afforded, and for $x \geq 10$, it is preferable to pay out the entire premium rate c as dividends (in addition to the proportional payments).

$V(x; 0.3, \beta)$						
x	$\beta = 0$	$\beta = 0.5$	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 3.5$
0	3.354	3.394	3.409	3.394	3.355	3.333
0.5	3.855	3.903	3.922	3.913	3.876	3.854
1	4.352	4.407	4.432	4.428	4.393	4.372
2	5.336	5.405	5.440	5.449	5.419	5.399
3	6.307	6.390	6.435	6.457	6.434	6.415
4	7.267	7.363	7.418	7.453	7.438	7.422
5	8.217	8.326	8.391	8.440	8.433	8.420
10	12.863	13.028	13.139	13.258	13.298	13.302
20	21.860	22.108	22.294	22.537	22.675	22.721

Table 4.2: Expected present value of dividends for different rates β with $\alpha = 1/3$, $\lambda = 1$, $c = 3.5$, $q = 0.3$ and $\delta = 0.05$.

In order to better understand the contribution of the proportional rate q and the constant rate β to the overall value of $V(x)$, we now decompose (4.6) as $V(x) := V_q(x) + V_\beta(x)$, where

$$V_q(x) := \mathbb{E}_x \left[\int_0^{\tau_x} e^{-\delta t} q X_t \, dt \right], \quad \text{and} \quad V_\beta(x) := \mathbb{E}_x \left[\int_0^{\tau_x} e^{-\delta t} \beta \, dt \right].$$

Along the same line of arguments as in Section 4.3 one can then derive

$$V_q(x) = A_q \, M\left(\frac{\delta}{q}, 1 + \frac{\lambda + \delta}{q}, z(x)\right) + V_{q_p}(x), \qquad x \geq 0, \tag{4.32}$$

and

$$V_\beta(x) = A_\beta \, M\left(\frac{\delta}{q}, 1 + \frac{\lambda + \delta}{q}, z(x)\right) + V_{\beta_p}(x), \qquad x \geq 0, \tag{4.33}$$

for respective constants A_q and A_β. We now proceed with an example to discuss the influence of the expected claim size $1/\alpha$ on the relative contribution of (4.32) and (4.33) to the total expected dividend payouts (4.6). Consider the following constellation of parameters: $\lambda = 1, c = 5, q = 0.5, \beta = 1$ and $\delta = 0.05$ so that the dividend rate at time t is given by $0.5X_t + 1$. Hence, the linear term qX_t constitutes the dominant term in the dividend rate if $X_t > 2$. Figure 4.2 displays the ratio $V_q(x)/V(x)$ for $\alpha = 1/3$ (solid line) and $\alpha = 1/4$ (dashed line). First, we observe that the ratio $V_q(x)/V(x)$ is increasing in x. This is in line with intuition since for larger initial capital x, the proportion of dividends from the linear term is larger; the concrete value of that ratio, however, reflects the occupation time of the various levels of the process. This is also illustrated by the different response of the ratio $V_q(x)/V(x)$ on changing the average claim size, depending on the range of x.

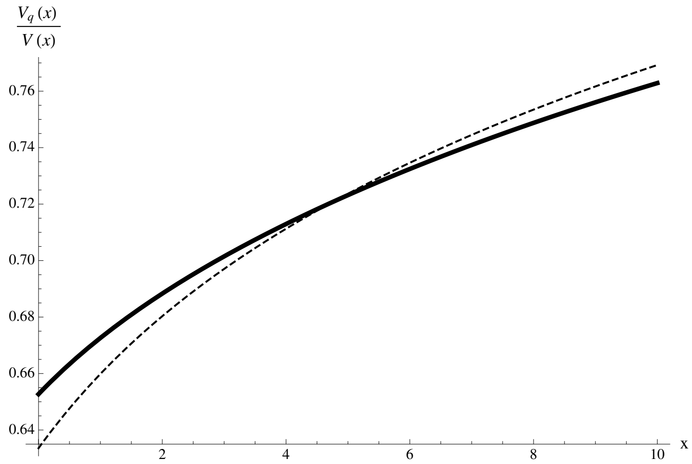

Figure 4.2: Ratio $V_q(x)/V(x)$ for $\alpha = 1/3$ (solid line) and $\alpha = 1/4$ (dashed line)

4.7.2 Optimal parameters

In a next step, it is natural to ask which combination of parameters q and β maximizes the expected present value of dividends until ruin. Let $\Theta = \{(q, \beta) : q > 0, \beta \in [0, c]\}$. With each pair $(q, \beta) \in \Theta$, we associate $V(x; q, \beta) := V(x)$. The optimization problem then consists in finding a pair (q^*, β^*) such that

$$V(x; q^*, \beta^*) = \max_{(q,\beta)\in\Theta} V(x; q, \beta), \tag{4.34}$$

for a given initial capital $x \geq 0$. In view of (4.17), such an optimization problem has to be approached numerically (here we used respectives routines in Mathematica). Table 4.3 displays the optimal parameters and resulting optimal dividend values for the case $\alpha = 1/3, \lambda = 1, c = 3.5$ and $\delta = 0.05$ and different initial capital values x. For illustration purposes we also depict the corresponding values for $\delta = 0.07$ in parentheses. One can observe that $q^*(x)$ increases in x (note that $q^*(x)$ is chosen as a function of initial capital x, but by construction then kept fixed throughout the life-time of the process, i.e. not a function of current surplus value). Furthermore, we always have $\beta^*(x) = 0$, i.e. a constant dividend rate does not contribute favorably to the compromise between profitability and length (lifetime) of the payments (this is also the case for other parameter values in numerical experiments). An interpretation of the latter is that since such a constant rate would be applied at all capital levels, the survival when close to ruin is more important than the payment of immediate dividends, which is somewhat in line with the philosophy behind dividend barrier strategies, cf. Section 4.7.3. For the higher discount rate $\delta = 0.07$, $q^*(x)$ changes drastically and even becomes infinity for large x, mimicking lump sum payments of a barrier strategy with barrier at level zero. Note from (4.4) that for $(c - \beta^*(x))/q^*(x) \geq x$, the drift of the process X_t will never be positive, cf. Figure 4.3.

x	$V(x; q^*, \beta^*)$	$q^*(x)$	$\beta^*(x)$
0	3.426 (3.279)	0.751 (3.789)	0.000 (0.000)
0.5	3.939 (3.780)	0.756 (3.871)	0.000 (0.000)
1	4.449 (4.280)	0.768 (4.088)	0.000 (0.000)
2	5.461 (5.279)	0.806 (4.896)	0.000 (0.000)
3	6.466 (6.276)	0.860 (6.413)	0.000 (0.000)
4	7.465 (7.274)	0.927 (9.502)	0.000 (0.000)
5	8.460 (8.272)	1.008 (18.227)	0.000 (0.000)
10	13.406 (13.271)	1.719 (∞)	0.000 (0.000)
20	23.334 (23.271)	31.623 (∞)	0.000 (0.000)

Table 4.3: Maximal expected present value of dividends and optimal pairs (q^*, β^*) for $\alpha = 1/3$, $\lambda = 1$, $c = 3.5$ and $\delta = 0.05$ ($\delta = 0.07$).

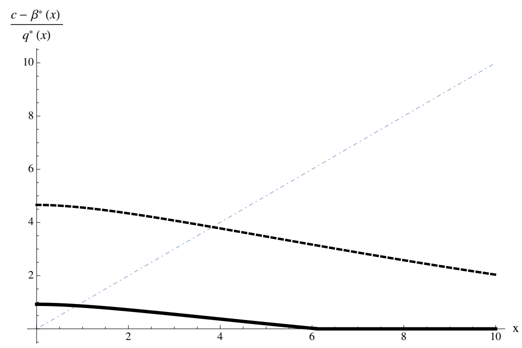

Figure 4.3: $\frac{c-\beta^*(x)}{q^*(x)}$ for $\delta = 0.05$ (dashed line) and $\delta = 0.07$ (solid line).

4.7.3 Comparison with the optimal barrier strategy

From classical results one knows that the optimal dividend strategy in a compound Poisson model with exponential claims is a barrier strategy (cf. Gerber [72]), if the sole criterion is the profitability. It is hence instructive to compare our optimal expected dividend payouts $V(x; q^*, \beta^*)$ according to the affine dividend strategy with $V_{b^*}(x)$, the one under the optimal barrier strategy b^*. Recall that

$$V_{b^*}(x) = \begin{cases} \frac{h(x)}{h'(b^*)}, & 0 \le x \le b^*, \\ x - b + \frac{h(b^*)}{h'(b^*)}, & x > b^*, \end{cases}$$

where $h(u) = (r+\alpha)e^{rx} - (s+\alpha)e^{sx}$, $r > 0$ and $s < 0$ are the roots of the characteristic equation

$$c\xi^2 + (\alpha c - (\lambda + \delta))\,\xi - \alpha\delta = 0,$$

and

$$b^* = \frac{1}{r-s} \ln \frac{s^2(s+\alpha)}{r^2(r+\alpha)}.$$

Table 4.4 compares the resulting $V_{b^*}(x)$ to the dividend payout of the optimal affine strategy $V(x; q^*, \beta^*)$ for the case $\alpha = 1/3, \lambda = 1, c = 3.5$ and $\delta = 0.05$ (in which case $b^* = 3.26$). Knowing that the barrier strategy is optimal among all admissible strategies, it is quite remarkable to observe how close one gets to this optimal value $V_{b^*}(x)$ by the best affine strategy.

x	$V_{b^*}(x)$	$V(x; q^*, \beta^*)$	$q^*(x)$	$\beta^*(x)$
0	3.437	3.426	0.751	0.000
$0.5\,b^*$	5.232	5.223	0.795	0.000
b^*	7.000	6.994	0.893	0.000
$1.5\,b^*$	8.764	8.749	1.034	0.000
$2\,b^*$	10.527	10.496	1.226	0.000
$3\,b^*$	14.055	13.981	1.854	0.000
$5\,b^*$	21.110	20.977	7.668	0.000

Table 4.4: Comparison of the expected present value of dividends under affine and barrier dividend strategies for initial capital values x with $\alpha = 1/3$, $\lambda = 1$, $c = 3.5$, $q = 0.3$ and $\delta = 0.05$.

A next question in this context is then how sensitive the performance of the affine dividend strategy is when varying $(q, \beta) \in \Theta$. To get an impression on that, Figure 4.4 depicts the contour lines $\{(q, \beta) \in \Theta : V(x; q, \beta) = aV_{b^*}(x)\}$, i.e. those parameter values for which we achieve a certain percentage a of the optimal dividend barrier strategy. Here $x = 10$ and $a \in [0.98, 0.9942]$. The red area for instance consists of all pairs (q, β) leading to at least 99.4% of $V_{b^*}(10)$. The size of that area is quite remarkable, showing that one can achieve quite convincing performance for a variety of (q,β)-values. The pair $(q,\beta)=(1.719,0)$ maximizes $V(10;q,\beta)$ and yields 99.5% of $V_{b^*}(10)$. One also sees that the gradient becomes larger as one moves towards smaller q-values indicating that $V(10;q,\beta)$ is sensitive to changes in q for smaller q.

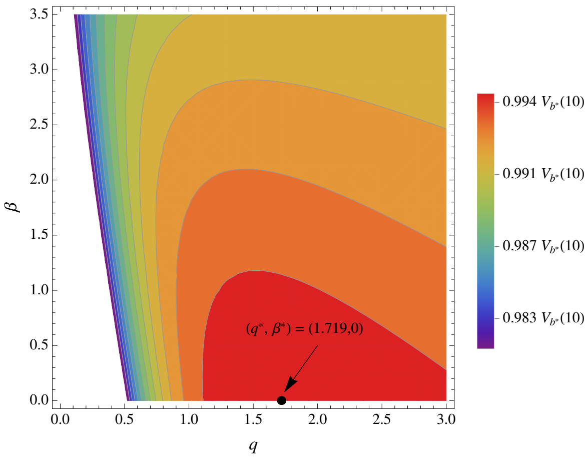

Figure 4.4: Contour lines for $x=10, \alpha=1/3,\ \lambda=1,\ c=3.5,\ q=0.3$ and $\delta=0.05$.

4.7.4 Dividend payments versus expected ruin time

Since one motivation to introduce an affine dividend strategy was the increased lifetime of the process, and since we have seen in the previous section that the performance of this strategy gets quite close to the one of optimal barrier strategies, it is now interesting to see to what extent the expected ruin time is improved using such an affine dividend strategy.

Let T_x^b the time to ruin with a barrier at level b. The Laplace transform of T_x^b for exponentially distributed claim amounts in the compound Poisson model is then given by

$$\mathbb{E}\left[e^{-\delta T_x^b}\right]=\begin{cases}\frac{(s+\alpha)(r+\alpha)}{\alpha}\left[\frac{re^{rb}e^{sx}-se^{sb}e^{rx}}{r(r+\alpha)e^{rb}-s(s+\alpha)e^{sb}}\right] & 0\le x\le b,\\ \frac{(s+\alpha)(r+\alpha)}{\alpha}\left[\frac{e^{(r+s)b}(r-s)}{r(r+\alpha)e^{rb}-s(s+\alpha)e^{sb}}\right] & x>b,\end{cases}$$

see e.g. [93, Equ.6.3]). We then have

$$\mathbb{E}\left[T_x^b\right]=-\frac{d}{d\delta}\mathbb{E}\left[e^{-\delta T_x^b}\right]\Big|_{\delta=0}.$$

Let $\mathbb{E}\left[\tau_{x;q,\beta}\right]$ be the expected ruin time under the affine dividend strategy $(q, \beta) \in \Theta$ (cf. Proposition 4.5.2) and consider the following constrained optimization problem:

$$\max_{(q,\beta)\in\Theta} \quad \mathbb{E}\left[\tau_{x;q,\beta}\right]$$
$$\text{subject to} \quad V(x;q,\beta) = aV_{b^*}(x),$$

where $a \in (0, 1)$. Note that the resulting optimal values q^* are not the same as the ones in the previous section, whereas β^* turns out to be again always zero. Figure 4.5 depicts the ratio $\mathbb{E}\left[\tau_{x;q^*,\beta^*}\right]/\mathbb{E}\left[T_x^{b^*}\right]$ for different initial capital values x as function of the performance factor a, for the same parameters $\alpha = 1/3, \lambda = 1, c = 3.5$ and $\delta = 0.05$. To match a higher required performance level a, one has to select larger values in the set Θ, which causes a reduction of the expected time to ruin $\mathbb{E}\left[\tau_{x;q^*,\beta^*}\right]$. Hence, the ratio $\mathbb{E}\left[\tau_{x;q^*,\beta^*}\right]/V_{b^*}(x)$ is monotone decreasing in a. For $x = 2$, we have that for $a = 0.95$, selecting the best pair $(q^*,\ \beta^*)$ roughly doubles the expected life time of the portfolio, whilst for $a = 0.99$, the improvement factor is still 1.33. For $x = 4$, it is worth noticing that under the horizontal dividend strategy (with $b^* = 3.257$), an immediate dividend payment occurs, leading to a reduced expected ruin time $\mathbb{E}\left[T_{b^*}^{b^*}\right]$. Here the affine dividend strategy then compares even more favorably in terms of lifetime of the process. However, this trend is not preserved for ever higher va lues of x. For illustrative purposes, we consider $x = 50$, where clearly there is a considerable initial dividend payment under the optimal barrier strategy, contributing to a major extent to the overall value $V_{b^*}(50)$. Matching this performance under an affine dividend strategy for some large factor a, say 0.99, requires to increase $(q^*,\ \beta^*)$ to an extent that the ratio $\mathbb{E}\left[\tau_{x;q^*,\beta^*}\right]/\mathbb{E}\left[T_x^b\right]$ is then even below 1 (see dotted line at $a = 0.99$).
Figure 4.6 depicts the resulting maximizer q^* for $x = 2$ (solid line), $x = 4$ (dashed line) and $x = 50$ (dotted line) as a function of a. For $x = 2$ and $x = 4$, the choice of q^* is almost identical. However, for $x = 50$, a significant non-linear increase of q^* is needed for higher values of a to make up for the large initial lump sum payment under the optimal barrier strategy.

Altogether, one sees that a suitably chosen affine dividend strategy can lead to almost as large values for the expected discounted dividend payments, while leading to considerably improved safety, measured in terms of expected ruin time of the portfolio. Note that the chosen numerical values of the discount rate δ are quite high, and smaller values of δ can lead to an even better performance of the affine strategy relative to the optimal barrier strategy.

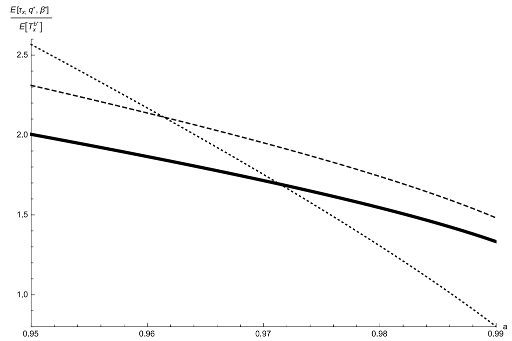

Figure 4.5: Ratio $\mathbb{E}\left[\tau_{x;q^*,\beta^*}\right]/\mathbb{E}\left[T_x^{b^*}\right]$ for $x=2$ (solid line), $x=4$ (dashed line) and $x=50$ (dotted line).

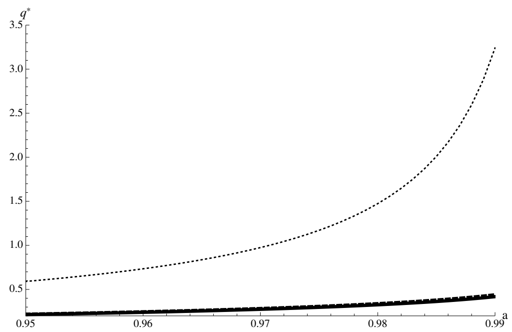

Figure 4.6: Maximizer q^* for $x=2$ (solid line), $x=4$ (dashed line) and $x=50$ (dotted line).

Consider now the optimal affine dividend strategy in the sense of (4.34). It is now a natural question to study how for such a performance level (measured in terms of total dividends value), the respective expected lifetimes under both strategies compare to each other if one is now allowed to vary the barrier level b. For that purpose, consider the following optimization problem:

$$\begin{aligned} \max_{b\geq 0} \quad & \mathbb{E}\left[T_x^b\right] \\ \text{subject to} \quad & V_b(x) = V(x;q^*,\beta^*), \end{aligned}$$

where the maximizer will be denoted by $\tilde{b}$. Table 4.5 gives the ratio $\mathbb{E}[T_x^{\tilde{b}}]/\mathbb{E}[\tau_{x;q^*,\beta^*}]$ for different capital values of x.

x	$\mathbb{E}[T_x^{\tilde{b}}]/\mathbb{E}[\tau_{x;q^*,\beta^*}]$	$\tilde{b}$
2	1.264	4.336
4	1.21	4.169
50	5.392	6.167

Table 4.5: Ratio $\mathbb{E}[T_x^{\tilde{b}}]/\mathbb{E}[\tau_{x;q^*,\beta^*}]$ for different values of x, $\alpha = 1/3$, $\lambda = 1$, $c = 3.5$ and $\delta = 0.05$.

It is interesting to observe that an appropriate choice of the barrier level leads to an improvement of the portfolio's expected lifetime in comparison to the optimal affine strategy, in particular for high capital values (here $x = 50$) for which the optimal affine strategy mimicks a barrier strategy at level zero, i.e. $q^* = \infty$. However, it should be kept in mind that this comes at the expense of an uneven dividend stream over time. Furthermore, since the optimal affine strategy is also profit oriented, this comparison between the two types of strategies does not reflect all aspects of the trade-off between profitability, safety and variability that are involved.

4.8 Conclusion

In this chapter, we studied affine dividend strategies in a classical compound Poisson risk model. Employing both analytical and probabilistic arguments, we derived explicit expressions for the expected dividend, the Laplace transform of the time to ruin and the expected ruin time in the case of exponentially distributed claims. The numerical results suggest that affine strategies lead to almost the same performance as the optimal barrier strategy in terms of expected total dividend values, but lead to, in many parameter settings, a considerably larger lifetime. Hence, in view of the compromise between profitability and safety in risk theory, such affine strategies certainly are an interesting alternative.

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Chapter 5

Affine dividend strategies in a Brownian risk model[1]

Abstract

We consider a Brownian risk model with affine dividend payments. In particular, we employ analytical techniques to derive closed-form expressions in terms of hypergeometric functions for the expected discounted dividends until ruin as well as the Laplace transform of the time to ruin. Moreover, we also study a variant of the model by allowing the surplus process to be invested at a possibly negative interest rate. Finally, we present a variety of numerical illustrations to compare the performance of affine and barrier dividend strategies and show that affine strategies can be a competitive alternative in certain situations including in the presence of negative interest rates.

5.1 Introduction

Since 1903 and the introduction of the collective risk model by Filip Lundberg to describe the time-evolution of the surplus of an insurance portfolio, the probability

[1]This chapter is based on the paper: Arian Cani. Affine dividend strategies in a Brownian risk model. Preprint

of ruin of such a portfolio has been a prime quantity to assess not only the stability but also the performance of an insurance company. However, infinite-time ruin can only be avoided if the surplus grows to infinity, which under practical considerations is typically unrealistic. This is why Bruno de Finetti [56] back in 1957 initiated an economicaly motivated change of focus and proposed a new performance measure to the actuarial community. He suggested that a rational way to prevent the surplus from going to infinity was to distribute part of it as dividends to shareholders. In particular, he proposed that the performance of an insurance portfolio should be quantified in terms of the maximal expected present value of all future dividends that will be distributed until ruin. Whereas de Finetti showed that for a simple random walk model, the optimal divdend strategy is of barrier type (whenever the surplus exceeds the barrier, all the excess is immediately paid out as dividends), researchers have since then used increasingly sophisticated mathematical tools to embed this optimality problem into a more general and realistic framework. In the classical Cramér-Lundberg model, Gerber [72] identified the so-called band strategy to be optimal, which for exponential claims reduces to a barrier strategy. This result was later re-derived by Schmidli [110] using the machinery of stochastic control theory. For a diffusion approximation, the optimal dividend problem was solved by Shreve et al. [112]. Many variants of the model were also studied in the literature, including reinsurance, investment, capital injections as well as modifications of the objective function and the addition of various constraints, see [21] and [9] for a general overview on dividend models in risk theory.

Interestingly, the barrier strategy often turns out to be optimal among all admissible payout strategies for a variety of risk models appearing in the literature (see Loeffen and Renaud [96] for the weakest currently known conditions on the risk process under which a barrier strategy is optimal). A considerable disadvantage of the barrier strategy is that it neglects the safety aspect of an insurance portfolio in the sense that every trajectory of the resulting surplus process leads to ruin. Another commonly raised criticism is that the resulting dividend flow may be very uneven over time and lead to long time periods where no dividends are paid at all (whenever the surplus is below the barrier). This issue was addressed by Avanzi and Wong [22] in the context of a Brownian risk model by specifiying a surplus-dependent dividend strategy with a mean-reversion tendency where the equilibrium level acts as a target payout ratio. The resulting dividend stream is hence smoothened over time, which is consistent with the existing empirical evidence that dividend smoothing behavior is prevalent across a wide spectrum of financial entities (see e.g. Michaely and Allen

[15], Brav et al. [34] and Larkin et al. [91] for related corporate finance literature). As emphasized in Avanzi et al. [23], the practice of dividend smoothing seems to be even more widespread in the insurance industry. In view of these aspects, Albrecher and Cani [7] studied an affine strategy with such a continuous dividend stream in the Cramér-Lundberg model.

Along this line of thought, in this chapter we consider affine dividend strategies in a Brownian risk model. As mentioned earlier, such dividend payout schemes were previously studied by Avanzi and Wong [22] in this setting in the case where the dividend rate is adjusted only according to the present surplus. While allowing for the possibility of paying at an additional constant rate is a rather straight-forward generalization of the model, we here propose a different and analytical approach for the calculation of quantities of interest such as the expected discounted dividends until ruin and the Laplace transform of the ruin time. It turns out that certain identities between special functions allow us to conclude that our results coincide with those obtained by Avanzi and Wong by means of probabilistic arguments. Moreover, we derive a closed-form expression for the expected time to ruin. Finally, we revisit the previous calculations under a possibly negative interest for the surplus which is of practical interest in the light of current economic conditions. A study in this direction can also be found in Eisenberg and Krühner [57] who examine optimal capital injections in a diffusion setting in presence of a negative discount rate.

The rest of the chapter is structured as follows. In Section 5.2, we give a mathematical formulation of the model and discuss some of its basic properties. Section 5.3 then gives an analytical approach to the solution of the second-orer differential equation satisfied by the expected discounted dividend payments and relates the obtained result to the existing literature. In Section 5.4, we compute the expected time to ruin from the corresponding Laplace transform using differentiation properties of hypergeometric functions with respect to their parameters. The obtained results are expressed in terms of generalized hypergeometric functions. Section 5.5 uses a simple variant of the model introduced in Section 5.2 to incorporate the possibility of investing the surplus at a possibly negative interest rate and provides expressions for the expected present value of dividend payouts depending on the interest rate value. Since it will serve for numerical comparisons, an explicit form for the total present value of dividends under a barrier strategy in the presence of negative interest rate is also given. Section 5.6 provides a detailed numerical analysis to determine which parameters are optimal and compare the resulting optimal affine strategy to the classical optimal barrier strategy (in the presence of negative interest rates). Finally, Section 5.7 contains some concluding remarks.

5.2 The model

Consider a company with initial capital $x \geq 0$ and assume that in the absence of dividend payments its surplus process at time t is modeled by

$$R_t = x + \mu t + \sigma W_t, \qquad t \geq 0, \tag{5.1}$$

where $\mu, \sigma > 0$ and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion. Suppose that the company pays dividends to its shareholders and let D_t denote the aggregate dividends paid up to time t. Thus, $X_t := R_t - D_t$ is the company's surplus after dividend payments. In the following, we will deal with affine dividend strategies, i.e.

$$dD_t = (qX_t + \beta)\, dt, \tag{5.2}$$

where $q > 0$ is a proportionality constant and $\beta \in [0, \mu]$ a constant rate. Then, the surplus X_t after distribution of dividends has dynamics described by a mean-reverting Ornstein-Uhlenbeck (OU) process, that is,

$$dX_t = (\mu - (qX_t + \beta))\, dt + \sigma dW_t, \qquad t \geq 0,$$

with $X_0 = x$. The idea of affine dividend strategies was first introduced by Avanzi and Wong [22] within the current Brownian risk model with $\beta = 0$. Recently, Albrecher and Cani [7] considered those affine strategies in the Cramér-Lundberg model. Applying Itô's Lemma to $e^{qt}X_t$ and integrating from 0 to t leads to the unique integral representation

$$X_t = \left(x - \frac{\mu - \beta}{q}\right) e^{-qt} + \frac{\mu - \beta}{q} + \sigma \int_0^t e^{-q(t-s)} dW_s, \qquad t \geq 0. \tag{5.3}$$

One observes that the process X_t is an exponentially weighted function of the past noise for which the influence of the initial capital decays exponentially in time. A typical sample of X_t is depicted on Figure 5.1, where the oscillation around the mean-reverting level $\frac{\mu-\beta}{q}$ is clearly visible. This mean-reverting property induces a smooth dividend stream over time, which is reflected by the almost linear time evolution of the associated aggregate dividends D_t.

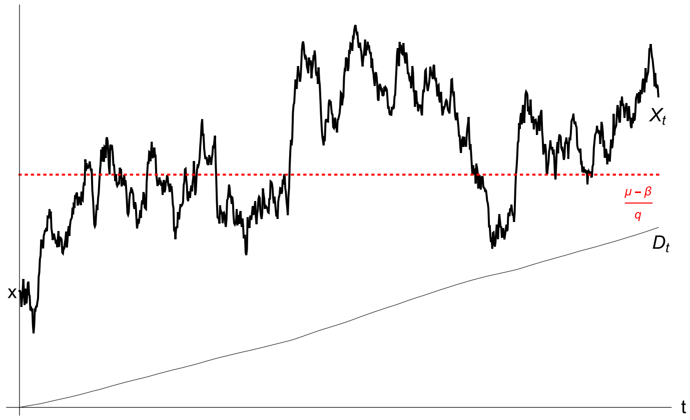


Figure 5.1: Sample path of X_t and its associated accumulated dividends D_t

A quantity that will be of main interest in the following considerations is the time to ruin of X_t, i.e.

$$\tau_x := \inf\{t \geq 0 : X_t = 0 \mid X_0 = x\}.$$

From Theorem 51.2 in Rogers and Williams [104], one has that for any $x \geq 0$, $\mathbb{P}(\tau_x < \infty) = 1$.

5.3 Expected discounted dividend payments

In this section, we are interested in studying the expected value of all dividend payouts until ruin

$$V(x) := \mathbb{E}_x \left[\int_0^{\tau_x} e^{-\delta t} \left(qX_t + \beta \right) dt \right], \qquad x \geq 0, \tag{5.4}$$

where $\delta > 0$ is the force of interest used for valuation. This problem was previously studied in the literature by Avanzi and Wong [22] for the case $\beta = 0$, in which case (5.4) will be denoted V_0. Specifically, using a simple probabilistic argument, the authors first re-expressed V_0 as

$$V_0(x) = d(x) - \mathbb{E}\left[e^{-\delta \tau_x}\right] d(0), \qquad x \geq 0, \tag{5.5}$$

where

$$d(x) := \mathbb{E}_x\left[\int_0^\infty e^{-\delta t} q X_t \, dt\right] \tag{5.6}$$

is the difference between the expected sum of discounted positive and negative dividend payments over an infinite time horizon. That is, the process X_t is continued after ruin. The specific form of (5.5) can be motivated as follows. The expected present value of dividends until ruin can be interpreted as the expected discounted sum of dividends (with negative dividends being understood as capital injections) without absorption at ruin minus a correction term which comprises the part of such dividends that are paid starting from the time of ruin and hence with zero initial capital. In a next step, the authors adjusted the Laplace transform of the first hitting time of an Ornstein-Uhlenbeck type process (cf. Alili et al. [14]) using an appropriate change of measure of the underlying process to derive

$$\mathbb{E}\left[e^{-\delta\tau_x}\right] = e^{\frac{1}{4}\left(u(x)^2-u(0)^2\right)} \frac{D_\nu\left(-u(x)\right)}{D_\nu\left(-u(0)\right)}, \qquad x \geq 0, \tag{5.7}$$

where $D_\nu(\cdot)$ denotes the parabolic cylinder function, $\nu = -\frac{\delta}{q}$ and $u(x) := \frac{\mu - qx}{\sigma}\sqrt{\frac{2}{q}}$. In view of (5.5), this led then to the following representation for V_0

$$V_0(x) = \frac{\mu\, q}{\delta(q+\delta)}\left(1 - e^{\frac{1}{4}\left(u(x)^2-u(0)^2\right)} \frac{D_\nu\left(-u(x)\right)}{D_\nu\left(-u(0)\right)}\right) + \frac{qx}{q+\delta}, \qquad x \geq 0. \tag{5.8}$$

They also noticed that V_0 can equivalently be expressed as the solution to the differential equation

$$\frac{\sigma^2}{2} V_0''(x) + (\mu - qx)V_0'(x) - \delta V_0(x) = -qx, \qquad x \geq 0, \tag{5.9}$$

which by a suitable change of coordinates can be transformed into another differential equation of hypergeometric type. However, an analytical approach to the solution of such a differential equation was left as an open problem in [22]. In the rest of this section, we use analytical tools to study the function V and solve (5.9) for the extended case $\beta \in [0, \mu]$.

We first establish some elementary properties of the function V. In the next proposition, we prove that V has a linear growth rate.

Proposition 5.3.1. *For $x \geq 0$, the function $V(x)$ admits the following bounds:*

$$\frac{qx}{q+\delta} \leq V(x) \leq \frac{qx}{q+\delta} + \frac{q(\mu-\beta)}{\delta(q+\delta)} + \frac{\beta}{\delta}. \tag{5.10}$$

Proof. To determine an upper bound for V, assume that the process X_t is allowed to continue after ruin. In particular, this means that whenever $X_t < -\frac{\beta}{q}$, negative dividends are paid (which can be interpreted as capital injections). Furthermore, denote with

$$v(x,t) := \mathbb{E}_x \left[\int_0^t e^{-\delta s} \left(qX_s + \beta \right) ds \right] \tag{5.11}$$

the difference between the expected sum of discounted dividend payments and discounted capital injections up to time t. Noting that the stochastic integral in (5.3) has expectation zero, we get by straightforward calculations that

$$v(x,t) = \frac{qx - (\mu - \beta)}{q + \delta} \left(1 - e^{-(q+\delta)t} \right) + \frac{\mu}{\delta} \left(1 - e^{-\delta t} \right).$$

The monotonicity of $v(x,t)$ in $t \in [0, \infty]$ follows from

$$\frac{\partial}{\partial t} v(x,t) = e^{-(q+\delta)t} \left(qx + \beta + \mu(e^{qt} - 1 \right) \geq 0.$$

Consequently, the almost-sure finiteness of τ_x implies that $V(x)$ is bounded from above by $v(x, \infty)$, which yields the desired result.

It remains now to determine a lower bound for V. For that purpose, define $f(x) := \frac{qx}{q+\delta}$ and consider an operator M acting on f defined as

$$Mf(x) := \mathcal{L}f(x) - \delta f(x) + qx + \beta,$$

where $x \geq 0$ and $\mathcal{L}f(x) := \frac{\sigma^2}{2} f''(x) + (\mu - (qx + \beta))\, f'(x)$ is the infinitesimal generator of the process (5.3). This operator can be rewritten in the more compact form

$$Mf(x) = \frac{(\mu - \beta)q}{q + \delta} + \beta \geq 0.$$

By Dynkin's formula,

$$e^{-\delta t} f(X_t) - f(x) - \int_0^t e^{-\delta s} \left[\mathcal{L}f(X_s) - \delta f(X_s) \right] ds$$

is a martingale with expectation zero. Replacing t by the bounded stopping time

$t \wedge \tau_x$ and taking expectations, we get

$$\mathbb{E}_x \left[e^{-\delta t \wedge \tau_x} f(X_{t \wedge \tau_x}) \right] = f(x) + \mathbb{E}_x \left[\int_0^{t \wedge \tau_x} e^{-\delta s} \left[\mathcal{L} f(X_s) - \delta f(X_s) \right] ds \right]. \tag{5.12}$$

Since $\mathcal{L} f(X_s) - \delta f(X_s) = \frac{(\mu - \beta) q}{q + \delta} - q X_s$, the integrand on the right-hand side is bounded from below by $e^{-\delta s} \left(\frac{(\mu-\beta)q}{q+\delta} - (qX_s + \beta) \right)$. In addition, because $X_t \leq \left(x - \frac{\mu - \beta}{q} \right) e^{-qt} + \frac{\mu - \beta}{q} + \sigma |W_t| := \bar{X}_t$, the term on the left-hand side is upper-bounded by

$$\mathbb{E} \left[e^{-\delta t} f(\bar{X}_t) \right] = e^{-(q+\delta)t} \frac{qx - (\mu - \beta)}{q + \delta} + e^{-\delta t} \frac{\mu - \beta + \sigma q \sqrt{\frac{2t}{\pi}}}{q + \delta}.$$

Consequently, letting $t \to \infty$ in (5.12) and envoking the monotone convergence theorem, we obtain

$$\mathbb{E}_x \left[\int_0^{\tau_x} e^{-\delta t} \left(q X_t + \beta \right) dt \right] \geq f(x) + \frac{q(\mu - \beta)}{q + \delta} \mathbb{E}_x \left[\int_0^{\tau_x} e^{-\delta t} dt \right],$$

which establishes the proposition. □

Proposition 5.3.2. *For $0 \leq y < x$, the function V satisfies*

$$\frac{q(x-y)}{q+\delta} \left(1 - \mathbb{E} \left[e^{-(q+\delta)\tau_y} \right] \right) \leq V(x) - V(y) \leq \frac{q(x-y)}{q+\delta} + V(x-y) \mathbb{E} \left[e^{-\delta \tau_y} \right]. \tag{5.13}$$

Proof. Let $0 \leq y < x$ and let X^x and X^y be the surplus processes starting at x and y with respective ruin times τ_x and τ_y. By a pathwise comparison of these processes, we get that $X_t^x(\omega) - X_t^y(\omega) = (x-y) e^{-qt}$ for each $\omega \in \Omega$ on condition that $t < \tau_y < \tau_x$.

Hence, we can write

$$\begin{aligned} V(x) - V(y) &= \mathbb{E} \left[\int_0^{\tau_y} e^{-\delta t} q (X_t^x - X_t^y) dt \right] + \mathbb{E} \left[\int_{\tau_y}^{\tau_x} e^{-\delta t} (q X_t^x + \beta) dt \right], \\ &= \mathbb{E} \left[\int_0^{\tau_y} e^{-(q+\delta)t} q (x-y) dt \right] + \mathbb{E} \left[\int_{\tau_y}^{\tau_x} e^{-\delta t} (q X_t^x + \beta) dt \right], \\ &\leq \int_0^{\infty} e^{-(q+\delta)t} q(x-y) dt + V(x-y) \mathbb{E} \left[e^{-\delta \tau_y} \right]. \end{aligned}$$

The last inequality holds due to the almost sure finiteness of τ_y in the first integral and due to the strong Markov property of the process X^x together with $X^x_{\tau_y}(\omega) = (x-y)e^{-q\tau_y(\omega)} \le x-y$ in the second integral.

Additionaly, we can derive

$$\begin{aligned} V(x) - V(y) &= \mathbb{E}\left[\int_0^{\tau_y} e^{-\delta t} q(X_t^x - X_t^y)dt\right] + \mathbb{E}\left[\int_{\tau_y}^{\tau_x} e^{-\delta t}(qX_t^x + \beta)dt\right], \\ &\ge \frac{q(x-y)}{q+\delta}\left(1 - \mathbb{E}\left[e^{-(q+\delta)\tau_y}\right]\right), \end{aligned}$$

which yields the desired result. □

As a direct consequence of the previous proposition, we get that V is continuous and monotone increasing on $[0, \infty)$.

5.3.1 Constructing an exact solution

As a function of the initial capital x, $V(x)$ satisfies the inhomogeneous second-order differential equation

$$\frac{\sigma^2}{2} V''(x) + (\mu - (qx+\beta))\, V'(x) - \delta V(x) = -(qx+\beta), \qquad x > 0, \tag{5.14}$$

together with the requirement $V(0) = 0$.

This characterization is based on the following heuristic argument. Consider an infinitesimal time interval dt. Then, the total expected discounted dividend payouts until ruin can be decomposed as the sum of an instantaneous dividend payment and its expected discounted continuation value

$$V(x) = (qx+\beta)dt + \mathbb{E}_x\left[e^{-\delta dt}\, V(X_{dt})\right]. \tag{5.15}$$

Because

$$X_{dt} = x + (\mu - (qx+\beta))\, dt + \sigma W_{dt},$$

using a Taylor expansion for $V(X_{dt})$ (and $e^{-\delta dt}$) allows us to rewrite (after some rearrangement) the right-hand side of (5.15) as

$$(qx+\beta)dt + (1-\delta dt)\,V(x) + (\mu - (qx+\beta))\,V'(x)dt + \frac{\sigma^2}{2}V''(x)dt + o(dt).$$

Hence, substituting this back into (5.15) and canceling all dt-terms yields (5.14).

We first look for a solution, say V_h, to the associated homogeneous differential equation of (5.14). Substituting $z := z(x) = \frac{(\mu-(qx+\beta))^2}{\sigma^2 q}$ and then defining $g(z) := V_h(x)$ in (5.14) yields *Kummer*'s confluent hypergeometric equation

$$zg''(z) + (b-z)g'(z) - ag(z) = 0, \qquad z \geq 0, \tag{5.16}$$

with parameters

$$a = \frac{\delta}{2q}, \qquad b = \frac{1}{2}.$$

Kummer's equation exhibits a regular singular point at $z = 0$ and an irregular singular point at $z = \infty$, which in the original coordinates correspond to $x = \frac{\mu-\beta}{q} \geq 0$ and $x = \infty$ respectively. Two linearly independent solutions to this differential equation are the Kummer's confluent hypergeometric function

$$M(a,b,z) = {}_1F_1(a,b,z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!}, \tag{5.17}$$

where $(a)_n := \frac{\Gamma(a+n)}{\Gamma(n)}$ is the Pochhammer symbol expressed in terms of the gamma function Γ, and Tricomi's confluent hypergeometric function

$$U(a,b,z) = \begin{cases} \frac{\Gamma(1-b)}{\Gamma(1+a-b)}M(a,b,z) + \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}M(1+a-b,2-b,z), & b \notin \mathbb{Z}, \\ \lim\limits_{\theta \to b} U(a,\theta,z), & b \in \mathbb{Z}. \end{cases} \tag{5.18}$$

Consequently, this leads to

$$V_h(x) = g(z) = \begin{cases} A_1\, M\left(\frac{\delta}{2q}, \frac{1}{2}, z(x)\right) + A_2\, U\left(\frac{\delta}{2q}, \frac{1}{2}, z(x)\right), & 0 \leq x \leq \frac{\mu-\beta}{q}, \\ A_3\, M\left(\frac{\delta}{2q}, \frac{1}{2}, z(x)\right) + A_4\, U\left(\frac{\delta}{2q}, \frac{1}{2}, z(x)\right), & x > \frac{\mu-\beta}{q}, \end{cases} \tag{5.19}$$

for arbitrary constants $A_i, i = 1, \ldots, 4$. The piecewise construction of V_h stems from the fact that the mapping $x \mapsto z(x)$ is not injective for $x \geq 0$. However, considering $z(x)$ separately over the domains $[0, \frac{\mu-\beta}{q}]$ and $\left(\frac{\mu-\beta}{q}, \infty\right)$ renders this mapping piecewise injective.

Having this in mind, the general solution to (5.14) takes the form

$$V(x) = V_h(x) + V_p(x),$$

where $V_p(x)$ is a particular solution to (5.14). Seeking a particular solution of the form $V_p(x) = Ax + B$, one gets

$$V_p(x) = \frac{qx}{q+\delta} + \frac{(\mu - \beta)q}{\delta(q+\delta)} + \frac{\beta}{\delta}, \qquad x \geq 0. \tag{5.20}$$

The next step is to determine the constants $A_i, i = 1, \ldots, 4$. The linear boundedness of V established in Proposition 5.3.1 requires examining the asymptotic behavior of the components of the general solution for large argument values. From classical results (cf. [3]), the respective asymptotic behaviors of the hypergeometric functions M and U, as $z \to \infty$, are given by

$$M(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-c} (1 + \mathcal{O}\left(|z^{-1}|\right)) \tag{5.21}$$

and

$$U(a, b, z) \sim z^{-a} (1 + \mathcal{O}\left(|z^{-1}|\right)). \tag{5.22}$$

In the original x-coordinates, as $x \to \infty$, this becomes

$$M\left(\frac{\delta}{2q}, \frac{1}{2}, z(x)\right) \sim \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}\right)} e^{z(x)} \left(\frac{|\mu - (qx + \beta)|}{\sigma\sqrt{q}}\right)^{\frac{\delta}{q}-1} \left(1 + \mathcal{O}\left(x^{-1}\right)\right) \tag{5.23}$$

and

$$U\left(\frac{\delta}{2q}, \frac{1}{2}, z(x)\right) \sim \left(\frac{|\mu - (qx + \beta)|}{\sigma\sqrt{q}}\right)^{-\frac{\delta}{q}} \left(1 + \mathcal{O}\left(x^{-1}\right)\right). \tag{5.24}$$

While (5.23) becomes unbounded exponentially fast as $x \to \infty$, (5.24) tends to 0. In view of the linear boundedness of V, we must have $A_3 = 0$. Next, from the initial condition $V(0) = 0$, we obtain

$$A_1\, M\left(\frac{\delta}{2q}, \frac{1}{2}, z(0)\right) + A_2\, U\left(\frac{\delta}{2q}, \frac{1}{2}, z(0)\right) = -\frac{(\mu-\beta)q}{\delta(q+\delta)} - \frac{\beta}{\delta}. \tag{5.25}$$

Assuming now that $V \in \mathcal{C}^2(\mathbb{R}_+)$ implies the following smooth-fit conditions at the regular singularity $\frac{\mu-\beta}{q}$:

$$V\left(\frac{\mu-\beta}{q}-\right) = V\left(\frac{\mu-\beta}{q}+\right), \tag{5.26}$$

$$V'\left(\frac{\mu-\beta}{q}-\right) = V'\left(\frac{\mu-\beta}{q}+\right), \tag{5.27}$$

$$V''\left(\frac{\mu-\beta}{q}-\right) = V''\left(\frac{\mu-\beta}{q}+\right). \tag{5.28}$$

The continuity condition (5.26) requires studying the behavior V in a small vicinity of the regular singularity $\frac{\mu-\beta}{q}$. For that matter, we use the asymptotic expansions of the confluent hypergeometric functions M and U at small argument values, which from [3] are respectively given by

$$\lim_{z\to 0} M(a,b,z) = 1 + \mathcal{O}(z), \tag{5.29}$$

and

$$\lim_{z\to 0} U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} + \mathcal{O}\left(|z|^{1-\Re(b)}\right), \qquad 0 < \Re(b) < 1. \tag{5.30}$$

In our case, this translates into

$$\lim_{x\to\frac{\mu-\beta}{q}} M\left(\frac{\delta}{2q}, \frac{1}{2}, z(x)\right) = 1 + \mathcal{O}\left(\left(x - \frac{\mu-\beta}{q}\right)^2\right)$$

and

$$\lim_{x\to\frac{\mu-\beta}{q}} U\left(\frac{\delta}{2q}, \frac{1}{2}, z(x)\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+\frac{1}{2}\right)} + \mathcal{O}\left(\left|x - \frac{\mu-\beta}{q}\right|\right).$$

As a result, the continuity requirement (5.26) at $\frac{\mu-\beta}{q}$ leads to

$$A_1 + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+\frac{1}{2}\right)} A_2 = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+\frac{1}{2}\right)} A_4. \tag{5.31}$$

Next, we examine the condition $V'\left(\frac{\mu-\beta}{q}-\right) = V'\left(\frac{\mu-\beta}{q}+\right)$ which guarantees that the derivative of V is continuous at $\frac{\mu-\beta}{q}$. Here, this boils down to $V_h'\left(\frac{\mu-\beta}{q}-\right) = V_h'\left(\frac{\mu-\beta}{q}+\right)$ which necessitates differentiating the functions M and U with respect to their argument. From [3], the differentiation formulas read

$$\frac{d}{dz}M(a,b,z) = \frac{a}{b}M\left(a+1,b+1,z\right)$$

and

$$\frac{d}{dz}U(a,b,z) = -a\,U\left(a+1,b+1,z\right).$$

For the sake of brevity, let $M(x) := M\left(\frac{\delta}{2q},\frac{1}{2},z(x)\right)$ and $U(x) := U\left(\frac{\delta}{2q},\frac{1}{2},z(x)\right)$. Then, we respectively obtain

$$\frac{d}{dx}M(x) = -\frac{2\delta\left(\mu-(qx+\beta)\right)}{\sigma^2 q}M\left(\frac{\delta}{2q}+1,\frac{3}{2},z(x)\right) \tag{5.32}$$

and

$$\frac{d}{dx}U(x) = \frac{\delta\left(\mu-(qx+\beta)\right)}{\sigma^2 q}U\left(\frac{\delta}{2q}+1,\frac{3}{2},z(x)\right). \tag{5.33}$$

Clearly, from (5.32), we get that $M(x)$ is continuously differentiable for all $x \geq 0$ with the property $M'\left(\frac{\mu-\beta}{q}\right) = 0$. On the other hand, $U(x)$ has a continuous derivative on $\mathbb{R}_+$ except at $x = \frac{\mu-\beta}{q}$, where it is not differentiable. This originates from the fact that the Tricomi function $U(a,b,z)$ is singular for $z=0$ if $\Re(b) \geq 1$ as it is the case in (5.33).

To fulfill condition (5.27), we have to investigate how (5.33) behaves as x approaches $\frac{\mu-\beta}{q}$. For that purpose, we need the following limiting behavior (cf. [3])

$$\lim_{z\to 0} U(a,b,z) = \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b} + \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + \mathcal{O}\left(|z|^{2-\Re(b)}\right), \qquad 1 \leq \Re(b) < 2, b \neq 1.$$

Consequently, we can write

$$\lim_{x\to\frac{\mu-\beta}{q}} U'(x) = \lim_{x\to\frac{\mu-\beta}{q}} \frac{\delta\left(\mu-(qx+\beta)\right)}{\sigma^2 q}\left(\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+1\right)}\frac{\sigma\sqrt{q}}{|\mu-(qx+\beta)|}+\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+\frac{1}{2}\right)}\right.$$
$$\left.+\mathcal{O}\left(\left|x-\frac{\mu-\beta}{q}\right|\right)\right)$$
$$= \operatorname{sgn}(\mu-(qx+\beta))\frac{\delta}{\sigma\sqrt{q}}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+1\right)}. \tag{5.34}$$

From the above, $V(x)$ is (continuously) differentiable at $\frac{\mu-\beta}{q}$ if and only if $A_2U'\left(\frac{\mu-\beta}{q}-\right) = A_4U'\left(\frac{\mu-\beta}{q}+\right)$. In view of (5.34), we must therefore have $A_2 = -A_4$.

Remark 5.3.1. *The functions $M(x)$ and $U(x)$ are both symmetric w.r.t. the vertical line at $\frac{\mu-\beta}{q}$ since $M\left(x+\frac{\mu-\beta}{q}\right)$ and $U\left(x+\frac{\mu-\beta}{q}\right)$ are even functions. However, in contrast to $M(x)$ which is continuously differentiable on $\mathbb{R}_+$, $U(x)$ has a corner point at $\frac{\mu-\beta}{q}$. Noting that $U\left(x+\frac{\mu-\beta}{q}\right)$ is an even function implies $U'\left(\frac{\mu-\beta}{q}-\right) = -U'\left(\frac{\mu-\beta}{q}+\right)$. Combining this last result with $M'\left(\frac{\mu-\beta}{q}\right) = 0$, it becomes evident that condition (5.27) can be rewritten only in terms of the coefficients A_2 and A_4, more particularly setting $A_2 = -A_4$ as shown above.*

Together with the initial condition (5.25) and the continuity requirement (5.31), the first-order smooth-fit condition yields a system of linear equations in terms of the coefficients A_1, A_2 and A_4. The solution of this system, which by construction also satisfies (5.28), is summarized in the following proposition.

Proposition 5.3.3. *For any $x \geq 0$, the sum of the expected discounted dividend payments up to the time of ruin in a diffusion model with affine dividend strategy* (5.2) *is given by*

$$V(x) = \begin{cases} C\left(\frac{2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+\frac{1}{2}\right)}M\left(\frac{\delta}{2q},\frac{1}{2},z(x)\right)-U\left(\frac{\delta}{2q},\frac{1}{2},z(x)\right)\right) & \\ +\frac{qx}{q+\delta}+\frac{(\mu-\beta)q}{\delta(q+\delta)}+\frac{\beta}{\delta}, & 0\leq x\leq \frac{\mu-\beta}{q}, \\ C\,U\left(\frac{\delta}{2q},\frac{1}{2},z(x)\right)+\frac{qx}{q+\delta}+\frac{(\mu-\beta)q}{\delta(q+\delta)}+\frac{\beta}{\delta}, & x>\frac{\mu-\beta}{q}, \end{cases}$$

where

$$C = \frac{\left(\frac{(\mu-\beta)q}{\delta(q+\delta)}+\frac{\beta}{\delta}\right)}{U\left(\frac{\delta}{2q},\frac{1}{2},z(0)\right)-\frac{2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+\frac{1}{2}\right)}M\left(\frac{\delta}{2q},\frac{1}{2},z(0)\right)},$$

and $z(x) = \frac{(\mu-(qx+\beta))^2}{\sigma^2 q}$.

We are now able to finalize the analytical characterization of V.

Proposition 5.3.4. *Suppose that* $\hat{V}$ *is a* $C^2((0,\infty))$ *positive linearly bounded solution to* (5.14) *in conjunction with the boundary condition* $\hat{V}(0) = 0$. *Then,* $\hat{V} = V$.

Proof. Applying Itô's Lemma to $e^{-\delta t}\hat{V}$ with $t \geq 0$, one obtains

$$e^{-\delta t\wedge\tau_x}\hat{V}\left(X_{t\wedge\tau_x}\right) = V(x) + \int_0^{t\wedge\tau_x} e^{-\delta s}\left[\frac{\sigma^2}{2}\hat{V}''(X_s) + \Big(\mu - (qX_s+\beta)\hat{V}'(X_s) - \delta\hat{V}(X_s)\Big)\right]ds$$
$$+\sigma\int_0^{t\wedge\tau_x} e^{-\delta s}\hat{V}'(X_s)dW_s.$$

Consequently, taking the expectation on both sides leads to

$$\mathbb{E}_x\left[e^{-\delta t\wedge\tau_x}\hat{V}\left(X_{t\wedge\tau_x}\right)\right] + \mathbb{E}_x\left[\int_0^{t\wedge\tau_x} e^{-\delta s}\left(qX_s+\beta\right)ds\right] = \hat{V}(x). \tag{5.35}$$

Using the linear boundedness of $\hat{V}$, for the first term on the left-hand side of (5.35), we have

$$\mathbb{E}_x\left[e^{-\delta t\wedge\tau_x}\hat{V}\left(X_{t\wedge\tau_x}\right)\right] \leq \mathbb{E}_x\left[e^{-\delta t}\hat{V}(X_t)\right],$$
$$\leq \mathbb{E}_x\left[e^{-\delta t}\left(A+BX_t\right)\right],$$

for some $A, B > 0$. Consequently, letting $t \to \infty$ in (5.35), the first term on the left-hand side converges to zero by dominated convergence (since $X_t \leq \left(x - \frac{\mu-\beta}{q}\right)e^{-qt} + \frac{\mu-\beta}{q} + \sigma|W_t|$) and employing the monotone convergence theorem for the second term, we arrive at

$$\mathbb{E}_x\left[\int_0^{\tau_x} e^{-\delta s}\left(qX_s+\beta\right)ds\right] = \hat{V}(x),$$

which yields the desired result. □

The last proposition ensures that the total expected dividends value (5.4) is the unique $C^2((0,\infty))$ positive linearly bounded solution to (5.14) together with the boundary condition $V(0) = 0$.

Remark 5.3.2. *An alternative representation of the function* V *can be given in terms of the parabolic cylinder function* $D_\nu(x)$. *Using the respective connections (cf. [3])*

$$D_\nu(x) = e^{-\frac{x^2}{4}} 2^{\frac{\nu}{2}} \left[\frac{2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)} M\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{x^2}{2}\right) - U\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{x^2}{2}\right)\right], \qquad x \leq 0,$$

and

$$D_\nu(x) = e^{-\frac{x^2}{4}} 2^{\frac{\nu}{2}} U\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{x^2}{2}\right), \qquad x \geq 0,$$

one can rewrite in view of Proposition 5.5.3:

$$V(x) = \frac{(\mu - \beta)q}{\delta(q+\delta)} \left(1 - e^{\frac{1}{4}(y(x)^2 - y(0)^2)} \frac{D_\nu(-y(x))}{D_\nu(-y(0))}\right) + \frac{qx}{q+\delta}, \qquad x \geq 0,$$

where $\nu = -\frac{\delta}{q}$ *and* $y(x) := \operatorname{sgn}(\mu - (qx+\beta))\sqrt{2z(x)} = \frac{\mu-(qx+\beta)}{\sigma}\sqrt{\frac{2}{q}}$. *This last expression coincides with the one obtained by Avanzi and Wong [22] for* $\beta = 0$ *given in (5.8). It is worth observing that due to the injectivity of the mapping* $x \mapsto y(x)$, *no piecewise construction for* V *is required.*

5.4 Time of ruin

In this section, we are going to discuss the expected time to ruin under the proposed affine dividend strategy. For this particular purpose, let us define the Laplace transform of the ruin time by

$$m_\delta(x) := \mathbb{E}\left[e^{-\delta\tau_x}\right].$$

While it is clear that this quantity can be expressed by replacing $u(x)$ by $y(x)$ in (5.7) which adds the possibility of paying dividends at a constant rate $\beta \geq 0$, we here derive it employing a simple analytical argument which allows us to connect the functions m_δ and V akin to the probabilisitic method used by Avanzi and Wong [22].

By virtually the same differential arguments as in Section 5.3, one can show that $m_\delta(x)$ satisfies the second-order homogeneous differential equation

$$\frac{\sigma^2}{2} m_\delta''(x) + (\mu - (qx+\beta))\, m_\delta'(x) - \delta m_\delta(x) = 0, \qquad x > 0,$$

with $m_\delta(0) = 1$. Comparing the last equation with (5.14), one observes that the two differ only in the inhomogeneous term and the boundary condition. Formally, by the boundedness of $m_\delta(x)$, we can write $m_\delta(x) = AV_h(x)$ for some constant A, where $V_h(x) = V(x) - V_p(x)$. Setting $x = 0$ leads to

$$m_\delta(0) = -AV_p(0),$$

that is

$$A = -V_p(0)^{-1}.$$

Proposition 5.4.1. *For any $x \geq 0$, the Laplace transform of the time tp ruin in a diffusion model with affine dividend strategy* (5.2) *is given by*

$$m_\delta(x) = \begin{cases} \dfrac{U\left(\frac{\delta}{2q},\frac{1}{2},z(x)\right) - \frac{2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+\frac{1}{2}\right)} M\left(\frac{\delta}{2q},\frac{1}{2},z(x)\right)}{U\left(\frac{\delta}{2q},\frac{1}{2},z(0)\right) - \frac{2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+\frac{1}{2}\right)} M\left(\frac{\delta}{2q},\frac{1}{2},z(0)\right)}, & 0 \leq x \leq \frac{\mu-\beta}{q}, \\[2ex] \dfrac{-U\left(\frac{\delta}{2q},\frac{1}{2},z(x)\right)}{U\left(\frac{\delta}{2q},\frac{1}{2},z(0)\right) - \frac{2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+\frac{1}{2}\right)} M\left(\frac{\delta}{2q},\frac{1}{2},z(0)\right)}, & x > \frac{\mu-\beta}{q}, \end{cases}$$

where $z(x) = \frac{(\mu-(qx+\beta))^2}{\sigma^2 q}$.

An immediate consequence of the representation $m_\delta(x) = AV_h(x)$ is that it provides a connection between the Laplace transform of the ruin time and the total expected discounted dividend payouts until ruin, namely,

$$V(x) = V_p(x) - m_\delta(x)V_p(0), \qquad x \geq 0. \tag{5.36}$$

A termwise comparison of (5.36) and (5.5) gives rise to an elegant probabilistic interpretation of the particular solution to (5.14), namely that it corresponds to the total expected discounted dividend payments (net of capital injections) if the process X_t is not absorbed at ruin time.

Let us now consider the expected ruin time

$$\mathbb{E}[\tau_x] = -\frac{d}{d\delta} m_\delta(x)\bigg|_{\delta=0}, \tag{5.37}$$

in more detail. In view of the form of $m_\delta(x)$, this necessitates differentiating the hypergeometric functions M and U with respect to the parameter a. For that purpose, we shall introduce the following notation:

$$\kappa(a,b,z) := \frac{d}{da} U(a,b,z).$$

Differentiating (5.18) with respect to a yields

$$\begin{aligned}\kappa(a,b,z) = &\frac{\Gamma(1-b)}{\Gamma(1+a-b)}\left[-\psi(1+a-b)M(a,b,z)+\frac{d}{da}M(a,b,z)\right] \\ &+\frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}\left[-\psi(a)M(1+a-b,2-b,z)+\frac{d}{da}M(1+a-b,2-b,z)\right],\end{aligned} \tag{5.38}$$

where $\psi(a) = \frac{d}{da}\log\Gamma(a)$ is the logarithmic derivative of the Gamma function, known as the digamma function. The function $\kappa(a,b,z)$ has a singularity at $a=0$, which can be removed by redefining

$$\kappa(0,b,z) := \lim_{a\to 0+}\kappa(a,b,z) = -\psi(1-b)+\lim_{a\to 0+}\frac{d}{da}M(a,b,z)+\Gamma(b-1)z^{1-b}M(1-b,2-b,z), \tag{5.39}$$

for which we used the fact that $\lim_{a\to 0+} M(a,b,z)=1$ and $\lim_{a\to 0+}\frac{\psi(a)}{\Gamma(a)}=-1$. While the first result is straightforward, the second can be justified as follows. We first re-express

$$\frac{\psi(a)}{\Gamma(a)} = -\frac{d}{da}\left(\frac{1}{\Gamma(a)}\right). \tag{5.40}$$

From the Weierstrass representation of the Gamma function, we have

$$\frac{1}{\Gamma(a)} = ae^{\gamma a}\prod_{n=1}^{\infty}\left(1+\frac{a}{n}\right)e^{-\frac{a}{n}},$$

where γ is the Euler-Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right).$$

A Taylor series expansion of the reciprocal Gamma function $1/\Gamma(a)$ around 0 (c.f. [102]) is

$$\frac{1}{\Gamma(a)} = a+\gamma a^2+\frac{1}{2}\left(\gamma^2-\frac{\pi^2}{6}\right)a^3+R(a), \tag{5.41}$$

where the remainder $R(a)$ vanishes (as well as its derivative w.r.t. a) as $a\to 0$. Replacing (5.41) in the right-hand side of (5.40) shows that

$$\lim_{a\to 0+}\frac{\psi(a)}{\Gamma(a)} = -1.$$

It now remains to evaluate the second term of (5.39) which requires computing the derivative of M with respect to a. The latter can be expressed in terms of the bivariate Kampé de Fériet function

$$F^{A,B,C}_{R,S,U}\left(\begin{matrix} a_1,\ldots,a_A \; b_1,\ldots,b_B \; c_1,\ldots,c_C \\ r_1,\ldots,r_R \; s_1,\ldots,s_S \; u_1,\ldots,u_U \end{matrix} x,y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{m+n}\prod_{j=1}^{B}(b_j)_m\prod_{j=1}^{C}(c_j)_n}{\prod_{j=1}^{R}(r_j)_{m+n}\prod_{j=1}^{S}(s_j)_m\prod_{j=1}^{U}(u_j)_n}\frac{x^m y^n}{m!\,n!}, \quad (5.42)$$

see [115, 61] for further details. More precisely, the connection is

$$\frac{d}{da}M(a,b,z) = \frac{z}{b}F^{1,2,1}_{2,1,0}\left(\begin{matrix} a+1 & 1,a & 1 \\ 2,b+1 & a+1 & - \end{matrix} z,z\right),$$

where the empty product indicated by the solid horizontal line is interpreted to be unity (see Section 1.5.1). Taking now the limit $a \to 0$ in the last expression, we remark that only the term $m=0$ contributes to the first sum in (5.42). This yields

$$\lim_{a\to 0+}\frac{d}{da}M(a,b,z) = \frac{z}{b}\,{}_2F_2\left(1,1;2,b+1;z\right) = \frac{d}{da}M(a,b,z)\bigg|_{a=0}, \quad (5.43)$$

where the ${}_2F_2$ generalized hypergeometric function is defined through

$${}_2F_2\left(a_1,a_2;b_1,b_2;z\right) := \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n}{(b_1)_n(b_2)_n}\frac{z^n}{n!}.$$

A substitution of the last result in (5.38) finally gives

$$\kappa(0,b,z) := -\psi(1-b) + \frac{z}{b}\,{}_2F_2\left(1,1;2,b+1;z\right) + \Gamma(b-1)z^{1-b}M(1-b,2-b,z). \quad (5.44)$$

Differentiating $m_\delta(x)$ w.r.t. δ together with some rearrangement respectively yields

$$\frac{d}{d\delta}m_\delta(x) = \frac{1}{2q}\left[\eta(\delta,x) - m_\delta(x)\eta(\delta,0)\right]$$

for $0 \le x \le \frac{\mu-\beta}{q}$ and

$$\frac{d}{d\delta}m_\delta(x) = \frac{1}{2q}\left[\frac{U\left(\frac{\delta}{2q},\frac{1}{2},z(x)\right)\eta(\delta,0) - \kappa\left(\delta,\frac{1}{2},z(x)\right)}{U\left(\frac{\delta}{2q},\frac{1}{2},z(0)\right) - \frac{2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+\frac{1}{2}\right)}M\left(\frac{\delta}{2q},\frac{1}{2},z(0)\right)}\right]$$

for $x > \frac{\mu-\beta}{q}$, where we have defined

$$\eta(\delta,x) := \frac{\kappa\left(\delta,\frac{1}{2},z(x)\right) + \frac{2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+\frac{1}{2}\right)}\left(\psi\left(\frac{\delta}{2q}+\frac{1}{2}\right)M\left(\frac{\delta}{2q},\frac{1}{2},z(x)\right) - \frac{d}{d\delta}M\left(\delta,\frac{1}{2},z(x)\right)\right)}{U\left(\frac{\delta}{2q},\frac{1}{2},z(0)\right) - \frac{2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2q}+\frac{1}{2}\right)}M\left(\frac{\delta}{2q},\frac{1}{2},z(0)\right)}.$$

We can now evaluate the last two expressions at $\delta = 0$ in the sense of (5.37) with the help of (5.43) and (5.44). Recalling that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ leads to the following proposition.

Proposition 5.4.2. *For any $x \geq 0$, the expected time of ruin in a diffusion model with affine dividend strategy* (5.2) *is given by*

$$\mathbb{E}\left[\tau_x\right] = \begin{cases} \frac{\sqrt{\pi}}{q}\left(\sqrt{z(0)}M\left(\frac{1}{2},\frac{3}{2},z(0)\right) - \sqrt{z(x)}M\left(\frac{1}{2},\frac{3}{2},z(x)\right)\right) & \\ +\frac{1}{q}\left(z(0)\,{}_2F_2\left(1,1;\frac{3}{2},2,\ z(0)\right) - z(x)\,{}_2F_2\left(1,1;\frac{3}{2},2,\ z(x)\right)\right), & 0 \leq x \leq \frac{\mu-\beta}{q}, \\ \frac{\sqrt{\pi}}{q}\left(\sqrt{z(0)}M\left(\frac{1}{2},\frac{3}{2},z(0)\right) + \sqrt{z(x)}M\left(\frac{1}{2},\frac{3}{2},z(x)\right)\right) & \\ +\frac{1}{q}\left(z(0)\,{}_2F_2\left(1,1;\frac{3}{2},2,\ z(0)\right) - z(x)\,{}_2F_2\left(1,1;\frac{3}{2},2,\ z(x)\right)\right), & x > \frac{\mu-\beta}{q}. \end{cases}$$

where $z(x) = \frac{(\mu-(qx+\beta))^2}{\sigma^2 q}$.

5.5 Analysis with an interest rate

5.5.1 Present value of dividends under affine strategies

We would like now to examine affine dividend strategies under the assumption that the surplus is invested continuously at a constant interest force i. The dynamics of the resulting surplus process, denoted by $(\tilde{X}_t)_{t\geq 0}$, is then given by

$$d\tilde{X}_t = \left(\mu - \left((q-i)\tilde{X}_t + \beta\right)\right)dt + \sigma dW_t, \qquad t \geq 0,$$

with $\tilde{X}_0 = x$. This leads to the unique (integral) representation

$$\tilde{X}_t = \begin{cases} \left(x - \frac{\mu-\beta}{q-i}\right)e^{-(q-i)t} + \frac{\mu-\beta}{q-i} + \sigma\int_0^t e^{-(q-i)(t-s)}dW_s, & q \neq i, \\ x + (\mu-\beta)t + \sigma W_t, & q = i. \end{cases}$$

The associated expected present value of dividends until ruin can then be defined as

$$\tilde{V}(x) := \mathbb{E}_x\left[\int_0^{\tilde{\tau}_x} e^{-\delta t}\left(q\tilde{X}_t + \beta\right)\right]dt, \qquad x \geq 0,$$

where

$$\tilde{\tau}_x := \inf\{t \geq 0 : \tilde{X}_t = 0 \mid \tilde{X}_0 = x\}.$$

By analogy to Section 5.3, $\tilde{V}(x)$ can be characterized as the solution to the second-order differential equation

$$\frac{\sigma^2}{2}\tilde{V}''(x) + (\mu - ((q-i)x + \beta))\,\tilde{V}'(x) - \delta\tilde{V}(x) = -(qx+\beta), \qquad x > 0, \tag{5.45}$$

subject to the boundary condition $\tilde{V}(0) = 0$.

The general solution to (5.45) will be of the form $\tilde{V}(x) = \tilde{V}_h(x) + \tilde{V}_p(x)$, where $\tilde{V}_h$ is the general solution of the related homogeneous differential equation and $\tilde{V}_p$ a particular solution. Looking for a particular solution of the form $\tilde{V}_p(x) = Ax + B$, we obtain

$$\tilde{V}_p(x) = \frac{qx}{q-i+\delta} + \frac{(\mu-\beta)q}{\delta(q-i+\delta)} + \frac{\beta}{\delta}, \qquad x \geq 0.$$

On the other hand, the form of $\tilde{V}_h$ and hence $\tilde{V}$ will differ depending on the value of i. More concretely, we will distinguish between three different cases.

Case $i < q$

Suppose first that $i < q$. Choosing $g(\tilde{z}) := \tilde{V}_h(x)$ together with the variable change $\tilde{z} := \tilde{z}(x) = \frac{(\mu-((q-i)x+\beta))^2}{\sigma^2(q-i)}$ yields Kummer's differential equation

$$\tilde{z}g''(\tilde{z}) + (b - \tilde{z})g'(\tilde{z}) - ag(\tilde{z}) = 0, \qquad \tilde{z} \geq 0, \tag{5.46}$$

with parameters

$$a = \frac{\delta}{2(q-i)}, \qquad b = \frac{1}{2}.$$

Using the same techniques as in Section 5.3, including the asymptotic considerations and smooth-pasting conditions, one arrives at the following solution:

Proposition 5.5.1. *For any $x \geq 0$, the sum of the expected discounted dividend payments up to the time of ruin in a diffusion model with force of interest $i < q$ and affine dividend* *strategy* (5.2) *is given by*

$$\tilde{V}(x) = \begin{cases} \tilde{C}\left(\frac{2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2(q-i)}+\frac{1}{2}\right)} M\left(\frac{\delta}{2(q-i)}, \frac{1}{2}, \tilde{z}(x)\right) - U\left(\frac{\delta}{2(q-i)}, \frac{1}{2}, \tilde{z}(x)\right)\right) & \\ +\frac{qx}{q-i+\delta} + \frac{(\mu-\beta)q}{\delta(q-i+\delta)} + \frac{\beta}{\delta}, & 0 \leq x \leq \frac{\mu-\beta}{q-i}, \\ \tilde{C}\, U\left(\frac{\delta}{2(q-i)}, \frac{1}{2}, \tilde{z}(x)\right) + \frac{qx}{q-i+\delta} + \frac{(\mu-\beta)q}{\delta(q-i+\delta)} + \frac{\beta}{\delta}, & x > \frac{\mu-\beta}{q-i}, \end{cases}$$

where

$$\tilde{C} = \frac{\left(\frac{(\mu-\beta)q}{\delta(q-i+\delta)} + \frac{\beta}{\delta}\right)}{U\left(\frac{\delta}{2(q-i)}, \frac{1}{2}, \tilde{z}(0)\right) - \frac{2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{2(q-i)}+\frac{1}{2}\right)} M\left(\frac{\delta}{2(q-i)}, \frac{1}{2}, \tilde{z}(0)\right)},$$

and $\tilde{z}(x) = \frac{(\mu-((q-i)x+\beta))^2}{\sigma^2(q-i)}$.

Case $i = q$

In the case $i = q$, (5.45) simplifies to a second-order non-homogeneous differential equation with constant coefficients, i.e. $\tilde{V}_h$ is a solution to

$$\frac{\sigma^2}{2}\tilde{V}_h''(x) + (\mu-\beta)\tilde{V}_h'(x) - \delta\tilde{V}_h(x) = 0, \qquad x \geq 0. \tag{5.47}$$

The general solution to (5.47) can then be written as

$$\tilde{V}_h(x) = C_1 e^{rx} + C_2 e^{sx},$$

for some constants C_1, C_2 where

$$r = \frac{-(\mu-\beta) + \sqrt{(\mu-\beta)^2 + 2\sigma^2\delta}}{\sigma^2} > 0,$$

and,

$$s = \frac{-(\mu-\beta) - \sqrt{(\mu-\beta)^2 + 2\sigma^2\delta}}{\sigma^2} < 0,$$

are respectively the positive and negative roots of the characteristic equation

$$\frac{\sigma^2}{2}\xi^2 + (\mu-\beta)\xi - \delta = 0.$$

The linear boundedness of $\tilde{V}$ implies $C_1 = 0$ while C_2 is determined by the initial condition $\tilde{V}(0) = 0$. Summarizing, we obtain:

Proposition 5.5.2. *For any $x \geq 0$, the sum of the expected discounted dividend payments* *up to the time of ruin in a diffusion model with force of interest $i = q$ and affine dividend strategy* (5.2) *is given by*

$$\tilde{V}(x) = \left(\frac{(\mu-\beta)q}{\delta^2} + \frac{\beta}{\delta}\right)(1 - e^{sx}) + \frac{qx}{\delta}, \qquad x \geq 0,$$

where $s = \frac{-(\mu-\beta)-\sqrt{(\mu-\beta)^2+2\sigma^2\delta}}{\sigma^2}$.

Case $i > q$

It remains now to examine the case $i > q$ with the additional requirement that $i < q + \delta$ which ensures that $\tilde{V}$ remains finite. As for $i < q$, this translates into solving Kummer's equation (5.46) over the domain $\tilde{z} < 0$. From $\tilde{z}'(x) = -\frac{2((\mu-\beta)-(q-i)x)}{\sigma^2} < 0$ for all $x \geq 0$, it follows that $\tilde{z}(x)$ is monotone decreasing with maximum value $\tilde{z}(0) = \frac{(\mu-\beta)^2}{\sigma^2(q-i)} < 0$. As a consequence and in contrast to the case $i < q$, the mapping $x \mapsto \tilde{z}(x)$ is now injective and (5.46) exhibits no regular singular point on $x \geq 0$. The general solution to (5.46) takes the form of

$$\tilde{V}_h(x) = C_1\, M\left(\frac{\delta}{2(q-i)}, \frac{1}{2}, \tilde{z}(x)\right) + C_2\, e^{\tilde{z}(x)}\, U\left(\frac{1}{2}\left(1 - \frac{\delta}{q-i}\right), \frac{1}{2}, -\tilde{z}(x)\right),$$

for $x \geq 0$ and some arbitrary constants C_1 and C_2. Note here that the independent pair $M(a, b, z)$ and $e^z U(b-a, b, -z)$ needs to be chosen to produce a real-valued solution. The linear boundedness of $\tilde{V}$ requires investigating the asymptotic behavior of this particular pair for both large negative and positive arguments. Starting with the Kummer function M, it is well-known (cf. [3]) that for $z \to -\infty$,

$$M(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}\left(1 + \mathcal{O}\left(|z^{-1}|\right)\right).$$

In our particular case, for large x, this reads

$$M\left(\frac{\delta}{2(q-i)}, \frac{1}{2}, \tilde{z}(x)\right) \sim \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\left(1 - \frac{\delta}{q-i}\right)\right)}\left(-\tilde{z}(x)\right)^{\frac{\delta}{2(i-q)}}\left(1 + \mathcal{O}(x^{-1})\right). \tag{5.48}$$

For the Tricomi function U, from (5.22) we get that for large x

$$e^{\tilde{z}(x)} U\left(\frac{1}{2}\left(1 - \frac{\delta}{q-i}\right), \frac{1}{2}, -\tilde{z}(x)\right) \sim e^{\tilde{z}(x)}\left(-\tilde{z}(x)\right)^{\frac{1}{2}\left(\frac{\delta}{q-i}-1\right)}\left(1 + \mathcal{O}(x^{-1})\right). \tag{5.49}$$

While by the condition $i < \delta + q$, (5.48) becomes sup-linearly unbounded as $x \to \infty$, (5.24) tends to 0 exponentially fast. The linear boundedness of $\tilde{V}$ can thus only be satisfied if we set $C_1 = 0$. The remaining constant C_2 is determined by the initial condition $\tilde{V}(0) = 0$, so that we arrive at the following result:

Proposition 5.5.3. *For any $x \geq 0$, the sum of the expected discounted dividend payments up to the time of ruin in a diffusion model with force of interest $q < i < q + \delta$ and affine dividend strategy* (5.2) *is given by*

$$\tilde{V}(x) = \left(\frac{(\mu - \beta)q}{\delta(q - i + \delta)} + \frac{\beta}{\delta} \right) \left(1 - \frac{e^{\tilde{z}(x)}\, U\left(\frac{1}{2}\left(1 - \frac{\delta}{q-i}\right), \frac{1}{2}, -\tilde{z}(x)\right)}{e^{\tilde{z}(0)}\, U\left(\frac{1}{2}\left(1 - \frac{\delta}{q-i}\right), \frac{1}{2}, -\tilde{z}(0)\right)} \right) + \frac{qx}{q - i + \delta}, \qquad x \geq 0,$$

where $\tilde{z}(x) = \frac{(\mu - ((q-i)x + \beta))^2}{\sigma^2(q-i)}$.

5.5.2 Optimal dividends in the presence of negative interest rates

From results in Shreve et al. [112], it is well-known that when the surplus is modeled by a Brownian motion with drift and earns interest at a constant force $i \in \mathbb{R}$, then, if it exists, an optimal way of paying out dividends in the sense of maximizing the expected discounted dividends until ruin is according to a barrier strategy. While the case $i > 0$ was studied by Gerber et al. [37], due to the non-injective transformation needed to convert the original differential equation satisfied by the expectation of the total dividends value into Kummer's confluent hypergeometric equation, their calculations do not carry over to negative interest rates $i < 0$. Therefore, in order to compare the performance of the best affine strategy with the optimal barrier strategy for $i < 0$, we first need to derive an expression for the latter. To that end, we take a step back to the original process (5.1) and assume that it is continuously invested at a force of interest $i < 0$, so that in the absence of dividend payments, it has the following dynamics:

$$d\tilde{R}_t = (\mu + i\tilde{R}_t)dt + \sigma dW_t, \qquad t \geq 0,$$

with $\tilde{R}_0 = x$. Note that such a process is of Ornstein-Uhlenbeck type with mean-reverting level $-\frac{\mu}{i} > 0$. Denote by $\tilde{V}_b(x)$ the expectation of the discounted dividends until ruin, considered as a function of the initial capital x if the barrier strategy with parameter $b \geq 0$

is applied. As a function of x, $\tilde{V}_b(x)$ satisfies the second-order differential equation

$$\frac{\sigma^2}{2}\tilde{V}_b''(x) + (\mu + ix)\tilde{V}_b'(x) - \delta\tilde{V}_b(x) = 0, \qquad 0 < x < b, \tag{5.50}$$

in conjunction with the boundary conditions

$$\tilde{V}_b(0) = 0, \qquad \tilde{V}_b'(b) = 1. \tag{5.51}$$

To solve (5.50), consider the auxiliary differential equation

$$\frac{\sigma^2}{2}h''(x) + (\mu + ix)h'(x) - \delta h(x) = 0, \qquad x > 0, \tag{5.52}$$

combined with the boundary condition $h(0) = 0$. Because the function $h(x)$ is uniquely determined up to a constant factor, we obtain in view of (5.50)-(5.51) the well-known characterization

$$\tilde{V}_b(x) = \frac{h(x)}{h'(b)}, \qquad 0 \le x \le b. \tag{5.53}$$

Furthermore, by the nature of the barrier strategy,

$$\tilde{V}_b(x) = x - b + \tilde{V}_b(b), \qquad x > b.$$

Choosing $y(z) := h(x)$ together with the variable change $z := z(x) = -\frac{(\mu+ix)^2}{i\sigma^2} > 0$ converts (5.52) into Kummer's differential equation

$$zy''(z) + (b - z)\, y'(z) - ay(z) = 0, \qquad z > 0,$$

with parameters

$$a = -\frac{\delta}{2i}, \qquad b = \frac{1}{2}.$$

The general solution of such equation can then be written as

$$h(x) = y(z) = \begin{cases} A_1\, M\left(-\frac{\delta}{2i}, \frac{1}{2}, z(x)\right) + A_2\, U\left(-\frac{\delta}{2i}, \frac{1}{2}, z(x)\right), & 0 \le x \le -\frac{\mu}{i}, \\ A_3\, M\left(-\frac{\delta}{2i}, \frac{1}{2}, z(x)\right) + A_4\, U\left(-\frac{\delta}{2i}, \frac{1}{2}, z(x)\right), & x > -\frac{\mu}{i}, \end{cases}$$

for arbitrary constants $A_i, i = 1, \ldots 4$. Again here, as in (5.19), the non-injectivity of the mapping $x \mapsto z(x)$ necessitates a piecewise construction. From the initial condition $h(0) = 0$, it follows that

$$A_1\, M\left(-\frac{\delta}{2i}, \frac{1}{2}, z(0)\right) = -A_2\, U\left(-\frac{\delta}{2i}, \frac{1}{2}, z(0)\right). \tag{5.54}$$

Next, requiring $\tilde{V}_b(x)$ to be continuous in x imposes $h(x)$ to be continuous at $-\frac{\mu}{i}$. Using similar arguments as in Section 5.3, this translates into

$$A_1 + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{\delta}{2i}+\frac{1}{2}\right)} A_2 = A_3 + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{\delta}{2i}+\frac{1}{2}\right)} A_4 \tag{5.55}$$

Note the similarity to (5.31). In addition, because we also want $V_b'(x)$ to be continuous in x, we must set $A_2 = -A_4$ (cf. Section 5.3, particularly Remark 5.3.1). Without loss of generality, choosing $A_1 = 1$ and combining (5.54) and (5.55), we obtain in view of (5.53) that

$$\tilde{V}_b(x) = \begin{cases} \frac{h(x)}{h'(b)}, & 0 \le x \le b, \\ x - b + \frac{h(b)}{h'(b)}, & x > b, \end{cases} \tag{5.56}$$

where

$$h(x) = \begin{cases} M\left(-\frac{\delta}{2i}, \frac{1}{2}, z(x)\right) - \frac{M\left(-\frac{\delta}{2i}, \frac{1}{2}, z(0)\right)}{U\left(-\frac{\delta}{2i}, \frac{1}{2}, z(0)\right)} U\left(-\frac{\delta}{2i}, \frac{1}{2}, z(x)\right), & 0 \le x \le -\frac{\mu}{i}, \\ \overline{C} M\left(-\frac{\delta}{2i}, \frac{1}{2}, z(x)\right) + \frac{M\left(-\frac{\delta}{2i}, \frac{1}{2}, z(0)\right)}{U\left(-\frac{\delta}{2i}, \frac{1}{2}, z(0)\right)} U\left(-\frac{\delta}{2i}, \frac{1}{2}, z(x)\right), & x > -\frac{\mu}{i}, \end{cases}$$

with

$$\overline{C} = \left(1 - \frac{2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{\delta}{2i}+\frac{1}{2}\right)} \frac{M\left(-\frac{\delta}{2i}, \frac{1}{2}, z(0)\right)}{U\left(-\frac{\delta}{2i}, \frac{1}{2}, z(0)\right)}\right),$$

and $z(x) = -\frac{(\mu+ix)^2}{i\sigma^2}$.

For a given initial surplus $\tilde{R}_0 = x$, let denote by b^* the value of b which maximizes $\tilde{V}_b(x)$. In view of (5.56), the first-order condition is given by

$$h''(b^*) = 0, \tag{5.57}$$

the solution of which has to be determined numerically. Hence, the corresponding optimal barrier $b = b^*$ maximizes $\tilde{V}_b(x)$ independently of x. From

$$\tilde{V}_b''(x) = \frac{h''(x)}{h'(b)}, \qquad 0 \le x \le b,$$

condition (5.57), it follows that

$$\tilde{V}_{b^*}''(b^*) = 0.$$

From a geometrical point of view, choosing $b = b^*$ renders the second derivative of $\tilde{V}_b(x)$ continuous at $x = b$. If we now let $x = b = b^*$ in (5.50) and make use of the second boundary condition in (5.51), we find that

$$\mu + ib^* - \delta \tilde{V}_{b^*}(b^*) = 0,$$

which leads to

$$\tilde{V}_{b^*}(b^*) = \frac{\mu + ib^*}{\delta}.$$

Since $i < 0$, $\tilde{V}_{b^*}(b^*)$ can be interpreted as the difference between two perpetuities, one with payment rate μ and another with rate $-ib^*$. In addition, because $\tilde{V}_{b^*}(b^*) > 0$, we must have

$$b^* < -\frac{\mu}{i}.$$

Hence, by reflecting the surplus process at the optimal barrier b^*, the mean-reverting property is not preserved. This is not surprising in the sense that if setting $b^* > -\frac{\mu}{i}$ would have been optimal, then whenever the surplus is between $-\frac{\mu}{i}$ and b^*, it is dragged back to $-\frac{\mu}{i}$ with a negative drift and possibly no dividends being paid, which is not reasonable.

5.6 Numerical illustrations

5.6.1 Optimal parameters

We now examine which combination of parameters (q, β) maximizes the expected present value of dividend payouts until ruin. Formally, let $\Theta = \{(q, \beta) : q > 0, \beta \in [0, \mu]\}$. To each pair $(q, \beta) \in \Theta$, we associate $V(x; q, \beta) := V(x)$. The optimization problem is then to determine an optimal pair (q^*, β^*) such that

$$V(x; q^*, \beta^*) = \max_{(q,\beta)\in\Theta} V(x; q, \beta), \tag{5.58}$$

for an initial capital $x > 0$. By the nature of the function V (c.f. Proposition 5.5.3), a numerical approach to this optimization problem is required. The following numerical results were obtained using the routines of Mathematica. Table 5.1 displays the resulting maximal expected dividend values together with its corresponding maximizers for $\delta = 0.05, \mu = 2, \sigma = 1$ and different initial capitals. For the sake of comparison, results for $\sigma = 2$ are given in parentheses. It is worth mentioning that by construction the optimal parameters (q^*, β^*) are chosen only as a function of the initial capital x, i.e., they are kept at their respective level throughout the portfolio's lifetime. First, we observe that $q^*(x)$ is increasing in x. Furthermore, it turns out that $\beta^*(x) = 0$ for all capital levels, i.e., it is preferable not to pay dividends at a constant rate to maximize the present value of dividends. A possible explanation of the latter is that close to ruin, extending the portfolio's lifetime (and therefore the length of the dividend payments) is more important than paying out more dividends immediately, hence choosing $\beta^* = 0$ allows the process to drift away from the ruin zone in a faster way.

For the increased volatility $\sigma = 2$, the length of the dividend payments is reduced since ruin happens sooner in some sense. As a response, it becomes optimal to lower the rate at which dividends are paid, which offsets to some extent the initial decrease of the portfolio's lifetime caused by the larger volatility. In the concrete example, an increase in volatility affects negatively the total dividend values for all initial capital values.

x	$V(x;q^*,\beta^*)$	$q^*(x)$	$\beta^*(x)$
0.2	18.878 (5.326)	0.462 (0.189)	0.000 (0.000)
0.4	27.712 (9.707)	0.476 (0.189)	0.000 (0.000)
0.6	32.063 (13.328)	0.492 (0.190)	0.000 (0.000)
0.8	34.343 (16.337)	0.506 (0.191)	0.000 (0.000)
1	35.630 (18.851)	0.519 (0.192)	0.000 (0.000)
2	37.944 (26.685)	0.552 (0.198)	0.000 (0.000)
5	40.958 (34.344)	0.576 (0.217)	0.000 (0.000)
10	45.616 (39.613)	0.591 (0.236)	0.000 (0.000)
20	54.890 (48.406)	0.612 (0.256)	0.000 (0.000)

Table 5.1: Maximal expected present value of dividends and optimal pairs (q^*, β^*) for $\delta = 0.05, \mu = 2$ and $\sigma = 1$ $(\sigma = 2)$.

5.6.2 Comparison with the optimal barrier strategy

For a diffusion risk process of the form (5.1), Shreve et al. [112] showed that the so-called *barrier strategy* maximizes the expected present value of dividends until ruin. It is therefore of interest to compare the dividend values achieved under the best affine strategy $V(x;q^*,\beta^*)$ with the one under the optimal barrier b^*. The latter which we denote by $V_{b^*}(x)$ takes the form

$$V_{b^*}(x) = \begin{cases} \frac{h(x)}{h'(b^*)}, & 0 \le x \le b^*, \\ x - b^* + \frac{h(b^*)}{h'(b^*)}, & x > b^*, \end{cases} \tag{5.59}$$

where $h(x) := e^{rx} - e^{sx}$ with $r > 0$ and $s < 0$ being the roots of the characteristic equation

$$\frac{\sigma^2}{2}\xi^2 + \mu\xi - \delta = 0,$$

and

$$b^* = \frac{1}{r-s} \log\left(\frac{s^2}{r^2}\right).$$

Results on the distribution of the time to ruin with a barrier at level b, which we here denote by T_x^b, were first given by Cox and Miller [51]; see also Gerber and Shiu [70]. In particular, the Laplace tansform of T_x^b is of the form

$$\mathbb{E}\left[e^{-\delta T_x^b}\right] = \begin{cases} \frac{re^{-s(b-x)}-se^{-r(b-x)}}{re^{-sb}-se^{-rb}}, & 0 \le x \le b, \\ \frac{r-s}{re^{-sb}-se^{-rb}}, & x > b, \end{cases} \tag{5.60}$$

from which we obtain

$$\mathbb{E}\left[T_x^b\right] = \begin{cases} \frac{\sigma^2}{2\mu^2}\left(e^{\frac{2\mu b}{\sigma^2}} - e^{\frac{2\mu(b-x)}{\sigma^2}} - \frac{2\mu x}{\sigma^2}\right), & 0 \le x \le b, \\ \frac{\sigma^2}{2\mu^2}\left(e^{\frac{2\mu b}{\sigma^2}} - 1 - \frac{2\mu b}{\sigma^2}\right), & x > b. \end{cases} \tag{5.61}$$

Table 5.2 compares the resulting dividend values under the optimal barrier strategy and the best affine strategy for $\delta = 0.05, \mu = 3, \sigma = 0.5$ and different capitals x (in which case $b^* = 0.605$). The results for $\sigma = 2$ (in which case $b^* = 5.899$) are given in parentheses. One remarks that for $\sigma = 0.5$ the resulting dividend payout obtained with the optimal affine strategy is very close to the one under the optimal barrier strategy, which is quite remarkable. In particular, this means that if the surplus is subject to low volatility (this also holds true for other low volatility levels in numerical experiments), one can achieve a total dividend value close to the optimal one by a continuous and smooth dividend flow. In the presence of a larger volatility, here $\sigma = 2$, the relative performance of the best affine dividend strategy turns out to be slightly diminished in comparison to the optimal barrier strategy, a potential explanation of which may originate from the fact that in the affine case, due to the augmented fluctuations of the surplus and hence higher risk of early ruin, one is further penalized for paying in a continuous fashion and not fully cashing in at high surplus levels, which is the philosophy behind the barrier strategy. It is also interesting to observe that to reduce the risk of early ruin, the optimal mean-reverting level $\frac{\mu}{q^*(x)}$ (since $\beta^*(x) = 0$) is roughly multiplied by a factor of ten when the volatility increases from $\sigma = 0.5$ to $\sigma = 2$, which is in a certain sense comparable to the augmentation in the optimal barrier level b^* from 0.605 to 5.899.

x	$V_{b^*}(x)$	$V(x; q^*, \beta^*)$	$q^*(x)$	$\beta^*(x)$
0.2	59.109 (14.424)	58.173 (12.575)	2.845 (0.311)	0.000 (0.000)
0.4	59.791 (25.122)	59.328 (21.957)	3.124 (0.313)	0.000 (0.000)
0.605	60.000 (33.250)	59.579 (29.173)	3.163 (0.315)	0.000 (0.000)
1	60.395 (43.401)	59.976 (38.450)	3.178 (0.321)	0.000 (0.000)
2	61.395 (53.648)	60.963 (48.826)	3.189 (0.338)	0.000 (0.000)
5.899	65.294 (60.000)	64.804 (56.858)	3.211 (0.374)	0.000 (0.000)
10	69.395 (64.101)	68.843 (60.838)	3.230 (0.387)	0.000 (0.000)
20	79.395 (74.101)	78.693 (69.928)	3.271 (0.404)	0.000 (0.000)

Table 5.2: Comparison of the expected present value of dividends under the optimal affine and barrier dividend strategies for initial capitals x with $\delta = 0.05, \mu = 3$ and $\sigma = 0.5$ ($\sigma = 2$).

A next question is to compare the respective expected times to ruin under both strategies when the respective parameters are chosen in an optimal way from a profitability perspective. Table 5.3 illustrates that when the volatility is small, the barrier strategy offers an additional safety component for most initial capital values, which is here measured by a prolongated expected portfolio's lifetime. The underlying reason is that if a surplus process with low volatility is close to ruin, it will very likely return to the optimal barrier under a barrier strategy since no dividends are paid, whereas under an affine strategy, because the drift is continuously reduced by dividend payments, it is more difficult to drift away from the ruin level, eventually precipitating the process to fall into ruin. Note that when the initial capital is small (here $x = 0.2$), then because the drift reduction through $q^*(x)$ is not too large and having in mind the results from Table 5.2, an affine dividend strategy appears a noteworthy compromise between safety and profitability. In particular, the surplus process lives on average twice as long under an affine strategy than under a barrier strategy for a very comparable profitability. For an increased volatility level $\sigma = 2$, $q^*(x)$ decreases to an extent that the affine strategy leads to a longer expected porfolio's lifetime up to $x = 2$, from which the resulting optimal $q^*(x)$-values are such the dividend payments are concentrated over a smaller time horizon, hence leading to shorter expected times to ruin relative to the ones under the optimal barrier strategy. Finally, it is worth remarking that the expected time to ruin under the optimal affine strategy is non-monotone in x, namely first increasing and decreasing afterwards, which is due to the respective weighting with which x and $q^*(x)$ enter the formula given in Proposition 5.4.2.

x	$\mathbb{E}\left[T_x^{b^*}\right]$	$\mathbb{E}\left[\tau_{x,q^*,\beta^*}\right]$
0.2	28'070.554 (401.163)	56'263.974 (757.748)
0.4	28'301.502 (698.335)	17'770.244 (1272.311)
0.605	28'303.336 (923.458)	15'312.565 (1594.138)
1	28'303.336 (1202.311)	14'473.051 (1820.584)
2	28'303.336 (1470.324)	13'904.044 (1576.075)
5.899	28'303.336 (1545.875)	12'814.809 (882.947)
10	28'303.336 (1545.875)	11'967.660 (714.545)
20	28'303.336 (1545.875)	10'373.325 (554.924)

Table 5.3: Comparison of the expected time to ruin under the optimal affine and barrier strategies for initial capital levels x with $\delta = 0.05, \mu = 3$ and $\sigma = 0.5$ ($\sigma = 2$).

5.6.3 Analysis with a negative interest rate

In Section 5.5, we assumed that the surplus is invested continuously at a constant interest force i. Motivated by the negative interest rates that currently prevail in most of the major advanced economies, we now analyze their effects on the trade-off between size and length of the dividend payments. In the present context, a negative interest rate can be interpreted as a surplus-based fee that the insurer pays to keep its money in the bank. At this point, it is worth pointing out that having a negative interest is equivalent to using a proportionality constant $q - i$ of which finally a fraction $\frac{q}{q-i}$ is collected as dividend. In the next considerations, we shall be interested in the case established in Proposition 5.5.1 for $i \leq 0$. In the spirit of (5.58) and denoting $\tilde{V}(x; i, q, \beta) := \tilde{V}(x)$, we consider

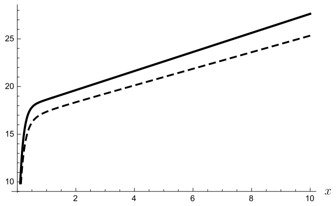
$$\tilde{V}(x; i, q^*, \beta^*) = \max_{(q,\beta)\in\Theta} \tilde{V}(x; i, q, \beta),$$

for $x > 0$. Obviously, the resulting optimal parameters q^* now depend on both x and i, whereas it turns out that β^* is always zero. Table 5.4 lists the pairs $\left(\tilde{V}(x; i, q^*, \beta^*), q^*\right)$ for different initial capital levels x and (negative) interest rates i setting $\delta = 0.05, \mu = 2$ and $\sigma = 1$. As expected, the optimal dividend values decrease for lower (more negative) interest rates. Moreover, one can observe that for all initial capitals x, a decrease in interest rate levels leads to a slight increase of the optimal q^*-values. This is line with intuition since a negative interest rate has an exponentially decaying depreciation effect on the surplus over time, one has to increase $q^*(x)$ so that more dividends are paid in the early stages of the portfolio's lifetime, i.e., when they are the most valuable with regards to the force of interest δ. An increase in $q^*(x)$ (which obviously leads to shorter expected time to ruin) also constitutes an interesting psychological appeal inherent to affine dividend strategies in the context of negative interest.

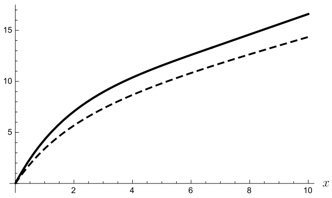
Optimal combinations $\left(\tilde{V}(x;i,q^*,0),q^*\right)$						
x	$i=-0.05$	$i=-0.04$	$i=-0.03$	$i=-0.02$	$i=-0.01$	$i=0.00$
0.2	(17.049,0.509)	(17.389,0.504)	(17.740,0.497)	(18.103,0.489)	(18.481,0.478)	(18.878,0.462)
0.4	(25.082,0.518)	(25.573,0.513)	(26.079,0.507)	(26.602,0.500)	(27.145,0.490)	(27.712,0.476)
0.6	(29.090,0.528)	(29.648,0.524)	(30.222,0.519)	(30.813,0.512)	(31.425,0.504)	(32.063,0.492)
0.8	(31.225,0.538)	(31.813,0.534)	(32.417,0.530)	(33.037,0.524)	(33.678,0.517)	(34.343,0.506)
1	(32.454,0.546)	(33.055,0.543)	(33.671,0.539)	(34.304,0.534))	(34.956,0.528)	(35.630,0.519)
2	(34.725,0.571)	(35.339, 0.570)	(35.967,0.567)	(36.609,0.563)	(37.267,0.558)	(37.944,0.552)
5	(37.618,0.595)	(38.256,0.593)	(38.908,0.590)	(39.575,0.587)	(40.258,0.582)	(40.958,0.576)
10	(41.994,0.613)	(42.686,0.610)	(43.393,0.607)	(44.116,0.603)	(44.856,0.598)	(45.616,0.591)
20	(50.686,0.638)	(51.488,0.635)	(52.308,0.631)	(53.147,0.626)	(54.006,0.620)	(54.890,0.612)

Table 5.4: Maximal expected present value of dividends and optimal q^* values for different (negative) interest rates and $\delta = 0.05, \mu = 2$ and $\sigma = 1$.

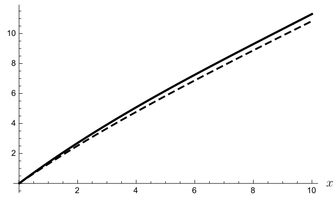
Let us now compare the expected present value of dividends under affine and barrier strategies in the presence of negative interest rates. For expository purposes, let $\mu = 1$ and $\delta = 0.05$. Figure 5.2 depicts the maximal total dividend values under both strategies as a function of the initial capital x for various negative interest rates and volatilities. Interestingly, despite the fact that both an increase of σ and a decrease of i, i.e. a more negative interest rate, have a negative impact on the portfolio's lifetime, their respective effect on the optimal barrier b^* differs. While b^* is increasing in σ, it appears to be decreasing for smaller interest rates. A potential explanation is that since a negative interest rate enters as an exponentially decaying function over time in the risk process (in contrast to σ which comes as a linear term), it is preferable in view of the discounting to pay out more dividends early so to alleviate the depreciating effect of a negative interest. When comparing the two strategies, one observes that they lead to very comparable dividend values for small initial capital values which is quite intuitive since for such capital levels more trajectories of the risk process are ruined before reaching the barrier and do not lead to any dividend payments under the barrier strategy, while they have a positive value under an affine scheme, which altogether has a more important weight in the expected value of all dividends. On the other hand, for larger initial capital values, one is penalized for not being able to make lump sum payments under affine strategies, which in the context of negative interest rates is an important qualitiative property. Note that for very large values of x, the performance of the respective optimal strategies will be very similar, since the initial lump sum payment in the barrier case (representing a major portion of the overall dividend value) is mimicked by a large value q^* in the affine strategy.



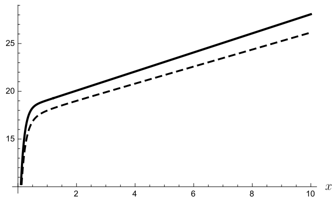
(a) $i = -0.05$, $\sigma = 0.5$; $b^* = 1.184$



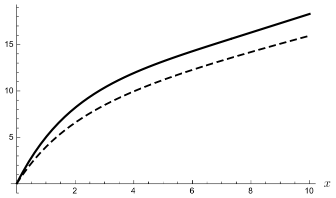
(b) $i = -0.05$, $\sigma = 2$; $b^* = 6.700$



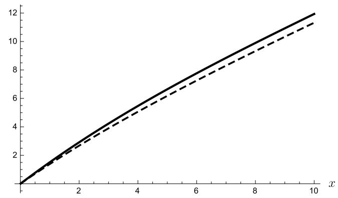
(c) $i = -0.05$, $\sigma = 5$; $b^* = 9.350$



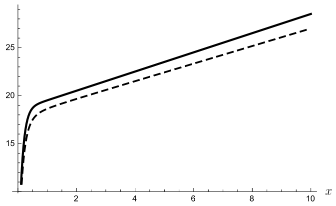
(d) $i = -0.03$, $\sigma = 0.5$; $b^* = 1.207$



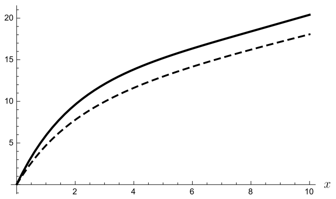
(e) $i = -0.03$, $\sigma = 2$; $b^* = 7.317$



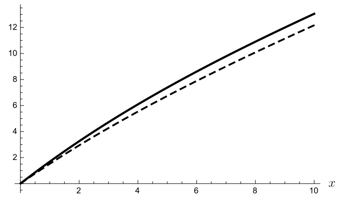
(f) $i = -0.03$, $\sigma = 5$; $b^* = 11.289$



(g) $i = -0.01$, $\sigma = 0.5$; $b^* = 1.237$



(h) $i = -0.01$, $\sigma = 2$; $b^* = 8.000$



(i) $i = -0.01$, $\sigma = 5$; $b^* = 14.078$

Figure 5.2: Comparison of the total dividend values under the optimal affine strategy (dashed line) and the optimal barrier strategy (solid line) as a function of the initial capital x for different negative interest rates and volatilities.

5.7 Conclusion

In this chapter, we derived an explicit formula for the expected dividends and the Laplace transform of the time to ruin for an affine dividend strategy in a diffusion setup using an analytical approach. This solves an open problem stated in Avanzi and Wong [22] and could be achieved by a refined study of special functions appearing in this context. As a by-product, an explicit expression for the expected ruin time could be derived. We then extended the study to allow for negative interest rates. The effects of the latter were studied in more detail indicating that for higher volatility of the underlying diffusion process, the optimal affine and the optimal barrier strategy lead to a very similar performance despite the very different nature of the two resulting processes.

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