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Analysis of Solvency Capital on a Multi-Year Basis

Master Thesis

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Reviewer

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List of Abbreviations

CIR Cox-Ingersoll-Ross

CDF cumulative distribution function

FDB future discretionary benefits

iid independent and identically distributed

LSMC least squares Monte Carlo

ORSA own risk solvency assessment

SCR solvency capital requirement

1. Introduction

1.1. Motivation

Since the year of 2016, the regulatory framework Solvency II (cf. [Eur09], [Eur15], [Eur16]) is in place in the European Union. It marks the beginning of a new era where all insurance and reinsurance companies are measured against the same new standards in terms of their risk management. Namely, specific regulations are laid out in order to better protect the policyholders and to ensure comparability among the companies. The change of focus from book values to a market-consistent valuation as well as the manner of computation of the solvency capital, that is now determined on the basis of a wide range of risks the companies are exposed to, are significant novelties. Since Solvency II is a relatively recent directive, it remains to be seen how strongly the European insurance companies are affected by the increase in regulatory supervision. Will the companies manage to meet the required amounts of solvency capital both at present and in the future? What are the crucial factors that determine the answer?

1.2. Objective

The objective of this thesis is to shed some light on the two questions asked above. As there are clearly no simple answers to such complex questions, we can only cover some fundamental aspects. In particular, we are interested in how the capital requirement is put together in Solvency II and how certain contract features influence the outcome. For this purpose, we perform the calculations needed to derive the future distributions of the solvency capital requirement (SCR) of two simple model companies. The companies differ in their kinds of guarantees and profit participation schemes, for which we compare the results over the entire lifetime of the contracts.

1.3. Literature Review

There are only few papers that deal with the projection of a company's solvency position into the future. In this section, we list the most prominent of those that study companies with participating life insurance contracts. This overview is meant to provide some insight into the kind of research that has already been conducted on this particular topic.

First, we mention [BM99], which models an insurance company in the UK, that offers participating life insurance contracts with features typical for the country. The contracts include a guarantee, reversionary bonus and a target terminal bonus rate, but consider neither the mortality of the policyholders nor any expenses. Thus, the focus lies on the effect of the investment returns on the solvency position. Remarkably, [BM99] was published in 1999, which was several years before the process of discussing and introducing the current directive, Solvency II, even started. Hence, [BM99] does not study the company's solvency in accordance with Solvency II but rather the ratio of assets to liabilities in terms of market values. As soon as the ratio of assets to liabilities drops below one, the company is said to be insolvent as it fails to meet its liabilities. For the asset side of the balance sheet, [BM99] relies on the Wilkie investment model and also incorporates dynamic asset switching. The company's cash flows are projected over a time period of 40 years in order to compare the ratios and the probability of insolvency of companies with different terminal bonus rates and asset allocation strategies.

Next, we look at [BPJ14] and [VD12], which both model insurance companies in France, complying with the regulation of Solvency II. While [BPJ14] chooses to set up its model in such a way that fair valuation can be carried out through closed formulas, [VD12] does not aim to completely avoid the complexity of nested simulations but addresses techniques to lower the computational effort. [BPJ14] considers a French participating contract including surrender by directly modeling the accumulation rate of the contract. In particular, the authors distinguish between the hedgeable and non-hedgeable parts of the accumulation rate, for which they define several probability measures. The short rate process is based on the Vasicek model, while the asset value stems from the Black-Scholes model. As the authors aim to provide a suitable framework for the own risk solvency assessment (ORSA), they limit their projection of the company's balance sheet to the next five years and study the values of the SCR and the solvency ratio for a number of simulations. They depict the cumulative distribution function (CDF) of the available free surplus as well as quantiles of the solvency ratio. The fair valuation in closed form is achieved by finding the factor that, when multiplied with the mathematical reserve, results in the best estimate of liabilities,

representing the time value of options.
[VD12] works on how the concept of solvency under Solvency II can be adapted to a multi-year time horizon. First, the authors extend the definitions of the SCR and a company's solvency to future time points and horizons, then they discuss the concept of nested simulations and how existing approximation techniques such as curve fitting and least squares Monte Carlo (LSMC) can be adjusted to fit a multi-year time horizon. The different techniques are applied to a standard French saving portfolio including profit sharing, a target crediting rate and dynamic lapses. Similarly to [BPJ14], the balance sheet is projected for five years since the techniques are meant to support the ORSA. However, [VD12] relies on the standard formula of Solvency II to compute the SCR. Specifically, the authors cover the risks arising from the stock index and the interest rates.
The next group of papers, [BG15], [Ber16] and [BPK16], is mainly concerned with German contracts, although the latter also includes several other European countries and the specific regulations therein. Further, it should be noted that the last two papers use the standard formula of Solvency II in order to compute the SCR. As German regulations for minimum profit participation are based on book values, [BG15] simultaneously models a book value and a market value balance sheet. The model company is assumed to only offer endowment contracts with yearly premiums. Mortality, surrender and expenses are not taken into account. However, the model contains different cohorts of contracts, which allows the study of different levels of guarantees and maturities within one portfolio. This is of great importance because the guarantees had to be reduced in the last few years due to the prolonged period of low interest rates, but insurers still have those high guarantees in their books. Moreover, [BG15] models the interest rate reserve ("Zinszusatzreserve"(ZZR)) that is required by German regulators. The term structure of interest rates is derived from the Cox-Ingersoll-Ross (CIR) model, while the modeled stocks and real estate each follow a geometric Brownian motion. Remarkably, the market value of liabilities is simplified and only includes the return guaranteed to policyholders, which does not agree with the definition of the best estimate in Solvency II. [BG15] provides numerical results for various capital market scenarios, which are defined by their long-term interest rate in the CIR model, and various initial leverage ratios of the company. The balance sheets are projected over a period of ten years.
The model in [Ber16] extends the model of [BG15]. The author adds an annuitization option at maturity and mortality dynamics following an improved Lee-Carter framework. Furthermore, he introduces adverse selection regarding annuitization and a demand function, which links the amount of new policies to the prevailing level of the guarantees offered

to policyholders. Besides, the short rate is now modeled through the Vasicek model, which allows for negative interest rates. In the framework of [Ber16], the SCR is computed via the standard formula and captures interest rate risk, spread risk, equity risk as well as longevity risk. The author does not only assess the solvency ratio, but also focuses on profitability in terms of the return on equity for a projection period of 20 years.
[BPK16] is again an extension of the models in [BG15] and [Ber16]. The authors consider companies that are located in either Germany, France, Italy, the Netherlands or Spain, which of course affects the regulations the companies are exposed to. Moreover, some companies only offer life insurance with endowment, annuity or term life business or several of these lines of business, while there are insurance groups that have a share in both life and non-life business. The SCR is again computed via the standard formula including both the market module and the life module. Finally, the projection of the balance sheet is performed for a period of ten years, for which the authors study the return on assets, the return on equity and the solvency ratio for companies with different lines of business and various capital market scenarios. For instance, the adverse scenarios allow for lower interest rates and a higher volatility of the processes. The exact choice of the parameters stems from a calibration on data including the financial crisis in 2008.
All of the papers listed in this review describe the evolution of a company's solvency position. While [BPJ14] and [VD12] concentrate on the projection of the solvency position in itself, the other papers specifically study the influence of the capital market, the company's asset allocation rules, leverage ratios and certain model parameters on the solvency position of the company. In particular, [BG15], [Ber16] and [BPK16] focus on the negative effects of prolonged low interest rate periods. The number of risk factors included in the models varies. For instance, [BPJ14] and [VD12] distinguish themselves in that they include surrender. [Ber16] and [BPK16] model mortality dynamics, which are neglected in the other papers.
In contrast, this thesis serves to understand and compare the general traits of two kinds of guarantees and profit participation mechanisms (maturity guarantee and cliquet-style guarantee) as they distinctly influence a company's solvency over a period of 20 years. We make sure that other specifications such as the initial balance sheet and the asset allocation are identical for both models in order to be able to link observations to the specific kind of guarantee. Although [BPK16] also addresses different insurance products, the authors do not study different endowment contracts as we do, but model endowment, annuity, term life and even non-life products. Most importantly, the focus differs from ours. [BPK16] investigates how different capital market scenarios affect companies of dif-

fering product diversity in different countries. As several model parameters are bound to the country in which the company operates, it is difficult to trace back specific contract features. Furthermore, we target the entire distribution of the solvency indicators. For instance, [BPJ14], [VD12] and [BG15] only capture and visualize a few quantiles. Finally, our models are based on closed-form valuation formulas just like the model in [BPJ14]. However, the setup is completely different from [BPJ14] in that we specify a bonus mechanism and then assume a model for the assets, whereas [BPJ14] incorporates the bonus by directly modeling the accumulation rate of the contract.

1.4. Structure

In chapter two, we begin by introducing the two models which provide the basis for all the analyses. We specify the design of the contracts including the guarantees and profit participation mechanisms. Then, we define the stress scenario, which is essential for the derivation of the SCR, in chapter three. In chapter four, we discuss the simulation of possible future outcomes and study the distribution of the companies' solvency figures over the entire lifetime of the contracts. Chapter five contains a sensitivity analysis on parameters reflecting the volatility, the guaranteed interest rate and the risk-free interest rate, respectively. Finally, we conclude this thesis in chapter six.

2. The Models

In this chapter, we describe the two models on which all further analyses are based. One of the models focuses on a maturity guarantee and will be referred to as the “maturity-model”, whereas the “cliquet-model” targets a cliquet-style guarantee. For each model, we discuss the company’s initial situation, the type of guarantee as well as the market-consistent valuation of the contract, and fix the model parameters.
It should be noted that both models are chosen especially for their tractability, an accurate representation of reality is secondary. For instance, we consider neither mortality, expenses, stochastic interest rates nor policyholders’ behavior such as surrender. As a result, both models prove to have closed-form solutions for the valuation of market values. This feature reduces the efforts needed for simulation significantly and allows us to concentrate on the deep analysis of the SCR and the solvency ratio (ratio of eligible own funds to SCR). The study of more complex models is left for future research.

2.1. Maturity-Model

The maturity-model is based on a so-called maturity or point-to-point guarantee as described in [BD97] and [GJ02]. This means that the insurance company guarantees a payment depending only on the market at inception and maturity of the contract but not on the development of the market on a yearly basis. Note that, for simplicity, we neither adapt the framework for stochastic interest rates of [BD97] nor the barrier option framework of [GJ02]. Furthermore, both [BD97] and [GJ02] assume equity holders to have a limited liability, which is not suitable for our purpose. When analyzing stress scenarios in order to compute a company’s solvency ratio, cases of negative equity are of special interest.

2.1.1. The Company

We assume the maturity-company to start off with a simplified balance sheet at $t = 0$ as in Figure 2.1. This simplified balance sheet consists of only three positions, one position on

Assets	Liabilities
A_0	$E_0 = (1 - \alpha)A_0$
	$L_0 = \alpha A_0$
A_0	A_0

Figure 2.1.: Simplified initial balance sheet

the asset side and two positions on the liability side. Note that we will refer to the asset side by "assets" or even the singular form "asset". Since we are interested in an economic balance sheet when analyzing solvency issues, we choose A_0, E_0 and L_0 to reflect the initial market values of assets, equity and liabilities. The equity position represents the residual value between assets and liabilities and thus coincides with the company's own funds as specified in article 87 of [Eur09]. As we do not incorporate the concept of the risk margin as in [Eur15, 37], E_0 and L_0 are exactly the time zero values of future payments to equity holders and policyholders, respectively.
This initial state of the company can be understood as the moment when equity holders and policyholders come together and agree upon an insurance contract. That is to say, E_0 is the equity holders' initial investment to get the company started and L_0 is the single premium paid by policyholders to purchase the insurance coverage. Together, equity holders and policyholders establish the initial balance sheet total A_0. The parameter α determines the ratio between own funds and liabilities.
For the evolution of the assets we adopt the standard Black-Scholes model. That is, we assume a continuous-time model on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We have a deterministic risk-free interest rate r and an asset process $(A_t)_{t \geq 0}$, which evolves according to a geometric Brownian motion, i.e. logarithmic returns are normally distributed. Under the real-world measure $\mathcal{P}$, the asset process has a drift of μ, which is the sum of the risk-free interest rate r and a risk premium λ, and a constant volatility of σ. $\mathcal{Q}$ denotes the risk-neutral measure under which the discounted asset process is a martingale. This

measure will be needed for valuation purposes, its existence is presumed. In brief,

$$dA_t = \mu\, A_t\, dt + \sigma\, A_t\, dW_t^{\mathcal{P}} \quad \text{under } \mathcal{P}, \tag{2.1}$$
$$dA_t = r\, A_t\, dt + \sigma\, A_t\, dW_t^{\mathcal{Q}} \quad \text{under } \mathcal{Q}, \tag{2.2}$$

where $W^{\mathcal{P}}$ and $W^{\mathcal{Q}}$ are standard Brownian motions under the measures $\mathcal{P}$ and $\mathcal{Q}$, respectively. Furthermore, we assume that the market is complete and free of arbitrage, transaction costs, tax effects and other constraints. The asset can be traded in any amount and shortselling is allowed.

2.1.2. The Guarantee

Now, we move on to the description of the insurance contract. The maturity-company offers its policyholders a guaranteed interest rate of r_G during the entire course of the contract, i.e. up to maturity T. Thus, policyholders are guaranteed a sum of

$$L_T^G = L_0 \cdot e^{r_G T}.$$

In addition, the company includes its policyholders in favorable developments of the assets by granting a terminal bonus of

$$\delta \left[\alpha A_T - L_T^G\right]^+,$$

where the plus operator of a number z is defined as $[z]^+ := \max(z, 0)$. The parameter δ is chosen such that the contract is fair at inception. By "fair" we mean that the single premium paid up front, $L_0 = \alpha A_0$, coincides with the time zero value of future payments to policyholders. Clearly, the terminal bonus is positive if the proportion of the assets that the policyholders contributed at time zero (α) is greater than the guaranteed sum L_T^G. Note that the terminal bonus is in fact a European call on the asset process. This can easily be seen when taking α out of the plus operator. We hereby get

$$\delta \left[\alpha A_T - L_T^G\right]^+ = \delta\alpha \left[A_T - \frac{L_T^G}{\alpha}\right]^+,$$

which obviously is the proportion $\delta\alpha$ of a European call on the asset process with maturity T and strike price L_T^G/α. We keep this in mind for when we get to the valuation of the contract. The terminal bonus represents the future discretionary benefits (FDB). The term "FDB" indicates that the terminal bonus is only assigned to policyholders at maturity and

only in the case of a favorable market development. A precise definition of the FDB is provided by [Eur15, 6,22].
Equity holders receive the residual of the asset value A_T and the payoff to policyholders. On the whole, we obtain the following payoffs to policyholders and equity holders at maturity, respectively.

$$\Psi_L(A_T) = L_T^G + \delta \left[\alpha A_T - L_T^G\right]^+ \tag{2.3}$$

$$\Psi_E(A_T) = A_T - \Psi_L(A_T) = A_T - L_T^G - \delta \left[\alpha A_T - L_T^G\right]^+ \tag{2.4}$$

2.1.3. Valuation of the Contract

In this section, we are looking for the market values of liabilities and own funds at time t, which are simply the time t values of the payoffs to policyholders and equity holders (2.3 and 2.4), respectively. Pricing is always performed under the risk-neutral measure $\mathcal{Q}$. Namely, the time t values are the discounted conditional expectations of the payoffs under $\mathcal{Q}$. For $0 \leq t \leq T$, we obtain as time t values

$$\begin{aligned} V_L(A_t, t) &= e^{-r(T-t)} \mathbb{E}_t^{\mathcal{Q}} \Big[\Psi_L(A_T) \Big] \\ &= \underbrace{e^{-r(T-t)} L_T^G}_{\text{Guarantee}} + \underbrace{\delta \alpha \, C\left(A_t, t, \frac{L_T^G}{\alpha} \right)}_{\text{FDB}}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} V_E(A_t, t) &= e^{-r(T-t)} \mathbb{E}_t^{\mathcal{Q}} \Big[\Psi_E(A_T) \Big] = A_t - V_L(A_t, t) \\ &= A_t - e^{-r(T-t)} L_T^G - \delta \alpha \, C\left(A_t, t, \frac{L_T^G}{\alpha} \right), \end{aligned} \tag{2.6}$$

where $C(x, t, K)$ denotes the call price at time t of a European call on an underlying valued x at time t and with strike price K. According to the Black-Scholes formula (see e.g. [Shr10, 220]),

$$C(x, t, K) = x \, \Phi\Big(d_1(x, t, K)\Big) - K \, e^{-r(T-t)} \Phi\Big(d_2(x, t, K)\Big), \tag{2.7}$$

where

$$d_1(x, t, K) = \frac{\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma \sqrt{T - t}}, \tag{2.8}$$

$$d_2(x, t, K) = d_1(x, t, K) - \sigma \sqrt{T - t} \tag{2.9}$$

and Φ is the CDF of the standard normal distribution.
Note that these valuation formulas only depend on the time t value A_t and not on the entire path of the asset process. Therefore, we can set up an economic balance sheet for any time t as long as the value of the asset process at time t (A_t) is known. Namely, the balance sheet items E_t and L_t are set to be equal to the time t values $V_E(A_t, t)$ and $V_L(A_t, t)$, respectively. The computational effort is minimal since all of the above are closed-form formulas.

2.1.4. Fair Contracts

Now that valuation formulas are established, we can formally describe the notion of a fair contract. As mentioned earlier, we choose the parameter δ, which declares to what extent policyholders get to participate in potential profits, in such a way that the contract is fair at inception. This implies that the other parameters of this model, $\alpha, A_0, \sigma, T, r, r_G$, are already determined. Then, we choose δ such that the single premium paid by policyholders equals the time zero value of the payoffs to policyholders. That is to say, we choose δ such that

$$L_0 = \alpha A_0 = V_L(A_0, 0).$$

Substituting the valuation formula (2.5) into the equation and solving for δ yields

$$\alpha A_0 = e^{-rT} L_T^G + \delta\alpha\, C\left(A_0, 0, \frac{L_T^G}{\alpha}\right) \quad \Leftrightarrow \quad \delta = \frac{\alpha A_0 - e^{-rT} L_T^G}{\alpha\, C\left(A_0, 0, \frac{L_T^G}{\alpha}\right)}. \tag{2.10}$$

2.1.5. Choice of Parameters

In the previous section, we discussed the correct choice of the parameter δ once the other parameters are already fixed. Now, we assign specific values with which all further analyses will be carried out. A summary of the parameters is given in Table 2.1. The maturity-company begins with a balance sheet total of 100. Since α is set to be 0.75, this total is composed of own funds of 25 and liabilities of 75. The contracts are in place for $T = 20$ years, during which the company guarantees its policyholders an interest rate of $r_G = 0.02$. Concerning the asset process under the real-world measure $\mathcal{P}$ (cf. (2.1)), we consider a volatility of $\sigma = 0.1$ and a mean rate of return of $\mu = 0.05$, which is made up of a risk-free interest rate of $r = 0.03$ and a risk premium of $\lambda = 0.02$.

Time zero value of the assets	A_0	$= 100$
Composition of the liabilities	α	$= 0.75$
Maturity	T	$= 20$
Volatility	σ	$= 0.1$
Risk-free interest rate	r	$= 0.03$
Risk premium	λ	$= 0.02$
Drift of the asset process	μ	$= r + \lambda = 0.05$
Guaranteed interest rate	r_G	$= 0.02$

Table 2.1.: Choice of parameters

Plugging these values into the earlier developed formula (2.10), we find the fair value of δ to be 0.68. Another consequence of the values set in Table 2.1 is the following amount of the guaranteed sum that policyholders can count on at maturity.

$$L_T^G = L_0 \cdot e^{r_G T} = \alpha A_0 \cdot e^{r_G T} = 111.89$$

2.2. Cliquet-Model

In the cliquet-model, we consider cliquet-style guarantees as discussed in [MP03] in the case without bonus account. However, instead of starting with an equity position of zero as in [MP03], equity holders of our company contribute to the initial balance sheet total in the same way as in the maturity-model. Cliquet-style guarantees differ from a maturity guarantee in the sense that bonus is credited to policyholders on a yearly basis. As a result, years with high asset returns cannot offset years with low returns. It matters by which path the asset process reaches its final value.

2.2.1. The Company

For the cliquet-company, we assume the exact same initial balance sheet as for the maturity-company (cf. Figure 2.1). Again, we deal with only three simplified positions in the balance sheet, which reflect the market values of assets, own funds and liabilities. The ratio between the initial values of own funds and liabilities, E_0 and L_0, is again controlled by the parameter α. Moreover, we assume that the cliquet-company invests its money just like the maturity-company. Therefore, the evolution of the asset process is identical to the one in the maturity-model (cf. (2.1) and (2.2)).

We basically choose the same setup for the maturity-model and the cliquet-model in order to make the two companies comparable. The idea is that, if the companies only differ by the type of guarantee they offer, we can analyze and compare the different characteristics of these guarantees.

2.2.2. The Guarantee

As mentioned before, the cliquet-company offers cliquet-style guarantees. As a consequence, the company credits bonus to policyholders on a yearly basis. Each year, policyholders are guaranteed a rate of return of g. Additionally, they get to participate in high returns of the asset process above the guaranteed level. Put together, the policyholders' account is multiplied in year t by

$$e^{g+\beta(\rho_t-g)^+},$$

where β determines the proportion of the surplus that is credited to policyholders. Similarly to the parameter δ in the maturity-model, β is chosen in such a way that the contract is fair at inception. ρ_t denotes the logarithmic return of the asset process between time $t-1$ and time t. Therefore, the evolution of the asset process between time $t-1$ and t can be written in terms of ρ_t as

$$A_t = A_{t-1} \cdot e^{\rho_t}.$$

Recursive application of this formula yields

$$A_t = A_0 \cdot e^{\sum_{i=1}^{t} \rho_i}.$$

Note that, for this type of guarantee, the guaranteed interest rate g is not only applied to the guaranteed accrual but also to bonus payments of earlier years. This means that the base to which the guarantee g is applied is raised by bonus payments. Clearly, this is a significant difference to the maturity guarantee where bonus is only credited at maturity. In order to get an explicit form of the logarithmic returns ρ_t, we first study the form of the asset process. Looking back at (2.1) and (2.2) and solving these stochastic differential equations, we obtain an explicit form of the asset process at time t.

$$A_t = A_0 \cdot \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t^{\mathcal{P}}\right\} \quad \text{under } \mathcal{P} \tag{2.11}$$

$$A_t = A_0 \cdot \exp\left\{\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t^{\mathcal{Q}}\right\} \quad \text{under } \mathcal{Q} \tag{2.12}$$

The logarithmic returns of the asset process can thus be expressed as

$$\rho_t = \ln\left(\frac{A_t}{A_{t-1}}\right) = \mu - \frac{\sigma^2}{2} + \sigma \underbrace{\left(W_t^{\mathcal{P}} - W_{t-1}^{\mathcal{P}}\right)}_{\sim\mathcal{N}(0,1)} \quad \text{under } \mathcal{P}, \tag{2.13}$$

$$\rho_t = \ln\left(\frac{A_t}{A_{t-1}}\right) = r - \frac{\sigma^2}{2} + \sigma \underbrace{\left(W_t^{\mathcal{Q}} - W_{t-1}^{\mathcal{Q}}\right)}_{\sim\mathcal{N}(0,1)} \quad \text{under } \mathcal{Q}. \tag{2.14}$$

Since increments of the Brownian motion are independent and stationary, log returns of different years are independent and identically distributed (iid).
On the whole, this leads to the following payoffs to policyholders and equity holders at maturity, respectively.

$$\Psi_L(\rho_{[0,T]}) = L_0 \cdot e^{\sum_{i=1}^{T}\left(g+\beta(\rho_i-g)^+\right)} = \alpha A_0 \cdot e^{\sum_{i=1}^{T}\left(g+\beta(\rho_i-g)^+\right)} \tag{2.15}$$

$$\Psi_E(\rho_{[0,T]}) = A_T - \Psi_L(\rho_{[0,T]}) = A_T - \alpha A_0 \cdot e^{\sum_{i=1}^{T}\left(g+\beta(\rho_i-g)^+\right)} \tag{2.16}$$

$\rho_{[0,T]}$ is a vector containing all the log returns of the asset process from time zero up to maturity. The distributions of the log returns appearing in (2.15) and (2.16) are to be understood under the real-world measure $\mathcal{P}$. Representations under the risk-neutral measure $\mathcal{Q}$ as in (2.12) and (2.14) are only important for valuation purposes.

2.2.3. Valuation of the Contract

In this section, we compute the market values of liabilities and own funds for all time points $0 \leq t \leq T$. We obtain these time t values by evaluating the discounted conditional expectations of the payoffs (2.15) and (2.16) under the risk-neutral probability measure $\mathcal{Q}$. First, we focus on the value of the liabilities at time zero, then at any time t within the lifetime of the contract. The value of the own funds can subsequently be determined as the residual between the market values of assets and liabilities.
For the time zero value of the liabilities, we follow the approach of [MP03] and compute

$$
\begin{aligned}
V_L(0) &= e^{-rT}\,\mathbb{E}^{\mathcal{Q}}\left[\Psi_L(\rho_{[0,T]})\right] = e^{-rT}\,\mathbb{E}^{\mathcal{Q}}\left[\alpha A_0\, e^{\sum_{i=1}^{T}\left(g+\beta(\rho_i-g)^+\right)}\right] \\
&= \alpha A_0\, e^{-rT}\,\mathbb{E}^{\mathcal{Q}}\left[\prod_{i=1}^{T} e^{g+\beta(\rho_i-g)^+}\right] = \alpha A_0\, e^{-rT}\prod_{i=1}^{T}\mathbb{E}^{\mathcal{Q}}\left[e^{g+\beta(\rho_i-g)^+}\right] \\
&= \alpha A_0\left(e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[e^{g+\beta(\rho_1-g)^+}\right]\right)^T,
\end{aligned} \tag{2.17}
$$

where the second last step is justified by the independence of the log returns and the last step by their identical distribution. In order to solve for the expression in parentheses, further rearrangements are necessary.

$$
\begin{aligned}
e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[e^{g+\beta(\rho_1-g)^+}\right] &= e^{-r}\,e^{g}\,\mathbb{E}^{\mathcal{Q}}\left[e^{\max(\beta\rho_1,\beta g)-\beta g}\right] \\
&= e^{-r}\,e^{(1-\beta)g}\,\mathbb{E}^{\mathcal{Q}}\left[\max\left(e^{\beta\rho_1},e^{\beta g}\right)\right] \\
&= e^{-r}\,e^{(1-\beta)g}\,\mathbb{E}^{\mathcal{Q}}\left[\left(e^{\beta\rho_1}-e^{\beta g}\right)^+ + e^{\beta g}\right] \\
&= e^{(1-\beta)g}\left[\underbrace{e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[\left(e^{\beta\rho_1}-e^{\beta g}\right)^+\right]}_{(*)} + e^{\beta g - r}\right]
\end{aligned} \tag{2.18}
$$

We note that the expression marked by $(*)$ is simply the time zero value of a European call on an underlying valued $e^{\beta\rho_1}$ at maturity $T=1$ with strike price $e^{\beta g}$. This allows us to find an explicit form of $(*)$ via the Black-Scholes formula (2.7). However, we first need to work out the time zero value and the volatility of the underlying. The time zero value is

$$
\begin{aligned}
e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[e^{\beta\rho_1}\right] &= e^{-r}\,\mathbb{E}^{\mathcal{Q}}\Bigg[\underbrace{e^{\beta\left(r-\frac{\sigma^2}{2}\right)+\beta\sigma\left(W_1^{\mathcal{Q}}-W_0^{\mathcal{Q}}\right)}}_{\sim\mathcal{LN}\left(\beta\left(r-\frac{\sigma^2}{2}\right),\,\beta^2\sigma^2\right)}\Bigg] \\
&= e^{-r}\,e^{\beta\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{2}\beta^2\sigma^2} \\
&= e^{(\beta-1)\left(r+\frac{1}{2}\beta\sigma^2\right)},
\end{aligned} \tag{2.19}
$$

where we used the form of ρ_1 under the risk-neutral measure $\mathcal{Q}$ as given in (2.14). Clearly, the volatility of the underlying is $\beta\sigma$. Plugging these values into the Black-Scholes formula (2.7), we arrive at

$$
\begin{aligned}
e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[\left(e^{\beta\rho_1}-e^{\beta g}\right)^{+}\right] &= e^{(\beta-1)\left(r+\frac{1}{2}\beta\sigma^2\right)}\,\Phi\left(\frac{(\beta-1)\left(r+\frac{1}{2}\beta\sigma^2\right)-\beta g+r+\frac{1}{2}\beta^2\sigma^2}{\beta\sigma}\right)\\
&\quad - e^{\beta g-r}\,\Phi\left(\frac{(\beta-1)\left(r+\frac{1}{2}\beta\sigma^2\right)-\beta g+r-\frac{1}{2}\beta^2\sigma^2}{\beta\sigma}\right)\\
&= e^{(\beta-1)\left(r+\frac{1}{2}\beta\sigma^2\right)}\,\Phi\left(\frac{r-g-\frac{1}{2}\sigma^2+\beta\sigma^2}{\sigma}\right) - e^{\beta g-r}\,\Phi\left(\frac{r-g-\frac{1}{2}\sigma^2}{\sigma}\right).
\end{aligned}
\tag{2.20}
$$

Finally, we insert expression (2.20) into (2.18) and the result again into (2.17) to obtain

$$
V_L(0)=\alpha A_0\left(e^{(1-\beta)\left(g-r-\frac{1}{2}\beta\sigma^2\right)}\,\Phi\left(\frac{r-g-\frac{1}{2}\sigma^2+\beta\sigma^2}{\sigma}\right)+e^{g-r}\,\Phi\left(\frac{g-r+\frac{1}{2}\sigma^2}{\sigma}\right)\right)^{T}.
\tag{2.21}
$$

Now that the time zero value of the liabilities is computed, the time t value for some $0\le t\le T$ can easily be derived as

$$
\begin{aligned}
V_L\left(\rho_{[0,t]},t\right) &= e^{-r(T-t)}\,\mathbb{E}_t^{\mathcal{Q}}\left[\Psi_L(\rho_{[0,T]})\right] = e^{-r(T-t)}\,\mathbb{E}_t^{\mathcal{Q}}\left[\alpha A_0\, e^{\sum_{i=1}^{T}\left(g+\beta(\rho_i-g)^+\right)}\right]\\
&= \alpha A_0\, e^{\sum_{i=1}^{t}\left(g+\beta(\rho_i-g)^+\right)}\, e^{-r(T-t)}\,\mathbb{E}_t^{\mathcal{Q}}\left[\prod_{i=t+1}^{T} e^{g+\beta(\rho_i-g)^+}\right]\\
&= \alpha A_0\, e^{\sum_{i=1}^{t}\left(g+\beta(\rho_i-g)^+\right)}\, e^{-r(T-t)}\prod_{i=t+1}^{T}\mathbb{E}^{\mathcal{Q}}\left[e^{g+\beta(\rho_i-g)^+}\right]\\
&= \alpha A_0\, e^{\sum_{i=1}^{t}\left(g+\beta(\rho_i-g)^+\right)}\left(e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[e^{g+\beta(\rho_1-g)^+}\right]\right)^{T-t}\\
&= \alpha A_0\, e^{\sum_{i=1}^{t}\left(g+\beta(\rho_i-g)^+\right)}\left(e^{(1-\beta)\left(g-r-\frac{1}{2}\beta\sigma^2\right)}\,\Phi\left(\frac{r-g-\frac{1}{2}\sigma^2+\beta\sigma^2}{\sigma}\right)\right.\\
&\qquad\left.+\,e^{g-r}\,\Phi\left(\frac{g-r+\frac{1}{2}\sigma^2}{\sigma}\right)\right)^{T-t},
\end{aligned}
\tag{2.22}
$$

where we take the first t log returns out of the conditional expectation since they are known at time t. Subsequently, we can drop the condition on the expectation as the log returns from $t+1$ to T are independent of the filtration at time t. Again, we use the independence and identical distribution of the log returns. In the last step, we simply plug

in the expression that was derived earlier. We do not distinguish between the guaranteed part and the FDB because the two of them cannot be told apart as easily as it was the case for the maturity-model.
Finally, the time t value of the own funds is the residual between assets and liabilities.

$$\begin{aligned} V_E\left(\rho_{[0,t]}, t\right) &= A_t - V_L\left(\rho_{[0,t]}, t\right) \\ &= A_t - \Bigg[\alpha A_0\, e^{\sum_{i=1}^{t}\left(g+\beta(\rho_i-g)^+\right)} \left(e^{(1-\beta)\left(g-r-\frac{1}{2}\beta\sigma^2\right)}\, \Phi\left(\frac{r-g-\frac{1}{2}\sigma^2+\beta\sigma^2}{\sigma}\right)\right. \\ &\qquad\qquad \left. + e^{g-r}\, \Phi\left(\frac{g-r+\frac{1}{2}\sigma^2}{\sigma}\right)\right)^{(T-t)}\Bigg] \end{aligned} \tag{2.23}$$

2.2.4. Fair Contracts

In this section, we seek to find the expression of β, the parameter determining the policyholders' degree of participation in the yearly surplus, that ensures a fair starting point of the contract. We assume that all of the other parameters, $\alpha, A_0, \sigma, T, r, g$, are already fixed. Then, we choose β such that the single premium paid by policyholders coincides with the time zero value of the liabilities. By (2.21), we arrive at

$$\begin{aligned} L_0 &= \alpha A_0 = V_L(0) \\ \Leftrightarrow \alpha A_0 &= \alpha A_0 \left(e^{(1-\beta)\left(g-r-\frac{1}{2}\beta\sigma^2\right)}\, \Phi\left(\frac{r-g-\frac{1}{2}\sigma^2+\beta\sigma^2}{\sigma}\right) + e^{g-r}\, \Phi\left(\frac{g-r+\frac{1}{2}\sigma^2}{\sigma}\right)\right)^T \\ \Leftrightarrow\ & e^{(1-\beta)\left(g-r-\frac{1}{2}\beta\sigma^2\right)}\, \Phi\left(\frac{r-g-\frac{1}{2}\sigma^2+\beta\sigma^2}{\sigma}\right) + e^{g-r}\, \Phi\left(\frac{g-r+\frac{1}{2}\sigma^2}{\sigma}\right) = 1. \end{aligned} \tag{2.24}$$

Clearly, we do not get an explicit form of β. However, this condition on β can be solved numerically by using the Newton-Raphson method, which is implemented in the software R via the uniroot function.
Once β is chosen such that the contract is fair at inception, i.e. (2.24) holds, the valuation formulas (2.21), (2.22) and (2.23) simplify to

$$V_L(0) = \alpha A_0, \tag{2.25}$$

$$V_L\left(\rho_{[0,t]}, t\right) = \alpha A_0\, e^{\sum_{i=1}^{t}\left(g+\beta(\rho_i-g)^+\right)}, \tag{2.26}$$

$$V_E\left(\rho_{[0,t]}, t\right) = A_t - \alpha A_0\, e^{\sum_{i=1}^{t}\left(g+\beta(\rho_i-g)^+\right)}. \tag{2.27}$$

Note that these valuation formulas only depend on already realized log returns of the asset process. It seems that, since the value of future returns on the policyholders' account is set to one, valuation can be achieved by considering only the past up to time t. In particular, market values are independent of the maturity T. On the contrary to this characteristic of the cliquet-model, valuation in the maturity-model (cf. (2.5) and (2.6)) does not only lean on the value of the asset process at time t but also on the future. We might say that, for valuation purposes in the maturity-model, we need to look ahead from time point t onwards. Specifically, market values depend on the time to maturity, $T - t$.

2.2.5. Choice of Parameters

For the choice of the parameters other than β, we again refer to Table 2.1. We use the same parameters as in the maturity-model in order to keep the two companies comparable. The companies are intended to only differ in the type of guarantee they offer. For instance, the parameters α and A_0 ensure that both companies start with the exact same balance sheet. Due to the distinct types of guarantees, we expect the companies to develop differently. While the asset side develops identically as specified by the asset process, the positions on the liability side will vary. We will study the effects of the two types of guarantees on the insurer's balance sheet and solvency over time.

Only one parameter is not yet specified, namely the guaranteed interest rate g in the cliquet-model. For the choice of g, we assume that the cliquet-company is aware that crediting bonus on a yearly basis is riskier than merely offering a terminal bonus. After all, the guaranteed rate has to be applied to already credited bonus even if the asset process develops unfavorably thereafter. As a result, the cliquet-company grants a guaranteed interest rate of $g = 0.01$, a lower rate than the maturity-company with $r_G = 0.02$.

This choice of parameters leads to a fair β of 0.41.

3. Derivation of the SCR

In this chapter, we define the stress scenario on which we base all calculations for obtaining the SCR. Since we do not consider mortality, lapse, stochastic interest rates or any other risk factors, there will be only one single stress scenario, namely the one concerning the asset process. Consequently, we can avoid the use of an aggregation formula and the assumption of specific correlation coefficients between different risk factors.
Then, we discuss the calculation of the SCR and the solvency ratio for each of the two models as implied by the stress scenario.

3.1. The Stress Scenario

The SCR is defined as the minimum amount of capital with which a company is able to survive a bicentenary event on a one-year time horizon (cf. [Eur09, 7]). Thus, the required amount is specified by the Value-at-Risk of the one-year loss of the own funds at a level of 99.5% (cf. [Eur09, 51]). For the computation of the SCR, there are basically two methods that are approved by [Eur09]. Either the company has to directly analyze the distribution of its own funds (internal model) or it has to use the prespecified stress scenarios (standard model).
For our analyses, we define a stress scenario by the bicentenary event of the asset process on a one-year time horizon, the 0.5%-quantile of the distribution over one year. Figure 3.1 illustrates our approach. Positioned at some time point t, we are looking for the stressed value of the assets denoted by $\tilde{A}_t$. For that purpose, we consider the distribution of the assets over the next year up to time $t+1$ and observe the 0.5%-quantile of A_{t+1}. Note that we study the distribution from t to $t+1$ under the real-world probability measure $\mathcal{P}$. This is important because we are interested in the value of the assets that will actually be observable at $t+1$. The risk-neutral measure $\mathcal{Q}$ only comes into play when we value payoffs. Once we have taken note of the 0.5%-quantile of A_{t+1}, we need to discount it one

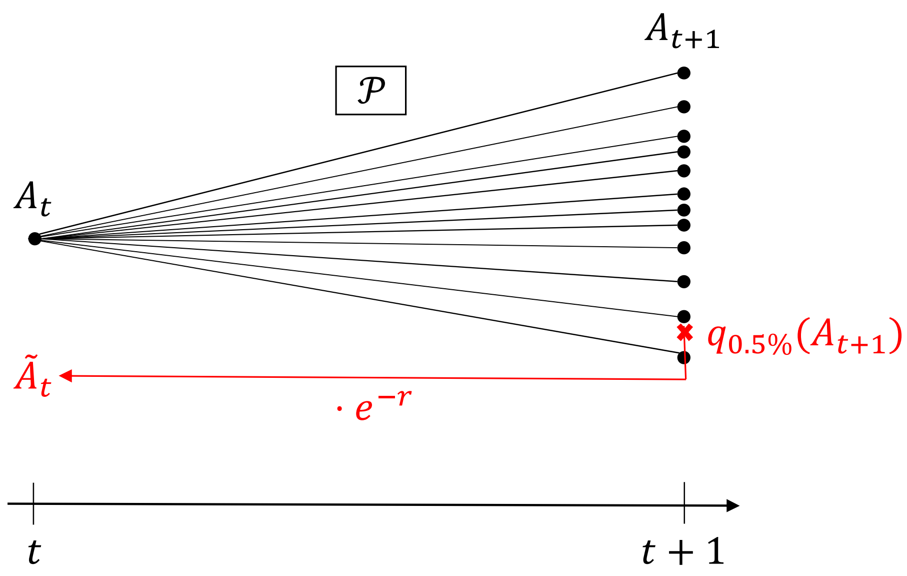


Figure 3.1.: Derivation of the stress scenario

year back to time t in order to obtain the stressed value $\tilde{A}_t$. This discount is necessary since we assume an instantaneous occurrence of the stress. In other words, the stress scenario implies that the value of the assets A_t drops to $\tilde{A}_t$ right after time t.
Next, we use the distribution of the assets under the real-world measure as in (2.1) to derive the stress on the assets mathematically. From the dynamics (2.1) and the explicit form of the assets (2.11), we gather the following relation between A_t and A_{t+1}.

$$\begin{aligned} \frac{A_{t+1}}{A_t} &= \exp\left\{\mu - \frac{\sigma^2}{2} + \sigma\Big(W^{\mathcal{P}}_{t+1} - W^{\mathcal{P}}_t\Big)\right\} \\ \Leftrightarrow A_{t+1} &= A_t \cdot \exp\left\{\mu - \frac{\sigma^2}{2} + \sigma\,\underbrace{\left(W^{\mathcal{P}}_{t+1} - W^{\mathcal{P}}_t\right)}_{\sim\mathcal{N}(0,1)}\right\} \end{aligned} \tag{3.1}$$

Note that we are looking for the stressed value $\tilde{A}_t$ as seen from time t. Hence, we condition on the filtration $\mathcal{F}_t$. The evolution of the asset process up to time t and, most importantly, the value A_t are known. Since the increment of the standard Brownian motion in (3.1) is standard normally distributed and the exponential function is a monotonically increasing function, the 0.5%-quantile of A_{t+1} is easily found to be

$$q_{0.5\%}(A_{t+1}) = A_t \cdot \exp\left\{\mu - \frac{\sigma^2}{2} + \sigma \cdot q_{0.5\%}\Big(\mathcal{N}(0,1)\Big)\right\}.$$

Since the mean rate of return μ is the sum of the risk-free rate r and the risk premium λ, we arrive at a stressed value of A_t of

$$\begin{aligned}\tilde{A}_t &= q_{0.5\%}(e^{-r}A_{t+1}) = e^{-r}\, q_{0.5\%}(A_{t+1}) \\ &= A_t \cdot \underbrace{\exp\left\{\lambda - \frac{\sigma^2}{2} + \sigma \cdot q_{0.5\%}\Big(\mathcal{N}(0,1)\Big)\right\}}_{=:sf_A}. \end{aligned} \tag{3.2}$$

Thus, we obtain the stressed value of the assets for any time t by multiplying the original value by a constant stress factor sf_A as defined in (3.2).
For our choice of parameters (cf. Table 2.1), we get a stress factor of roughly 0.78. Therefore, the stress scenario is equivalent to an immediate decline of the assets by about 22%.

3.2. Calculation of SCR and Solvency Ratio

As mentioned before, the SCR is the minimum capital needed in order to stay solvent, i.e. maintain a position with nonnegative own funds, throughout a bicentenary event. Figure 3.2 depicts the change in the balance sheet when a stress occurs at some time t. In

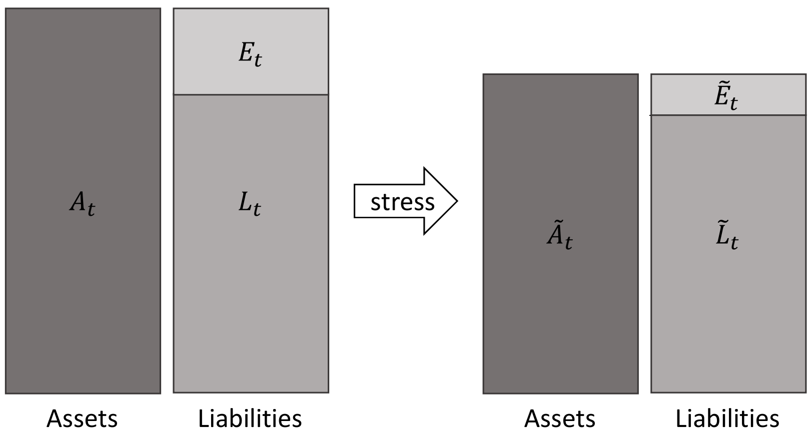


Figure 3.2.: Impact of the stress scenario on the balance sheet

the case of a stress on the assets, the balance sheet total plummets according to the stress factor in (3.2). The new value of the assets is simply

$$\tilde{A}_t = sf_A \cdot A_t.$$

Clearly, the market values of own funds and liabilities will also have to change since they sum up to the value of the assets. Taking into account the decline in assets, we reevaluate own funds and liabilities and call the new values $\tilde{E}_t$ and $\tilde{L}_t$. Exact formulas for the two models will be provided shortly. Once the items of the balance sheet in the stress scenario are established, we can easily infer the SCR. Namely, as in the standard formula, we define the SCR to be the decline in own funds caused by the stress, that is

$$SCR_t = E_t - \tilde{E}_t.$$

The solvency ratio is the ratio between own funds and SCR.

$$Sol_t = \frac{E_t}{SCR_t}$$

Naturally, the insurer aims for ratios above 100% as these imply that the own funds are greater than the capital requirement. If ratios fall below 100%, a supervising institution intervenes and imposes regulatory measures. However, we will not discuss such measures. Instead, we assume that the insurer borrows money to reach a ratio of 100% if he finds himself in a position of a lower ratio. Further details will be given later on.
The next two sections deal with the calculation of the SCR and the solvency ratio within the frameworks of our two models. For that purpose, we have to consider the calculation of $\tilde{E}_t$, the reevaluated own funds in the stress scenario, for both models.

3.2.1. Maturity-Model

For the maturity-model, we recall that the valuation formula of the liabilities (2.5) only depends on the value of the asset process at time t. If a stress occurs, the value of the assets drops instantly. Therefore, the value of the liabilities in the stress scenario can be derived by adjusting the value of the assets in the valuation formula. We obtain

$$\begin{aligned} \tilde{L}_t &= V_L(\tilde{A}_t, t) = V_L(sf_A \cdot A_t, t) \\ &= e^{-r(T-t)} L_T^G + \delta\alpha\, C\left(sf_A \cdot A_t, t, \frac{L_T^G}{\alpha}\right). \end{aligned} \tag{3.3}$$

Naturally, the value of the guarantee, $e^{-r(T-t)} L_T^G$, is not affected by a stress on the assets. This means that the guaranteed part of the liabilities cannot absorb any part of the stress. The value of the FDB, however, diminishes since a drop in the value of the assets reduces the policyholders' prospects of receiving a terminal bonus. Next, we obtain the value of the own funds in the stress scenario by computing the difference between stressed assets and stressed liabilities.

$$\tilde{E}_t = \tilde{A}_t - \tilde{L}_t = sf_A \cdot A_t - e^{-r(T-t)} L_T^G - \delta\alpha\, C\left(sf_A \cdot A_t, t, \frac{L_T^G}{\alpha}\right) \tag{3.4}$$

So then, the SCR at time t equals

$$\begin{aligned} SCR_t &= E_t - \tilde{E}_t \\ &= A_t - e^{-r(T-t)} L_T^G - \delta\alpha\, C\left(A_t, t, \frac{L_T^G}{\alpha}\right) \\ &\quad - \left[sf_A \cdot A_t - e^{-r(T-t)} L_T^G - \delta\alpha\, C\left(sf_A \cdot A_t, t, \frac{L_T^G}{\alpha}\right)\right] \\ &= (1 - sf_A)A_t - \delta\alpha \left[C\left(A_t, t, \frac{L_T^G}{\alpha}\right) - C\left(sf_A \cdot A_t, t, \frac{L_T^G}{\alpha}\right)\right]. \end{aligned} \tag{3.5}$$

By studying the SCR, we basically study how a stress on the asset side of the balance sheet transfers to the liability side, keeping a special focus on the own funds. The first term in (3.5) describes the decline in assets caused by the stress scenario. For the SCR, which is simply the decline in own funds, the first term is diminished by the decline in the value of the FDB. In other words, as the call price describing the value of the FDB decreases, it captures some of the impact of the stress and thereby relieves the own funds. Finally, we arrive at the following solvency ratio.

$$\boxed{Sol_t = \frac{E_t}{SCR_t} = \frac{A_t - e^{-r(T-t)} L_T^G - \delta\alpha\, C\left(A_t, t, \frac{L_T^G}{\alpha}\right)}{(1 - sf_A)A_t - \delta\alpha \left[C\left(A_t, t, \frac{L_T^G}{\alpha}\right) - C\left(sf_A \cdot A_t, t, \frac{L_T^G}{\alpha}\right)\right]}} \tag{3.6}$$

Note that the ratio is a function of the asset value at time t and therefore not path-dependent. It can easily be computed once the value of the assets is known.

We also introduce a second solvency figure, the excess coverage, which is defined as own funds minus SCR. Thus, we study the relationship between own funds and SCR both on a relative scale as well as an absolute scale. Since the SCR is the difference between the own funds and the stressed value of the own funds, the excess coverage simplifies to the stressed value of the own funds, that is

$$E_t - SCR_t = E_t - (E_t - \tilde{E}_t) = \tilde{E}_t.$$

The formula for $\tilde{E}_t$ was already given in (3.4). Like the solvency ratio, the excess coverage of the maturity-model is not path-dependent. Note that we could alternatively work with the relative excess coverage instead of the absolute one, i.e. the difference of own funds and SCR divided by the value of the assets. However, the implications are the same as for the absolute excess coverage.

3.2.2. Cliquet-Model

For the cliquet-model, the reevaluation of the own funds in the stress scenario appears to be a little more complicated than for the maturity-model. Again, we start by computing the new value of the liabilities, $\tilde{L}_t$. The own funds $\tilde{E}_t$ can then be deduced as the residual between the stressed asset value $\tilde{A}_t$ and $\tilde{L}_t$.
Since the stress on the assets occurs right after time t, log returns up to time t are not affected. However, the mean of the log return from time t to $t+1$ is shifted downwards due to the stress. We denote this adjusted log return by $\tilde{\rho}_{t+1}$. In particular,

$$\begin{aligned}\tilde{\rho}_{t+1} &= \ln(sf_A) + \ln\left(\frac{A_{t+1}}{A_t}\right) \\ &= \ln(sf_A) + r - \frac{\sigma^2}{2} + \sigma\left(W_{t+1}^{\mathcal{Q}} - W_t^{\mathcal{Q}}\right) \end{aligned} \tag{3.7}$$

under the risk-neutral probability measure $\mathcal{Q}$. Since the increment of the Brownian motion is standard normally distributed, the adjusted log returns follow a normal distribution with mean $\ln(sf_A) + r - \frac{\sigma^2}{2}$ and volatility σ. Note that $\ln(sf_A)$ is negative because the stress factor is smaller than one. Thus, the adjusted returns indeed have a lower mean than the original returns (cf. (2.14)). As the log returns of the asset process are iid and the stress factor is a constant independent of time, the adjusted log returns are iid as well.
The value of the liabilities in the stress scenario is equal to

$$\begin{aligned}\tilde{L}_t &= e^{-r(T-t)}\,\mathbb{E}_t^{\mathcal{Q}}\left[\alpha A_0\, e^{\sum_{i=1}^{t}\left(g+\beta(\rho_i-g)^+\right)}\, e^{g+\beta(\tilde{\rho}_{t+1}-g)^+}\, e^{\sum_{i=t+2}^{T}\left(g+\beta(\rho_i-g)^+\right)}\right] \\ &= \underbrace{\alpha A_0\, e^{\sum_{i=1}^{t}\left(g+\beta(\rho_i-g)^+\right)}}_{(1)}\; \underbrace{e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[e^{g+\beta(\tilde{\rho}_{t+1}-g)^+}\right]}_{(2)}\; \underbrace{e^{-r(T-t-1)}\,\mathbb{E}^{\mathcal{Q}}\left[e^{\sum_{i=t+2}^{T}\left(g+\beta(\rho_i-g)^+\right)}\right]}_{(3)}, \end{aligned} \tag{3.8}$$

where we use the independence of the log returns. Term (1) equals L_t as can be seen from (2.26). As for term (3), we refer to (2.18), (2.20) and (2.24) to obtain

$$e^{-r(T-t-1)}\,\mathbb{E}^{\mathcal{Q}}\left[e^{\sum_{i=t+2}^{T}\left(g+\beta(\rho_i-g)^+\right)}\right]=\Big(\underbrace{e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[e^{g+\beta(\rho_1-g)^+}\right]}_{=1}\Big)^{T-t-1}=1.$$

It remains to compute the value of term (2). Of course, we expect to arrive at a value lower than one since it is much less likely for the policyholders to receive a bonus in the year of the stress compared to other years. For the calculation, we proceed similarly to section 2.2.3, i.e. as in [MP03].

$$\begin{aligned}
e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[e^{g+\beta(\tilde\rho_{t+1}-g)^+}\right] &= e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[e^{g+\beta(\tilde\rho_1-g)^+}\right]=e^{-r}\,e^{g}\,\mathbb{E}^{\mathcal{Q}}\left[e^{\max(\beta\tilde\rho_1,\beta g)-\beta g}\right]\\
&= e^{-r}\,e^{(1-\beta)g}\,\mathbb{E}^{\mathcal{Q}}\left[\max\left(e^{\beta\tilde\rho_1},e^{\beta g}\right)\right]\\
&= e^{-r}\,e^{(1-\beta)g}\,\mathbb{E}^{\mathcal{Q}}\left[\left(e^{\beta\tilde\rho_1}-e^{\beta g}\right)^+ + e^{\beta g}\right]\\
&= e^{(1-\beta)g}\Big[\underbrace{e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[\left(e^{\beta\tilde\rho_1}-e^{\beta g}\right)^+\right]}_{(*)}+e^{\beta g-r}\Big],
\end{aligned}\tag{3.9}$$

where $(*)$ is the time zero value of a European call on an underlying with value $e^{\beta\tilde\rho_1}$ at maturity $T=1$ and strike price $e^{\beta g}$. In preparation for applying the Black-Scholes formula (2.7), we derive the time zero value of the underlying.

$$\begin{aligned}
e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[e^{\beta\tilde\rho_1}\right] &= e^{-r}\,\mathbb{E}^{\mathcal{Q}}\Big[\underbrace{e^{\beta\left(\ln(sf_A)+r-\frac{\sigma^2}{2}\right)+\beta\sigma\left(W_1^{\mathcal{Q}}-W_0^{\mathcal{Q}}\right)}}_{\sim\mathcal{LN}\left(\beta\left(\ln(sf_A)+r-\frac{\sigma^2}{2}\right),\,\beta^2\sigma^2\right)}\Big]\\
&= e^{-r}\,e^{\beta\left(\ln(sf_A)+r-\frac{\sigma^2}{2}\right)+\frac{1}{2}\beta^2\sigma^2}\\
&= e^{\beta\cdot\ln(sf_A)}\,e^{(\beta-1)\left(r+\frac{1}{2}\beta\sigma^2\right)},
\end{aligned}\tag{3.10}$$

where we use the representation of the adjusted log returns as in (3.7). We note that the volatility of the underlying is $\beta\sigma$. Therefore, the Black-Scholes formula (2.7) yields

$$
\begin{aligned}
&e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[\left(e^{\beta\tilde{\rho}_1}-e^{\beta g}\right)^+\right]\\
&= e^{\beta\cdot\ln(sf_A)}\,e^{(\beta-1)\left(r+\frac{1}{2}\beta\sigma^2\right)}\,\Phi\left(\frac{\beta\cdot\ln(sf_A)+(\beta-1)\left(r+\frac{1}{2}\beta\sigma^2\right)-\beta g+r+\frac{1}{2}\beta^2\sigma^2}{\beta\sigma}\right)\\
&\quad - e^{\beta g-r}\,\Phi\left(\frac{\beta\cdot\ln(sf_A)+(\beta-1)\left(r+\frac{1}{2}\beta\sigma^2\right)-\beta g+r-\frac{1}{2}\beta^2\sigma^2}{\beta\sigma}\right)\\
&= e^{\beta\cdot\ln(sf_A)}\,e^{(\beta-1)\left(r+\frac{1}{2}\beta\sigma^2\right)}\,\Phi\left(\frac{\ln(sf_A)+r-g-\frac{1}{2}\sigma^2+\beta\sigma^2}{\sigma}\right)\\
&\quad - e^{\beta g-r}\,\Phi\left(\frac{\ln(sf_A)+r-g-\frac{1}{2}\sigma^2}{\sigma}\right).
\end{aligned}
\tag{3.11}
$$

Plugging this into (3.9), we get the following value for term (2).

$$
\begin{aligned}
&e^{-r}\,\mathbb{E}^{\mathcal{Q}}\left[e^{g+\beta(\tilde{\rho}_{t+1}-g)^+}\right]\\
&= e^{(1-\beta)g}\left[e^{\beta\cdot\ln(sf_A)+(\beta-1)\left(r+\frac{1}{2}\beta\sigma^2\right)}\,\Phi\left(\frac{\ln(sf_A)+r-g-\frac{1}{2}\sigma^2+\beta\sigma^2}{\sigma}\right)\right.\\
&\qquad\left.+\,e^{\beta g-r}\,\Phi\left(\frac{g-r+\frac{1}{2}\sigma^2-\ln(sf_A)}{\sigma}\right)\right]\\
&= e^{\beta\cdot\ln(sf_A)+(1-\beta)\left(g-r-\frac{1}{2}\beta\sigma^2\right)}\,\Phi\left(\frac{\ln(sf_A)+r-g-\frac{1}{2}\sigma^2+\beta\sigma^2}{\sigma}\right)\\
&\quad + e^{g-r}\,\Phi\left(\frac{g-r+\frac{1}{2}\sigma^2-\ln(sf_A)}{\sigma}\right) =: sf_L,
\end{aligned}
\tag{3.12}
$$

which we denote as sf_L, the stress factor on the liabilities resulting from a stress on the assets. Collecting terms (1), (2) and (3), (3.8) becomes

$$\tilde{L}_t = sf_L \cdot L_t. \tag{3.13}$$

Similar to the change in asset value in the stress scenario, the change in the value of the liabilities can be expressed by the constant factor sf_L. For our choice of parameters (cf. Table 2.1), sf_L is about 0.98. Thus, when a stress occurs, the value of the liabilities shrinks by roughly two percent. This drop is accounted for by the diminished prospects of a bonus in the year of the stress. Bonus payments of subsequent years, however, are not affected. Consequently, the drop of two percent in the liabilities is rather small. In other words, the liabilities do not contribute much to absorbing the stress. Most of the decline

will hit the own funds.
The value of the own funds in the stress scenario ($\tilde{E}_t$) follows immediately from $\tilde{A}_t$ and $\tilde{L}_t$.

$$\tilde{E}_t = \tilde{A}_t - \tilde{L}_t = sf_A \cdot A_t - sf_L \cdot L_t \tag{3.14}$$

Therefore, the SCR of the cliquet-company amounts to

$$\begin{aligned} SCR_t &= E_t - \tilde{E}_t \\ &= (A_t - L_t) - (sf_A \cdot A_t - sf_L \cdot L_t) \\ &= (1 - sf_A)A_t - (1 - sf_L)L_t. \end{aligned} \tag{3.15}$$

Again, the first term of the SCR equals the decline in assets caused by the stress scenario. The SCR, the decline in own funds, is a little smaller than the decline in assets. Namely, two percent of the liabilities are available to absorb some of the stress. All of the remaining reduction, however, hits the own funds.
Finally, the solvency ratio corresponds to

$$\boxed{Sol_t = \frac{E_t}{SCR_t} = \frac{A_t - L_t}{(1 - sf_A)A_t - (1 - sf_L)L_t}} \tag{3.16}$$

Since the value of the liabilities (L_t) is path-dependent, the solvency ratio is path-dependent as well.
The excess coverage coincides with the value of the stressed own funds, the formula was given in (3.14). Like for the solvency ratio, the path-dependence of the liabilities is passed on to the excess coverage.

4. Analysis of Future SCRs and Solvency Ratios

In this chapter, we study and compare the solvency positions of the two companies at inception and at future time points ($t > 0$). For time points after time zero, the quantities of interest depend on the random behavior of the assets. In order to analyze the distributions, we simulate paths of the asset process under the real-world probability measure $\mathcal{P}$. The choice of probability measure is an important issue, for which we recall that the risk-neutral measure $\mathcal{Q}$ is only needed for valuation purposes. We obtain empirical distributions by computing the SCR and the solvency ratio for each simulated path at every point in time. It is indeed necessary to simulate entire paths of the asset process since the liabilities of the cliquet-model are path-dependent.

To keep our approach transparent, we begin by simulating only one single path of the asset process and performing the corresponding calculations. Then, we proceed to simulating a larger number of paths, which allows us to analyze empirical distributions as well as dependencies of SCR and solvency ratio on the value of the assets, the only source of randomness in our models. Moreover, we consider the development of the shortfall probability and the expected shortfall. By "shortfall" we relate to the case in which the SCR exceeds the own funds. At the end of this chapter, we discuss the way we treat such shortfalls. We will assume that a company that cannot cover its SCR borrows the remaining difference between own funds and SCR for a period of one year. The loss arising from the cost of capital involved is then paid from a separate account. However, for now, it suffices to know that we continue a simulation even if the SCR cannot be met and even if the own funds are negative at some point in time.

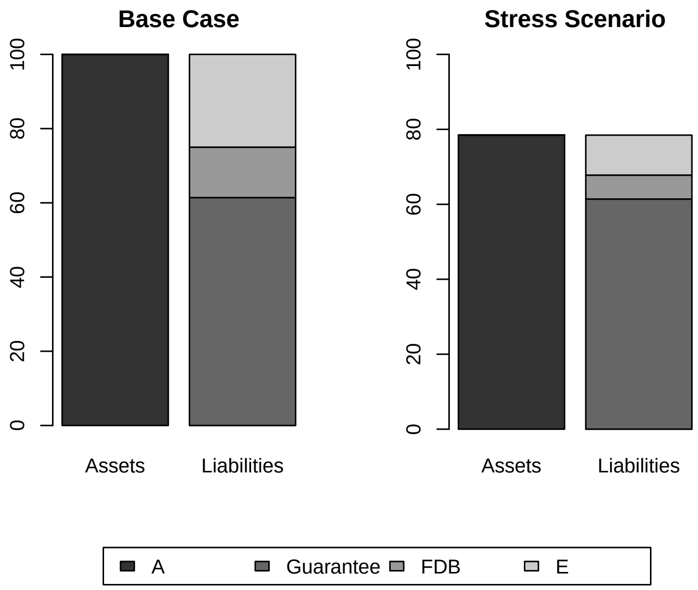
All simulations are implemented through the software R. The corresponding code can be found in the appendix.

4.1. Initial Solvency Position

In this section, we begin by having a look at the initial solvency of the two companies. Since the value of the assets is known at time zero, we can directly apply the formulas derived in the previous chapter. The solvency figures are fully defined by (3.6), (3.4) and (3.16), (3.14).

4.1.1. Maturity-Model

Figure 4.1 shows the impact of the stress scenario on the initial balance sheet of the maturity-company. As discussed before, the decline in assets leads to a decline in own



	Base Case	Stress Scenario
A_0	100	78.46
Guarantee	61.40	61.40
FDB	13.60	6.38
E_0	25	10.67

Figure 4.1.: The effect of the stress scenario at time zero, maturity-company

funds as well as FDB, while the time zero value of the guarantee, $e^{-rT}L_T^G$, cannot be changed. The specific values result from our choice of parameters as in Table 2.1 and the valuation formula (3.4). We arrive at a SCR of

$$SCR_0 = E_0 - \tilde{E}_0 = 25 - 10.67 = 14.33$$

and a solvency ratio of

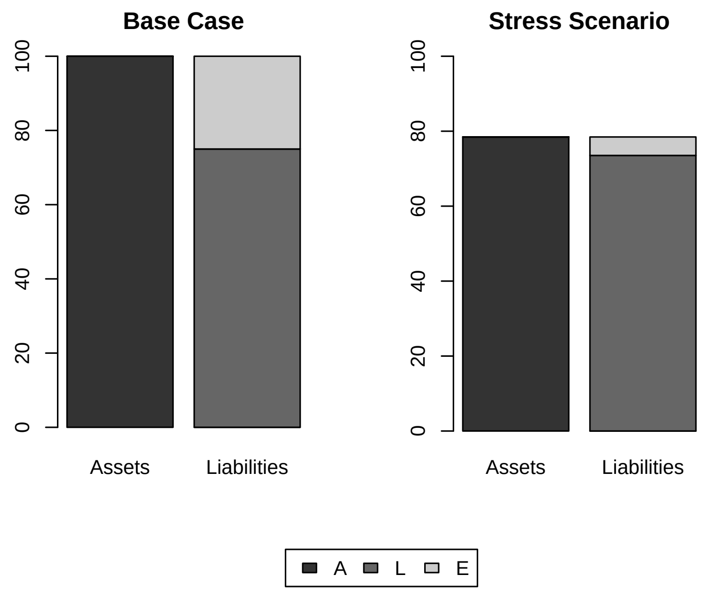
$$Sol_0 = \frac{E_0}{SCR_0} = \frac{25}{14.33} = 174\%.$$

Remarkably, the maturity-company starts with a solvency ratio far above the required 100%. For the excess coverage, we obtain

$$E_0 - SCR_0 = 25 - 14.33 = 10.67.$$

4.1.2. Cliquet-Model

For the cliquet-model, we recall that our setup leads to an identical initial balance sheet as in the maturity-model. However, the stress scenario affects the cliquet-company differently as shown in Figure 4.2. We arrive at these values via the formulas (3.13) and (3.14).



	Base Case	Stress Scenario
A_0	100	78.46
L_0	75	73.53
E_0	25	4.93

Figure 4.2.: The effect of the stress scenario at time zero, cliquet-company

Clearly, the value of the liabilities barely decreases in view of the stress scenario. As a result, the own funds in the stress scenario are significantly lower than those in the maturity-model. We obtain a SCR of

$$SCR_0 = E_0 - \tilde{E}_0 = 25 - 4.93 = 20.07,$$

which is considerably higher than the SCR of the maturity-company. Since both companies begin with the same amount of own funds, the cliquet-company has a lower solvency ratio, namely

$$Sol_0 = \frac{E_0}{SCR_0} = \frac{25}{20.07} = 125\%,$$

which is still above 100%. Although the cliquet-company grants its policyholders a lower guaranteed interest rate than the maturity-company, the type of guarantee causes a lower solvency ratio. The excess coverage amounts to

$$E_0 - SCR_0 = 25 - 20.07 = 4.93.$$

4.2. Simulating One Path

We seek to simulate one entire path of the asset process, i.e. $A_0, A_1, \ldots, A_{19}$. Note that we stop at $t = 19$ since SCR and solvency ratio are not defined for $t = 20$ due to the termination of the contracts. Instead of simulating A_0 up to A_{19} independently by drawing from the lognormal distribution of the assets as specified in (2.11), we need to consider the log returns as in (2.13) to obtain one consistent path. Randomly picking 19 independent realizations of the standard normal distribution and plugging these into (2.13) in place of the increment of the Brownian motion, we attain one possible realization of the log returns $\rho_1, \rho_2, \ldots, \rho_{19}$. From the log returns, we can deduce the following values of the asset process recursively.

$$A_1 = A_0 \cdot e^{\rho_1}, \quad A_2 = A_1 \cdot e^{\rho_2}, \quad \ldots, \quad A_{19} = A_{18} \cdot e^{\rho_{19}},$$

where A_0 is the initial value of the assets known at time zero. Figure 4.3 shows one possible path of the assets. While the dotted line shows the interest rate guaranteed by the maturity-company with regard to the initial value of the assets, the dashed lines represent the interest rate guaranteed by the cliquet-company with regard to the current value of the assets at each time point. For the guarantee given by the maturity-company, the solvency does not depend on the evolution of the assets but only on the value at time t. That is to say, it suffices if the assets generate the guaranteed interest rate on average. Although the sample path represented in Figure 4.3 ends one year before maturity, we can already claim that the guarantee will be met almost surely for this path. As a result, we expect the maturity-company to pay a terminal bonus. On the contrary, the cliquet-company grants a guaranteed rate on a yearly basis. For every year in which the path of the asset process stays below the dashed line, the cliquet-company has to grant its policyholders

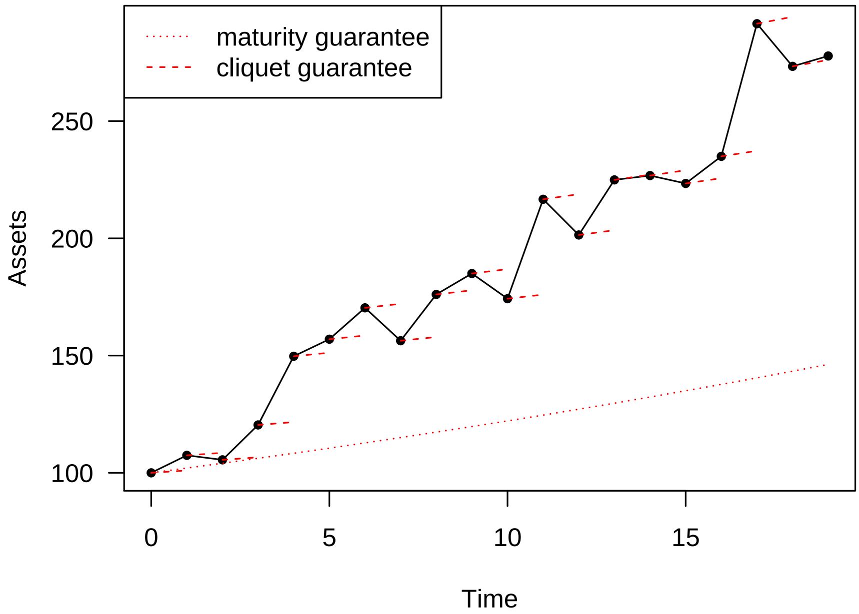


Figure 4.3.: Sample path of the asset process

the guaranteed interest although the return on the assets cannot compensate for such an increase in value. This is the case, for example, in the second year. The value of the assets declines and thus does not cover the guarantee. However, if the evolution of the assets exceeds the dashed line as in the first year, the cliquet-company credits a bonus.
All relevant values up to time $t = 8$ are rounded to two decimal places and listed in Table 4.1. Additionally, Figure 4.4 shows the composition of the asset values from own funds and liabilities. The market values of liabilities and own funds are obtained according to the valuation formulas (2.5), (2.6) and (2.26), (2.27). For the maturity-model, liabilities and own funds rise and drop according to the development of the assets. Recall that the market value of the liabilities is composed of a guaranteed part and the value of the FDB. While the value of the guaranteed part is deterministic and only depends on the time to maturity, the value of the FDB, i.e. the terminal bonus, additionally follows the rise and decline of the assets. For the cliquet-model, we notice that the own funds again follow the movements of the asset process. However, the value of the liabilities increases every year since the cliquet-company has to credit at least the guaranteed rate. It seems that the maturity guarantee is more flexible as the market value of liabilities can decrease in years of low asset returns. However, in years of high asset returns, the liability position in the cliquet-model does not increase as fast as in the maturity-model, which compensates for the missing flexibility to some extent.

Time (t)	0	1	2	3	4	5	6	7	8
Log Returns		0.07	-0.02	0.13	0.22	0.05	0.08	-0.09	0.12
Assets	100	107.46	105.54	120.42	149.71	156.98	170.36	156.34	176.07
Maturity-Model									
E_t	25	28.64	26.54	34.55	49.95	53.05	59.27	51.16	60.69
L_t	75	78.82	79.00	85.88	99.76	103.93	111.10	105.17	115.38
SCR_t	14.33	15.00	15.19	16.21	18.05	18.69	19.73	19.05	20.40
Sol_t	174%	191%	175%	213%	277%	284%	300%	269%	298%
$E_t - SCR_t$	10.67	13.64	11.35	18.33	31.90	34.36	39.54	32.11	40.30
Cliquet-Model									
E_t	25	29.75	27.06	37.07	58.03	62.94	72.53	57.52	71.70
L_t	75	77.71	78.49	83.35	91.68	94.04	97.83	98.81	104.37
SCR_t	20.07	21.62	21.19	24.30	30.45	31.97	34.78	31.73	35.88
Sol_t	125%	138%	128%	153%	191%	197%	209%	181%	200%
$E_t - SCR_t$	4.93	8.13	5.86	12.77	27.58	30.97	37.76	25.79	35.82

Table 4.1.: Values of the sample path

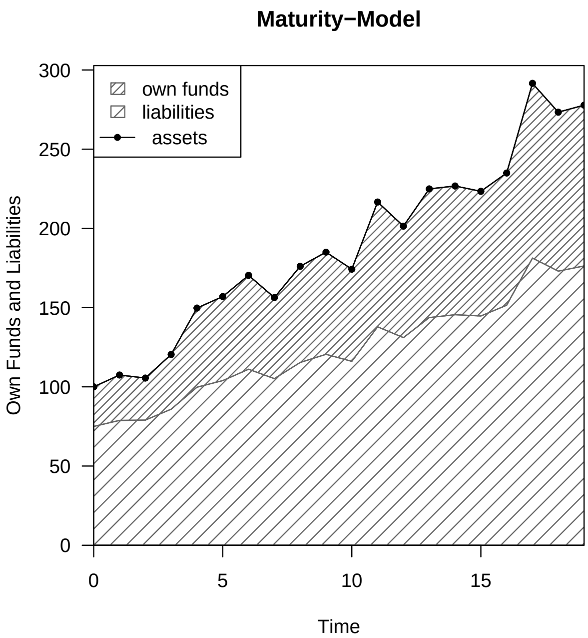


Figure 4.4.: Composition of the asset values from own funds and liabilities

It should be noted that the level of the SCR by itself does not provide enough information to assess the solvency of a company. A high value of the SCR does not pose a problem if the own funds are high as well. Therefore, we always need the own funds together with the SCR in order to draw meaningful conclusions. SCRs and solvency ratios are computed in accord with the formulas of the previous chapter (3.5), (3.6) and (3.15), (3.16). In order to compare the solvency of both models for this particular sample path of the assets, we plot the own funds together with the SCR over the entire lifetime of the contracts in Figure 4.5. The difference of these two quantities is the excess coverage. We also add the solvency ratio to the plot to compare the two companies in terms of their absolute coverage (excess coverage) as well as their relative coverage (solvency ratio). Both

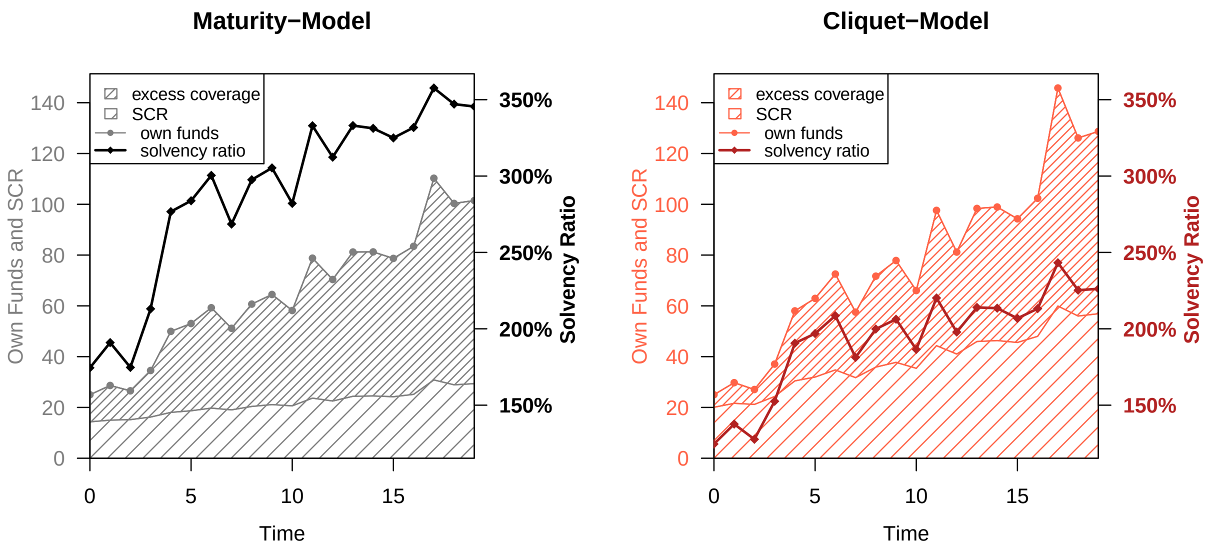


Figure 4.5.: Absolute and relative coverage of the SCR

solvency figures are directly affected by the fluctuation of the assets. In general, rising asset values cause an increase in solvency and vice versa. This behavior is reasonable since sufficiently increasing assets ensure that the guarantee is covered and should thus result in high solvency. Decreasing assets, however, are an indication that equity holders will have to pay for the guarantees. Figure 4.5 shows that the solvency position presents itself differently depending on which solvency figure we consider, which justifies the earlier introduction of the second solvency indicator, the excess coverage.

In relative terms, that is considering the solvency ratio, the two models seem to be very different. The ratios, which are already apart at inception, grow even further apart over time. However, when considering the coverage in absolute numbers, the two models do not seem to differ as much. Especially at the time towards maturity, their solvency positions appear to be rather similar for this sample path. In short, the solvency ratio indicates diverging solvency positions, whereas the excess coverage indicates converging solvency. Checking Table 4.1, we observe that the own funds and the SCR vary stronger in the case of the cliquet-model, whether studied on their own or in terms of the excess coverage. As a result of the assets' increase on average, the cliquet-model manages to catch up to the maturity-model regarding the excess coverage as shown in Figure 4.5. However, since the cliquet-model has to deal with higher capital requirements and since these are located in the denominator of the solvency ratio, it is the ratio of the maturity-model that rises faster. Hence, both ratios rise at a different speed and the maturity-model extends its initial lead over the course of the contracts. The dependence of these two solvency figures on the value of the assets will be studied more in detail later on.

4.3. Time Point Analysis

Now that we have gone through the simulation of one particular path, we proceed to generating a larger number of paths in the style of a Monte Carlo simulation. Specifically, we consider $100,000$ independent paths of the asset process in our analysis. Since we required 19 realizations of a standard normal distribution for one path, we are now in need of $19 \cdot 100,000 = 1,900,000$ realizations. Right before we start, we set the seed of the random number generator to one to create reproducible paths. We arrive at $100,000$ solvency figures for each time point from $t = 1$ to $t = 19$ and for each model. There are different possibilities to organize and analyze this information. For instance, we distinguish between a time point analysis as carried out in this section and a time period analysis performed later on.

As an example, we mainly focus on a time point halfway through the contract, namely $t = 10$. But when suitable, we also include other time points to attain further insights and learn about the sensitivity of our results with respect to the time point. We consider the empirical distributions of the solvency figures (solvency ratio and excess coverage) and study the dependencies of these figures on the value of the assets.

Before having a look at the solvency figures themselves, we depict the empirical distributions and dependencies of the different components, that is liabilities, own funds and SCR, on the asset value. In order to analyze the empirical distributions, we simulate $100,000$ paths of the asset process and obtain $100,000$ values of each quantity. Histograms of these values display the empirical distributions and enable us to compare the two different models. Note that we use the same sample paths for both models in order to ensure comparability. Also, we crop the histograms because realizations at the far end of the distributions are very rare and thus highly dependent on the generated random numbers. This omission of extreme values allows for a better representation of the main part of the distribution.

For the analysis of the dependencies on the asset value, we draw 500 random asset paths under the real-world probability measure and then plot the quantities of interest against the value of the assets to obtain scatter plots. We again focus on time point ten, which is halfway through the contracts. These plots allow us to study the dependencies between the quantities and the value of the assets, which is the only source of randomness our models are subjected to. For each quantity, we integrate the results of both models into one plot. We hereby attain a deeper understanding of how the models function and differ from each other in terms of their solvency positions.

4.3.1. Liabilities

Figure 4.6 shows the empirical distribution and the scatter plot of the liabilities. We notice

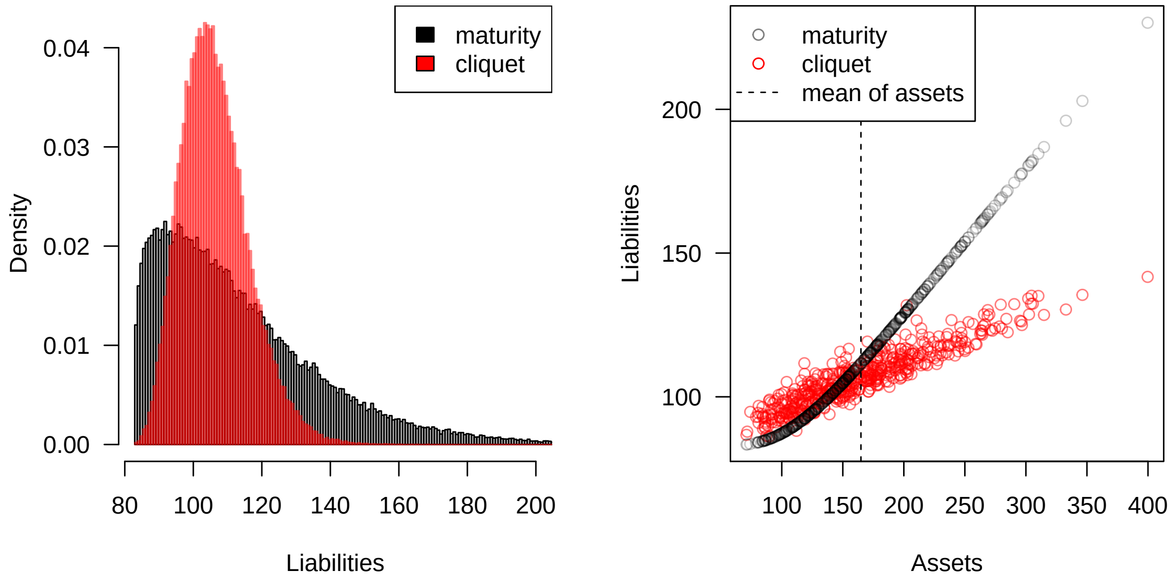


Figure 4.6.: Empirical distribution and scatter plot of the liabilities at time ten

that the values of the maturity-company vary more than those of the cliquet-company. As we observed in the previous section, the market value of the liabilities rises faster for the maturity-company once the assets develop favorably. This observation is linked to the fact that the maturity-company grants its policyholders a higher participation rate in the surplus than the cliquet-company. The 99.9%-quantile of the liabilities in the maturity-company is about 239, while the corresponding quantile of the cliquet-company lies at around 146. Clearly, the value of the liabilities is restricted below by the guarantee given to the policyholders and thus cannot drop beneath a certain level. Namely, the market value of the liabilities does not fall below 82.9 for both companies, which matches the value of the guarantee if no bonus is credited (cf. (4.1)). We need to mention that the observed results are to some extent dependent on the time point we have chosen. In fact, the value of the guarantee in the case of no bonus payments, i.e. the liability position for constantly low asset returns, happens to coincide for both models at $t = 10$.

$$e^{-r(T-t)} L_T^G = e^{-r(T-t)} \alpha A_0 \, e^{r_G T} = \alpha A_0 \, e^{(r_G - r)T} \, e^{rt} = \alpha A_0 \, e^{gt} = 82.9, \tag{4.1}$$

where the last equality holds since $(r_G - r)T + rt = (r_G - r)2t + rt = (2r_G - r)t = gt$ for $t = 10$. To have a comparison with other time points, we present the empirical distributions and scatter plots for time one in Figure 4.7 and for time 19 in Figure 4.8. For time one, we observe that the liabilities of the cliquet-model peak at the value that matches the case in which only the guaranteed rate is granted. Values lower than $\alpha A_0 \, e^{g} = 75.75$ are not possible. All realizations of returns below the guarantee result in the minimum value of the liabilities of 75.75. The lower bound of the maturity-model is smaller, namely $e^{-r \cdot 19} L_T^G = 63.27$. At time point 19, the liabilities of the maturity-model cannot drop below the value of the guarantee, which is $e^{-r} L_T^G = 108.58$. For the cliquet-model, however, we arrive at a bound of $\alpha A_0 \, e^{g \cdot 19} = 90.69$. Clearly, the bound of the cliquet-model can only be reached if the asset returns stay below the guaranteed rate for each of the 19 years.

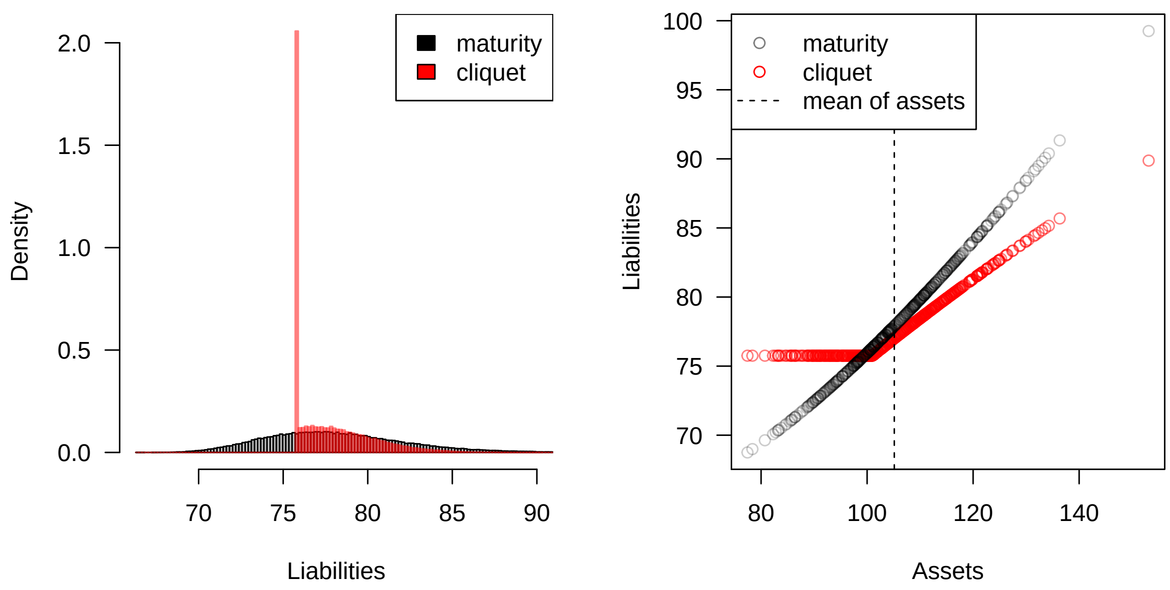


Figure 4.7.: Empirical distribution and scatter plot of the liabilities at time one

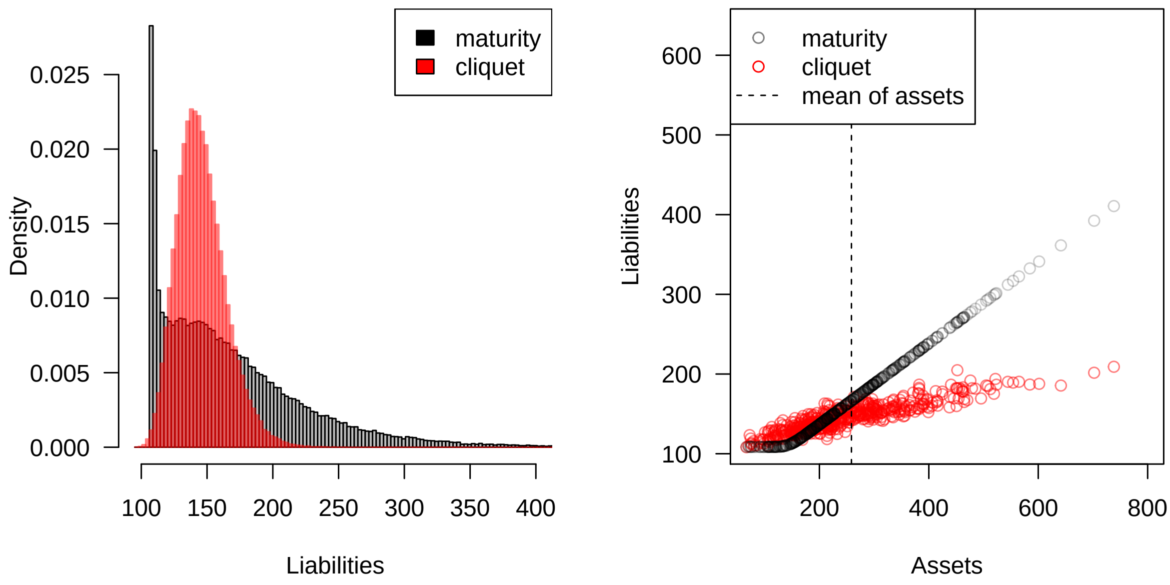


Figure 4.8.: Empirical distribution and scatter plot of the liabilities at time 19

For the maturity-model, it suffices to draw a path of the asset process that fails to exceed the value of the guarantee at time 19. Of course, this is far more likely than hitting the bound of the cliquet-model (probability of 3.8% in comparison to $4.4 \cdot 10^{-7}\,\%$).

Looking at the scatter plot of Figure 4.6, we notice that the points belonging to the maturity-model follow a function, while the points of the cliquet-model form some sort of cloud. This observation is attributed to the fact that the market value of the liabilities of the maturity-model is fully determined by the value of the assets at that specific time point, i.e. the liabilities are not path-dependent. The relation between liabilities and assets was given in (2.5). On the contrary, the valuation formula of the cliquet-model depends on the path of the asset process. In other words, the same asset values at time t can entail different values of liabilities for the cliquet-model depending on the particular paths by which the asset values were reached. The dashed line represents the mean of the assets at time $t = 10$. As the assets are lognormally distributed and thus right-skewed, most of the realizations cluster around the mean and to the left of it, while there are only few realizations at the far right of the x-axis.

For small asset values, the values of the liabilities of the maturity-model are mainly below those of the cliquet-model as they primarily consist of the initially guaranteed part. As shown in (4.1), the values of the liabilities of both models match at time ten if absolutely no bonus is credited, i.e. for constantly low returns on the assets. However, an asset path reaching a low value at time ten may have experienced single years of high asset returns. Such years would immediately raise the value of the liabilities of the cliquet-model as bonus is credited right away. Thus, this explains why many liability values of the cliquet-model are located slightly above the corresponding ones of the maturity-model for small asset values. For increasing asset values, we notice that the slope of the liability values increases in the case of the maturity-model, while the trend of the cliquet-model seems to be constant. Consequently, the liabilities of the two models diverge and, for large asset values, those of the maturity-model are considerably higher than those of the cliquet-model.

In order to find an explanation for the different behaviors of the two models, we study the trends observable in the scatter plot of Figure 4.6. We begin with the maturity-model by recalling the valuation formula for the liabilities given in (2.5) for general t and taking the derivative with respect to the value of the assets A_t. Hereby, we obtain the slope

$$\frac{\partial L_t^M}{\partial A_t} = \frac{\partial \left(e^{-r(T-t)} L_T^G + \delta\alpha\, C\Big(A_t, t, \frac{L_T^G}{\alpha}\Big)\right)}{\partial A_t} = \delta\alpha\, \frac{\partial\, C\Big(A_t, t, \frac{L_T^G}{\alpha}\Big)}{\partial A_t} = \delta\alpha\, \Phi\left(d_1\Big(A_t, t, \frac{L_T^G}{\alpha}\Big)\right), \tag{4.2}$$

where L_t^M marks the value of the liabilities of the maturity-model at time t. The derivative of the first term clearly is zero. It remains to take the derivative of the call price with respect to the value of the underlying. As this is precisely the delta of the call option, we refer to [Shr10, 159] for the last equality. d_1 is given in (2.8). δ, α as well as the CDF of the standard normal distribution all take values between zero and one. Therefore, the slope of the liabilities of the maturity-model is also in $[0, 1]$. In essence, the market value of the liabilities increases with the assets because the value of the terminal bonus, the FDB, increases. The rise in liabilities cannot surpass the rise in assets since it is not certain if the policyholders will receive a bonus and even if they receive one, they are only entitled to a fraction of the assets. For the second derivative, we get

$$\frac{\partial^2 L_t^M}{\partial A_t^2} = \delta\alpha\, \frac{\partial\, \Phi\left(d_1\Big(A_t, t, \frac{L_T^G}{\alpha}\Big)\right)}{\partial A_t} = \delta\alpha\, \phi\left(d_1\Big(A_t, t, \frac{L_T^G}{\alpha}\Big)\right) \frac{1}{\sigma\sqrt{T-t}}\, \frac{1}{A_t},$$

where ϕ is the probability density function of the standard normal distribution. As expected by looking at Figure 4.6, the second derivative of the liabilities of the maturity-model is positive or, equivalently, the slope is increasing in the value of the assets. This behavior is linked to the fact that the payment of a terminal bonus becomes more and more certain when the assets rise. That is to say, the uncertainty in the call option, which stalls the increase of the liabilities, diminishes and finally becomes negligible with rising asset values. Remarkably, the increase of the slope ceases when the assets rise and we find the slope to converge.

Next, we consider the limiting cases of relatively low and high asset values with reference to the convergence of the call price. This will allow us to deal with simplified formulas and thereby provide us with a deeper understanding of the above mentioned convergence of the slope and the maturity-model in general. If the assets take on very small values, then the call is expected to expire out of the money, i.e. it pays out zero at maturity. Policyholders do not receive a terminal bonus. In this case, the call price is roughly zero.

$$C\left(A_t, t, \frac{L_T^G}{\alpha}\right) \approx 0 \tag{4.3}$$

The value of the liabilities of the maturity-model as given in expression (2.5) reduces to the guaranteed part, namely

$$L_t^M = e^{-r(T-t)} L_T^G,$$

which clearly does not depend on the assets. Checking with Figure 4.6, we notice that the liabilities of the maturity-model indeed tend to the constant $e^{-r(T-10)} L_T^G = 82.9$ for small asset values. This marks the lower bound of the liabilities as they can never drop below the value of the guarantee.

If the asset value is very large, however, the call is expected to expire in the money and thus resembles a forward contract. There will most likely be a positive payment of $A_T - L_T^G/\alpha$ at maturity. The call price can then be approximated by

$$C\left(A_t, t, \frac{L_T^G}{\alpha}\right) \approx A_t - e^{-r(T-t)} \frac{L_T^G}{\alpha}. \tag{4.4}$$

Substituting this approximation into the valuation formula of the maturity-model in (2.5) leads to a value of the liabilities of

$$\begin{aligned} L_t^M &= e^{-r(T-t)} L_T^G + \delta\alpha \left(A_t - e^{-r(T-t)} \frac{L_T^G}{\alpha}\right) \\ &= e^{-r(T-t)} L_T^G + \delta\alpha\, A_t - \delta\, e^{-r(T-t)} L_T^G \\ &= \delta\alpha\, A_t + (1-\delta) e^{-r(T-t)} L_T^G. \end{aligned}$$

For this approximation, we determine the slope of the liabilities to be $\delta\alpha = 0.51$ by taking the derivative with respect to the assets A_t. Thus, if the terminal bonus can be assumed to

be positive due to high asset values, the liabilities of the maturity-model tend to exhibit a constant slope. That is, for high asset values, the liabilities are linear in the assets, which matches Figure 4.6. An increase of one unit in the assets amounts to an increase of 0.51 in the liabilities since the policyholders participate in favorable asset developments according to their initial contribution to the balance sheet total (α) and the participation rate ensuring a fair contract (δ).
Figure 4.9 provides values of the call price at time ten and their approximation as in (4.3) and (4.4) for several asset values. In particular, we consider values of the assets reaching from 70 to 400 since this is the range we observe in our simulated paths. From the table and the graph, we can infer the range of asset values that allows for a reasonable approximation. This specifies the earlier vague description of very small and very large asset values.

A_t	$C\left(A_t, t, \frac{L_T^G}{\alpha}\right)$	approximation
70	0.92	0
80	2.36	0
90	4.87	0
100	8.62	0
150	43.07	39.48
160	51.97	49.48
170	61.20	59.48
200	90.03	89.48
300	189.50	189.48
400	289.48	289.48

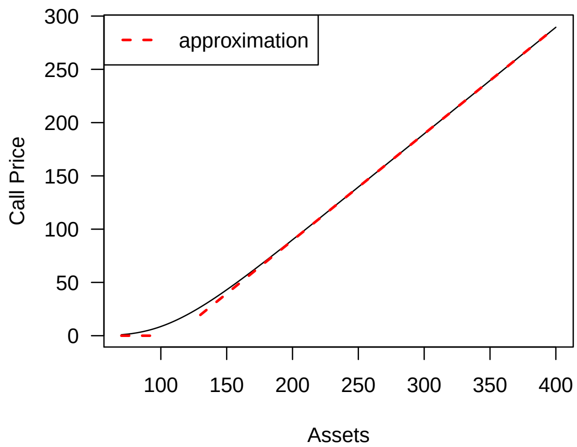


Figure 4.9.: Approximation of the call price at time ten for different asset values

For the cliquet-model, we are dealing with a whole cloud of points. Even though we cannot compute a slope as for the maturity-model, the cloud clearly follows some kind of trend, which we would like to determine. For that purpose, recall the valuation formula for the liabilities of the cliquet-model given in (2.26) for some general time t. The value of the liabilities always stays above a lower bound of

$$L_t^C = \alpha A_0 \, e^{gt} \tag{4.5}$$

due to the guarantee granted at inception. L_t^C stands for the value of the liabilities of the cliquet-company at time t. For $t = 10$, this lower bound happens to be 82.9. The equivalence to the lower bound of the maturity-model is a characteristic of this specific time point as shown in (4.1). Clearly, this value is fixed by the guarantee and does not depend on the asset value A_t. Paths of the asset process with consistently low returns lead to a market value of the liabilities that is close to the lower bound. For making out the trend of the liabilities, however, we consider the case in which the returns of the assets are consistently high, i.e. bonus is granted each year. In such cases, the plus operator conveniently disappears and the value of the liabilities is simplified to

$$\begin{aligned} L_t^C &= \alpha A_0 \, e^{\sum_{i=1}^t \left(g+\beta(\rho_i - g)\right)} = \alpha A_0 \, e^{(1-\beta)gt} \left(e^{\sum_{i=1}^t \rho_i}\right)^\beta \\ &= \alpha A_0^{1-\beta} \, e^{(1-\beta)gt} \Big(\underbrace{A_0 \, e^{\sum_{i=1}^t \rho_i}}_{=A_t} \Big)^\beta = \alpha \left(A_0 \, e^{gt}\right)^{1-\beta} A_t^\beta . \end{aligned} \tag{4.6}$$

α is part of this formula since it indicates how much the policyholders contributed to the company's balance sheet total at inception of the contracts. β controls to what extent the policyholders get to participate in asset returns above the guaranteed level. It was chosen such that the contracts were fair at inception. The chances that this approximation is close to the actual value of the liabilities increase when considering very high asset values. On the far right side of the x-axis in Figure 4.6, asset values are so high that the corresponding paths have to mostly consist of returns above the guaranteed rate. This necessity explains why the cloud is less dispersed compared to smaller asset values. At the same time, it presents the opportunity to compute a slope. Taking the derivative of (4.6), we obtain

$$\frac{\partial L_t^C}{\partial A_t} = \alpha \left(A_0 \, e^{gt}\right)^{1-\beta} \beta \, A_t^{\beta-1}.$$

This formula yields slopes roughly between 0.14 and 0.17 for asset values between 300 and 400, which correspond to the high end of the asset range we observe in the scatter plot of Figure 4.6. We hereby get a rough idea of the underlying trend across all asset values. Alternatively, we could simply run a linear regression in order to work out the trend of the cloud. This second approach leads to a trend of about 0.17 as well.

4.3.2. Own Funds

Next, we move on to the analysis of the own funds. Figure 4.10 shows the empirical distribution and the scatter plot of the own funds. We notice that the values of the own funds are less widespread for the maturity-company than for the cliquet-company. While the difference on the left tail of the distributions appears to be minor, the cliquet-company

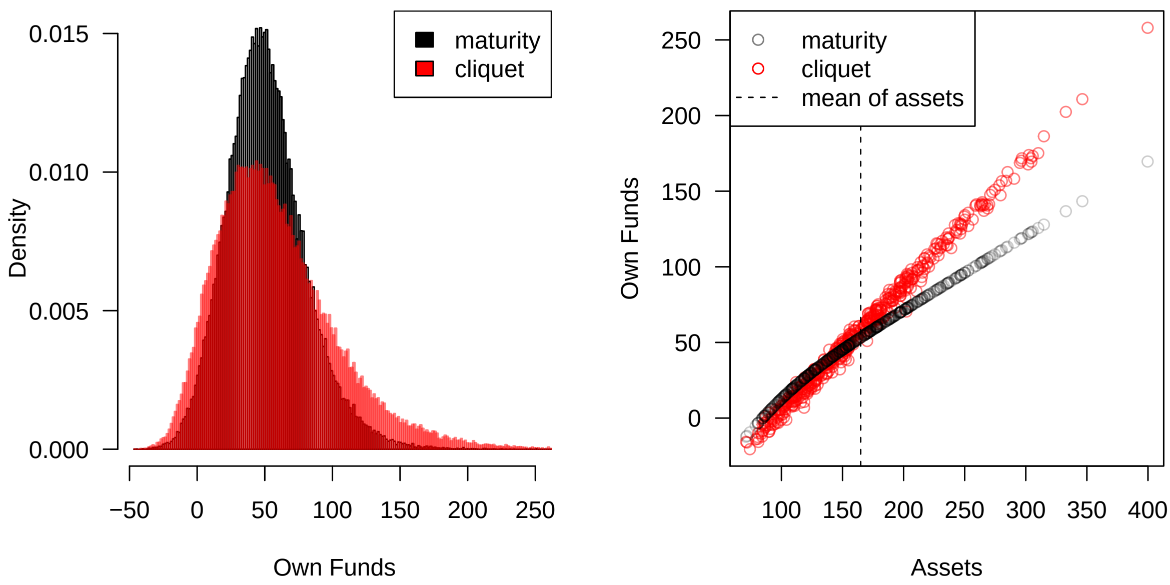


Figure 4.10.: Empirical distribution and scatter plot of the own funds at time ten

clearly has a heavier right tail in terms of its own funds. Thus, the two companies are rather similar in their own funds in bad times, i.e. when own funds are low, but differ significantly in good times, i.e. when own funds are high. We will encounter this effect more often when we extend our analysis to other key figures. Indeed, it makes sense intuitively since low own funds are linked to a low performance of the assets, in which case both companies merely grant the guaranteed interest rate. The dissimilarity in the design of the bonus system becomes secondary. However, when the assets perform well, both companies are obliged to make high bonus payments. It is important to note that, due to differing model setups and thus valuation formulas, this effect is more or less distinct depending on the time point (cf. Figure 4.11). Furthermore, the risk of insolvency is greater for the cliquet-company than for the maturity-company at time ten. Specifically, 2.5% of the underlying paths result in negative own funds for the maturity-company, in contrast to 5.4% for the cliquet-company.

For the analysis of the scatter plot in Figure 4.10, we recall that the own funds are simply the residual between asset value and value of the liabilities. Therefore, in general, the conclusions drawn for the liabilities can be reversed to match the own funds. Consequently, the values of the own funds of the cliquet-model lie below those of the maturity-model for small asset values. Moreover, the trend of the own funds of the cliquet-model seems to be constant, while the slope of the maturity-model decreases. This leads to diverging values of own funds when considering high values of the assets. In these cases, the cliquet-company

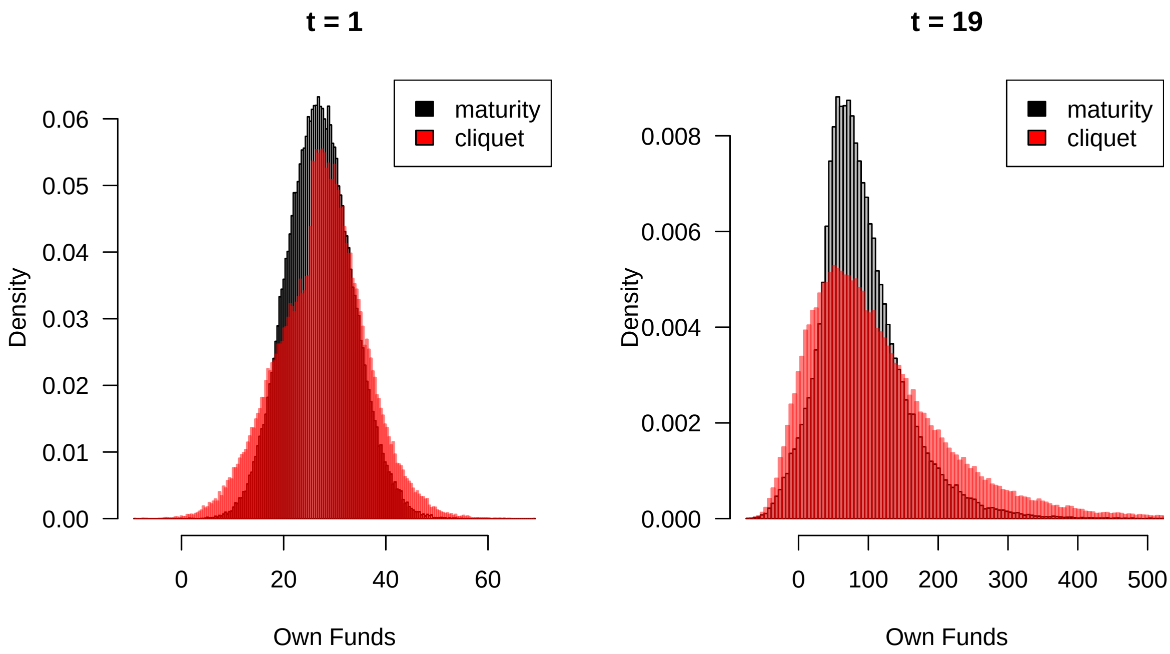


Figure 4.11.: Empirical distributions of the own funds at time one and 19

holds considerably more own funds than the maturity-company.

Similarly to (4.2), the slope of the own funds of the maturity-model amounts to

$$\frac{\partial E_t^M}{\partial A_t} = \frac{\partial\Big(A_t - e^{-r(T-t)}L_T^G - \delta\alpha\, C\big(A_t, t, \frac{L_T^G}{\alpha}\big)\Big)}{\partial A_t} = 1 - \delta\alpha\,\frac{\partial\, C\big(A_t, t, \frac{L_T^G}{\alpha}\big)}{\partial A_t}$$
$$= 1 - \delta\alpha\,\Phi\left(d_1\left(A_t, t, \frac{L_T^G}{\alpha}\right)\right), \tag{4.7}$$

which is again in $[0,1]$. The own funds increase together with the assets. However, they cannot increase more than the assets because the liabilities increase as well. We compute the second derivative to be

$$\frac{\partial^2 E_t^M}{\partial A_t^2} = -\delta\alpha\,\frac{\partial\,\Phi\Big(d_1\big(A_t, t, \frac{L_T^G}{\alpha}\big)\Big)}{\partial A_t} = -\delta\alpha\,\phi\left(d_1\left(A_t, t, \frac{L_T^G}{\alpha}\right)\right)\frac{1}{\sigma\sqrt{T-t}}\,\frac{1}{A_t},$$

which is negative. Therefore, as observed earlier, the slope of the own funds of the maturity-model is indeed decreasing with rising asset values.

For the maturity-model, we consider again the limiting cases of the call price as in (4.3) and (4.4). In the case of very small asset values, we can express the own funds of the maturity-model as

$$E_t^M = A_t - e^{-r(T-t)} L_T^G.$$

The value of the assets is only reduced by the value of the guarantee as we assume that there will not be a terminal bonus for the policyholders. Thus, an increase of the assets by one unit directly leads to an equivalent increase of the own funds. If the assets cannot cover the guarantee, then the own funds are negative and we declare the company to be insolvent. This is the case for some of the low asset values shown in Figure 4.10.
Very large asset values result in own funds of about

$$\begin{aligned} E_t^M &= A_t - e^{-r(T-t)} L_T^G - \delta\alpha \left(A_t - e^{-r(T-t)} \frac{L_T^G}{\alpha} \right) \\ &= A_t - e^{-r(T-t)} L_T^G - \delta\alpha\, A_t + \delta\, e^{-r(T-t)} L_T^G \\ &= (1-\delta\alpha) A_t - (1-\delta) e^{-r(T-t)} L_T^G. \end{aligned}$$

Clearly, the slope with respect to A_t equals $1 - \delta\alpha = 0.49$. If the terminal bonus is expected to be positive due to high asset values, the own funds rise by 0.49 units for every additional unit of the assets. The slope declined in contrast to the above case of very small asset values since a part of the good asset development now has to be passed on to policyholders.
For the cliquet-model, we consider again the two extremes of no bonus and bonus each year up to time t, respectively. As the liabilities of the cliquet-model always stay above $\alpha A_0\, e^{gt}$ (cf. (4.5)), the own funds cannot exceed

$$E_t^C = A_t - \alpha A_0\, e^{gt}. \tag{4.8}$$

Realizations of the own funds may get close to this limit for small asset values. Large values of the assets, however, can only be attained by surpassing the guaranteed rate of return and thus crediting bonus in some of the years. Therefore, the own funds stay well below the limit if the asset values are large. We are interested in the linear trend of the own funds with respect to the assets. As the slope of the limiting value (4.8) equals one, the trend we are looking for is expected to be smaller than one. If the returns on the assets are consistently high, i.e. bonus is credited every year, the own funds of the cliquet-model are simply

$$E_t^C = A_t - \alpha A_0\, e^{\sum_{i=1}^{t} \left(g + \beta(\rho_i - g)\right)} = A_t - \alpha \left(A_0\, e^{gt} \right)^{1-\beta} A_t^{\beta}, \tag{4.9}$$

where the computation steps are identical to those of (4.6). The derivative of (4.9) is

$$\frac{\partial E_t^C}{\partial A_t} = 1 - \alpha \left(A_0\, e^{gt}\right)^{1-\beta} \beta\, A_t^{\beta-1},$$

which yields slopes between 0.83 and 0.86 for asset values at the high end of the asset range of the scatter plot in Figure 4.10 (asset values of 300 to 400).
Performing a linear regression leads to an estimated trend of 0.83. As own funds and liabilities add up to the value of the assets, the trends of the two quantities with respect to the assets add up to one.

4.3.3. SCR

Next, we study the empirical distribution and the scatter plot of the SCR, which are shown in Figure 4.12. Considering the empirical distribution, we observe that the value

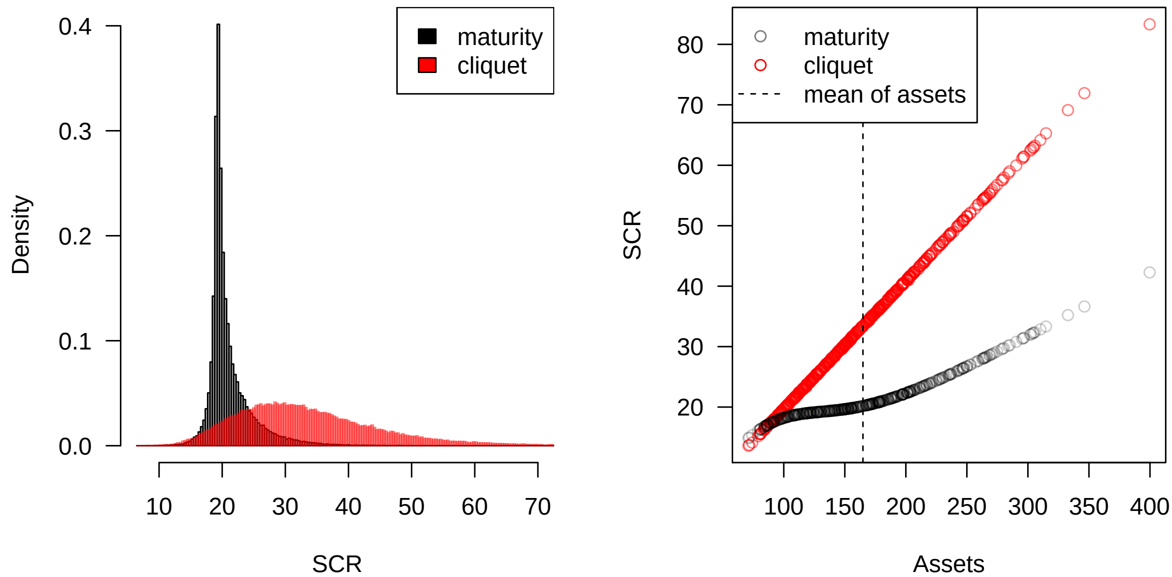


Figure 4.12.: Empirical distribution and scatter plot of the SCR at time ten

of the SCR varies more in the case of the cliquet-model as compared to the maturity-model. While about 70% of the realizations lead to a SCR between 18 and 22 for the maturity-company, the distribution of the SCR of the cliquet-company is spread out over a wider range of values and peaks at around 30. On the low end, the companies have similar SCRs, namely the 0.1%-quantile is 12.5 for the maturity-company and 10.9 for the

cliquet-company.
From the scatter plot in Figure 4.12, we notice that the SCR of the maturity-model is uniquely determined by the value of the assets at time ten via the formula provided earlier in (3.5). As for liabilities and own funds, the SCR of the cliquet-model is spread around a general trend since it depends on the entire path of the asset process as given in (3.15). However, the level of dispersion appears to be much lower for the SCR when compared to liabilities and own funds. Furthermore, we note that the levels of the SCR of both models are very similar for small asset values, while they diverge for large asset values. Since the SCR of the maturity-model tends to a smaller slope in comparison to the trend of the cliquet-model, the SCR of the cliquet-model is higher in most cases with the gap increasing in the asset value.
For a more rigorous analysis, we turn to the corresponding formulas specifying the SCR and begin with the maturity-model. We find the slope observable in Figure 4.12 by taking the derivative of the SCR with respect to the assets. Since the SCR is simply the difference in own funds caused by the stress scenario, it suffices to take the derivative of the own funds as well as the stressed own funds. The derivative of the own funds was computed earlier in (4.7). As the own funds are a function of the asset value and the stressed own funds are a function of the stressed asset value ($sf_A \cdot A_t$), the derivative of the own funds after the influence of the stress scenario, denoted as $\tilde{E_t^M}$, can be inferred to be

$$\frac{\partial \tilde{E_t^M}}{\partial A_t} = sf_A \cdot \left[1 - \delta\alpha\, \Phi\left(d_1\Big(sf_A \cdot A_t, t, \frac{L_T^G}{\alpha} \Big) \right) \right] \tag{4.10}$$

by application of the chain rule. Joining (4.7) and (4.10), the derivative of the SCR becomes

$$\begin{aligned}
\frac{\partial\, SCR_t^M}{\partial A_t} &= \frac{\partial \left(E_t^M - \tilde{E_t^M} \right)}{\partial A_t} = \frac{\partial E_t^M}{\partial A_t} - \frac{\partial \tilde{E_t^M}}{\partial A_t} \\
&= 1 - \delta\alpha\, \Phi\left(d_1\Big(A_t, t, \frac{L_T^G}{\alpha} \Big) \right) - sf_A \cdot \left[1 - \delta\alpha\, \Phi\left(d_1\Big(sf_A \cdot A_t, t, \frac{L_T^G}{\alpha} \Big) \right) \right] \\
&= 1 - sf_A - \delta\alpha \left[\Phi\left(d_1\Big(A_t, t, \frac{L_T^G}{\alpha} \Big) \right) - sf_A \cdot \Phi\left(d_1\Big(sf_A \cdot A_t, t, \frac{L_T^G}{\alpha} \Big) \right) \right]. \qquad (4.11)
\end{aligned}$$

This formula is difficult to interpret as it contains not only the CDF of the normal distribution but also the function d_1 as specified in (2.8). Since both of these functions are monotonically increasing in their first arguments and the stress factor lies in $(0, 1)$, the square brackets are positive and the third term of (4.11) is negative. Thus, the sign of the

derivative of the SCR depends on the size of the third term compared to the first part, namely $1 - sf_A = 0.22$. Later on, we will see that the slope can indeed be negative for certain values of the assets.
In order to simplify the formulas and to get a better idea of the sensitivity of the SCR with respect to the assets, we again consider the limiting cases of very small and very large asset values. For very small asset values, we approximate the call price by zero as in (4.3). Clearly, if the asset value is small enough to ensure a good approximation (cf. Figure 4.9), then the same applies to the stressed asset value as it is even smaller. This leads to a SCR of

$$\begin{aligned} SCR_t^M &= E_t^M - \tilde{E_t^M} \\ &= A_t - e^{-r(T-t)} L_T^G - \left(sf_A \cdot A_t - e^{-r(T-t)} L_T^G \right) \\ &= (1 - sf_A) \cdot A_t. \end{aligned}$$

In essence, the SCR of the maturity-model is linear in the assets for small asset values. The corresponding slope is $1 - sf_A = 0.22$ because the SCR, which is the difference in own funds, equals the difference in assets if a stress occurs and the chance of a bonus is already ruled out due to low asset values. In this case, the market value of the liabilities cannot absorb any of the stress since it merely consists of the guarantee that cannot be altered. Therefore, an increase of one unit in the assets translates into an increase of 0.22 units in the SCR as higher asset values result in higher absolute losses in the case of a stress.
For very large asset values, we use the approximation in (4.4). Since the stress scenario reduces the value of the assets, we need to check if the value of the stressed assets is large enough for a good approximation. If this is the case, then the SCR is about

$$\begin{aligned} SCR_t^M &= E_t^M - \tilde{E_t^M} \\ &= (1 - \delta\alpha) A_t - (1 - \delta)\, e^{-r(T-t)} L_T^G - \left((1 - \delta\alpha)\, sf_A \cdot A_t - (1 - \delta)\, e^{-r(T-t)} L_T^G \right) \\ &= (1 - \delta\alpha)(1 - sf_A) \cdot A_t, \end{aligned}$$

which implies a slope of $(1 - \delta\alpha)(1 - sf_A) = 0.11$. Thus, if the payment of a terminal bonus is basically certain due to high asset values, then an increase of one unit in the assets translates into an increase of 0.11 units in the SCR. This increase in SCR is lower than in the above case of no bonus because the existence of a terminal bonus means that a possible loss in assets due to stress is partly absorbed by the bonus. Equity holders do not have to compensate for the loss themselves because the value of the policyholders' bonus diminishes. As policyholders are entitled to a proportion of $\delta\alpha$ of the call, the factor concerning equity holders and, more specifically, the SCR is $1 - \delta\alpha$. These limiting cases explain the slope of the SCR of the maturity-model for extreme values of the assets. For values of the assets for which the call price lies somewhere between (4.3) and (4.4), the effects described above intermingle. The liabilities absorb some of the stress, however, not as effectively as for large asset values because the assets may develop in such a way that the terminal bonus turns out to be zero and all of the loss has to be covered by the equity holders after all.

Next, we study the SCR of the cliquet-model. Recall that a sudden loss in assets as modeled by the stress scenario results in an almost equal decrease in own funds. In particular, the SCR of the cliquet-model is given by

$$SCR_t^C = (1 - sf_A)A_t - (1 - sf_L)L_t^C,$$

which was derived in (3.15). We can immediately see that the first term is linear in the assets with slope $1 - sf_A = 0.22$. The value of the liabilities including the associated dispersion is only factored in by about two percent. This explains why we do not see a cloud of values for the SCR as in the cases of liabilities and own funds. Furthermore, the parallels to the maturity-model now become apparent. Except for the additional minor term taking into account the liabilities of the cliquet-model, $(1-sf_L)L_t^C$, the SCRs of both models are simply $(1 - sf_A)A_t$ for small asset values. While the representation does not change for the cliquet-model when the value of the assets increases, the maturity-model is affected by the terminal bonus, which allows for a lower SCR.

Additionally, we provide scatter plots for the time points one and 19 in Figure 4.13 to investigate the bend in the SCR of the maturity-model. As the earlier discussion on the SCRs was led for some general time point t, it applies to the time points one and 19 as well. We notice that the bend becomes more distinct towards maturity, which is why we focus on the time point 19. For time 19, the slope of the SCR of the maturity-model tends to $1 - sf_A = 0.22$ and $(1 - \delta\alpha)(1 - sf_A) = 0.11$ for the two limiting cases as represented in the plot by dotted lines. On the low end of the assets, the slope pretty much coincides with the trend of the SCR of the cliquet-model. We are now interested in the bend of the SCR of the maturity-model that is observable between asset values of about 100 and 200. In contrast to time ten, the slope of the SCR of the maturity-model at time 19 even grows negative for asset values in this region. That is to say, the negative term at the end of expression (4.11) outweighs the positive first part of $1-sf_A = 0.22$. The property that the

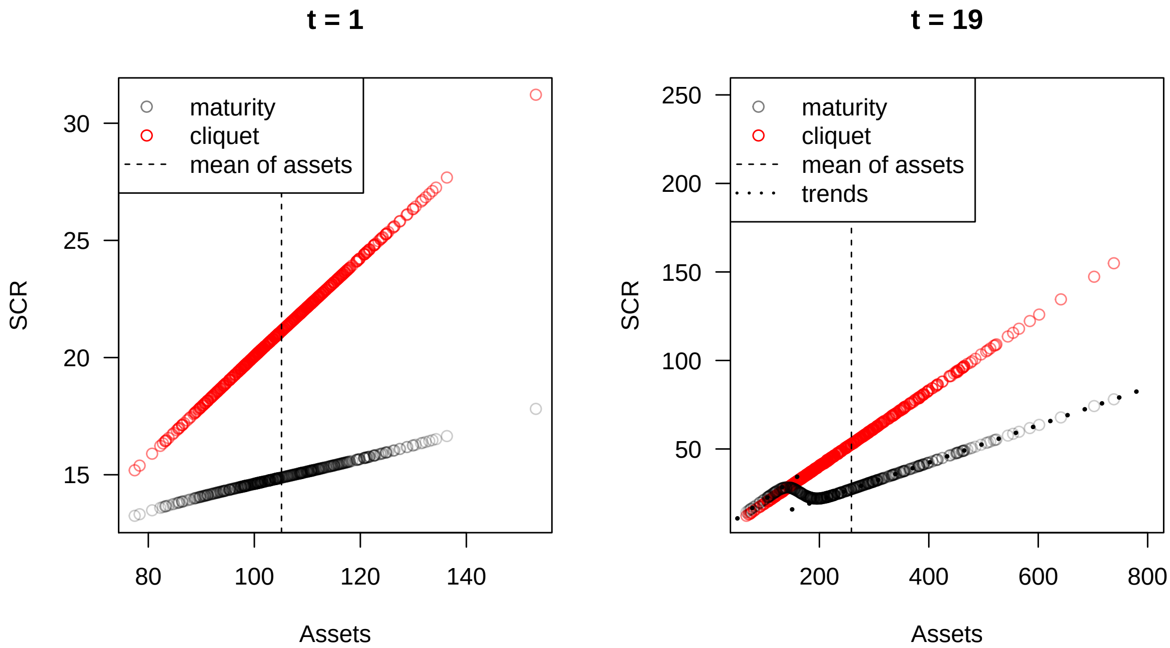


Figure 4.13.: Scatter plots of the SCR at time one and 19

slope decreases from its initial value of 0.22 is related to the fact that the terminal bonus, as it becomes more likely, can take over some of the stress related loss in assets, thereby relieving the own funds. The bend, i.e. the change between decreasing and increasing SCR, stems from the bend observable in the own funds in Figure 4.10. For certain asset values, the value of the own funds is already to the right of the bend changing to a lower slope, while the value of the stressed own funds is still to the left of the bend with a higher slope. Once both positions pass the bend, i.e. a certain asset value is passed, they increase at the same rate, which leads to the constant increase of SCR as in our limiting case for large asset values.

For times closer to maturity, the range of asset values likely to be observed at maturity diminishes. When comparing two different points in time, the later time point has a wider range of asset values for which a terminal bonus is either almost certain or almost impossible. This leaves a smaller range of asset values to form the bend, making it more distinct. In other words, the approximation via the limiting cases can be extended further into the bend when time passes on, which causes more abrupt transitions.

4.3.4. Solvency Ratio

Next, we have a look at the empirical distribution and the scatter plot of the solvency ratio in Figure 4.14. In general, the plots indicate that the cliquet-company finds itself in

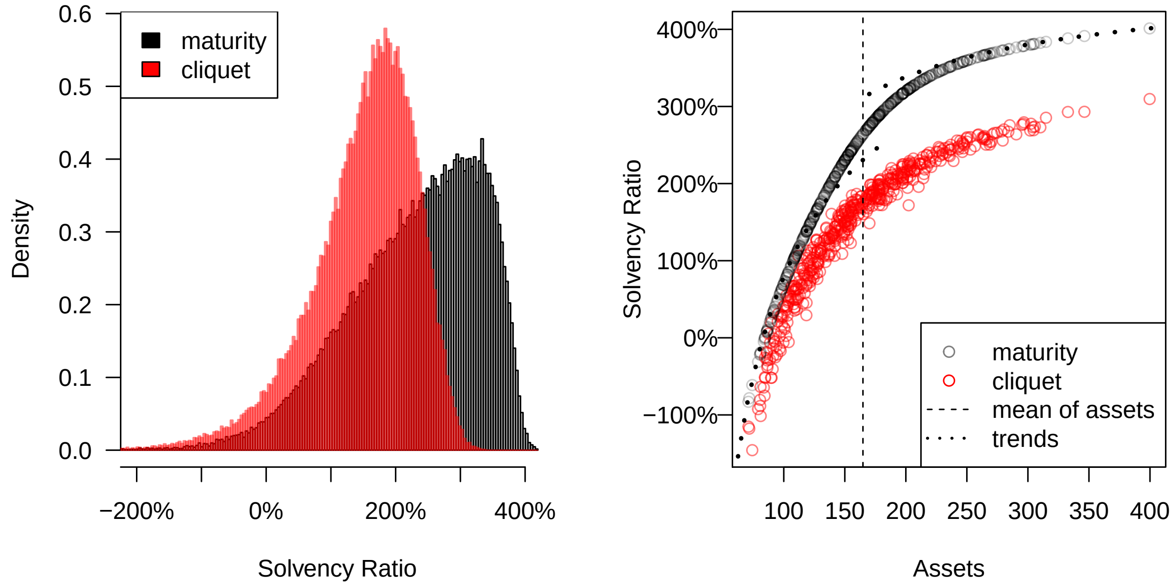


Figure 4.14.: Empirical distribution and scatter plot of the solvency ratio at time ten

a weaker solvency position than the maturity-company. The distribution of the solvency ratio is skewed to the left. Consequently, there is a huge downside potential, even more so for the cliquet-company than for the maturity-company. Also, the maturity-company allows for higher solvency ratios when considering the most favorable paths. In essence, we are interested in the amount of paths that lead to a solvency ratio below 100%. While 11.9% of the paths are critical in this sense for the maturity-company, it is 23.2% of the paths for the cliquet-company.

When evaluating the scatter plot in Figure 4.14, we notice that the ratios of both models are closer together for asset values at the lower end of the axis than for high asset values. Furthermore, it seems that the ratio of the cliquet-model always stays below the one of the maturity-model. The ratios rise with increasing value of the assets, but the rate at which they rise diminishes. As the trend of the ratio of the cliquet-model decreases faster than the one of the maturity-model, the ratios drift apart such that the gap amounts to about 100 percentage points for large asset values. Nevertheless, the difference between the ratios varies widely since the values of the cliquet-model form a cloud. That is, the difference depends on the exact path of the asset process. As the dispersion of the cliquet-company's SCR was minimal, the cloud in the solvency ratio clearly originates from the cloud observable in the own funds. Figure 4.15 shows the scatter plots for the time points one and 19 for comparison.

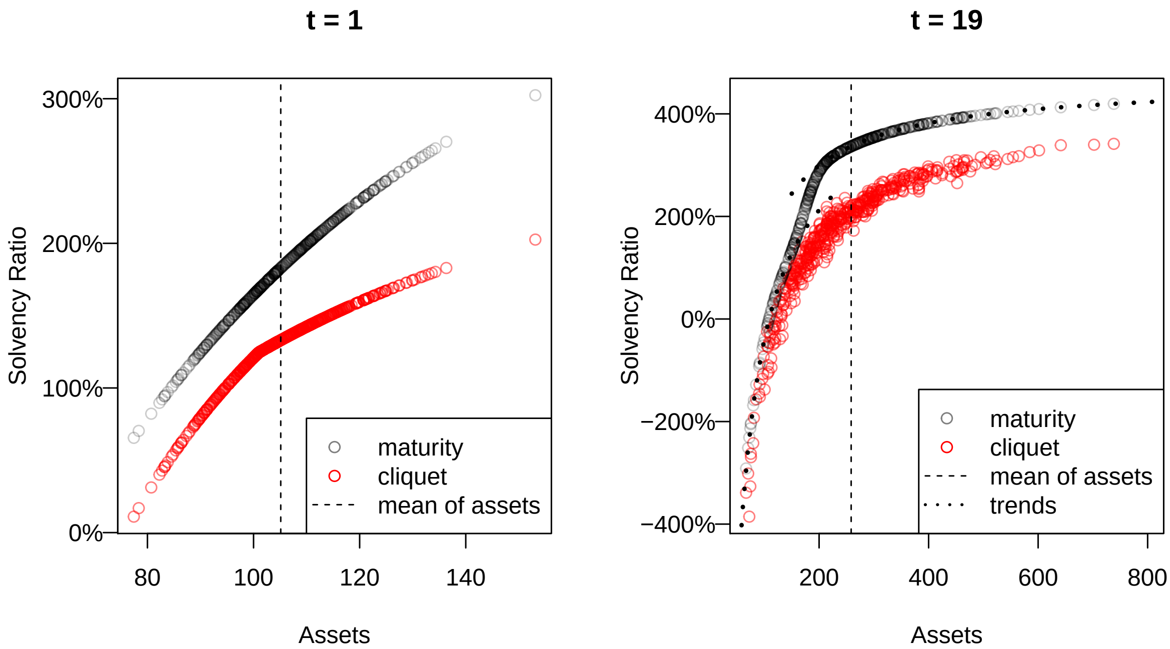


Figure 4.15.: Scatter plots of the solvency ratio at time one and 19

For the analysis of the maturity-model, we take the derivative of the solvency ratio as given in (3.6) with respect to the assets. As the derivatives of own funds and SCR were computed earlier in (4.7) and (4.11), we can simply put the results together to obtain

$$\frac{\partial\, Sol_t^M}{\partial A_t} = \frac{\partial\, \frac{E_t^M}{SCR_t^M}}{\partial A_t} = \frac{\frac{\partial E_t^M}{\partial A_t} \cdot SCR_t^M - E_t^M \cdot \frac{\partial\, SCR_t^M}{\partial A_t}}{\left(SCR_t^M\right)^2}$$
$$= \frac{\left[1 - \delta\alpha\, \Phi\left(d_1\left(sf_A \cdot A_t, t, \frac{L_T^G}{\alpha}\right)\right)\right] sf_A \cdot E_t^M - \left[1 - \delta\alpha\, \Phi\left(d_1\left(A_t, t, \frac{L_T^G}{\alpha}\right)\right)\right] \tilde{E_t^M}}{\left(E_t^M - \tilde{E_t^M}\right)^2}, \tag{4.12}$$

where E_t^M is specified in (2.6) and $\tilde{E_t^M}$ in (3.4). Clearly, this formula is hard to interpret, which is why we again consider the limiting cases of very small and large asset values. They are depicted in the scatter plot of Figure 4.14 as dotted lines. Using the approximation in

(4.3), the solvency ratio of the maturity-model is roughly

$$Sol_t^M = \frac{A_t - e^{-r(T-t)} L_T^G}{(1-sf_A)A_t} = \frac{1}{1-sf_A} - \frac{e^{-r(T-t)} L_T^G}{(1-sf_A)A_t} \tag{4.13}$$

for small asset values. Hence, the ratio tends towards negative infinity when the assets tend towards zero. The derivative of this approximation is

$$\frac{\partial\, Sol_t^M}{\partial A_t} = \frac{\partial \left(\frac{1}{1-sf_A} - \frac{e^{-r(T-t)} L_T^G}{(1-sf_A)A_t} \right)}{\partial A_t} = \frac{e^{-r(T-t)} L_T^G}{(1-sf_A)A_t^2} \, .$$

As expected, the slope is positive and decreases with rising value of the assets. This means that a slight increase in the assets can make a lot of difference for the solvency ratio when the assets are low, but the effect gets less prominent when the assets increase. For large asset values, we employ the approximation in (4.4) and arrive at

$$Sol_t^M = \frac{(1-\delta\alpha)A_t - (1-\delta)\, e^{-r(T-t)} L_T^G}{(1-\delta\alpha)(1-sf_A)A_t} = \frac{1}{1-sf_A} - \frac{(1-\delta)\, e^{-r(T-t)} L_T^G}{(1-\delta\alpha)(1-sf_A)A_t} \, ,$$

which tends to $1/(1-sf_A) = 464\%$ as the value of the assets increases. Therefore, the solvency ratio of the maturity-model can never exceed 464% no matter how well the assets develop. We compute the derivative of this approximation to be

$$\frac{\partial\, Sol_t^M}{\partial A_t} = \frac{\partial \left(\frac{1}{1-sf_A} - \frac{(1-\delta)\, e^{-r(T-t)} L_T^G}{(1-\delta\alpha)(1-sf_A)A_t} \right)}{\partial A_t} = \frac{(1-\delta)\, e^{-r(T-t)} L_T^G}{(1-\delta\alpha)(1-sf_A)A_t^2} \, ,$$

which is again positive and decreasing for rising asset values. Note that the approximation for large asset values assumes a positive terminal bonus, which means that bonus to policyholders can be cut if the assets develop unfavorably in future years. The equity holders would not have to bear the loss themselves. As a result, the level of the solvency ratios is generally higher than for the approximation in which we assume the assets to be too low for granting a terminal bonus.
For the cliquet-model, we recall the composition of the solvency ratio given in (3.16).

$$Sol_t^C = \frac{A_t - L_t^C}{(1-sf_A)A_t - (1-sf_L)L_t^C}$$

We will not derive the trend that describes the direction of the cloud, but rather point out a resemblance to formula (4.13), which approximates the ratio of the maturity-model for small asset values. When discussing the SCR of the cliquet-model in the previous section, we discovered that the SCR is mainly driven by $(1 - sf_A)A_t$ because the second term, $(1 - sf_L)L_t^C$, is minor in comparison. If we simply neglect the second term in the representation of the SCR, then the solvency ratio of the cliquet-model takes on the form

$$Sol_t^C \approx \frac{1}{1 - sf_A} - \frac{L_t^C}{(1 - sf_A)A_t}.$$

This formula is identical to the one of the maturity-model in (4.13) if the assets developed so poorly that the cliquet-company only gave the guaranteed interest rate each year. Namely, $L_t^C = \alpha A_0\, e^{gt}$ equals $e^{-r(T-t)}L_T^G$ in this case at time ten as shown in (4.1). This explains why the ratio of the cliquet-model can be very similar to the ratio of the maturity-model for certain paths of the asset process that result in low asset values at time ten. Specifically, paths in which the cliquet-company grants little or no bonus lead to similar solvency ratios. Once the value of the assets increases, the chances of drawing a path that yields no bonus to policyholders disappear. Consequently, the value of the liabilities is greater than in the case of no bonus and the solvency ratios lie below the approximation (4.13).

To sum up, the ratios of both models can be approximated by (4.13) if bonus payments are not considered. Naturally, very low asset values lead to such a scenario where no bonus is granted. Once we add bonus payments, however, the ratios of the cliquet-model fall below the initial approximation (4.13) and those of the maturity-model are lifted to a higher level than (4.13). It seems that the ratio of the cliquet-model is inferior to the one of the maturity-model mainly due to the high values of the SCR of the cliquet-model, which have great influence as they impact the ratio in the denominator. Although the cliquet-company has more own funds than the maturity-company for high asset values, the greater SCR compared to the maturity-company still causes a smaller ratio.

4.3.5. Excess Coverage

Finally, we arrive at the analysis of the excess coverage. Figure 4.16 depicts the empirical distribution and the scatter plot. Again, the plots indicate that the cliquet-company is inferior to the maturity-company in terms of its solvency position, at least when considering the expected excess coverage. Once we study the tails of the distributions, however, the excess coverage leads up to different implications than the solvency ratio. The distributions of the excess coverage appear to be right-skewed. Although the coverage of the cliquet-

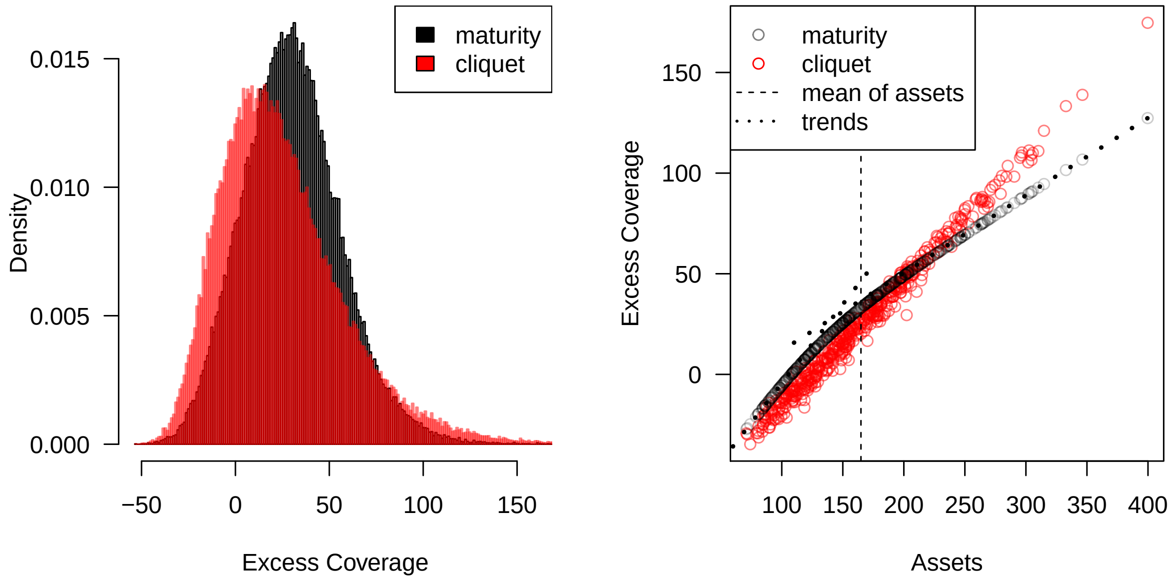


Figure 4.16.: Empirical distribution and scatter plot of the excess coverage at time ten

company peaks to the left of the maturity-company, it has a heavier right tail. In other words, the cliquet-company is expected to have a worse solvency position but, at the same time, its upside potential is higher. If the assets develop favorably, the cliquet-company can report a higher coverage in absolute terms than the maturity-company. Remarkably, both companies exhibit a similar level of excess coverage for the least favorable paths. The paths leading to a negative excess coverage correspond, of course, to the paths resulting in solvency ratios below 100%.

From the scatter plot in Figure 4.16, we infer that the excess coverage of the maturity-model is greater than the one of the cliquet-model for low asset values. Nevertheless, the cliquet-model can reach similar coverage depending on the exact path of the asset process. The dispersion originates from the dispersion observed in the values of the own funds. For high asset values, the situation is reversed in comparison to the solvency ratio. While the slope of the excess coverage of the maturity-model seems to decrease, the trend of the cliquet-model seems to remain constant. This results in a higher coverage for the cliquet-model from a certain value of the assets onwards with the difference in coverage increasing. In contrast to the solvency ratio, the excess coverage suggests that the cliquet-company's solvency position is actually better than the one of the maturity-company when dealing with high asset values at time ten. As an example for other time points, we add the scatter plots at time one and 19 in Figure 4.17.

We begin with the analysis of the excess coverage of the maturity-model. As part of

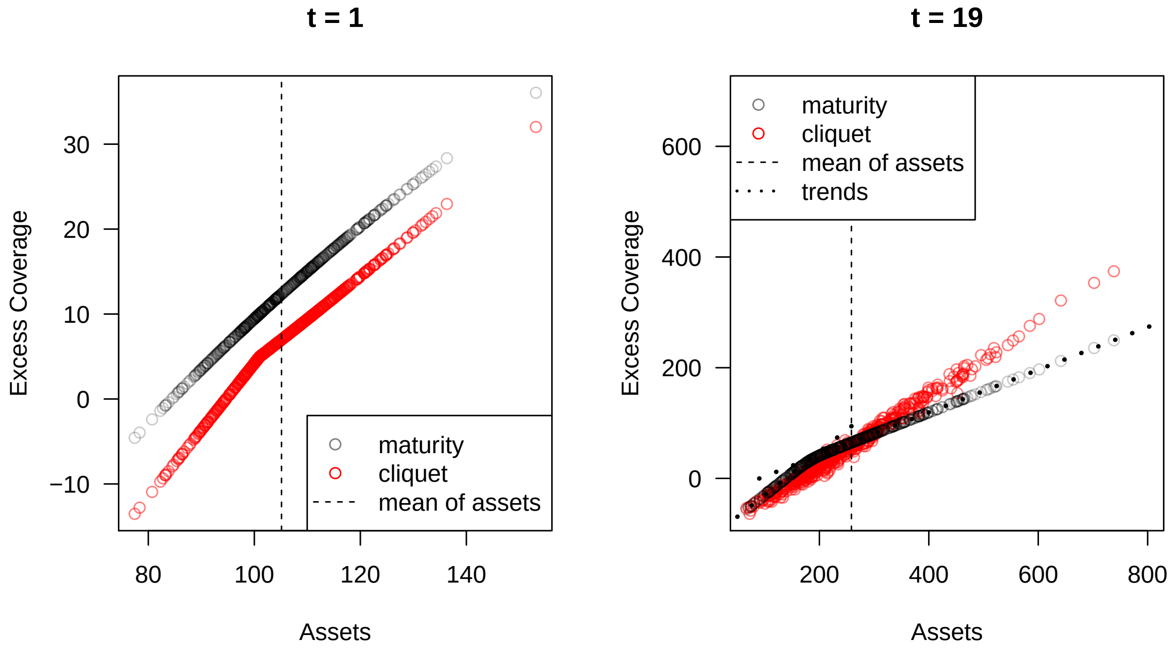


Figure 4.17.: Scatter plots of the excess coverage at time one and 19

studying the SCR, we already computed the derivative of the stressed own funds, which are exactly the excess coverage. We recall expression (4.10).

$$\frac{\partial \tilde{E}_t^M}{\partial A_t} = sf_A \cdot \left[1 - \delta\alpha\, \Phi\left(d_1\left(sf_A \cdot A_t, t, \frac{L_T^G}{\alpha}\right)\right)\right]$$

This slope is positive since the CDF takes values in $[0, 1]$. Furthermore, we notice the derivative to be decreasing with rising asset values, as was to be expected by observation of Figure 4.16, since the d_1 function as well as the CDF are monotonically increasing in their first arguments.

For very small asset values, we replace the call price in the representation of the stressed own funds in (3.4) by zero, which gives an approximation of

$$\tilde{E}_t^M = sf_A \cdot A_t - e^{-r(T-t)} L_T^G. \tag{4.14}$$

Clearly, the derivative of this approximation of the excess coverage equals $sf_A = 0.78$. If the asset value is so low that a terminal bonus can basically be ruled out, then a change in the stressed assets will solely affect the own funds. The value of the liabilities consists only of the guarantee, which cannot be touched. For large asset values, we employ the approximation of the call price in (4.4) and thereby obtain

$$\tilde{E_t^M} = (1 - \delta\alpha)\, sf_A \cdot A_t - (1 - \delta)\, e^{-r(T-t)} L_T^G.$$

The slope is $(1 - \delta\alpha)\, sf_A = 0.39$ in this case. On the contrary to the approximation for small asset values, we now count on the terminal bonus. Consequently, a change in the value of the stressed assets affects the own funds less than before since parts of the change are spent on the value of the liabilities. In essence, the slopes of the excess coverage in the limiting cases are the same as those of the own funds multiplied by the stress factor. This is due to the fact that the excess coverage simply equals the stressed own funds.
For the cliquet-model, an expression of the stressed own funds is given in (3.14) as

$$\tilde{E_t^C} = sf_A \cdot A_t - sf_L \cdot L_t^C.$$

This formula is very close to the approximation of the excess coverage of the maturity-model in (4.14). Namely, the factor sf_L is close to one and the value of the liabilities L_t^C equals $e^{-r(T-t)} L_T^G$ if no bonus has been granted up to time ten (c.f. (4.1)). As a result, the excess coverage of the cliquet-model at time ten can come close to the one of the maturity-model for paths in which the asset returns are consistently low over the first ten years. For higher asset values, L_t^C is larger than $e^{-r(T-t)} L_T^G$ due to bonus allocation to policyholders. Thus, the coverage of the cliquet-model stays below the approximation (4.14) for large asset values. As seen during the analysis of liabilities and own funds, there are several methods to obtain a rough trend of the cliquet-company's excess coverage. A simple linear regression, for instance, yields a trend of 0.62. Since the slope of the maturity-company's coverage changes from 0.78 to 0.39 as the effect of the bonus mechanism sets in, the cliquet-model presents higher absolute coverage values than the maturity-model for large values of the assets.
When comparing the scatter plot of the own funds in Figure 4.10 directly to the scatter plot of the excess coverage, which is the stressed own funds, in Figure 4.16, we observe a similar pattern. However, the points of the two models intersect only at a larger asset value in the case of the excess coverage. This behavior is accounted for by the fact that the excess coverage deals with the smaller, stressed asset value. Hence, larger asset values are needed compared to the case of the own funds in order to ensure a significant value of the terminal bonus.
On the whole, solvency ratio and excess coverage lead to slightly different conclusions. While the maturity-company seems to have a clear advantage over the cliquet-company

in terms of its solvency when using the ratio as the benchmark, the assessment of the companies' solvency positions becomes more ambiguous when dealing with the excess coverage. The cliquet-company can indeed display higher upside potential than the maturity-company, meaning that it can achieve higher values of absolute coverage if the assets produce high returns. Hence, it matters whether own funds and SCR are put into relation on a relative or an absolute basis. Especially when the SCR takes on particularly low or high values as in the tails of the distribution, the situation depicted by the quotient can differ from the one based on the difference. It is important to be aware of this fact because otherwise results can be misleading.

4.4. Time Period Analysis

In this section, we move on from the analysis of one single time point to an overview of the companies' solvency across the entire course of the contracts. Specifically, we study the evolutions of the solvency ratio and the excess coverage for each company. Moreover, we are interested in the shortfall probability across the years, which is the probability that the company's own funds cannot cover the SCR, i.e. the solvency ratio is below 100%. We consider the shortfall probability separately for each year. It should be noted that a shortfall is not equivalent to the insolvency of the company. In the case of a shortfall, the company still exists but cannot meet its capital requirement.

4.4.1. Quantile Plots of the Solvency Figures

In section 4.2, we simulated one single path of the asset process and computed the corresponding solvency figures. Now, we are aiming to not just depict one single realization but the distribution of the figures. For that purpose, we set the seed of the random number generator to one and simulate $100,000$ asset paths for each time point. This gives us $100,000$ solvency ratios and values of excess coverage for every point in time from $t = 1$ up to $t = 19$. To concentrate all that information, we compute empirical quantiles for each year and plot their evolution over the time axis. We consider the 1%-, 5%-, 10%-, 25%-, 50%-, 75%-, 90%-, 95%- and 99%-quantiles. As the 25%-quantile, for instance, should lie between the $0.25 \cdot 100,000 = 25,000$th and $25,001$st largest realization, we use the smoothed empirical estimate as defined in [KPW12]. Namely, the q-quantile of a sample of size n is estimated to be

$$(1-h)\, x_{(j)} + h\, x_{(j+1)},$$

where

$$j = \lfloor q(n+1) \rfloor \quad \text{and} \quad h = q(n+1) - j.$$

The operator $\lfloor \cdot \rfloor$ picks the largest integer that is smaller or equal to its argument. $x_{(j)}$ denotes the j-th order statistic, i.e. it stands for the j-th largest realization. Resuming our example of the 25%-quantile of a random variable X, the calculation becomes

$$q_{25\%}(X) = 0.75\, x_{(25{,}000)} + 0.25\, x_{(25{,}001)}$$

since $0.25 \cdot 100{,}001 = 25{,}000.25$. As our sample size of $100{,}000$ is rather large, it is of minor relevance whether we use this smoothed empirical estimate or simply $x_{(25{,}000)}$ as the 25%-quantile.

Figure 4.18 shows the results for the solvency ratio of the two companies. The asterisks indicate the different initial values of 174% and 125%. The observations we made when

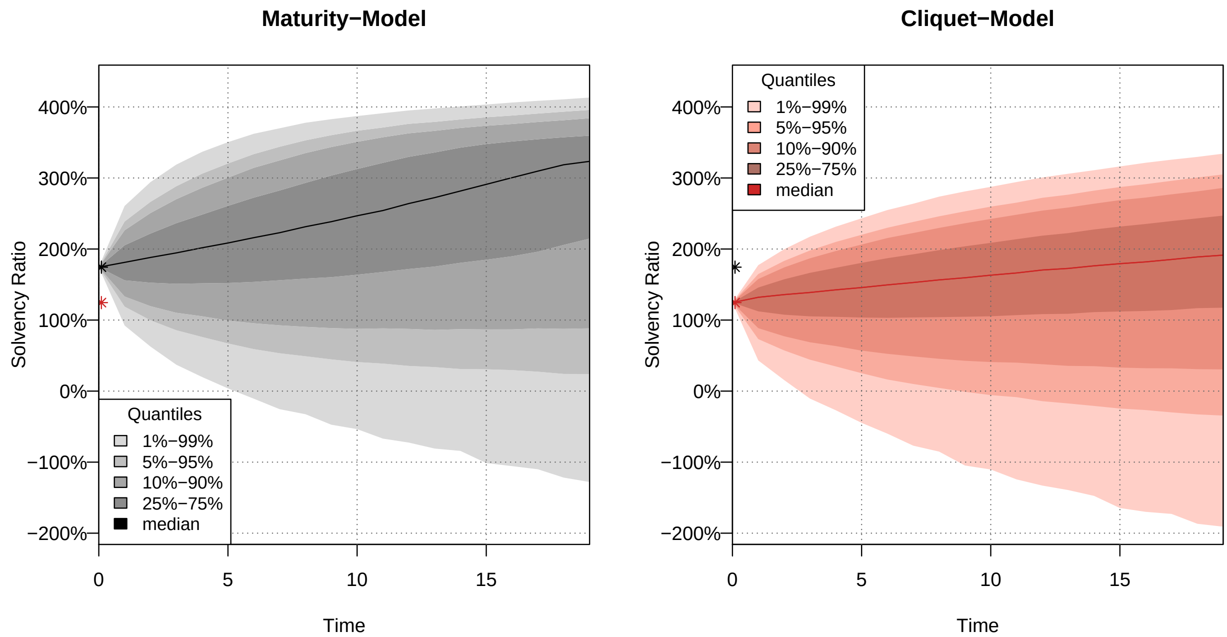


Figure 4.18.: Quantile plots of the solvency ratio

considering one single time point are also manifested in these plots. In fact, the profiles of these quantile plots at time point ten coincide with the histograms in Figures 4.14 and 4.16. For instance, we observe that high ratios of 300% or 400% are more likely in the

maturity-model. While the ratios of both models are skewed to the left, the cliquet-model has a heavier left tail. Thus, very low ratios are more likely to occur in the cliquet-model. Also, the probability for the ratio to fall below the mark of 100% is greater for the cliquet-model than the maturity-model at any point in time. Still, the ratios of the two models appear to be fairly similar when the assets develop unfavorably, i.e. for low ratios. Additionally, we are now able to inspect the changes in the ratios over time. We notice that the medians increase. After starting off with a higher ratio by almost 50%, the median of the maturity-model has a greater slope and always stays above the one of the cliquet-model. If we view the distance between the one percent and the 99%-quantile as an indicator for the uncertainty of the ratio, then the ratio of the maturity-model is more uncertain than the one of the cliquet-model for the first few years after the inception of the contracts. The difference in the uncertainty appears to diminish towards maturity.
Next, we take a look at the excess coverage to compare the conclusions regarding the companies' solvency positions with those drawn by means of the solvency ratio. Figure 4.19 shows the quantile plots for both companies. Again, the asterisks mark the initial

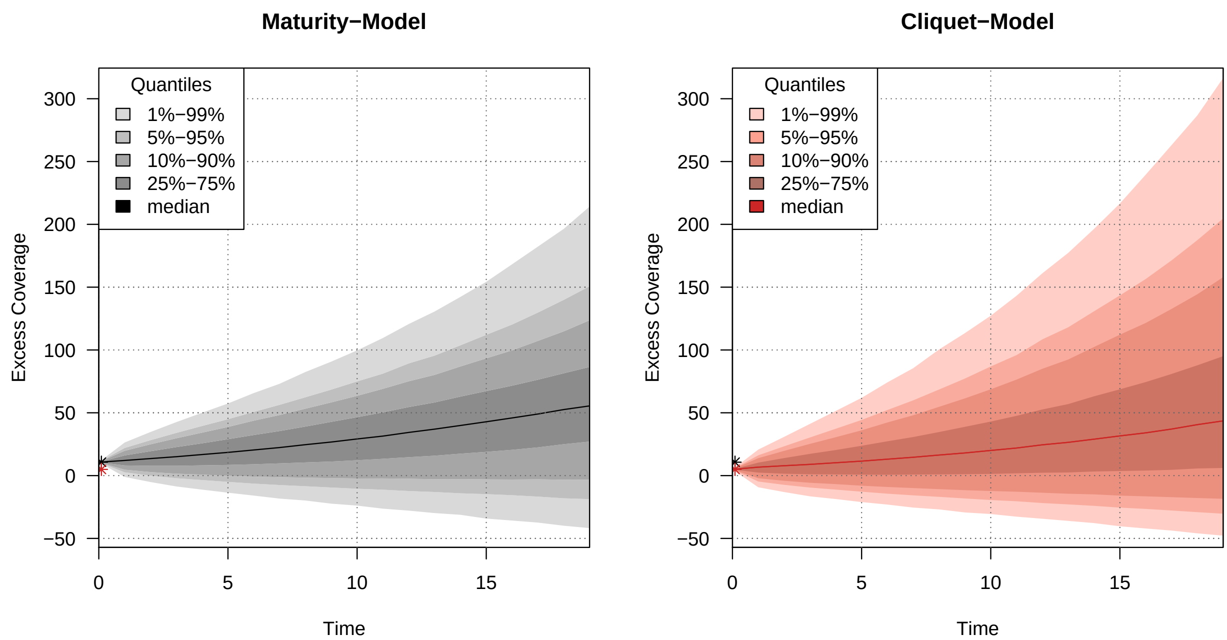


Figure 4.19.: Quantile plots of the excess coverage

values, namely 10.67 and 4.93. Thus, the excess coverage confirms the stronger position of the maturity-model at inception. The medians are increasing for both models with the median of the maturity-model staying above the one of the cliquet-model throughout the course of the contracts. Naturally, the spread between the one percent and the 99%-

quantile widens over time just as it was the case for the solvency ratios. The value of the coverage becomes more uncertain the further we look into the future. As the excess coverage equals the difference between own funds and SCR, its value drops below zero once the SCR exceeds the own funds. Hence, these situations are equivalent to those in which the solvency ratio falls below 100%. When comparing the coverage of the two models, we notice that the low ends of the distributions are similar over all the years. For the high ends, however, the 99%-quantile of the cliquet-model rises faster than its counterpart in the maturity-model. Thus, the depicted quantiles spread over a wider range of values for the cliquet-model as compared to the maturity-model. Moreover, very high values of coverage are more likely to be achieved in the cliquet-model.
For both solvency figures, the median of the maturity-model lies above the median of the cliquet-model at all points in time, which suggests a stronger solvency of the maturity-model. However, the behavior in the tails of the distributions varies according to which figure we consider. While the maturity-model clearly has a heavier right tail in terms of the solvency ratio, the cliquet-model has a heavier right tail for the excess coverage after the first few years. For low values, we note that the cliquet-model has a heavier left tail for both of the figures. The gap between the 1%-quantiles of the two models appears to be roughly as big as the difference in the companies' initial solvency.

4.4.2. Shortfall Probability and Expected Shortfall

We are now interested in the probability of a shortfall, by which we refer to the event in which the solvency ratio lies below 100%, that is we are looking for

$$\mathbb{P}[Sol_t < 100\%] = \mathbb{P}[E_t < SCR_t].$$

A rough estimate can be inferred through Figure 4.18 by determining how much of the distribution falls short of 100%. For a more exact derivation of the probabilities, we employ the following approach. Using the same simulations of the asset path as for the quantile plots, we are provided with $100,000$ values of own funds, SCRs and, if combined, solvency ratios per time point. We can easily compute empirical probabilities based on these simulations. At each point in time, we simply count the number of paths for which the own funds are not sufficient to cover the SCR and divide the amount by the total number of paths, which is $100,000$ in our case. The results are depicted in Figure 4.20 together with results of the expected shortfall, which is the expectation of the solvency ratio given that it lies below 100%, that is

$$\mathbb{E}[Sol_t | Sol_t < 100\%].$$

Thus, the expected shortfall gives us an idea of how bad the ratio really is if the SCR is not met. Empirically, we obtain this quantity by taking the arithmetic mean of the ratios among all paths with a ratio below 100%. It is important to remember that simulations

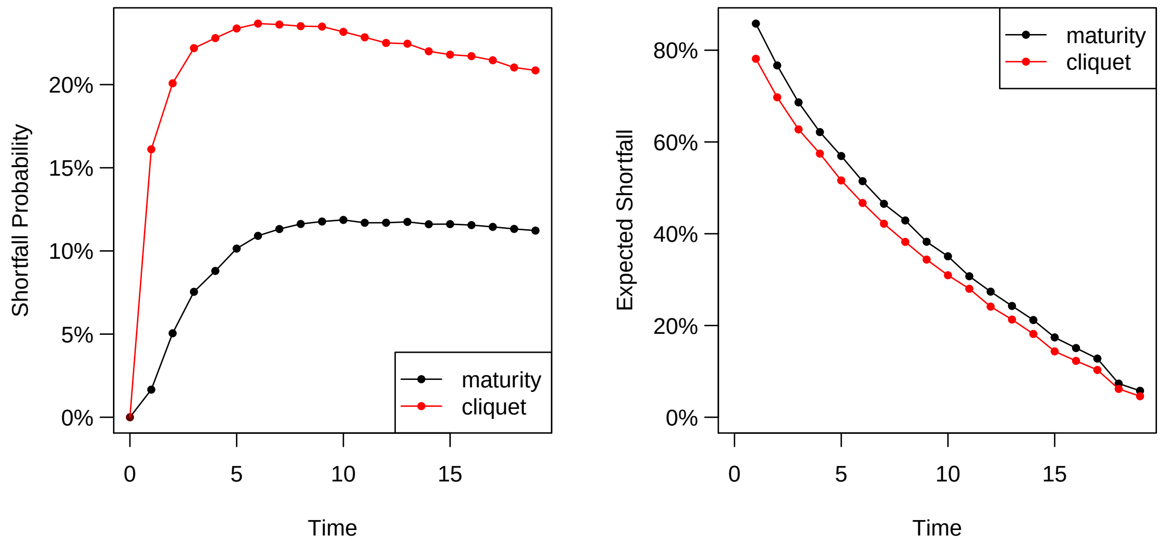


Figure 4.20.: Shortfall probability and expected shortfall

are not stopped prematurely even if the own funds do not meet the capital requirement. How we deal with such a situation is explained in the next section.

For both models, the shortfall probability equals zero at inception since the initial solvency ratios are above 100%. The companies do not start with a shortfall, which is known through the value of the assets at time zero. At time one, the shortfall probability rises since the values of own funds and SCR are now random and a shortfall cannot be ruled out. Within the 100,000 simulated paths, there are some for which the own funds are not sufficient to cover the SCR. We notice that the shortfall probability jumps to about 16% for the cliquet-model, while it increases more moderately for the maturity-model to a value of about two percent. The probability rises until time ten for the maturity-model and time six for the cliquet-model, then it decreases. In the maturity-model, the downward trend is very slight. Namely, the empirical shortfall probability levels off between eleven and twelve percent as the time to maturity diminishes. The decrease in the shortfall probability of the cliquet-model is more prominent than the one of the maturity-model. The probability

drops from 24% back to 21%. All in all, the shortfall probability of the maturity-model is constantly below the probability of the cliquet-model. Thus, the cliquet-company is more susceptible to not being able to cover its SCR. This observation was to be expected as the cliquet-company already starts off in a less comfortable position in view of its solvency. The difference in the shortfall probabilities is of the order of about ten percent. While the shortfall probabilities may appear large at first glance, we recall that the shortfall is defined by the own funds not covering the SCR and does not necessarily lead to the insolvency of the company. In fact, a shortfall merely means that the probability of insolvency is above 0.5% for the year thereafter. Of course, it also involves regulatory intervention.
Next, we consider the expected shortfall, which is also depicted in Figure 4.20. The plots begin at time one since the shortfall probability is zero at inception, which causes the expected shortfall to not be defined. For both models, the expected shortfall at time one is around 80%. That is to say, if either company's solvency ratio drops below the required mark of 100% at time one, then we expect its ratio to be around 80%. Therefore, the company's ratio misses its requirement by about 20%. The values of the expected shortfall drop the further we look into the future towards maturity and arrive at around five percent at time 19. This makes sense intuitively because the uncertainty increases with every year we look ahead. Consequently, we find more and more asset paths leading to solvency ratios far below 100%. It should be noted that the expected shortfall does not drop below zero percent. Therefore, we can expect the company to have positive own funds, i.e. to still be solvent, even if it fails to meet its SCR. When comparing the two models, the expected shortfall of the cliquet-model always lies below the one of the maturity-model. This is not a surprise if we recall the quantile plots of the solvency ratio in Figure 4.18. The quantiles we depicted of the cliquet-model are always below those of the maturity-model. However, the difference in the expected shortfall of the two models seems to diminish over time.

4.5. Cost of Capital

In this section, we explain how a company of our models acts if it experiences a year in which its own funds do not suffice to cover its SCR. Instead of modeling some kind of regulatory intervention, we simply assume that the affected company borrows the missing money for a period of one year on the capital market. Figure 4.21 illustrates our approach for the case of a shortfall at time t. The company borrows the difference between SCR and

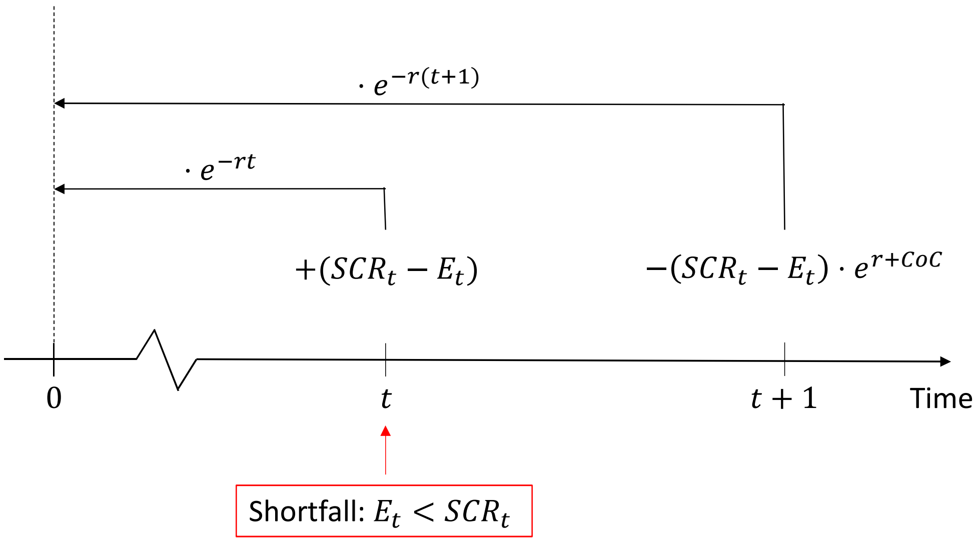


Figure 4.21.: The process of borrowing money in the case of a shortfall

own funds at time t in order to be able to report a solvency ratio of 100%. One year later, at time $t+1$, this loan has to be paid back including interest. Of course, the lender asks for an interest rate that is higher than just the risk-free rate since he intends to make a profit from providing the money. Besides, the lender will charge a premium for the default risk he is taking. We call the additional interest rate beyond the risk-free rate the cost of capital rate (CoC) and set it to 6%. For simplicity, we assume that all costs arising from borrowing money are paid from a separate account within the company, that is not reflected in the balance sheet. Furthermore, the borrowed money is kept separately and earns the risk-free rate. It does not increase the value of the company's assets. Also, it should be noted that the cost of capital we encounter in this section has no relation to the risk margin as specified in [Eur15, 37].

In order to study the cost arising from borrowing money whenever the solvency ratio fails to reach the requirement of 100%, we consider the present value of all future cost of capital. Clearly, the present value is a random quantity that depends on the realization of the asset process. While it may be large if the yearly returns on the assets are low, it is zero if the returns are high enough to always produce solvency ratios above 100%. To first obtain an expression of the discounted cost of capital in the case of a shortfall at time t, we return to Figure 4.21. The arrows above the timeline indicate that we discount the cash flows to time zero. Thus, we arrive at a cost of

$$(SCR_t - E_t)\ e^{r+CoC}\, e^{-r(t+1)} - (SCR_t - E_t)\ e^{-rt} = e^{-r(t+1)}\ (SCR_t - E_t)\left(e^{r+CoC} - e^{r}\right).$$

By the end of the year, the company has to pay back the amount it borrowed, the risk-free rate as well as the additional interest beyond the risk-free rate on a one-year period. We generalize this expression by adding a plus operator to check for whether we are dealing with a shortfall or not. Then, we sum up over all time points to get the present value of the future cost of capital, namely

$$PV_{CoC} = \sum_{t=0}^{T-1} \left(e^{-r(t+1)} \left(SCR_t - E_t \right)^{+} \left(e^{r+CoC} - e^{r} \right) \right). \tag{4.15}$$

To get the empirical distribution of the present value under the real-world measure $\mathcal{P}$, we simulate $100,000$ asset paths under $\mathcal{P}$. For each asset path, we compute the corresponding present value of the cost of capital according to (4.15) for both models. The empirical CDFs of each $100,000$ values are shown in Figure 4.22. Additionally, we list some key

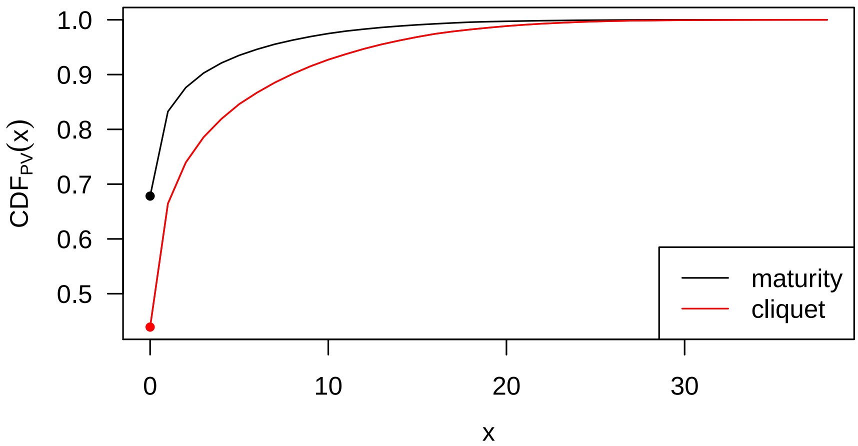


Figure 4.22.: Empirical CDF of the present value of future cost of capital

figures in Table 4.2. The CDF of the present value of the maturity-model begins at a value of 0.68, while the one of the cliquet-model starts at 0.44. These values represent the probability that the present value of the future cost of capital is zero since the plus operator in (4.15) ensures that only nonnegative values can be attained. Thus, the chance of not generating any cost of capital resulting from borrowed money, which is equivalent to keeping a solvency ratio of more than 100% throughout the course of the contracts, is about 24% higher in the maturity-model than in the cliquet-model. Although the CDF of the cliquet-model rises faster, the CDF of the maturity-model is still always closer to one and reaches the value one first. Hence, the present value of the future cost of capital of the cliquet-model has a heavier right tail when compared to the one of the maturity-model.

	Maturity-Model	Cliquet-Model
mean(PV)	0.96	2.24
volatility(PV)	2.78	4.42
$q_{99.9\%}(PV)$	23.74	28.78
$\mathbb{E}[PV \mid PV > 0]$	2.97	4.00
$\mathbb{P}(PV = 0)$	0.68	0.44
$\mathbb{P}(PV \leq 1)$	0.83	0.66
$\mathbb{P}(PV \leq 2)$	0.88	0.74
$\mathbb{P}(PV \leq 5)$	0.94	0.85

Table 4.2.: Key figures regarding the present value of future cost of capital (PV)

This means that the present value of the cliquet-model is more likely to take on high values. In fact, all strictly positive values are more probable to occur in the cliquet-model than in the maturity-model. Mean, volatility and the other figures given in Table 4.2 also suggest that the values of the maturity-model are more concentrated at zero than those of the cliquet-model. On the whole, the cliquet-model displays a higher necessity to borrow money in order to fulfill the solvency requirements. This is consistent with the preceding analyses of this chapter. The cliquet-company's solvency position is simply inferior to the one of the maturity-company.

5. Sensitivity Analysis

This chapter focuses on how the choice of certain parameters influences our results of the companies' solvency. In particular, we vary the volatility of the asset process (σ), the guaranteed interest rate (r_G and g) and the risk-free interest rate (r), respectively, while all other parameters remain unchanged. Then, we study the effects on the distribution of the solvency ratio by viewing quantile plots similar to those in section 4.4.1. These plots represent the distribution of the ratio for every single year as well as its evolution over the course of the contracts. For plots depicting the sensitivity of the excess coverage, we refer to the appendix.

5.1. Volatility

Figure 5.1 shows the quantile plots of the solvency ratio for three different volatility levels. We compare the level of 10%, which was used in all our earlier analyses, with the cases of lower volatility at 5% and higher volatility at 15%, respectively. The quantiles included are those at 5%, 25%, 75% and 95% of the distribution as well as the median. Clearly, a change in the volatility has a great impact on the models. Since the asset process evolves differently, the magnitude of the stress factor is affected. According to expression (3.2), the stress factor of volatility levels of 5% and 15% is 0.90 and 0.69, respectively. As expected, the stress scenario implies a more drastic drop in the assets for higher volatility levels since the bicentenary event is more severe if the assets on the whole are more volatile. Furthermore, the participation rates δ and β are impacted as shown in Table 5.1. In general, lower volatility levels lead to higher solvency ratios. This effect can be observed for both models and already at time zero. Also, the uncertainty in the ratio appears to increase with the volatility of the assets. Comparing the two models suggests that the maturity-model is superior in terms of its solvency ratio no matter which level of volatility prevails. For the volatility level of 15%, we notice that the solvency ratio of the cliquet-company already fails to reach the required 100% at inception of the contracts.

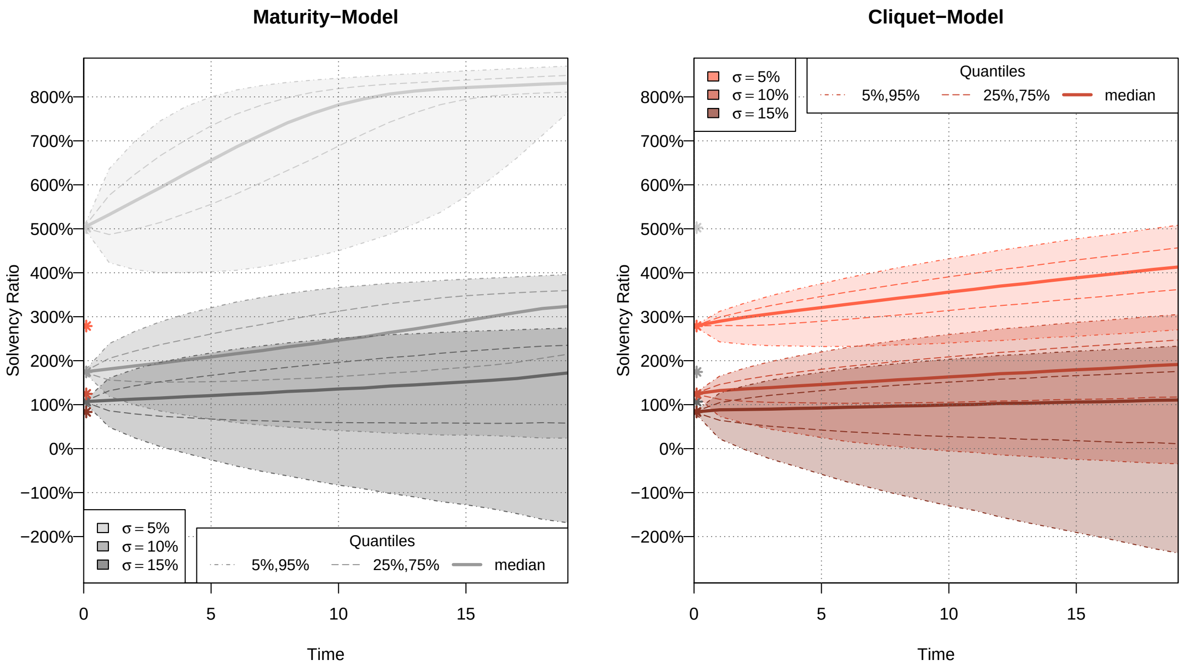


Figure 5.1.: Sensitivity of the solvency ratio regarding the volatility

	Maturity-Model	Cliquet-Model
σ	δ	β
5%	0.90	0.64
10%	**0.68**	**0.41**
15%	0.53	0.30

Table 5.1.: Participation rates for our choice of volatility levels

5.2. Guaranteed Interest Rate

Next, we vary the guaranteed interest rates of both models to range between zero, one and two percent. The resulting quantile plots are presented in Figure 5.2. Adjusting the guaranteed interest rate affects the participation rates δ and β, which regulate how much the policyholders profit from the companies' generated surplus. Naturally, an increase in the guaranteed interest rate raises the value of the contract from the policyholders' perspective and thus has to be balanced with a decrease in the corresponding participation rate in order to maintain the fairness of the contract. In particular, the parameters matching our choice of guaranteed rates are given in Table 5.2. Note that the guaranteed rates of two percent in the maturity-model and one percent in the cliquet-model correspond to the base case we studied in previous chapters.

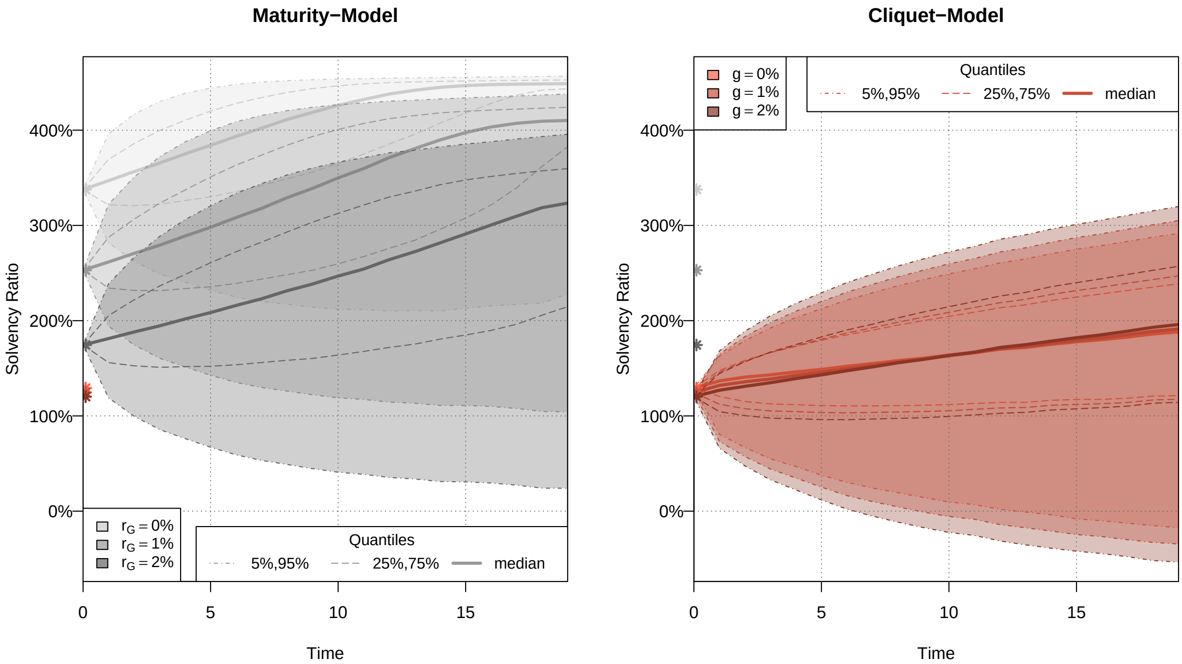


Figure 5.2.: Sensitivity of the solvency ratio regarding the guaranteed interest rate

Maturity-Model		**Cliquet-Model**	
r_G	δ	g	β
0%	0.97	0%	0.55
1%	0.90	**1%**	**0.41**
2%	**0.68**	2%	0.23

Table 5.2.: Participation rates for our choice of guarantees

For the maturity-model, we observe that lower guarantees immediately lead to higher solvency ratios. This reasonable feature is also present for the initial solvency ratios of the cliquet-model. However, a change in the guaranteed interest rate does not seem to affect the solvency ratio of the cliquet-model as much as the ratio of the maturity-model. For the cliquet-model, the distribution of the ratio hardly varies between each of the guaranteed rates. Merely the range between the 5%- and the 95%-quantile increases slightly with an increase in the guarantee. When assessing the reasons for the differing sensitivity of the two models with respect to the guaranteed rate, we recall the different structures of the SCR (cf. (3.5) and (3.15)) as these certainly play an important role.

$$SCR_t^M = (1 - sf_A)A_t - \delta\alpha \left[C\left(A_t, t, \frac{L_T^G}{\alpha}\right) - C\left(sf_A \cdot A_t, t, \frac{L_T^G}{\alpha}\right)\right]$$
$$SCR_t^C = (1 - sf_A)A_t - (1 - sf_L)L_t^C$$

While the SCR of the maturity-model significantly depends on the guaranteed rate r_G and the participation rate δ, the SCR of the cliquet-model is mainly based on the stress factor, which is completely insensitive to the guaranteed rate.

5.3. Risk-Free Interest Rate

Figure 5.3 depicts quantile plots of the solvency ratio for different choices of the risk-free interest rate, namely two, three and four percent. Recall that the risk-free rate of three percent corresponds to the base case. Similarly to the previous section where we varied

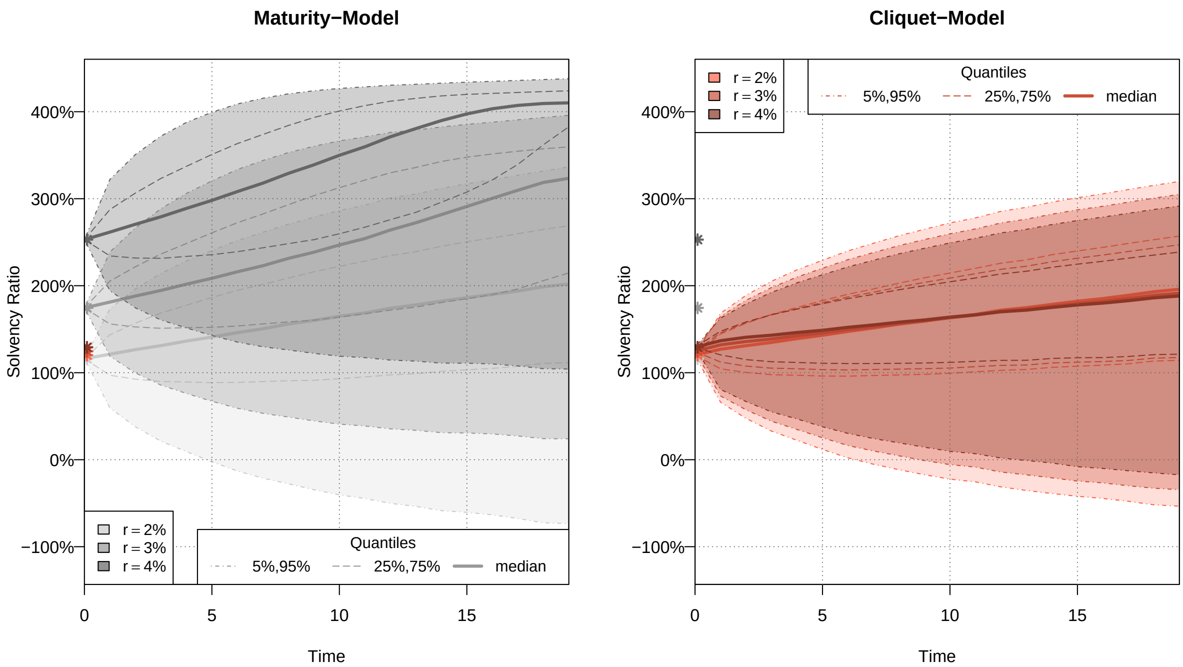


Figure 5.3.: Sensitivity of the solvency ratio regarding the risk-free interest rate

the guaranteed interest rate, a change in the risk-free rate r affects the participation rates δ and β but not the stress factor. The corresponding participation rates are provided in Table 5.3. As we use the guaranteed interest rates of the base case, the maturity-model

	Maturity-Model	Cliquet-Model
r	δ	β
2%	0	0.23
3%	**0.68**	**0.41**
4%	0.90	0.55

Table 5.3.: Participation rates for our choice of risk-free interest rates

guarantees two percent and the cliquet-model one percent. Thus, a risk-free rate of two percent represents a special case for the maturity-model since then the guaranteed interest rate equals the risk-free rate. As a result, policyholders are not included in the distribution of bonus payments, i.e. δ equals zero. The participation rates increase together with the risk-free rate because a higher risk-free rate devalues the arranged guaranteed interest rate from the policyholders' point of view. Hence, the contracts can only remain fair if value is added by extending the policyholders' participation in surplus.

According to Figure 5.3, higher values of the risk-free interest rate entail larger solvency ratios for the maturity-model. This observation is not surprising as the guaranteed rate r_G is more likely to be met if the risk-free rate is larger. However, the distribution of the solvency ratio of the cliquet-model appears to be only slightly influenced by a change in the risk-free rate. Higher values of the risk-free rate lead to a higher initial solvency and to a slightly less uncertain solvency ratio as per spread between the 5%- and 95%-quantile. Again, the distinct structures of the SCR (cf. (3.5) and (3.15)) are essential for understanding the differences in the sensitivity of the two models.

6. Conclusion

In this thesis, we studied the solvency of two companies according to Solvency II. We used two simple setups based on [BD97], [GJ02] and [MP03] for modeling different types of contracts. While the maturity-company offers a contract, which only credits surplus at the very end of the contract period, the contracts of the cliquet-company include cliquet-style guarantees. Since, in the latter case, surplus can be credited each year, the cliquet-model requires yearly analyses of the asset returns to determine the final payoff to policyholders. In order to ensure comparability between the companies and focus on the effects arising from the differing types of guarantees, we chose identical initial situations with regard to the balance sheets.

By means of simulation, we analyzed the distribution of the solvency ratio separately for every future point in time up to maturity as well as its development across the entire course of the contracts. We presented several possibilities of visualization. Additionally to the solvency ratio, we considered the excess coverage as an indicator for the companies' solvency. This allowed us to view the coverage of the SCR by the own funds both on a relative scale and an absolute scale.

We discovered that, in general, the maturity-model presents a stronger solvency than the cliquet-model. However, it is worthwhile to have a look at the excess coverage when it comes to high values of the assets, i.e. the right tail of the distribution. While the solvency ratio of the cliquet-model seems to be inferior to the one of the maturity-model for all realizations of the asset process, the coverage of the cliquet-model measured in absolute terms can exceed the one of the maturity-model if it comes to high asset values. For consistently low asset returns, both figures lead to the conclusion that the two models are fairly similar in terms of their solvency.

Clearly, the two models treated in this thesis are heavily simplified versions of what can be found in reality. For instance, the constant risk-free interest rate r is not consistent with the real world and could be replaced by stochastic interest rates to reflect the existing uncertainty. This issue as well as further modifications of the models regarding the design of the guarantees and the profit participation mechanisms are left for future research.

A. Sensitivity of the Excess Coverage

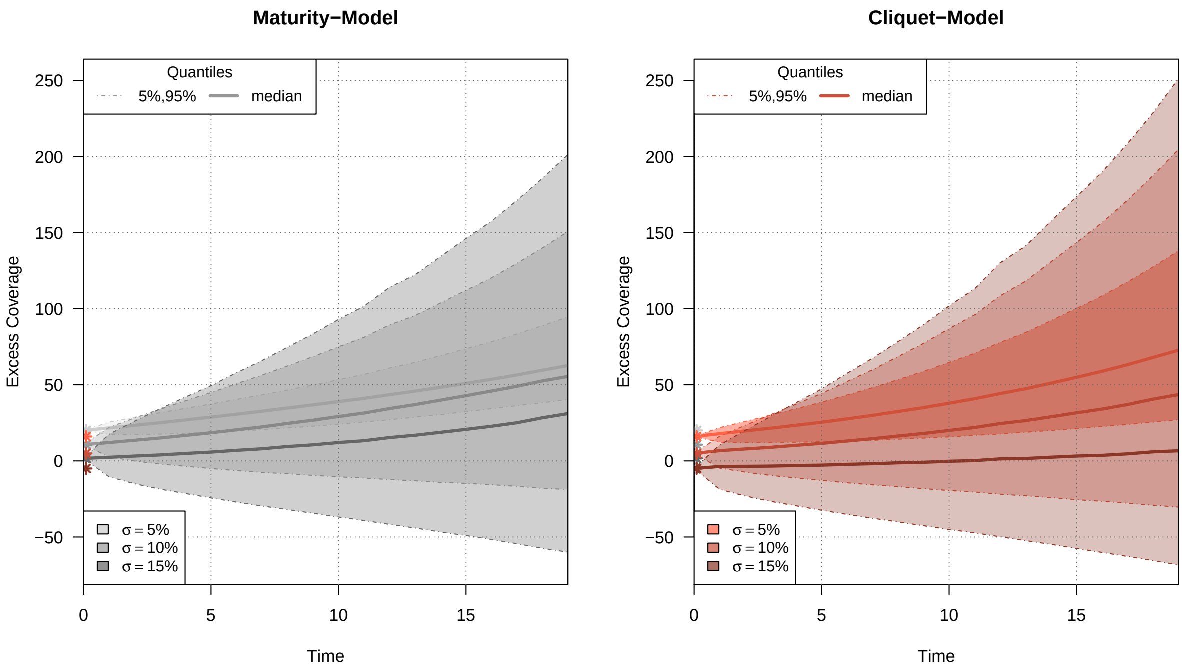


Figure A.1.: Sensitivity of the excess coverage regarding the volatility

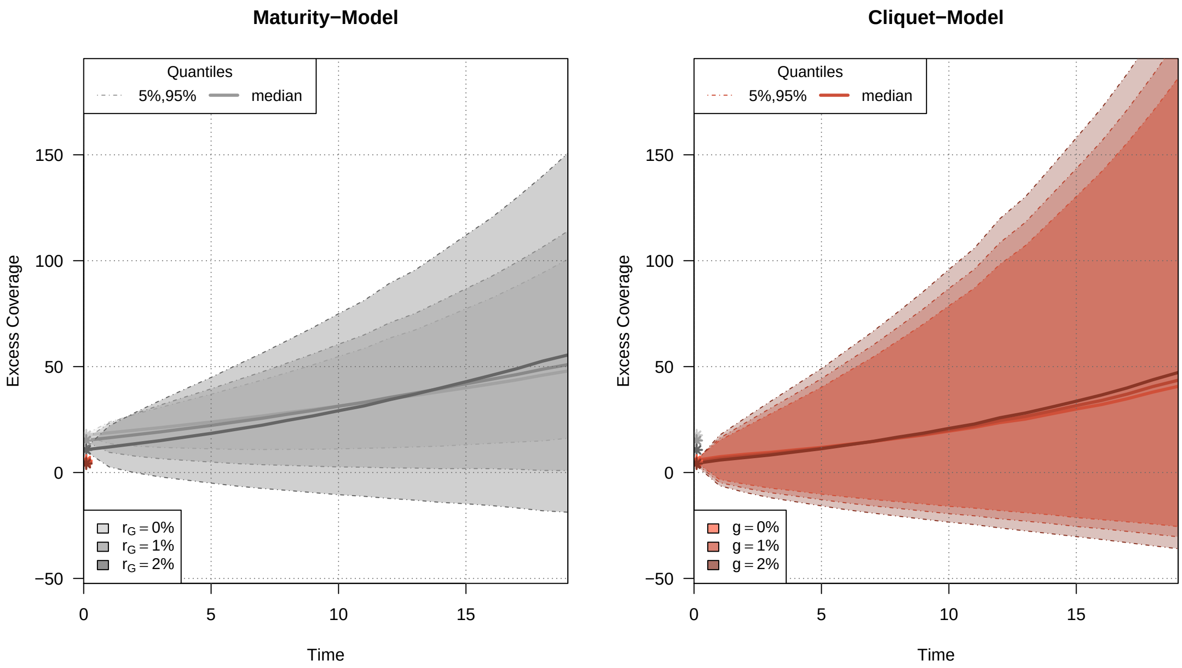


Figure A.2.: Sensitivity of the excess coverage regarding the guaranteed interest rate

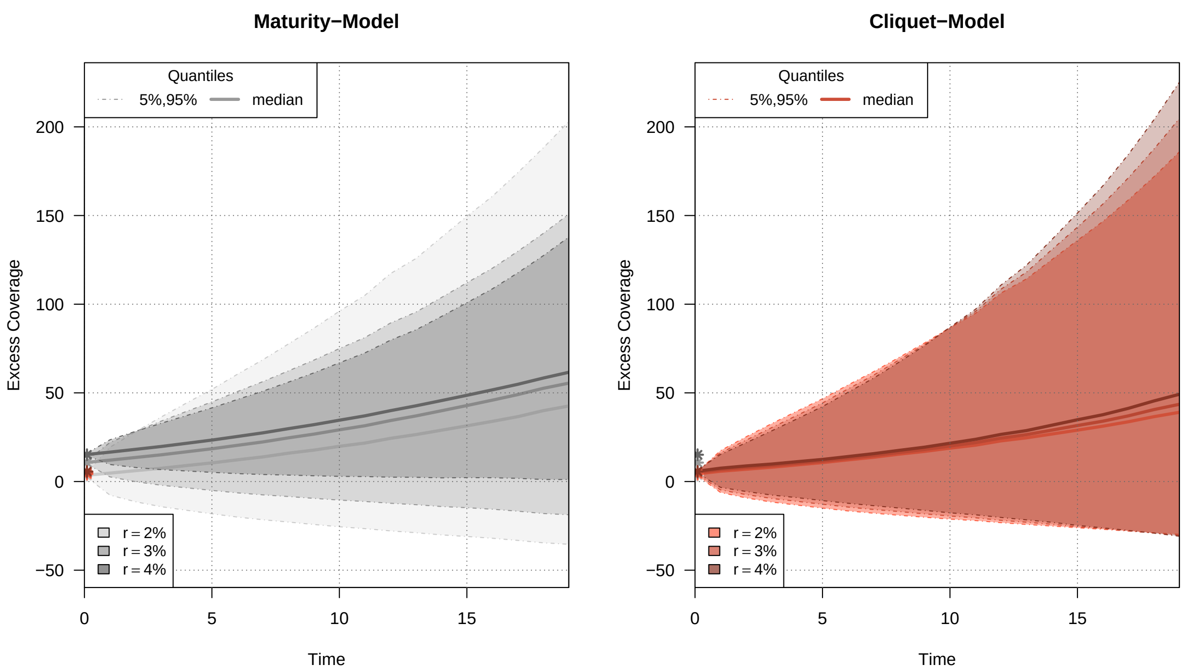


Figure A.3.: Sensitivity of the excess coverage regarding the risk-free interest rate

B. R-Code

B.1. Valuation Formulas

```
###parameters
A_0 = 100
alpha = 0.75
sigma = 0.1 #volatility
T = 20 #time to maturity
r = 0.03 #risk-free interest rate
lambda = 0.02 #risk premium
mu = r + lambda #mean rate of return under P
CoC = 0.06 #cost of capital rate

#parameters maturity-specific
r_G = 0.02 #guaranteed interest rate (maturity guarantee)
L_TG = alpha*A_0 * exp(r_G*T) #guaranteed payoff to policyholder at ...
    maturity

#parameters cliquet-specific
g = 0.01 #guaranteed interest rate (cliquet/rate of return guarantee)
#---------------------------------
###stress factor VaR(99,5%)
sf_A = exp((lambda-(sigma^2)/2)+sigma*qnorm(0.005)) #stress on the ...
    assets under P, discounting one time point
#---------------------------------
###valuation maturity-specific
#time t value of a European call
call <- function(x,t,K){
  d_1 = (log(x/K)+(r+(sigma^2)/2)*(T-t))/(sigma*sqrt(T-t))
  d_2 = d_1 - sigma*sqrt(T-t)
  return(x*pnorm(d_1) - K*exp(-r*(T-t))*pnorm(d_2))
}
```

```

#determine delta (to have a fair contract under Q at time t=0)
delta = (alpha*A_0 - exp(-r*T)*L_TG)/(alpha*call(A_0,0,L_TG/alpha))

###valuation of liabilities and own funds
#time t value of the liabilities
V_L_m <- function(A_t,t){
  delta*alpha*call(A_t,t,L_TG/alpha) + exp(-r*(T-t))*L_TG
}
#time t value of the own funds
V_E_m <- function(A_t,t){
  A_t-V_L_m(A_t,t)
}

###solvency ratio
#t=0
E_0_m = V_E_m(A_0,0)
L_0_m = V_L_m(A_0,0)
SCR_0_m = E_0_m - V_E_m(sf_A*A_0,0)
Sol_0_m = E_0_m/SCR_0_m
#in general
Solvency_m <- function(A_t,t){
  E_t = V_E_m(A_t,t)
  SCR_t = E_t-V_E_m(sf_A*A_t,t)
  #returns own funds, SCR and Sol at time t
  return(c(E_t,SCR_t,E_t/SCR_t))
}
#----------------------------------------
###valuation cliquet-specific
#determine beta (to have a fair contract under Q at time t=0)
func <- function(beta){
  exp((1-beta)*(g-r-0.5*beta*sigma^2)) * ...
      pnorm((r-g-0.5*sigma^2+beta*sigma^2)/sigma) + exp(g-r) * ...
      pnorm((g-r+0.5*sigma^2)/sigma)
}
beta <- uniroot(function(x)(func(x)-1),lower=0,upper=1,tol=10^(-10))$root

#values under stress
sf_L <- exp(beta*log(sf_A)+(1-beta) * (g-r-0.5*beta*sigma^2)) * ...
    pnorm((log(sf_A) + r - g - 0.5*sigma^2+beta*sigma^2)/sigma) + ...
    exp(g-r) * pnorm((g-r+0.5*sigma^2-log(sf_A))/sigma)

```

```
###valuation of liabilities and own funds
#time t value of the liabilities (t≥1 , rho is of length t)
#rho contains already realized log returns
V_L_c <- function(rho,t){
  alpha*A_0*exp(sum(g+beta*pmax(rho-g,0)))
}
#time t value of the own funds (t≥1 , rho is of length t)
V_E_c <- function(rho,t){
  A_t = A_0 * exp(sum(rho))
  return(A_t-V_L_c(rho,t))
}

###solvency ratio
#t=0
L_0_c = alpha*A_0
E_0_c = (1-alpha)*A_0
SCR_0_c = E_0_c - (sf_A*A_0-sf_L*L_0_c)
Sol_0_c = E_0_c/SCR_0_c
#in general (for t≥1 , rho is of length t)
Solvency_c <- function(rho,t){
  L_t = V_L_c(rho,t)
  A_t = A_0 * exp(sum(rho)) #no stress
  E_t = A_t-L_t
  SCR_t = E_t-(sf_A*A_t-sf_L*L_t)
  #returns own funds, SCR and Sol at time t
  return(c(E_t,SCR_t,E_t/SCR_t))
}
```

B.2. Valuation Along Simulated Paths

```
n = 500 #number of paths drawn
#n = 100000 #for plotting histograms and computing quantiles
t = 10 #time point to be analyzed

A_t <- rep(0,n)
#maturity
E_t_m <- rep(0,n)
SCR_t_m <- rep(0,n)
Sol_t_m <- rep(0,n)
```

```
#cliquet
E_t_c <- rep(0,n)
SCR_t_c <- rep(0,n)
Sol_t_c <- rep(0,n)

set.seed(1)
for(i in 1:n){
  rho <- mu-(sigma^2)/2 + sigma * rnorm(t,mean=0,sd=1) #length t, ...
      draw under P
  A_temp = A_0 * exp(sum(rho))
  A_t[i] = A_temp
  #maturity
  comp_m <- Solvency_m(A_temp,t)
  E_t_m[i] <- comp_m[1]
  SCR_t_m[i] <- comp_m[2]
  Sol_t_m[i] <- comp_m[3]
  #cliquet
  comp_c <- Solvency_c(rho,t)
  E_t_c[i] <- comp_c[1]
  SCR_t_c[i] <- comp_c[2]
  Sol_t_c[i] <- comp_c[3]
}
#maturity
L_t_m <- A_t-E_t_m
exc_cov_t_m <- E_t_m-SCR_t_m
#cliquet
L_t_c <- A_t-E_t_c
exc_cov_t_c <- E_t_c-SCR_t_c
```

B.3. Cost of Capital

```
n = 100000 #number of paths to be drawn
#save PV of future cost of capital per path drawn
cost_capital_m <- rep(0,n)
cost_capital_c <- rep(0,n)

set.seed(1)
for(i in 1:n){
  #draw a path of A under P (A_0,A_1,...,A_T-1)
```

```
  rho <- (mu-(sigma^2)/2) + sigma * rnorm(T-1) #length is T-1
  t <- seq(0,T-1,1)

  ###maturity
  sum_rho <- cumsum(rho)
  A <- c(A_0,A_0*mapply(exp,sum_rho)) #path of A, length is T
  comp_m <- Solvency_m(A,t) #length of comp is 3*T
  E_m <- comp_m[1:T] #E_0,...,E_T-1
  SCR_m <- comp_m[(T+1):(2*T)] #SCR_0,...,SCR_T-1
  short_cov_m <- pmax(SCR_m-E_m,0) #(SCR-E)+ (shortfall in coverage) ...
      for time points 0,...,T-1
  summand_m <- mapply(exp,(-r)*(t+1)) * short_cov_m * (exp(r+CoC)-exp(r))
  cost_capital_m[i] = sum(summand_m) #PV of future cost of capital ...
      for path i

  ###cliquet
  short_cov_c <- rep(0,T)
  short_cov_c[1] = max(SCR_0_c-E_0_c,0) #(SCR-E)+ (shortfall in ...
      coverage) for time point 0
  for(j in 1:(T-1)){
    comp_c <- Solvency_c(rho[1:j],j)
    E_c <- comp_c[1]
    SCR_c <- comp_c[2]
    short_cov_c[j+1] = max(SCR_c-E_c,0) #(SCR-E)+ (shortfall in ...
        coverage) for time points 1,...,T-1
  }
  summand_c <- mapply(exp,(-r)*(t+1)) * short_cov_c * (exp(r+CoC)-exp(r))
  cost_capital_c[i] = sum(summand_c) #PV of future cost of capital ...
      for path i
}
#----------------------------------
###empirical CDF
#maturity
x_m <- seq(0,ceiling(max(cost_capital_m)),1)
count_m <- hist(cost_capital_m,breaks=c(0,x_m),plot=F)$counts
cdf_m <- cumsum(count_m)/n
#cliquet
x_c <- seq(0,ceiling(max(cost_capital_c)),1)
count_c <- hist(cost_capital_c,breaks=c(0,x_c),plot=F)$counts
cdf_c <- cumsum(count_c)/n
```

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Ich bin mir bewusst, dass eine unwahre Erklärung rechtliche Folgen haben wird.

Ulm, den 15.03.2017

(Unterschrift)