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le : ____________________

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Titre : Market Value Margins for a Non-Life insurance company under Solvency II
Practical calculations under the Cost of Capital approach

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Market Value Margins for a Non-Life insurance company under Solvency II

Practical calculations under the Cost of Capital approach

Mouna DAYA-VIOSSAT

MOTS-CLES: marge de risque, approche coût du capital, QIS5, fair value

KEYWORDS: MVM, Cost of Capital approach, QIS5, fair value

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Abstract

In the Solvency II framework, Market Value Margin (MVM) comes in addition to the Best Estimate valuation of liabilities as an attempt to provide a market-consistent value of technical provisions.
In the "Cost of Capital" approach that has now been adopted, the MVM is defined as the present value of the current and future costs of capital required to support the liabilities until full run-off. The capital itself is defined as the amount of funds needed over a one-year time-horizon to ensure solvency within a 99.5th confidence level, and this is computed by the Value at Risk (VaR) of the Available Capital, which in turn depends on the MVM. Hence, a mutual dependency between Capital and MVM arises. Several simplifications and approximations, most ignoring the circularity, have been suggested in order to project future capital requirements in the MVM calculations. However, little research has been done to quantify these approximations. The subject of this thesis is to propose a set of analytical methods to derive an "exact formula" for MVM for a Non-Life insurance company, from a theoretical model first, and then fit this model to a given set of simulations of future cash-flows, obtained from real claims history data. The constraints are the ones imposed to date by CEIOPS in its interpretation of the texts of the European Directive.

Problématique

Dans le contexte Solvabilité II, la marge de risque vient compléter au passif le Best Estimate pour estimer le niveau des provisions techniques en valeur de marché.
Dans l'approche "Coût du Capital" qui a maintenant été retenue, la marge de risque est définie comme la valeur actuelle probable du coût des capitaux présent et futurs requis pour supporter le passif jusqu'à son extinction. Le capital lui-même est défini comme le montant des fonds nécessaires sur un horizon d'un an pour assurer la solvabilité de la compagnie avec un intervalle de confiance à 99.5%. Ce capital est calculé par la Valeur à Risque (VaR) du capital disponible qui lui-même dépend de la valeur de la marge de risque, d'où une relation de circularité. Plusieurs simplifications ont été proposées afin de projeter les besoins futurs en capital entrant dans la définition de la marge de risque, tout en ignorant le caractère circulaire de son calcul. Cependant, peu d'études ont été menées afin de quantifier ces approximations. L'objet de ce mémoire est de parvenir à trouver un ensemble de méthodes analytiques permettant d'obtenir une "formule exacte" de la marge de risque pour une société d'assurance non-vie, à partir d'un modèle théorique dans un premier temps, pour ensuite caler ce modèle à un jeu de simulations données de flux futurs, tout en respectant les contraintes imposées à ce jour par le CEIOPS dans son interprétation des textes de la directive européenne.

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Chapter 1 Background

This first Chapter describes the context in which the Market Value Margin (MVM) under study in this thesis has been evolving over the last few years. After introducing the concept of "fair value of liabilities", it will briefly set the scene of Solvency II and refer to the latest definition as specified in QIS5 technical specifications; it will then provide an overview of the approaches adopted under various frameworks, describing how the Cost of Capital approach gained popularity over the Percentile approach; finally, it will define several elements that need to be taken into consideration when calculating the MVM.

1.1 Solvency II framework and Fair Value of Liabilities

1.1.1 Solvency II overview

The current Solvency I framework, in place since the early 1970s, takes a historic view of risk assessment. It uses a set of simple factors to calculate capital requirements. Among its weaknesses is the lack of financial convergence at the international level and the fact that the capital required under this system is both calibrated at a very low level of prudence and also relatively risk insensitive.

Solvency II – scheduled to come into force in October 2012 – takes a prospective view for risk definition. It sets out new EU-wide requirements on capital adequacy and risk management for insurers with the aim of increasing policyholder protection. It seeks to implement solvency requirements that better reflect the risks that companies face and deliver a supervisory system that is consistent across all member states.

Solvency II is based on three "Pillars":

– Pillar 1: considers key quantitative requirements, including own funds, the calculation of technical provisions and the rules relating to the calculation of the Solvency II capital requirements (the Solvency Capital Requirement – SCR, and Minimum Capital Requirement – MCR), with the SCR calculated either through an approved full or partial internal model, or through the European standard formula approach with an option of Undertaking Specific Parameters (USP's). One of the main components of Pillar 1 is that the technical provisions should now include a Risk Margin;
– Pillar 2: deals with the qualitative aspects of a company's internal controls, but does include the ORSA (Own Risk Solvency Assessment) which may involve quantitative analysis on a different basis than the Pillar I assessment, for example it is likely to involve an aspect of shareholder focus rather than the purely policyholder focus of Pillar I. It sets out requirements for the governance and risk management of insurers, as well as for the effective supervision of insurers;
– Pillar 3: focuses on disclosure and transparency requirements.

Solvency II is being created in accordance with the Lamfalussy four-level process (cf. Appendix F). Throughout the preparation for the Framework Directive proposal, the European Commission regularly asks the Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS) for advice upon certain issues, through its "Calls for Advice". While developing their answers CEIOPS consults with the European insurance industry through consultation papers and public hearings, seeking input "on the preparation of its advices to the European Commission and the drafting of its own recommendations, guidelines and standards[1]". It has engaged in consultation with the industry to test the impact of proposed Solvency II regulations, since late 2005. Companies are involved in the process, via Quantitative Impact Studies (QIS), on a voluntary basis as well as contributing to public feedback from the industry on the advice published by CEIOPS.

[1] http://www.ceiops.eu/content/view/2/2/

To date, there have been 81 Consultation Papers (CPs) in three waves of advice and four completed QIS's (QIS5 is currently being run between August and November 2010, with an expected publication of a report on results by CEIOPS by April 2011).

A brief timeline of the project is presented below.

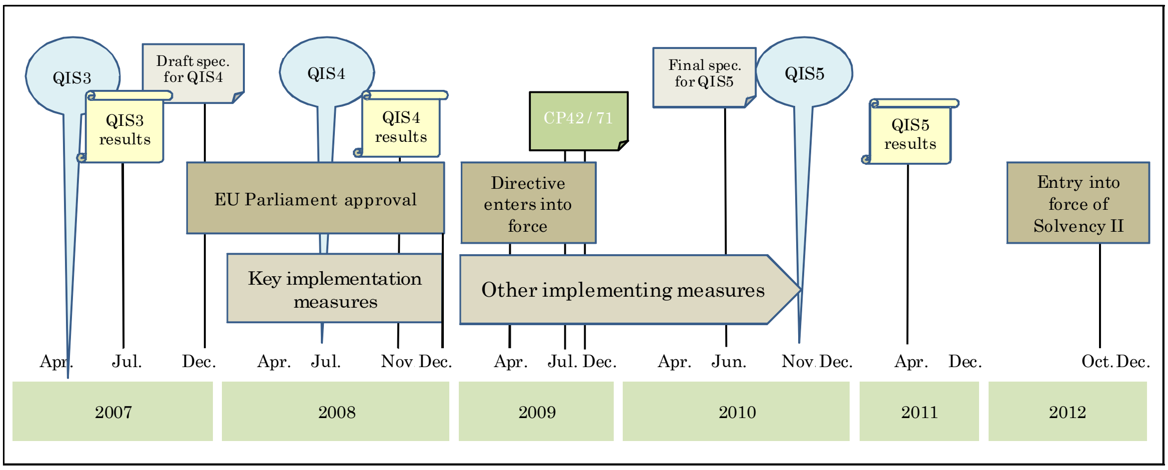

FIG. 1.1 – Solvency II project timeline

Under the prospective risk-approach underlying the Solvency II framework, better reflecting risks to which an insurance company is exposed implies obtaining a fair valuation of those risks. To achieve this, a market-consistent approach to risk valuation has been commonly sought for – as has been the trend in recent years – and this includes both sides of the balance sheet.

1.1.2 Market consistent economic (solvency) balance sheet

In the background of the MVM discussions within a Solvency II framework lies the market consistent economic (solvency) balance sheet on which Solvency assessment is built, with the capital requirements considering risks emanating from both sides of this balance sheet.
The diagram below (right) depicts its three main components (and their sub-components):

(i) the market value of assets (MVA),
(ii) the market consistent value of liabilities (MVL) consisting of the Best Estimate Liabilities (BEL) and the MVM for non-hedgeable risks and
(iii) Solvency Capital Requirement (SCR)

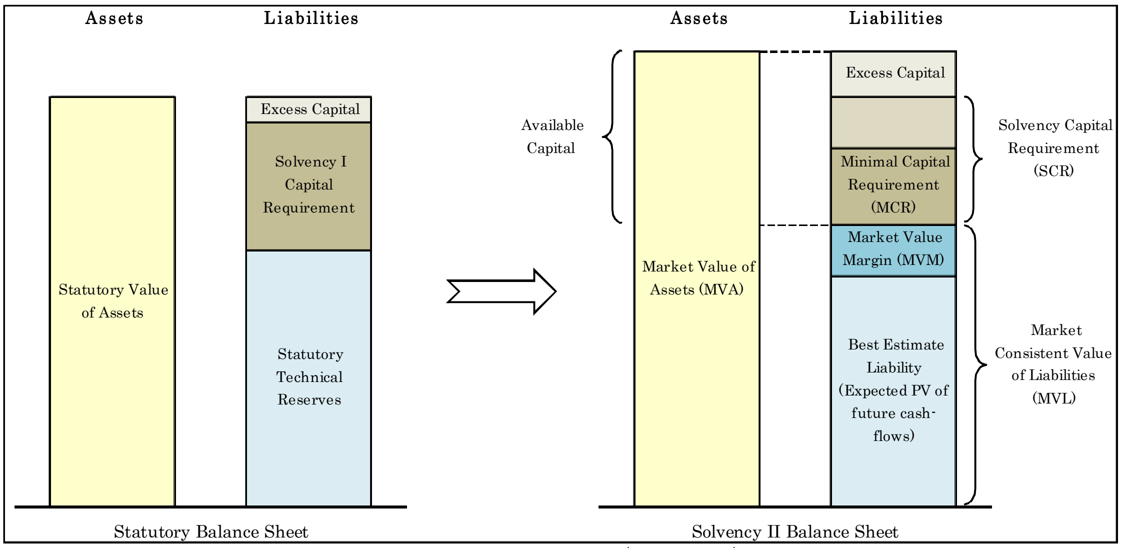

FIG. 1.2 – Statutory and economic (solvency) balance sheets

The economic balance sheet supersedes the Solvency I (or statutory) balance sheet shown above where implicit prudence was generally held in the Technical Reserves and Assets. Prudence is now made explicit in the MCR and SCR components.

MVA is relatively straightforward to conceive with assets being mostly valued at market prices already for those publishing IFRS balance sheets, whereas liabilities are often booked including prudential margins not necessarily linked with market information or best estimates. However, there is a general agreement that the balance sheet of an insurance company should be measured in a consistent way, and this is, among other things, what Solvency II is addressing through the market consistent valuation.

Another concept underpinning the scope affecting the MVM quantification is the types of risks affecting insurance liabilities:

– Hedgeable risks are the ones that can be hedged through financial instruments. The cost of hedging is given by the market value of those replicating instruments. The MVM is therefore implicitly embedded in their observed market prices.
– Non-hedgeable risks are risks for which a deep and liquid market is not available. They are risks for which a market price cannot be observed. Non-hedgeable risks include both financial and non-financial (underwriting) risks.

The figure below shows how risks affecting insurance liabilities can be broken down into four key components and gives examples of the types of risks that fall into each category (cf. [5]), covering both life and non-life activities.

Types of Risks

Sources of exposure		Hedgeable	Non-hedgeable
	Financial	- 10 year USD, EUR, Yen cash flow or interest rate option - 10 year equity option - Rational lapse behaviour	- 60 year USD, EUR, Yen cash flow or interest rate option - 15 yr emerging markets cash flow - 30 year equity option
	Non-financial	- Screen or exchange traded CAT risks - Actively traded securitised risks	- Most insurance risks, e.g. mortality, property, casualty, etc. - Irrational lapse behaviour

FIG. 1.3 – Types of risks affecting the liability cash-flows

Risks grouped in the first column can use a "Mark-to-market" approach and the MVM is thus implicit in the observed market prices. The second column captures "Mark-to-model" risks which thus require calculation of an explicit MVM.
Therefore, an explicit MVM is only applicable for non-hedgeable non-financial risks and (possibly) non-hedgeable financial risks, although for a non-life insurer the former are the most material.

1.1.3 Fair Value of Liabilities

Determining the fair valuation of assets and liabilities is a central component of solvency and financial reporting standards. While for the majority of assets used as investments for general insurance companies, the market value of assets (MVA) is determined by the capital markets, it is generally less straightforward to provide a fair value of liabilities. In most cases, insurance liabilities are not actively traded in deep and liquid markets and therefore the market consistent value cannot be determined directly from the capital markets. Some insurance liabilities have a readily obtainable market price through replication using tradable or synthetic financial instruments and the fair value is taken as the market value of the cost of setting up a replicating portfolio[2]. However, for the majority of non-life insurance liabilities, such securities do not exist. The market consistent value of liabilities (MVL) must therefore be explicitly calculated using market consistent valuation techniques.

Even where a market does exist for aspects of non-life liabilities (such as CBOT), attempts to derive the market values from approach such as Wang transforms have had limited success.

A commonly accepted description of 'fair value' of a liability is the price that a third party would charge to assume responsibility for the liability. The more risky the liability, the more compensation required by the third party. More formally, the market consistent value of liabilities is described as the price that would be paid in an 'arm's length' transaction between two knowledgeable and willing parties under normal business conditions. Unfortunately, the market price under any of the definitions given will vary between different third parties, potentially materially so.

Using the Best Estimate of the Liability (BEL) – i.e. the expected present value of future liability cash flows discounted using the risk-free rate – as an estimate for the 'fair value' would imply that those future cash-flows are also risk-free. Now, it is well acknowledged that this is not the case in that these do bear some risk. As a result, an insurance stakeholder under normal conditions will add a risk margin to allow for further uncertainty.

[2] The replicating portfolio or hedge portfolio is simply defined as the portfolio of assets that most closely matches the corresponding liability cash flows. In the absence of arbitrage, and if the liability cash flows could be matched exactly, the market consistent value of the liabilities will exactly equal the market value of the replicating portfolio.

In this context, the Market Value Margin (MVM) is defined as a risk margin in addition to the BEL required to manage the business on an ongoing basis. The two components add up to reach the fair value of the liability.

Another way to look at this is to bear in mind that one of the aims of Solvency II is to seek policyholders' protection and to be the founding principle of insurance companies' solvency. As such, an insurer should hold sufficient available capital today such that they would still have assets in excess of liabilities after the theoretical 1 in 200 event (99.5% confidence level) over the next year. Thus, the level of solvency capital held today must be sufficient to support the potential movement in the fair value of all liabilities and all assets under a stress scenario over a one-year time horizon, and this movement includes both that of the BEL and that of the MVM.

There have been several methods under study to quantify the MVM. The approach now favoured within Solvency II is referred to as the Cost of Capital (CoC) method, where the MVM is estimated by the present value of the expected cost of current and future Solvency Capital Requirement (SCR) for non-hedgeable risks to support the complete run-off of all liabilities.

1.2 Market Value Margin (MVM) in the Solvency II framework

1.2.1 Definition

"*The value of technical provisions shall be equal to the sum of a best estimate and a risk margin*"

"*The risk margin shall be such as to ensure that the value of the technical provisions is equivalent to the amount that insurance and reinsurance undertakings would be expected to require in order to take over and meet the insurance and reinsurance obligations*"[3]

The risk margin can be seen as a buffer above discounted best estimate cash flows, to protect against worse than expected outcomes. It is intended to represent the cost a third party would incur when purchasing the insurance portfolio in case of insolvency.

Throughout this document, the Market Value Margin will be referred to as MVM or Risk Margin interchangeably.

1.2.2 A brief history

The definition set out by the Commission leaves a lot of room on how to calculate the Market Value Margin.

Discussions around the appropriate approach to calculating MVMs progressed for some time at a European level under the Solvency II banner. The main approaches that were under consideration were the percentile approach and the Cost of Capital (CoC) approach (described in more detail further below). The main bodies entering into this reflection were CEIOPS[4], the CRO Forum[5], the CEA[6], and Groupe Consultatif[7].

[3] Level 1 text, cf. Appendix F
[4] Committee of European Insurance and Occupational Pensions Supervisors – the Level 3 Committee for the insurance and occupational pensions sectors under the so called "Lamfalussy Process"
[5] Chief Risk Officers Forum – a professional risk management group represented by CROs of the various members
[6] Comité Européen des Assurances – the European insurance and reinsurance federation
[7] Groupe Consultatif Actuariel Européen – representing the actuarial profession in discussion with the EU institutions

The main approaches are listed below along with a brief definition for the risk margin:
– Cost of Capital: the risk margin is the expected cost of holding current and future capital requirements to run off the existing liabilities
– Percentile: the risk margin is a set percentile of the discounted ultimate future payments less the BEL
– Explicit assumptions: a margin comes out on top of the BEL by selecting prudent explicit parameters in the reserving exercise (e.g. decrease discount rate)
– Implicit assumptions: the risk margin results in arbitrary prudence when assessing technical reserves (e.g. non-discounted reserves, prudent development factors, prudent initial expected loss ratios, prudent case estimates…)

It should be borne in mind that one of the goals of Solvency II is to provide consistent methodologies for all insurance policies, be it Life or Non-Life. In the initial discussions that took place around the approach to use for MVM calculation, Non-Life insurers tended to favour the percentile method, while the Life insurers favoured the CoC approach, mainly because many of the former already used percentiles to derive confidence intervals around technical provisions and the latter used a cost-of-capital approach when computing the European Embedded Value (E.E.V). However, the percentile approach was not easily implementable in Life insurance, mainly because of the prominence of asset liability matching (A.L.M) which makes it difficult, when considering a given percentile of the mathematical reserves, to strip out the non-financial element – which relates to the liability side solely – from the financial one. Existing methods were complex to use in practice and as such have been abandoned. In addition, some Non-Life insurers argued that a fixed percentile was not always consistent across all lines of business and less so with pricing where the percentile-level risk margin differs between a short-tail class and a long-tail class of business.

As a result and over the recent years, the Cost of Capital method has emerged as a strongly preferred methodology for this calculation and has already been adopted in certain jurisdictions as part of the framework for establishing capital requirements and measuring available capital. Several publications advocating this approach can be found. In their paper (cf. [7], 2006), Groupe Consultatif provides a comparison between the first two approaches (CoC and percentile) and what they call an "Assumption Approach" representing the current industry practices at that time. Their opinion was that only once the required level of capital has been defined, the CoC method has greater clarity over the other two. A discussion paper (cf. [5], 2006) demonstrates why it is also the CRO Forum's preferred approach, pointing out six main reasons (where the CoC approach (i) "supports appropriate risk management actions", (ii) "provides a more appropriate reflection of risk", (iii) "ensures a better response to a potential crisis in the insurance industry", (iv) "is easy to implement", (v) "is transparent, easily verifiable and understandable by the supervisor and other constituencies" and (vi) "it passes the "use test" envisioned in the Solvency II framework"). Similarly, the CEA's reasoning to recommend the CoC approach (cf. [6], 2006) is that it provides: (i) "consistency with the overall framework", (ii) "transparency", (iii) "verification and auditability", (iv) "homogeneous applications" and it has (v) "workable precedents". On the other hand, the Commission initially recommended the percentile approach (through QIS1 and QIS2, with a confidence interval set at the 90^{th} and at the 75^{th} percentile respectively – although QIS2 also invited participants to the alternative CoC approach to enable an assessment of the two methods) and later accepted in principle (in QIS3) the CoC favoured by the industry for its apparent ease of implementation.

In addition – and complementing on the "workable precedents" item mentioned above – certain regulatory regimes, namely the Swiss Solvency Test (SST) for instance uses a Cost of Capital approach in determining the MVM. Indeed, Switzerland (which is not a member of the EU and consequently not subject to Solvency II) led the way with their SST suggested by the Swiss Federal Office of Private Insurance (FOPI or OFAP[8]) which came into force since 2006. Mirroring the European Union Solvency II project that will be applicable for all its member states from 2012, the SST had been designed with a desire of compatibility with the then known Solvency II building blocks. It has thus acted as a pioneer on the CoC approach providing a useful industry test run. As a matter of fact, the Standard Formula developed under Solvency II uses much from the SST.

[8] Office Fédéral des Assurances Privées, Swiss supervisor of private insurance companies

The Cost of Capital approach had also already been used by the insurance industry for portfolio valuation or Embedded Value reporting and business transactions.

1.2.3 CoC vs. Percentile approach

This section more closely describes the two main approaches that have been suggested to assess the MVM.

1.2.3.1 The Percentile method

The percentile (or quantile) approach takes the perspective that the price required by a third party would be such as to ensure that the liabilities can be met with a predefined confidence level.

The method uses the underlying risk distribution to directly determine an aggregate fair value of the liability (the sum of the BEL and the MVM) as a predefined critical percentile of the distribution. Rather than calculating an explicit MVM as in the Cost of Capital methodology, the percentile method simply selects a specific percentile of the discounted ultimate future payments distribution. The difference between this amount and the BEL gives the MVM, as can be seen on the following chart:

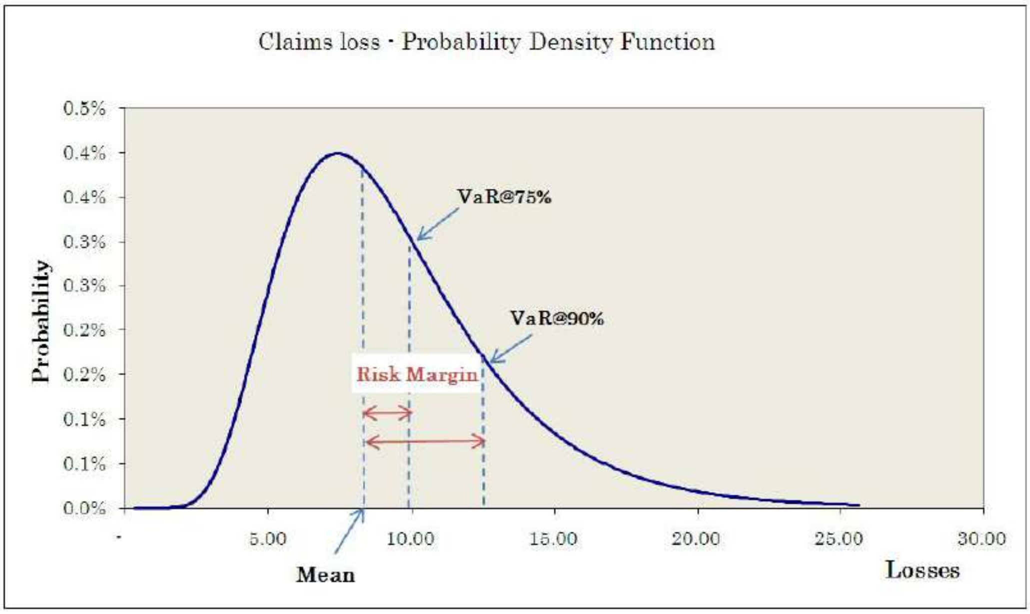

FIG. 1.4 – Percentile approach for risk margin

This approach was first described (for regulatory purposes) and prescribed by the Australian Regulator (APRA) in the Prudential Standard GPS 210 – *Liability Valuation for General Insurers*. It was also originally proposed for Solvency II purposes but has since lost support relative to the Cost of Capital methodology.

A number of drawbacks appeared for this approach to be consistent with the future solvency frameworks and financial reporting standards. First of all, setting the percentile level is necessarily arbitrary by nature (e.g. 90% under QIS1 and 75% under QIS2) and does not seem to show any link with any potential market price. Moreover, an appropriate level for the percentile would depend on the shape of the distribution. In arguing in favour of the Market Cost of Capital approach, the CRO Forum provides the following analysis to show that the percentile approach gives no consideration to the shape of the distribution, unlike the CoC method. The illustrative example below considers a long tailed (Gamma) skewed distribution and a short tailed (Normal) symmetric distribution, and compares the risk margin values as per these two alternative methods. The risk margin under the Market Cost of Capital approach (MCoC, although CoCM is also found in the literature) comes out to be at the 76th percentile in the first case (left hand side) and at the 56th percentile in the second case (right). This – if we believe the MCoC shows a better reflection of risk –

respectively leads to an understatement and an overstatement of the risk margin in the percentile approach.

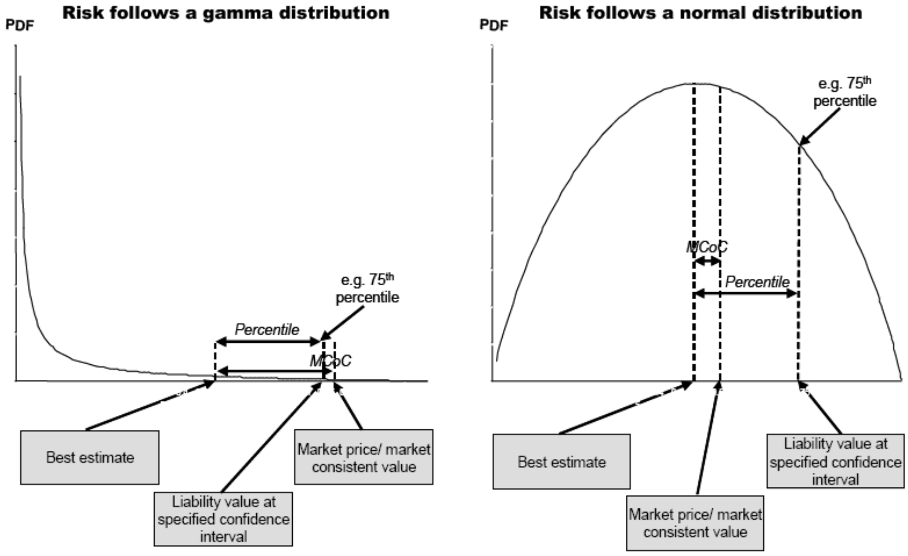

FIG. 1.5 – CRO Forum issues with percentile approach – (*source:* cf. [5])

Another limitation is that this approach heavily relies on actuarial judgements (when selecting the underlying volatility, distributions, aggregation levels, development factors, etc.), giving little transparency and auditability to regulators. Then, it requires significant data and analysis that may be a major concern for small and medium companies.

1.2.3.2 The CoC method

1.2.3.2.1 Definition

The CoC approach takes the perspective that sufficient capital is needed to be able to run-off the business. Here, the risk margin is estimated by the present value of the expected cost of current and future Solvency Capital Requirement (SCR) for non-hedgeable risks to support the complete run-off of all liabilities.

Schematically, the MCoC calculation can be carried out in 4 steps:
– First, project the expected SCR until all liabilities run-off. This puts into the equations the fact that an undertaking taking over the portfolio has to put up future regulatory capital $SCR(1)$, $SCR(2)$, … , $SCR(n-1)$ until the portfolios have run-off completely at time $t = n$;
– Second, multiply all current and future SCR by the Cost of Capital rate (CoC rate CoC or c). This capture the fact that the insurer selling the portfolio has to compensate the insurer taking over the portfolio for immobilizing future capital requirements;
– Third, discount everything to time 0;
– The sum then gives the CoC risk margin.

These steps result in the derivation of the MVM at time 0. They are illustrated in the chart below.

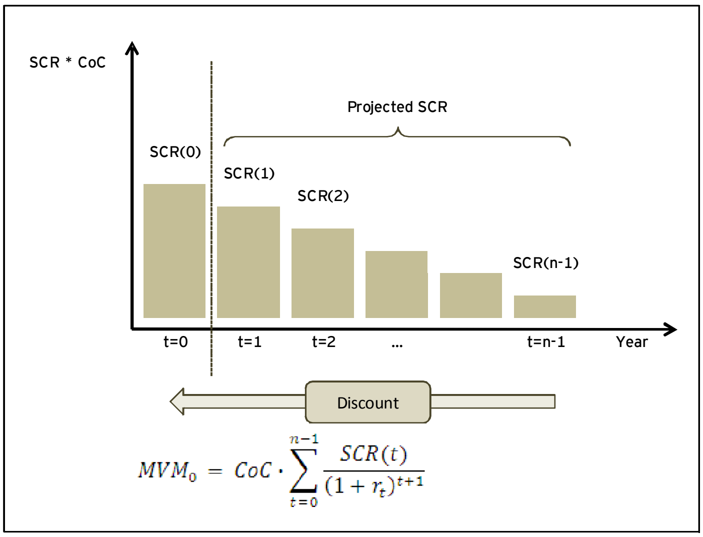

FIG.1.6 – General CoC MVM formulation

This is only the generic formulation of the Cost of Capital approach for the risk margin. The specifications as to what the SCR should include or exclude and what level of granularity is required according to CEIOPS are described in more detail in §1.2.4.

The time period under consideration spans from $t = 0$ (valuation date also assumed to be when the portfolio transfer takes place) to $t = n$ (assuming that the risk relating to the obligations will run off within n years).

Unlike the SST formulation of the CoC risk margin that ignores capital raised in the first year, the generic formulation under Solvency II here starts at $t = 0$, reflecting an instantaneous exit value approach. This was not always clearly defined until QIS4. The rationale for starting at $t = 0$ behind Solvency II is that capital would have to be held for all future years, including the first year.

In the current use of the CoC approach where the additional assumption of disregarding the circularity is made (cf. below), the pros that have been put forward in using this approach, as opposed to the percentile method, is its ease of implementation, where no stochastic modelling is required, the better risk sensitiveness and the more explicit link with the exit value theory.

1.2.3.2.2 The circularity issue

The SCR, in its extended formulation as implicitly given in the Directive, is computed as the Value at Risk (VaR) of the Available Capital. This Available Capital (cf. FIG. 1.2) depends on the SCR via the Risk Margin, as can be seen from the formula $MVM_0 = CoC \cdot \sum_{t=0}^{n-1} \frac{SCR(t)}{(1+r_t)^{t+1}}$ expressed in FIG.1.6. Put differently, the Risk Margin is described in terms of the SCR and the SCR depends on the potential movement in the Risk Margin which makes up part of the Market Value of Liabilities (MVL), hence a mutual dependency.

In order to solve this circularity, CEIOPS states[9] that "any reference to technical provisions within the calculations for the individual SCR modules of the Standard Formula is to be understood to exclude the risk margin".

Thus, it is assumed within the Standard Formula that the Risk Margin does not change under a stress scenario.

[9] QIS5 Technical Specifications (cf.68[3]) – SCR.1.3

A lot of public discussions have arisen around the CoC approach to MVM, acknowledging the inherent circularity in its definition. The option has generally been taken to ignore this circularity based on the intuitive assumption that given the relative size of the MVM in comparison of the total MVL, the impact of stressing the MVM as part of the MVL would be insignificant. This shortcut serves as a fundamental basis for all other resulting simplifications that have been suggested in this context, as will be described further.

However, very few analyses have been conducted to quantify the materiality of these approximations.

In addition, it should be worth mentioning that these conceptual simplifications are not reflected in market behavior. Indeed, in the case of a reinsurance cover for instance, the reinsurance costs for a book that has deteriorated over the previous year increase disproportionately, thus making sense to assume that if the entirety of the book were sold to a third party, they would also charge a much larger MVM in a distressed scenario than in more usual conditions.

A case study from some academic work ([9]) uses a claims model based on an underlying lognormal distribution and two sets of volatilities to solve the MVM. It considers the MVL along the three following assumptions under a stress scenario: the stress is applied to (i) the BEL only, (ii) the BEL + MVM and (iii) the MVM is determined by approximating the future SCRs in proportion to the projected BEL (this is known as the "proportional method" (e.g. as in the last three QIS's). The main result of this study is that it suggests that the proportional method systematically understates the capital base needed. In this case, CEIOPS states that "if this proves true – and a proportional method is still to be allowed as a simplification in the Solvency II context – a Cost-of-Capital rate higher than the rate of 6 per cent could be necessary in order to compensate for this bias."[10]

This thesis intends to add its own quantitative contributions to the issue by providing an estimate of what an exact solution to the issue would be as well as providing a measure of the suggested simplifications.

1.2.4 CPs 42 / 71 and QIS5 technical specifications

Consultation, advice and technical guidance on the now adopted Cost of Capital approach to assess the Market Value Margin can be found in Consultation Papers 42 and 71, and then further in the QIS5 technical specifications.

This section outlines the key messages from those papers as regards the risk margin, which will serve as a basis of comparison with the quantitative and qualitative results shown in this thesis.

1.2.4.1 CPs 42 / 71

CEIOPS' advice on the overall structure of the Risk Margin calculation through CP42 ("Draft CEIOPS' Advice for Level 2 Implementing Measures on Solvency II: Article 85(d) - Calculation of the Risk Margin"[11], 54 pages in total) and to a lesser extent through CP71 ("Draft CEIOPS" Advice for Level 2 Implementing Measures on Solvency II: SCR Standard Formula Calibration of non-life underwriting risk"[12]) essentially considers the following:

– the definition of the reference undertaking, including the assumptions that this undertaking has to fulfil: CP42 introduces the principle to assume, when calculating the fair value of liabilities, that they are being bought by a third party (the "reference undertaking") whose only liabilities will be the Line of Business (LoB) in question. The following assumptions hold in relation to the reference undertaking:

[10] CP42 (cf.[1]) – 3.115

[11] https://www.ceiops.eu/fileadmin/tx_dam/files/consultations/consultationpapers/CP42/CEIOPS-CP-42-09-L2-Advice-TP-Risk-Margin.pdf

[12] https://www.ceiops.eu/fileadmin/tx_dam/files/consultations/consultationpapers/CP71/CEIOPS-CP-71-09-Draft-L2-Advice-Calibration-of-the-non-life-underwriting-risk.pdf

- it will be exposed to underwriting risk (existing business only), counterparty default risk (ceded reinsurance and SPVs only), operational risk and unavoidable market risk. The internal models of the original undertaking (partial or full) can be used to measure these;
- it is to hold assets exactly equal to the sum of SCR, technical provisions and risk margin;
- the loss absorbing capacity of technical provisions in the reference undertaking corresponds to those of the original undertaking;
- the held assets can be assumed to be those that minimise the SCR and thus the risk margin;

It is to be assumed that no diversification benefits should arise between lines of business and that a single SCR will be calculated to cover all the risks within the reference undertaking.

- the calculation of the risk margin in accordance with the Cost of Capital approach;
- the calibration of the Cost-of-Capital rate: the rate of at least 6% is prescribed – however if the proportional method for allocating SCR to future years proves to systematically understate the risk margin then a rate greater than 6% may be necessary to compensate for this;
- the projection of the future SCRs related to the reference undertaking;
- simplification methods (covered in CP45 "Simplified methods and techniques to calculate technical provisions"[13])

Following industry comments on the consultation papers, the latest set of advice to date have been incorporated into the latest Quantitative Impact Studies QIS5. The core concepts and calculations remain, but some clarifications and amendments to the latest CPs were suggested.

1.2.4.2 QIS5

The §V.2.5 "Risk Margin" of the QIS5 Technical Specifications (cf. [3]) sets out the following:
- The definition of the risk margin and the general methodology for its calculation;
- The Cost-of-Capital rate to be applied in the risk margin calculations;
- The level of granularity regarding the risk margin calculations;
- Simplifications that may be applied in the risk margin calculations.

Among the noticeable amendments from the CEIOPS advice is the allowance for diversification between lines of business. Risk margins must still be calculated for each Solvency II line of business and so the whole account risk margin, taking account of diversification, must be allocated to each line of business. This allocation must recognise the contribution of each line of business to the overall SCR over the lifetime of the liabilities. No diversification credit is allowed for, however, between legal entities' risk margins in Group consolidation.

As with previous QIS's, these technical specifications were released with a number of accompanying spreadsheets and calibration tools[14] to assist undertakings in completing the exercise. The main ones are the "QIS5 Spreadsheet for solo entities" and "Helper Tabs", which assist in performing some intermediate calculations. The risk margin has its own separate accompanying Helper Tab spreadsheet[15], assisting participating companies in carrying out the risk margin calculations through a hierarchy of simplifications, including a "full calculation" when undertakings have derived the future SCRs for the different risk modules of the reference undertaking.

CEIOPS provides recommendations and guidance through industry consultation and feedback. It should be reminded that it has no regulatory powers as such but can rather be regarded as a powerful lobby Group to the Commission comprising the individual European regulators. However, CEIOPS is staffed by representatives from the regulators from each of the member states.

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[13] https://www.ceiops.eu/fileadmin/tx_dam/files/consultations/consultationpapers/CP45/CEIOPS-CP-45-09-L2%20Advice-TP-Simplifications.pdf

[14] https://www.ceiops.eu/en/fixed-width/consultations/qis/quantitative-impact-study-5/spreadsheets-and-it-tools/index.html

[15] https://www.ceiops.eu/fileadmin/tx_dam/files/consultations/QIS/QIS5/Spreadsheets&IT-Tools/23.09-update/H_Risk_Margin_201000921.xls

The European insurance industry includes a large number of small and medium-sized companies which would in practice face significant costs if they were required to interpret the European Directive on their own and develop complex internal models to calculate their solvency capital requirements. The proposed alternative of using simplifications such as the Standard Approach will allow them to calculate capital requirements with little extra cost, although the approach is likely to be slightly more conservative, reflecting its approximate nature and the fact that smaller firms have less diversification benefit within their portfolio. The same holds true for a number of key and specific concepts, such as, precisely, the Risk Margin calculation, where experience shows that most companies, including the ones using an internal model, will follow and comply (on a best-effort basis) with what has been set out in the latest QIS5. Indeed, according to the QIS4-report with respect to the use of simplifications and proxies[16] "*The majority, if not all, of undertakings (independently of their size) used simplifications to project the SCR for the purposes of calculating the risk margin. The risk margin proxy and helper tab for non-life were also extensively used by undertakings.*"

1.3 Framework comparison

1.3.1 Framework overview

This table gives a brief overview of the main frameworks in place.
The various concepts displayed for the capital measures such as "Capital base", "Risk measure", "Time Horizon" and "Risk exposure horizon" are described in more detail in section 1.4.

		Solvency II	SST[1] (Switzerland)	APRA[2] (Australia)	ICA[3] Non-Life (UK)	NAIC[4] (USA)	Rating agencies	IFRS Phase 2
Capital	Balance Sheet basis	Economic	Economic	Economic	No set definition	US Statutory	Varies	Economic
	Risk Measure	VaR on a 99.5% confidence level	Expected Shortfall on a 99% confidence level	VaR on a 99.5% confidence level	VaR on a 99.5% confidence level	n.a ---------------- RBC (Risk Based Capital) ---------------- Prescribed factors based on historic industry adverse development	S&P: VaR at ratings target	n.a
	Risk exposure	1 year	1 year	Ultimate	Ultimate	n.a	Ultimate	n.a
Risk Margin	CoC / Percentile / Implicit margin / Other	CoC approach on LoB level, including non-hedgeable market risk (QIS5)	CoC approach on total portfolio level, including non-hedgeable market risk	Percentile MVM= max(75^{th} - mean;0.5*Stdev)	No explicit margin ------------- Undiscounted reserves + UPR	No explicit margin ------------- Undiscounted reserves + UPR	Varies	3 methods: - percentile - CTE - CoC
	Capital base for MVM	SCR	SST	n.a	n.a	n.a	Ratings target	IFRS gives no guidance about what capital be costed if CoC route is chosen

[1] Philip Keller, Swiss Federal Office of Private Insurers, The Swiss Experience with Market Consistent Technical Provisions – Cost of Capital Approach, February 24, 2006
[2] Australian Prudential Regulation Authority (APRA), General Insurance Risk Margins Industry Report, 30 June 2004 (issued October 2005)
[3] Individual Capital Assessment
[4] National Association of Insurance Commissioners (NAIC) - Risk Base Capital (RBC) approach
n.a: "not applicable"

Table 1-1 – Overview of Capital and MVM approaches

1.3.2 Convergence between frameworks: IFRS 4 (Phase II) and Solvency II

Despite the fact that international accounting standards for insurers are now moving towards a fair value framework for financial reporting, it is very unlikely, given the current developments that IFRS and Solvency II will converge.

[16] QIS4 report sub-section 7.2.5, page 78 (https://www.ceiops.eu/fileadmin/tx_dam/files/consultations/QIS/CEIOPS-SEC-82-08%20QIS4%20Report.pdf)

All listed EU companies have been required to use IFRS[17] since 2005. The overall objective is to create a sound foundation for future accounting standards that are principles-based, internally consistent and internationally consistent. The IASB[18] (in a joint project with the US FASB[19]) is currently engaged in a project to devise and issue a comprehensive accounting standard that will address recognition, measurement, presentation, and disclosure requirements for insurance contracts, known as "IFRS 4 Phase II". The standard is expected to take effect in 2016, and implementation is likely to be 2013 or 2014. In the latest proposal[20], some clarifications have been made as regards the decomposition of the insurance liabilities but no clear guidance has emerged.

Specifically, there is still no clear agreement on the risk margin which is even a major topic of discussion between the IASB and FASB. The Exposure Draft permits three calculation methods – percentile, conditional tail expectation and Cost of Capital. This is in contrast with Solvency II which provides detailed guidance on the CoC methodology as a unique measure for MVM. Moreover, a "residual margin" is introduced, designed to eliminate any gains at the inception of the insurance contract, which is not required under Solvency II. The granularity for calculating the risk margin is also different. Diversification between lines of business is taken into account in the Solvency II risk margin whereas diversification benefit between portfolios is not allowed under IFRS 4 (Phase II), although "portfolios" are not defined. The CoC rate definition is likely to be different as well.

1.4 Other considerations

Some key elements underlying the Cost of Capital methodology need to be defined. As has been shown in "Table 1-1 – Overview of Capital and MVM approaches", there are a number of definitions and approaches for the capital and risk margin elements across frameworks and reporting purposes. This thesis fits into the scheme of the Solvency II framework as outlined by CEIOPS's work on recommendations and guidelines (referred to as the Level 2 implementing measures or the future Level 3 supervisory guidelines regarding the risk margin calculations).

1.4.1 Capital base

Under the risk based economic approach that serves as a basis for the Solvency II framework, the capital base used to reflect the risks faced by the undertaking assuming the insurance risk is the Solvency Capital Requirement (SCR). It is calibrated to a 1 in 200 probability of ruin (assuming a Value at Risk (VaR) measure[21]) over a one-year period and based on a comprehensive analysis of risks that should take into account both risk mitigation and diversification.

The current SCR can be calculated using (i) the "Standard Formula" calibrated by CEIOPS using a combination of stress and scenario tests and factor-based calculations, splitting risks into modules and sub-modules for capital purposes with an allowance for aggregation and diversification across the modules as laid out in the QIS's (cf. Appendix F), or (ii) an internal model that would reflect a firm's own risk profile and management approach more appropriately.

For the sole purposes of the risk margin calculation, however, the capital base definition, whether assessed through the Standard Formula or an internal model slightly differs from its conceptual definition, with a number of assumptions and simplifications suggested by CEIOPS to make it operational[22] through the Standard Formula.

The SCR here should capture the following:
– the underwriting risk;

[17] International Financial Reporting Standard
[18] International Accounting Standards Board
[19] Financial Accounting Standards Board
[20] The Insurance Contracts (Phase II) Exposure Draft (ED) was published on July 30th 2010
[21] The risk measure used by the SST for example is the Tail VaR at the 99th confidence level.
[22] QIS5 Technical Specifications (V.2.5) (cf. [3])

– the unavoidable market risk – although it is stated that[23] "For non-life insurance obligations and short-term and mid-term life insurance obligations the unavoidable market risk can be considered to be nil";
– the credit risk with respect to reinsurance contracts and special purpose vehicles;
– operational risk;

Moreover, it should be assumed that:
– underwriting risk related to new business is not included, with the exception of the bound but not incepted business;
– diversification between lines of business is recognised.
In this respect, the risk margin is to be calculated per line of business. In order to do so, QIS5 suggests to determine the risk margin for the whole business, allowing for diversification between lines of business, and then to allocate the amount to the lines of business according to their contribution to the overall SCR during the lifetime of the business.

The complete calculation of the risk margin also requires estimates of this SCR in all subsequent periods. The same definition and scope is then to be applied to these future SCRs.

1.4.2 Risk exposure horizon: 1 year vs. ultimate

The risk exposure horizon describes the timeframe over which the capital should be sufficient to cover for the risks that could emerge in a distressed scenario.

This should not be confused with the horizon over which the MVM captures the capital costs (in the CoC method). Indeed, whereas on the one hand the MVM reflects the capital cost of risks over the lifetime of the liabilities (until run-off), the risk exposure horizon, on the other hand, determines whether the capital base is intended to be sufficient to absorb adverse development for the whole run-off period (in which case capital is raised at inception (valuation date) and is gradually released on a yearly basis) or, alternatively, whether it is intended to provide capital sufficient to absorb adverse deviations from expectations just over the next year – and be further funded sequentially one year at a time until the business totally runs-off.

For Solvency II, the SCR is meant to cover one-year of deterioration, meaning that only "shocks" applied to the following year are considered. The following graph depicts, on the liability side of the economic balance sheet, how the capital (denoted Cap_0 below) funded at time $t = 0$ is adequate to restore the balance sheet to a fair value of liabilities at the end of a distressed first year, where both the BEL and the MVM are subject to a distressed scenario. Note that for illustrative purposes, only the reserve risk is taken into account.

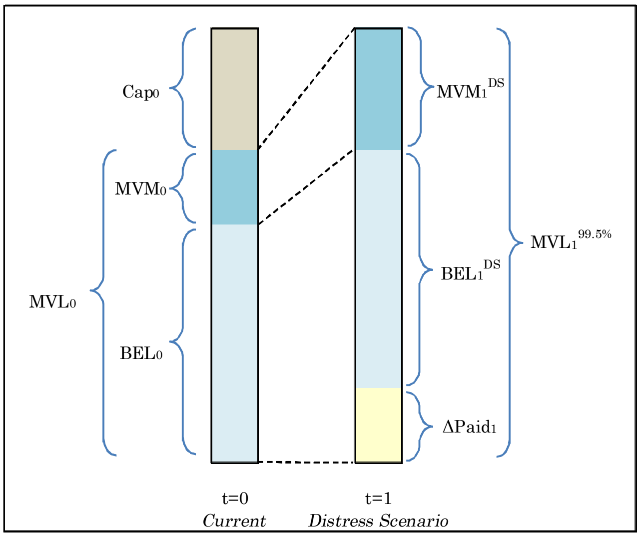


FIG. 1.7 – SCR and the one year horizon basis

[23] QIS5 Technical Specifications (cf. [3]) (TP.5.18)

It can be seen that the distressed scenario $MVL_1^{99.5\%}$ (at time $t = 1$) is the sum of $BEL_1{}^{DS}$ and $MVM_1{}^{DS}$ (i.e., the estimated BEL and MVM at the end of one year following a distress event at the 99.5th percentile) and the claims paid out during the year. The required capital is the difference between the distressed scenario MVL and the current MVL. It can also be expressed as the sum of the changes in the BEL and MVM as follows:

$$
\begin{aligned}
&Cap_0 = MVL_1^{99.5\%} - MVL_0 \\
&= \left(BEL_1{}^{DS} + MVM_1{}^{DS} + \Delta Paid_1\right) - \left(BEL_0 + MVM_0\right) \\
&= \left(BEL_1{}^{DS} - BEL_0\right) + \Delta Paid_1 - \left(MVM_0 - MVM_1{}^{DS}\right) \\
&= \Delta BEL + \Delta Paid_1 + \Delta MVM
\end{aligned}
$$

As mentioned above, the change in MVM is ignored in the QIS5 simplifications.

1.4.3 CoC rate

"The Cost-of-Capital rate used shall be equal to the additional rate, above the relevant risk-free interest rate, that an insurance or reinsurance undertaking would incur holding an amount of eligible own funds, […] equal to the Solvency Capital Requirement necessary to support the insurance and reinsurance obligation over the lifetime of that obligation"[24].

In order to be market consistent, the Cost of Capital rate component of the MVM needs to be calibrated appropriately. The CoC rate prescribed by CEIOPS is to be a fixed 6% over the risk-free rate, as inspired by the Swiss Solvency Test.
The set calibration is still subject to ongoing discussion. Additionally, as mentioned in "1.2.3.2.2 The circularity issue" CEIOPS states that a larger rate than 6% should be used if the proxies for computing MVM have a tendency to underestimate the SCR.

1.4.4 Discount factors

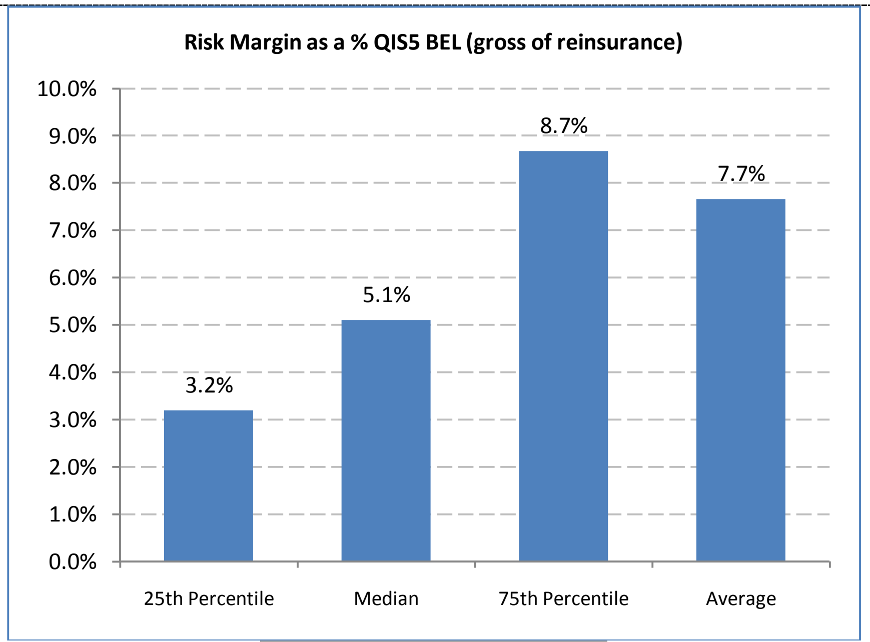
The risk-free interest rate term structure in the relevant currency at the valuation date should be used in the MVM and also when deriving the BEL.
The QIS5 package provides some risk-free interest rate term structures for a number of currencies, with the possibility to include 0%, 50%, 75% or 100% of the illiquidity premium. For General Insurance liabilities, discounting should be carried out with a 50% illiquidity premium. However, when discounting the future SCRs in the MVM calculation, the risk-free rate term structure should not include the illiquidity premium, on the basis that "[…] the reference undertaking may not be able to earn the illiquidity premium under the conditions of the transfer scenario"[25]. This illiquidity premium aspect and its relevance on discounting the liabilities will be ignored here.

1.4.5 Relative size

The size of the MVM for a Non-Life insurance company will vary according to several factors including the business mix, the company's size and maturity.

The following graphs provide some indicative results overview on the relative size of the MVM as calculated by Non-Life insurance companies in their QIS5 submission.

[24] Level 1 text Article 76 §5, cf.[4]
[25] QIS5 Technical Specifications (cf. [3]) (TP.5.10)



	Risk margin (as a % of QIS5 BEL)
St. Dev.	10%
Max	63%
Min	0%

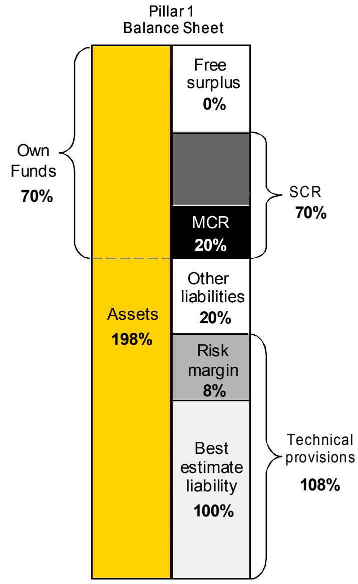


FIG. 1.8 – Risk Margin as a percentage of BEL – QIS5 survey results overview[26]

The maximum is 63%, caused by the impact of premiums receivable on the BEL.
On average, the MVM makes up 8% of the BEL (gross of reinsurance) on a Solvency II basis.

Having introduced the background and the main concepts underpinning the MVM calculation under the Cost of Capital approach as adopted in the Solvency II regime – and covered the inherent complexity governing its formulation, the following Chapter will now give an overview of its practical implementations in the form of proxies as suggested by CEIOPS first, and then through various methods that could be used to collapse the circularity issue and provide a better estimate of the MVM.
It will further introduce some notations and equations and study two analytical structures where "closed" formulas for the MVM have been derived. These two analytical structures will be the ones ultimately implemented in the third Chapter.

[26] Source: "Quantitative Impact Study 5 – Post-QIS5 insights Non Life, November 2010". This provides a QIS5 results benchmarking overview from a survey conducted by Ernst & Young on just over 60 solo entities across 15 European territories (including the UK).

Chapter 2 MVM in practice – attempts to approach an exact solution

In the Cost of Capital approach under study here, the Market Value Margin is the expected (discounted) cost of current and future capital requirements, with capital amounts defined in terms of the changes in economic balance sheets. One of the main challenges for determining the MVM under this approach is to determine future capital requirements. Most importantly, it needs to be acknowledged that with the future economic balance sheets being unknown, future capital requirements are random and similarly future MVM amounts are random. And the further away the time horizon is, the more uncertain any such calculation becomes – a situation often figuratively described as the "funnel of uncertainty". In practice, even with the circularity issue put aside, approximating expected future SCRs until run-off turns out to be the only way to go for most companies without investing significant time and resources.

One of the purposes of this thesis is to provide a "closed-form" solution given a specified modeled behavior for claims, within a specified framework. As such and for the purposes of demonstration and the sake of avoiding unnecessary complications, we will now restrict ourselves to the reserve risk component of the required capital. From this section onwards, the SCR will be that of a single line of business of a Non-Life insurer subject to the reserves risk only, and will simply be referred to as "Capital".

In this context, the future balance sheets and hence the MVM amounts will only depend on how the liabilities will evolve over time, which, from an estimation perspective at the time of calculation, is random. When writing the equations, it is thus essential to consider the information being used to define the distributions of the respective amounts that come into the picture.

The current and future Capital and MVM come under two general simultaneous equations:

$$Cap_{T|t} \sim 99.5\% \left[\frac{U_{T+1} + MVM_{T+1}}{1 + r_{(T+1,T)}} | \mathcal{F}_{T,t} \right] - \left(U_{T|t} + MVM_{T|t} \right) \tag{2-1}$$

$$MVM_{T|t} \sim CoC \cdot \sum_{i=T}^{n-1} \frac{E\left[Cap_i | \mathcal{F}_{T,t}\right]}{\left(1 + r_{(i+1,T)}\right)^{i-T+1}} \tag{2-2}$$

where:

– the notation $|\mathcal{F}_{\mathrm{T}}$ denotes that we are conditioning on information about claims development available at time T[27]. $\mathcal{F}_{\mathrm{T,t}}$ further denotes that we are ultimately conditioning on information available today at time t, and in general the information between t and T follows a random path

– $Cap_{T|t}$ and $MVM_{T|t}$ are the distributions of the Capital and MVM respectively required at time T, given the information available at time $t \leq T$, all in time-equal- T money

– $U_{T|t}$ is the discounted back-to-time T ultimate claims estimate distribution at time T, given the information available at time $t \leq \mathrm{T}$, i.e. $U_{T|t} = \text{discount yield curve} \times \text{payment pattern} \times \mathrm{E}\left(C_n - C_T \middle| \mathcal{F}_{T,t}\right) + C_T$, with C_T defined below

– C_t is the undiscounted cumulative claims payments as at time t (cf. exact formulation as stated in equation (2-5))

– $r_{(i,T)}$ denotes the relevant risk-free forward rate at time T for maturity i

– n is the time of complete maturity (i.e. $C_n = C_{n+1} = C_{n+2} = \cdots$)

– CoC is the Cost-of-Capital rate

[27] More formally, let $\{\mathcal{F}_t\}_{t\geq 0}$ be an increasing sequence of $\sigma-$algebras in a probability space $(\Omega, \mathcal{F}, P)$. Such sequences will be called filtrations.

$Cap_{T|t}$ and $MVM_{T|t}$ can also be expressed as follows:

$$Cap_{T|t} \sim 99.5\% \left[\frac{E[C_n|\mathcal{F}_{T+1}]+MVM_{T+1}}{1+r_{(T+1,T)}}|\mathcal{F}_{T,t}\right] - \left(E\left[C_n|\mathcal{F}_{T,t}\right] + MVM_{T|t}\right) \tag{2-3}$$

$$MVM_{T|t} \sim \frac{1}{1+r_{(T+1,T)}}\left(E\left[MVM_{T+1}|\mathcal{F}_{T,t}\right] + CoC \cdot Cap_{T|t}\right) \tag{2-4}$$

using $U_{T|t} = E\left[C_n|\mathcal{F}_{T,t}\right]$ and $U_{T+1}|\mathcal{F}_{T,t} = E[C_n|\mathcal{F}_{T+1}]|\mathcal{F}_{T,t}$ (and $E\left[E\left[C_n|\mathcal{F}_{T,t}\right]|\mathcal{F}_t\right] = E[C_n|\mathcal{F}_t]$) according to the Tower Property[28] ($t \leq T$)).
This means that the ultimate cumulative claims payments are a random variable conditioned to the time of the calculation and the time in the projection.

More formally, $E\left[C_n|\mathcal{F}_{T,t}\right]$ should be broken down into the following equation:

$$E\left[C_n|\mathcal{F}_{T,t}\right] = C_T + \sum_{k=T}^{n-1} \frac{(C_{k+1} - C_k)}{\left(1 + r_{(k+1,T)}\right)^{k-T+1}} = U_{T|t} \tag{2-5}$$

Bearing this in mind and for the sake of simplicity in the notations, we will use the notation $\boldsymbol{E\left[C_n|\mathcal{F}_{T,t}\right]}$ or $\boldsymbol{U_{T|t}}$ interchangeably in the following.

2.1 QIS5 proxies

The opening year SCR calculated with the Standard Formula does not require the MVM as an input. The current MVM (MVM_0) only appears as a new item in the economic balance sheet when moving from the "Solvency I valuation principle" balance sheet. This is a first approximation, disregarding the mutual relation between these two quantities. The second approximation resides in how the future capitals are projected without re-computing a full capital requirement for each year until all liabilities become completely extinct. QIS4 originally suggested a "proportional proxy" where the future capital amounts follow the run-off pattern of the obligations, by approximating the future capitals as a ratio of expected discounted outstanding liabilities at each future time period to the opening expected discounted outstanding liabilities.

The proportional proxy is still allowed in QIS5[29] which further introduces a hierarchy of simplifications (in increasing simplicity order) for projecting future SCRs. It also provides some representative examples of such simplifications for some of them, as described below when relevant to the scope of this study (where the notations have been transposed to those of this section). In all cases, only the current (opening) MVM (MVM_0) is tackled.

1. "Make a full calculation of all future SCRs without using simplifications"
2. "Approximate the individual risks or sub-risks within some or all modules and sub-modules to be used for the calculation of future SCRs"
3. "Approximate the whole SCR for each future year, e.g. by using a proportional approach"
   For example: $Cap_t = \frac{BEL_t}{BEL_0}.Cap_0,\ t = 1, \ldots, n-1$
4. "Estimate all future SCRs "at once", e.g. by using an approximation based on the duration approach"
   For example: $MVM_0 = \frac{CoC}{1+r(1,0)} Dur_{mod}(0).Cap_0$
   where $Dur_{mod}(0)$ is the modified duration of reference undertaking's (re)insurance obligations net of reinsurance at $t = 0$
5. "Approximate the risk margin by calculating it as a percentage of the best estimate"
   In this simplest proxy, the MVM is simply determined as a given percentage of the best estimate technical provisions, $MVM_0 = \alpha_{LoB}.BEL_0$ where α_{LoB} is a fixed percentage (provided) for a given LoB.

The results presented in the next Chapter will quantify proxies 3, 4 and 5 compared to an exact solution.

[28] The Tower Property is one of the properties of Conditional Expectation: If $\mathcal{F} \subset \mathcal{H}$ then $E(X|\mathcal{F}) = E(E(X|\mathcal{H})|\mathcal{F})$.
[29] QIS5 Technical Specifications (cf. [3]) (TP.5.32)

2.2 Approaches towards explicit solutions

A small number of approaches have been suggested to solving the problem of circularity without making use of any proxies. We briefly present the main ones along with their limitations.

A common resolution to the circularity is to solve it backwards, from the point in time where all the liabilities are run-off.

Specifically, at some point in time n, there is no further uncertainty (U_n is fully known so there is no further requirement to hold future solvency capital), consequently $MVM_n = 0$.
Using equations ((2-1) and (2-2)) we get:

$$Cap_{n-1|t} = 99.5\%\left[\frac{U_n}{1+r_{(n,n-1)}}|\mathcal{F}_{n-1,t}\right] - \left(U_{n-1|t} + MVM_{n-1|t}\right) \quad \text{(2-6)}$$

$$MVM_{n-1|t} = \frac{CoC}{1+r_{(n,n-1)}}.Cap_{n-1|t} \quad \text{(2-7)}$$

Combining these, we get,

$$Cap_{n-1|t} = \frac{1+r_{(n,n-1)}}{1+r_{(n,n-1)}+CoC}\left(99.5\%\left[\frac{U_{n|t}}{1+r_{(n,n-1)}}|\mathcal{F}_{n-1}\right] - U_{n-1|t}\right)$$

2.2.1 Recombining binomial tree: option pricing analogy

Very broadly speaking, a binomial tree is a data structure accessed beginning at the root node.
Binomial trees are often used in the pricing of financial derivatives, as a graphical representation of possible intrinsic values that an option may take at different nodes or time periods. The Binomial tree is a discretized description of geometric Brownian motion which is often used to describe asset behavior. The structure is a recombining tree where the asset S underlying the derivative (for instance, the stock price in the case of a stock option) is assumed to follow an evolution such that, in each period in time, it increases by a fixed proportion or decreases by another fixed proportion. These fixed proportions are labeled as the "up factor" and the "down factor". The tree traces out an approximation of all possible price histories of the underlying asset.

We could transpose this to the evolution of the ultimate claims estimates U_t, as depicted in the following tree model diagram:

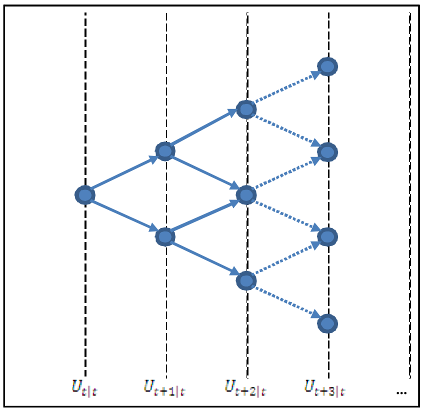


FIG. 2.1 – Option pricing analogy – recombining binomial tree

where $U_{T|t}$ with $(T \geq t)$ is the ultimate claims estimate at time T, given the information available at time $t \leq T$.

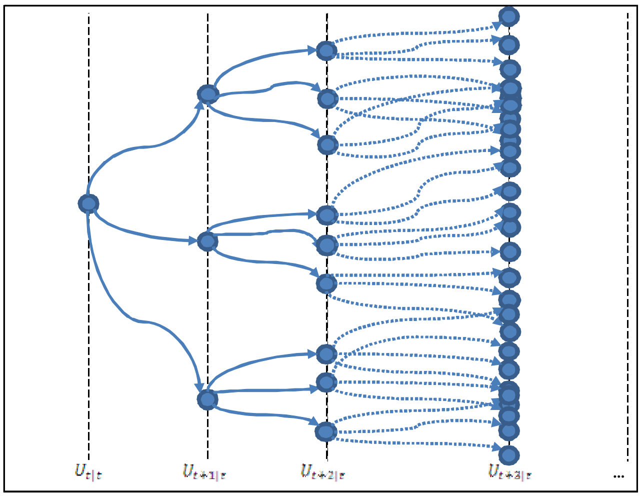
The binomial tree here would be the result of an optimizing computer algorithm, in the form of a mesh applied to an existing set of projected simulations of the ultimate claims distributions – which in turn could be derived by Bootstraping and allowing for process-error[30].

As mentioned previously, the solving process would imply a backwards recursion approach. For each node of the binomial mesh of $U_{T|t}$, the triplet $(U_{T|t}, Cap_{T|t}, MVM_{T|t})$ for $t \leq T \leq n$ is computed based on the next time step branch $(U_{T+1|t}, Cap_{T+1|t}, MVM_{T+1|t})$ using equations (2-1) and (2-2) and ultimately using the starting step $(U_{n|t}, Cap_{n|t}, MVM_{n|t}) = (U_{n|t}, 0,0)$, reflecting that when looking at the full progress of ultimate claims estimates of time, there is no further uncertainty at some point in time n.

Among the possible issues of this approach, the mesh fit will necessary be very approximate due to the small number of points to describe a whole distribution. Also, the more simulations we have, the more information will be lost in this process, making the fit, and the resulting solution more approximate. In addition, taking the 99.5th percentile of two points (i.e. in most cases the maximum of the two points) clearly is a simplified process that eliminates part of the information at hand.
In addition, this implementation would lead to an $O(n^2)$ algorithm, where n is the number of time steps.

2.2.2 Monte Carlo

We could extend the recombining binomial tree approach to a tree with p simulated branches and proceed in the same manner.
This could be obtained either through the use of a nested stochastic simulations model or through the result of an optimizing computer algorithm to fit a set of pre-determined simulations.



While the result is useful in that it allows the collapse of the circularity, a basic Monte Carlo implementation would lead to an $O(n^p)$ algorithm where n is the number of time steps and p the number of simulations. This, even with today's computers powers, is not reasonably implementable in practice.

2.2.3 Clustering

A variation of the Monte Carlo approach would be to group some of the nodes together, into clusters of increasing sizes when moving further into the time steps. This would come down to be very similar to a binomial tree algorithm, with more than just two branches (but less branches than a full Monte Carlo implementation). This would reduce the complexity of the algorithm but still make its implementation complex and time-consuming in practice.

[30] One particular Bootstrapping technique is covered in Appendix D

2.2.4 Iterative procedure

Another approach to allow the collapse of circularity would be to proceed by iteration, working backwards and back and forth until a solution converges. We first need to assume we have a proxy that makes sense. Using for instance the "proportional method" as defined in QIS5 (this will be described in more detail in §2.2.6.2.3), the following could be carried out:

Let us first write the first two years MVMs:

$$MVM_0 = CoC.\sum_{i=0}^{n-1} Cap_{i|0} \tag{2-8}$$

$$MVM_1 = CoC.\sum_{i=1}^{n-1} Cap_{i|1} \tag{2-9}$$

The current and future capital approximations, as seen as at time $t = 0$:

$$\begin{aligned} Cap_{i|0} &= 99.5\%\left[\Delta Paid_1 + R_{1|1} + MVM_1|\mathcal{F}_0\right] - R_{0|0} - MVM_0 && \text{if } i = 0 \\ Cap_{i|0} &= Cap_{0|0}\frac{R_{i|0}}{R_{0|0}} && \text{if } i \neq 0 \end{aligned} \tag{2-10}$$

The future capital approximations, as seen as at time $t = 1$:

$$Cap_{i|1} = Cap_{0|0}\frac{R_{i|1}}{R_{0|0}} \tag{2-11}$$

The reserves and paid claims distributions $R_{i|1}$ and $\Delta Paid_1$ are generally obtained by performing a "re-reserving" technique on each simulation of a bootstrapped one year-ahead completed triangle. This is also often called "actuary-in-the-box". The reserves $R_{i|0}$ are the deterministic reserves obtained using the payment pattern implied during the reserving exercise.

Combining these, we get:

$$Cap_{0|0} = 99.5\%\left[\Delta Paid_1 + R_{1|1} + CoC.\sum_{i=1}^{n-1} Cap_{0|0}\frac{R_{i|1}}{R_{0|0}}|\mathcal{F}_0\right] - R_{0|0} - CoC.\sum_{i=0}^{n-1} Cap_{0|0}\frac{R_{i|0}}{R_{0|0}} \tag{2-12}$$

$$Cap_{0|0} = 99.5\%\left[\Delta Paid_1 + R_{1|1} + CoC.\frac{Cap_{0|0}}{R_{0|0}}.\sum_{i=1}^{n-1} R_{i|1}\,|\mathcal{F}_0\right] - R_{0|0} - CoC.\frac{Cap_{0|0}}{R_{0|0}}.\sum_{i=0}^{n-1} R_{i|0} \tag{2-13}$$

where the capital at time $t = 0$ is self-defined.

We can now start the iteration process:

– Iteration 1: ${Cap_{0|0}}^{(1)} = 0$

– Iteration 2: ${Cap_{0|0}}^{(2)} = 99.5\%\left[\Delta Paid_1 + R_{1|1}|\mathcal{F}_0\right] - R_{0|0}$

– Iteration 3: ${Cap_{0|0}}^{(3)} = 99.5\%\left[\Delta Paid_1 + R_{1|1} + CoC.\frac{{Cap_{0|0}}^{(2)}}{R_{0|0}}.\sum_{i=1}^{n-1} R_{i|1}\,|\mathcal{F}_0\right] - R_{0|0} - CoC.\frac{{Cap_{0|0}}^{(2)}}{R_{0|0}}.\sum_{i=0}^{n-1} R_{i|0}$

and iterate k times until a solution converges, i.e until we have ${Cap_{0|0}}^{(k)} = {Cap_{0|0}}^{(k-1)}$.

Without developing further, the issue residing in this iterative method would be that convergence is not always assured. More research would need to be carried out to prove this and to provide any further qualitative comments on this approach. To the extent of my knowledge, this has not been carried out yet to date for the purposes of solving the MVM formulation.

2.2.5 Other approaches

Before tackling the approach ultimately used in this thesis – "analytical models", this section briefly describes other attempts found in the actuarial papers that have been suggested to solve the MVM issue for a Non-Life insurer.

In [12], the suggested approach is a generic distribution-free recursive procedure to calculate the MVM. However, it requires further modelling the "one-year hindsight estimate of the unpaid claims at the 99.5% confidence level" for each year until run-off. In [13], a Bayesian stochastic loss reserve model is applied to compute the Cost of Capital risk margin. This approach also involves generating several simulations at some stage.

The analytical approach proposed in this thesis does not require further simulations other than those already derived during most reserving exercises.

2.2.6 Analytical models

This section covers the approach ultimately used in this project. A simple claims process is studied, with a number of explicit assumptions made. Two scenarios will be explored, the first using a multiplicative structure and the second using a very simple additive one. A generic solution using a backwards recursion is obtained for each of these scenarios.
As will be seen in the case studies in the next Chapter, where simulated cash-flows of claims payments derived from actual Bootstrapped triangles or from outputs from internal models will be fitted to the specified models, we will be using projected discounted cash-flows, using the risk-free interest rate term structure as described in §1.4.4. For this reason, and for the sake of simplicity and clarity throughout the notations, the discounting will be removed from the equations from this point onwards.

2.2.6.1 General structure – Notations / Definitions / Assumptions

This section serves as a basis for the proofs described in "Appendix A – Proofs".

Let us start by defining a number of terms, considering one class of business.

– The cumulative payments C_t are the currently available data of the cumulative claims payments as at time t, throughout all origin periods, with the exclusion of New Business, as set out in §3.1.1. Note that the resulting incremental claims payments ($C_{k+1} - C_k$ for $k \geq t$) are used on a discounted basis. More formally, for $k \geq t$,we would have:

$$C_{k+1} - C_k = \frac{(C_{k+1} - C_k)^{undiscounted}}{\left(1 + r_{(k+1,t)}\right)^{k-t+1}} \tag{2-14}$$

But, as just mentioned, the following developments will be using the notation $C_{k+1} - C_k$.

– The reserves R_t as at time t are defined as a present value of future incremental payments:

$$R_t = \left(\sum_{k=t}^{n-1} (C_{k+1} - C_k) \,|\mathcal{F}_t\right) \tag{2-15}$$

However, with discounting removed from the notations while actual numerical incremental payments are discounted, this will further be expressed as follows:

$$R_t = (C_n|\mathcal{F}_t) - C_t \tag{2-16}$$

– The future losses L_t can be described in terms of the potential adverse deviations of the reserve estimation from expectations on an ultimate basis:

$$L_t = R_t - E(R_t) \tag{2-17}$$

It can be noted that the expected value and variance are as follows:

$$E(L_t|\mathcal{F}_t) = 0$$
$$Var(L_t|\mathcal{F}_t) = Var(R_t|\mathcal{F}_t)$$

– The 1-year loss deterioration L_t^1 describes the deterioration of the expected reserves over the next year:

$$L_t^1 = E(C_n|\mathcal{F}_{t+1}) - E(C_n|\mathcal{F}_t) \tag{2-18}$$

It can also be thought of in terms of the change in the ultimate claims assessment between the current year and the next one:

$$L_t^1 = C_{t+1} + E(R_{t+1}|\mathcal{F}_{t+1}) - \big(C_t + E(R_t|\mathcal{F}_t)\big)$$
$$L_t^1 = C_{t+1} + E(C_n - C_{t+1}|\mathcal{F}_{t+1}) - \big(C_t + E(C_n - C_t|\mathcal{F}_t)\big)$$
$$L_t^1 = E(C_n|\mathcal{F}_{t+1}) - E(C_n|\mathcal{F}_t)$$

As expected intuitively, we can note that all future 1-year loss deteriorations until run-off add up to the future losses L_t as can easily be shown below:

$$\sum_{k=t}^{n-1} L_k^1 = \sum_{k=t}^{n-1} C_{k+1} + E(R_{k+1}|\mathcal{F}_{k+1}) - \big(C_k + E(R_k|\mathcal{F}_k)\big)$$
$$\sum_{k=t}^{n-1} L_k^1 = C_n - C_t + E(R_n|\mathcal{F}_n) - E(R_t|\mathcal{F}_t)$$
$$\sum_{k=t}^{n-1} L_k^1 = R_t - E(R_t|\mathcal{F}_t)$$

with $R_n = 0$ and $C_n - C_t = R_t$. Thus, we have:

$$\sum_{k=t}^{n-1} L_k^1 = L_t$$

– The capital requirements at time t using the 99.5th percentile risk measure basis are initially defined as follows, on both risk-exposure horizon basis:

- On an *ultimate horizon* basis, we can write:

$$Cap_t = 99.5\%[L_t|\mathcal{F}_t] \tag{2-19}$$

$$Cap_t = 99.5\%[R_t|\mathcal{F}_t - E(R_t|\mathcal{F}_t)] = 99.5\%[(C_n|\mathcal{F}_t) - C_t] - E(C_n|\mathcal{F}_t) + E(C_t) \tag{2-20}$$

This assumes that companies have to hold enough capital to cover any potential future losses to run-off of the portfolio.

- On a *1-year horizon* basis, we now have:

$$Cap_t^1 = 99.5\%[L_t^1|\mathcal{F}_t] \tag{2-21}$$

$$Cap_t^1 = 99.5\%[E(C_n|\mathcal{F}_{t+1})|\mathcal{F}_t] - E(C_n|\mathcal{F}_t) \tag{2-22}$$

This assumes that companies have to hold enough capital to cover the one-year deterioration in the expected reserves.

Only the latter capital definition needs to be considered under Solvency II, Pillar I, as mentioned in §1.4.2. However, it now needs to be extended to include the 1-year deterioration of the risk margin on top of that of the best estimate of the liabilities.

Three (Cap_t^1, MVM_t) formulations can be explored.

Capital	MVM	Description
$Cap_t^{1A} = 99.5\%[L_t^1 \vert \mathcal{F}_t]$	$MVM_t^A = c\sum_{i=t}^{n-1} E\left[Cap_i^{1A} \vert \mathcal{F}_t\right]$	Expected Cost of Capital Risk Margin, with capital capturing the expected BEL deterioration only
$Cap_t^{1B} = 99.5\%[L_t^1 + MVM_{t+1}^B - MVM_t^B \vert \mathcal{F}_t]$	$MVM_t^B = c.Cap_t^{1B}$	Cost of Capital Risk Margin with capital defined in terms of MVM
$Cap_t^{1C} = 99.5\%[L_t^1 + MVM_{t+1}^C - MVM_t^C \vert \mathcal{F}_t]$	$MVM_t^C = c\sum_{i=t}^{n-1} E\left[Cap_i^{1C} \vert \mathcal{F}_t\right]$	Expected Cost of Capital, MVM defined in terms of expected cost of all future Capital requirements

Table 2-1 – Capital and Market Value Margins formulations

Clearly, the first two methods are proxies introduced to by-pass the inherent circularity. The third method is used in this thesis. The notation Cap_t^{1C} will thus be used in the following developments, as a reminder that (i) a one-year exposure horizon for the capital requirement is taken into account and that (ii) the capital formulation includes a shock on MVM unlike other formulations used in practice and described in Table 2-1 above.

– The Market Value Margin is thus expressed as follows:

$$MVM_t^C = c\sum_{i=t}^{n-1} E\left[Cap_i^{1C} | \mathcal{F}_t\right]$$

Let us start now briefly summarize the notations and assumptions that were already described above in the document.

– The Cost of Capital rate, $c = CoC$ will be taken as a fixed 6%.
– The Risk measure is the VaR at the 99.5th level[31]. The following notation – 99.5%[…] will be used to indicate the percentile amount. We will also use the notation $\phi = \Phi^{-1}(99.5\%)$ where $\Phi^{-1}(p)$ denotes the quantile function of the standard normal distribution of order p.
– Discounting is performed directly on the cash-flows

Having introduced all the notations, the two main equations to solve under the selected method are as follows:

$$Cap_t^{1C} = 99.5\%[L_t^1 + MVM_{t+1}^C - MVM_t^C | \mathcal{F}_t] \tag{2-23}$$

$$MVM_t^C = c\sum_{i=t}^{n-1} E\left[Cap_i^{1C} | \mathcal{F}_t\right] \tag{2-24}$$

The capital expression can further be developed as follows:

$$Cap_t^{1C} = 99.5\%[MVM_{t+1}^C + E(C_n|\mathcal{F}_{t+1}) - MVM_t^C - E(C_n|\mathcal{F}_t)|\mathcal{F}_t]$$

$$Cap_t^{1C} = 99.5\%\left[c\sum_{i=t+1}^{n-1} E\left[Cap_i^{1C}|\mathcal{F}_{t+1}\right] + E(C_n|\mathcal{F}_{t+1}) - c\sum_{i=t}^{n-1} E\left[Cap_i^{1C}|\mathcal{F}_t\right] - E(C_n|\mathcal{F}_t)|\mathcal{F}_t\right]$$

$$Cap_t^{1C} = 99.5\%\left[c\sum_{i=t+1}^{n-1} E\left[Cap_i^{1C}|\mathcal{F}_{t+1}\right] + E(C_n|\mathcal{F}_{t+1})|\mathcal{F}_t\right] - c\sum_{i=t}^{n-1} E\left[Cap_i^{1C}|\mathcal{F}_t\right] - E(C_n|\mathcal{F}_t)$$

Using the following:

$$E\left[Cap_t^{1C}|\mathcal{F}_t\right] = Cap_t^{1C}$$

[31] If we let $VaR(X)$ denote the Value at Risk at the 99.5th level for a loss X, i.e. the 99.5% quantile of the loss distribution, then if X is continuous then $VaR(X)$ is the solution to $Prob\{X \le VaR(X)\} = 99.5\%$

we get the iterative regime to solve for Cap_t^{1c}:

$$Cap_t^{1c} = \frac{1}{1+c}\left(99.5\%\left[c\sum_{i=t+1}^{n-1} E\left[Cap_i^{1c}|\mathcal{F}_{t+1}\right] + E(C_n|\mathcal{F}_{t+1})|\mathcal{F}_t\right] - c\sum_{i=t+1}^{n-1} E\left[Cap_i^{1c}|\mathcal{F}_t\right] - E(C_n|\mathcal{F}_t)\right) \quad (2\text{-}25)$$

The MVM will then be deduced using equation (2-24).

It can be seen that in order to overcome the underlying circularity, we need to solve this backwards, from time n when all the liabilities are run-off. The capital requirement is then null and so is the MVM, as there is no requirement to hold future solvency capital.

Therefore, if the liabilities mature after one time period (i.e. $t = n-1$) then for this to work, the following equation must hold (remembering that $Cap_n^{1c} = 0$):

$$Cap_{n-1}^{1c} = \frac{1}{1+c}\left(99.5\%[E(C_n|\mathcal{F}_n)|\mathcal{F}_{n-1}] - E(C_n|\mathcal{F}_{n-1})\right)$$

$$Cap_{n-1}^{1c} = \frac{1}{1+c}\left(99.5\%[C_n|\mathcal{F}_{n-1}] - E(C_n|\mathcal{F}_{n-1})\right) \quad (2\text{-}26)$$

Compared to equation (2-22), it can be seen that at time $t = n-1$ the capital allowing for the shock in the MVM – Cap_t^{1c} as we introduced it – is less than the capital ignoring it – Cap_t^1. Indeed, we have the following relationship:

$$Cap_{n-1}^{1c} = \frac{1}{1+c}Cap_{n-1}^1 \quad (2\text{-}27)$$

And this feature will hold for each $t \leq n-1$ as well, since Cap_t^{1c} includes the divisor $(1+c)$ which will act as a reducing factor to the capital that includes the change in MVM under a distressed scenario in contrast to the capital that does not allow for it (i.e. the one adopted in practice). This is because the MVM "replaces" the capital Cap_t^1, which can be seen if expressed as $MVM_{n-1}^C + Cap_{n-1}^{1c} = Cap_{n-1}^1$ derived by comparing the two capital formulations at time $t = n-1$.
It can be noted that the MVM under the SST does not benefit from this effect since there is no cost required on the current year capital.

Finally, the following needs to be solved:

$$\boxed{\begin{gathered} Cap_t^{1c} = \frac{1}{1+c}\left(99.5\%\left[c\sum_{i=t+1}^{n-1} E\left[Cap_i^{1c}|\mathcal{F}_{t+1}\right] + E(C_n|\mathcal{F}_{t+1})|\mathcal{F}_t\right] - c\sum_{i=t+1}^{n-1} E\left[Cap_i^{1c}|\mathcal{F}_t\right] - E(C_n|\mathcal{F}_t)\right) \\ \text{with } Cap_{n-1}^{1c} = \frac{1}{1+c}\left(99.5\%[C_n|\mathcal{F}_{n-1}] - E(C_n|\mathcal{F}_{n-1})\right) \\ MVM_t^C = c\sum_{i=t}^{n-1} E\left[Cap_i^{1c}|\mathcal{F}_t\right] \end{gathered}} \quad (2\text{-}28)$$

We will explore different scenarios of claims processes and solve these equations under each scenario. In this research project, the following theoretical structures were considered:
– a multiplicative structure using lognormal claims assumptions on development factors;
– an additive structure using normally distributed paid claims increments.

In the third section, we will fit our data to the models and see what analytical solutions for the MVM we would get if the actual claims were following the modelled processes, with the estimated parameters. We will also measure the goodness of fits and comment on the mesh selection.

2.2.6.2 Explored scenarios

This section provides the results under the explored models.
The detailed underlying analytical derivations of the results presented below are fully provided in "Appendix A – Proofs".

2.2.6.2.1 Multiplicative structure: lognormal assumptions

Here it is assumed that the cumulative paid claims follow the following regime:

$$\boxed{C_t = C_{t-1}.LnN(\tilde{\mu}_t, \tilde{\sigma}_t^2)}$$

for $t = 1, \ldots, n$, with n being the time where all liabilities run-off.

The assumed distribution is right skewed.
In [8], the LogNormals were assumed to be independent, which can equivalently be seen as assuming that the development factors are lognormally distributed and independent between different development periods. A dependency structure assumption is also added to this model, which imposes independent Gaussian Copulas between different time periods. This added dependency assumption takes away the Markovian property of the process, as now some form of momentum based on the past continues into the future.

The following graph shows the model's run-off of liabilities as well as its dependency structure between time periods:

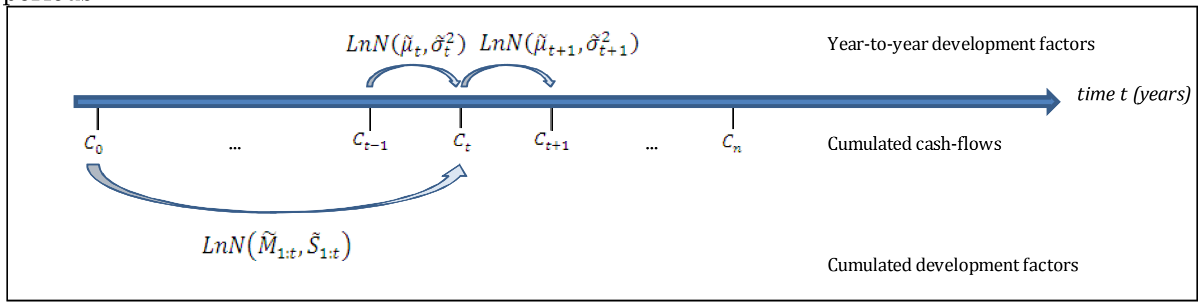


FIG. 2.2 – Lognormal structure

The theoretical developments and the notations used are fully described in Appendix A.1. After writing the equations and working backwards, the resulting lognormal regime can be written as:

$$\begin{gathered}
Cap_t{}^{1c} = C_t \frac{W_t - W_{t+1}}{Y_t} \\
\text{with} \\
W_t = (1 + cF_tY_t)W_{t+1} + F_tY_t \\
Y_i = e^{-\tilde{M}_{t+1:n} - \frac{1}{2}\tilde{S}_{t+1:n}} \\
F_t = \frac{1}{1+c} e^{\tilde{M}_{t+1:n}} \left(e^{\tilde{\sigma}_{t+1}\phi} e^{\frac{1}{2}\tilde{S}_{t+2:n}} - e^{\frac{1}{2}\tilde{S}_{t+1:n}} \right) \\
\tilde{M}_{t:n} = \sum_{k=t}^{n} \tilde{\mu}_k \\
\tilde{S}_{t:n} = \sum_{k=t}^{n} \sigma_k{}^2 + 2 \sum_{i=t}^{n-1} \sum_{j=i+1}^{n} \rho_{i,j}\sigma_i\sigma_j - \Delta_{1,v}
\end{gathered} \tag{2-29}$$

where $\Delta_{1,v}$ and $\tilde{\mu}_k$ are defined in Appendix A.3 and capture the past dependencies on which the current and future information are conditioned to.

$$\phi = \Phi^{-1}(99.5\%)$$

And our variable of interest:

$$\boxed{MVM_t{}^C = c\, Cap_t{}^{1c} + c\, C_t e^{\tilde{M}_{t+1}+\frac{1}{2}\tilde{S}_{t+1}} W_{t+1}} \tag{2-30}$$

Given the current losses and the lognormal parameters, the solutions for $Cap_t{}^{1c}$ and $MVM_t{}^C$ can be obtained by computing the W_t via the recursive formula, given above, starting from W_{n-1} downwards (with $W_n = 0$).

It is worth noting here that the $\left(Cap_t^{1A}, MVM_t^A\right)$ system formulation presented in "Table **2-1** – Capital and Market Value Margins formulations" and defined as the Expected Cost of Capital Risk Margin, with capital capturing the expected BEL deterioration only, have the following solution (cf. proofs in Appendix A):

$$\boxed{MVM_t^A = c.C_t e^{\tilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}} \left(\sum_{i=t}^{n-1} e^{\tilde{\sigma}_{i+1}\phi - \frac{1}{2}\left(\tilde{\sigma}_{i+1}^2 + 2\sum_{j=i+2}^{n} \rho_{i+1,j}\sigma_{i+1}\sigma_j\right) - \Delta_{1,u}} - (n-t) \right) \quad \text{for } t \in [\![0, n-1]\!]} \tag{2-31}$$

$$\boxed{Cap_t^{1A} = C_t e^{\tilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}} \left(e^{\tilde{\sigma}_{t+1}\phi - \frac{1}{2}\tilde{\sigma}_{t+1}^2} - 1 \right)} \tag{2-32}$$

This, alongside our "exact solutions" will provide us with a measure of the simplifying assumption made to ignore the change in the MVM over a 1-year horizon.

2.2.6.2.2 Additive structure:

The model being considered now assumes that a company is exposed to claims with the following known process:

$$\boxed{C_t = C_{t-1} + X(\Theta_t)}$$

where the $X(\Theta_t)$ are correlated, closed under addition (i.e. if $X(\Theta_k)$ and $X(\Theta_l)$ for $k, t \geq 1$ follow a given statistical law X, then $X(\Theta_k) + X(\Theta_l)$ will follow the same statistical law X), with Θ_t being a vector parameter. This is equivalent to assuming that the increments are distributed along the law of $X(\Theta_t)$ and correlated between different development periods.

The following graph shows the model's run-off of liabilities for the additive model in general.

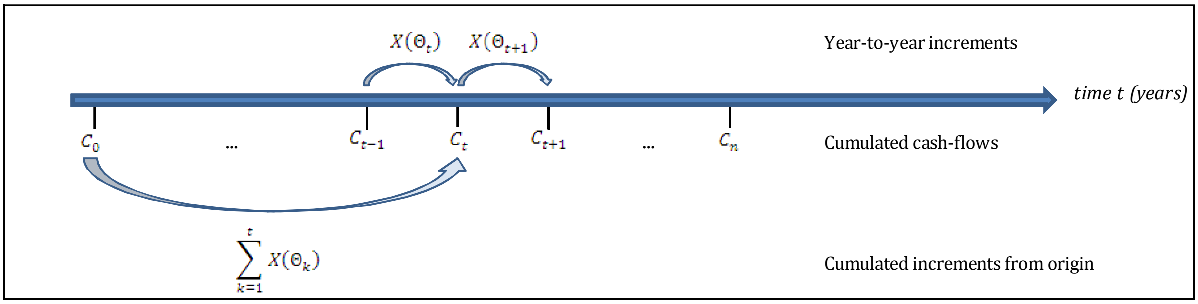


FIG. 2.3 – Additive structure

In this structure, and under the chosen solving assumptions, the circularity collapses and there is no need to work backwards. After writing the equations (cf. Appendix A.2), the resulting additive regime can be written as:

$$Cap_t^{1c} = \frac{F_{x(\Theta_{t+1})}{}^{-1}(99.5\%, \Theta_{t+1}) - E[X(\Theta_{t+1})]}{1+c} \tag{2-33}$$

And:

$$MVM_t{}^C = \frac{c}{1+c}\sum_{i=t}^{n-1}\left(F_{x(\Theta_{i+1})}{}^{-1}(99.5\%, \Theta_{i+1}) - E[X(\Theta_{i+1})]\right) \tag{2-34}$$

In the first stages of this research project, a first order Markov chain process for the claims structure was initially considered, assuming that the increments were independent between time periods. In this case, one of the apparent drawbacks of this structure is its lack of path-dependency. The future cash-flows, namely the claims increments $X(\Theta_t)$ from year $t-1$ to t, do not depend on past and current information to date, which, in its strict sense can seem unrealistic for a large number of real-life insurance products, in terms of their claims developments. For a single given claim, one would expect that how its amount develops over time is not totally independent of what has emerged to date, be it in the information obtained on the potential insured's liability or the assessment of its case estimate. Also, if we think of a Catastrophe event, it is most likely that the emergence of a Cat will cause change in the development profile of the affected class(es) of business when compared to a scenario where the Cat did not occur. More generally, market conditions and inflation, and for example a new courts jurisdiction will have a long-lasting effect on the claims developments.

It is the addition of a dependency structure between time periods that now makes the process path-dependent.

However, even without this added path-dependency, it should be acknowledged that the analytical structures described here should not be expected to model and reflect an actual insurance claims process.

Their ultimate purpose is to be used as meshes applied on a series of projected cash flows of claims payments that have been modelled elsewhere – either via a Bootstrap from a loss triangle or through an internal model as detailed in the next Chapter. As such, they should rather be seen as fictive analytical tools to solve a specific problem formulation, in this case, overcoming the inherent circularity in the currently accepted definition of the Market Value Margin.

That being said, the previous results can easily be transposed to the Normal distribution, explicitly showing the dependency structure:

– Normal assumptions
In this case, we have $\boxed{X(\Theta_t) = N(\tilde{\mu}_t, \tilde{\sigma}_t^2)}$ with $X(\Theta_t)$ defined as in (A.2- 1) in the Appendix.
The claims payment model becomes:

$$\boxed{C_t = C_{t-1} + N(\tilde{\mu}_t, \tilde{\sigma}_t^2)}$$

The graph for the normal model becomes:

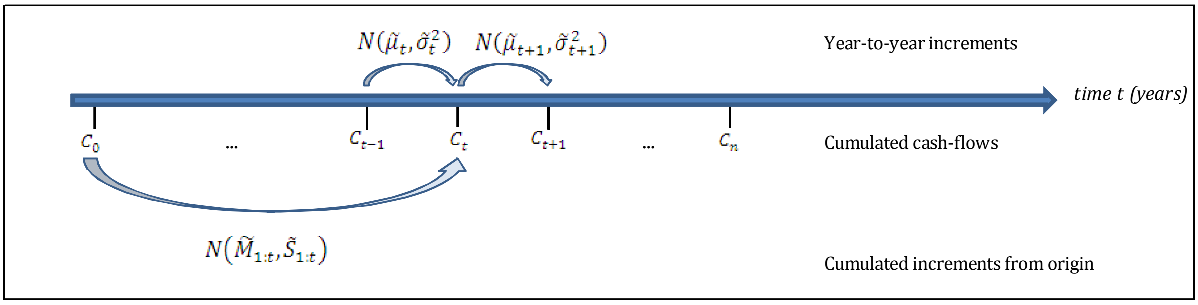


FIG. 2.4 – Normal structure

The resultant normal regime was simplified into:

$$Cap_t^{1C} = \frac{\phi \tilde{\sigma}_{t+1}}{1+c} \quad \text{with} \quad \phi = \Phi^{-1}[99.5\%] = 2.576 \tag{2-35}$$

Similarly,

$$MVM_t{}^C = \frac{c\phi}{1+c} \sum_{i=t}^{n-1} \tilde{\sigma}_{i+1} \tag{2-36}$$

An interesting feature in contrast to the lognormal structure is that the MVM result in the normal model moves from being a sum squares to a linear function of the standard deviation.

There are some specific limitations arising when using a Normal modelling for this particular claims process. Specifically, as the distribution does not allow for thick-tails, this could, especially in a Solvency context underestimate a potential distressed scenario. Also, normality allows for the possibility of negative cumulative payments.

The simplified solution for (Cap_t^{1A}, MVM_t^A) system formulation is as follows (cf. proofs in Appendix A):

$$\boxed{Cap_t^{1A} = \phi \tilde{\sigma}_{t+1}} \tag{2-37}$$

$$\boxed{MVM_t^A = c\sum_{i=t}^{n-1}\phi\tilde{\sigma}_{i+1}} \tag{2-38}$$

It can be noted that we have:

$$Cap_t{}^{1c} = \frac{Cap_t{}^{1_A}}{1+c} \tag{2-39}$$

and similarly

$$MVM_t^C = \frac{MVM_t^A}{1+c} \tag{2-40}$$

suggesting that the capital and resulting MVM allowing for a change in the MVM over a 1-year horizon is less than the one ignoring it as a simplification. In the case of the normal structure, the reduction is inversely proportional to the Cost of Capital factor, more precisely, it is exactly obtained by dividing the exact solution by $(1+c)$.

2.2.6.2.3 Transposition to QIS5 proxies

Assuming the capital requirement was assessed appropriately, the last three levels of simplifications for the MVM calculation as suggested for QIS5 and described in §2.1 – QIS5 proxies can each be put into equations along the two structures developed previously. Using the same notations, we get the following:

3. <u>QIS5 proxy 3 – "Proportional approach"</u>

In general, we have:

$$MVM_{t|t}^{QIS5-Proxy3} = cE\left(Cap_t^{1c}|\mathcal{F}_t\right)\left(1+\sum_{i=t}^{n-1}\frac{E(R_{i+1}|\mathcal{F}_t)}{E(R_t|\mathcal{F}_t)}\right), t=0,\dots,n-1 \tag{2-41}$$

where R_t is defined as in (2-16) and where $E(R_i|\mathcal{F}_t)$ is the conditional expected reserve defined as follows:

$E(R_i|\mathcal{F}_t) = E(C_n - C_i|\mathcal{F}_t)$, for $0 \le t \le i \le n-1$

At time $t = 0$ where the calculation takes place, the expected payment pattern over the run-off of the liabilities is used as an estimate of how the capital requirement will run-off over the lifetime of all liabilities.

$$MVM_{0|0}^{QIS5-Proxy3} = cE\left(Cap_0^{1c}|\mathcal{F}_0\right)\left(1+\sum_{i=0}^{n-1}\frac{E(R_{i+1}|\mathcal{F}_0)}{E(R_0|\mathcal{F}_0)}\right) \tag{2-42}$$

– Accompanied with the lognormal assumption of C_t, (2-41) becomes:

$$MVM_{t|t}^{QIS5-Proxy3|Ln} = cE\left(Cap_t^{1c}|\mathcal{F}_t\right)\left(1+\sum_{i=t}^{n-1}\frac{1-e^{-M_{i+2}-\frac{1}{2}S_{i+2}}}{1-e^{-M_{t+1}-\frac{1}{2}S_{t+1}}}\right) \tag{2-43}$$

using $E(R_i|\mathcal{F}_t) = C_t e^{M_{t+1}+\frac{1}{2}S_{t+1}}\left(1-e^{-M_{i+1}-\frac{1}{2}S_{i+1}}\right)$ for $0 \le t \le i \le n-1$

– Accompanied with the normal assumption of C_t, (2-41) now becomes:

$$MVM_{t|t}^{QIS5-Proxy3|N} = cE\left(Cap_t^{1c}|\mathcal{F}_t\right)\left(1+\sum_{i=t}^{n-1}\frac{M_{i+2}}{M_{t+1}}\right) \tag{2-44}$$

using $E(R_i|\mathcal{F}_t) = \sum_{k=i+1}^{n}\mu_k = M_{i+1}$ for $0 \le t \le i \le n-1$

4. <u>QIS5 proxy 4 – "Estimate all future SCRs "at once"</u>

Using the modified duration at each time t until the full run-off, we have:

$$MVM_{t|t}^{QIS5-Proxy4} = cE\left(Cap_t^{1c}|\mathcal{F}_t\right).Dur_{mod}(t) \quad \textbf{(2-45)}$$

where $Dur_{mod}(t)$ is the modified duration of the reference undertaking's (re)insurance obligations net of reinsurance at time $0 \leq t \leq n-1$.

5. <u>QIS5 proxy 5 – "Approximate the risk margin by calculating it as a percentage of the best estimate"</u>

$$MVM_{t|t}^{QIS5-Proxy5} = \alpha_{LoB}.E(R_t|\mathcal{F}_t) \quad \textbf{(2-46)}$$

For a participating non-life insurance undertaking, the technical specifications provide the following percentages of the best estimate, on which the risk margin calculations should be based for the lines of business:

Lines of business	Per cent of the BE
Direct insurance and accepted proportional reinsurance:	
Medical expenses	8.5%
Income protection	12.0%
Workers' compensation	10.0%
Motor vehicle liability	8.0%
Motor, other classes	4.0%
Marine, aviation and transport	7.5%
Fire and other damage	5.5%
General liability – Third party liability	10.0%
Credit and suretyship	9.5%
Legal expenses	6.0%
Assistance	7.5%
Miscellaneous non-life insurance	15.0%
Accepted non-proportional reinsurance:	
Health business	17.0%
Property business	7.0%
Casualty business	17.0%
Marine, aviation and transport business	8.5%

Table 2-2 – QIS5 proxy 5: percentages of the BEL to approximate the Risk Margin

This last section provided "closed-form" analytical solutions for the MVM, as exact solutions for two simple claims structures or as proxies such as the ones suggested for QIS5. With these theoretical results at hand and two sets of "real" claims development triangles, we can move onto the next Chapter which will ultimately compare the results of the MVM estimates along the QIS5 proxies on the one hand, and along the application of an analytical "mesh" on the other hand.

Chapter 3 Application: MVM Model

The end developments of the previous Chapter showed how to calculate a "closed-form" solution for the Market Value Margin if the claims were to follow two simple analytical structures. Building on this, this Chapter will now transpose those results into finding an estimate for more general claims models, provided the outputs of these can be extracted in the form of a set of simulations of future cash-flows.
The approach is to apply a "mesh" on the sets of simulations – obtained either by some Bootstrapping or by an internal model – by fitting those outputs to either of the two analytical models that will act as "supra" models to describe the initial models more concisely for the sole purpose of the "closed-form" computation of the MVM.

An interesting advantage of this approach is that in general no additional simulations are required for the sole purpose of computing the MVM. A set of projected cash-flows is in practice usually at hand from work carried out elsewhere, as many companies for example derive ranges in their reserving exercises or, if using an internal model for Solvency II, its outputs could be used. This then avoids having to perform simulations on simulations to capture conditional predictive distributions of claims.

In order to get a better overview of how the different processes described in this Chapter piece together, the following graph depicts the layout of this section.

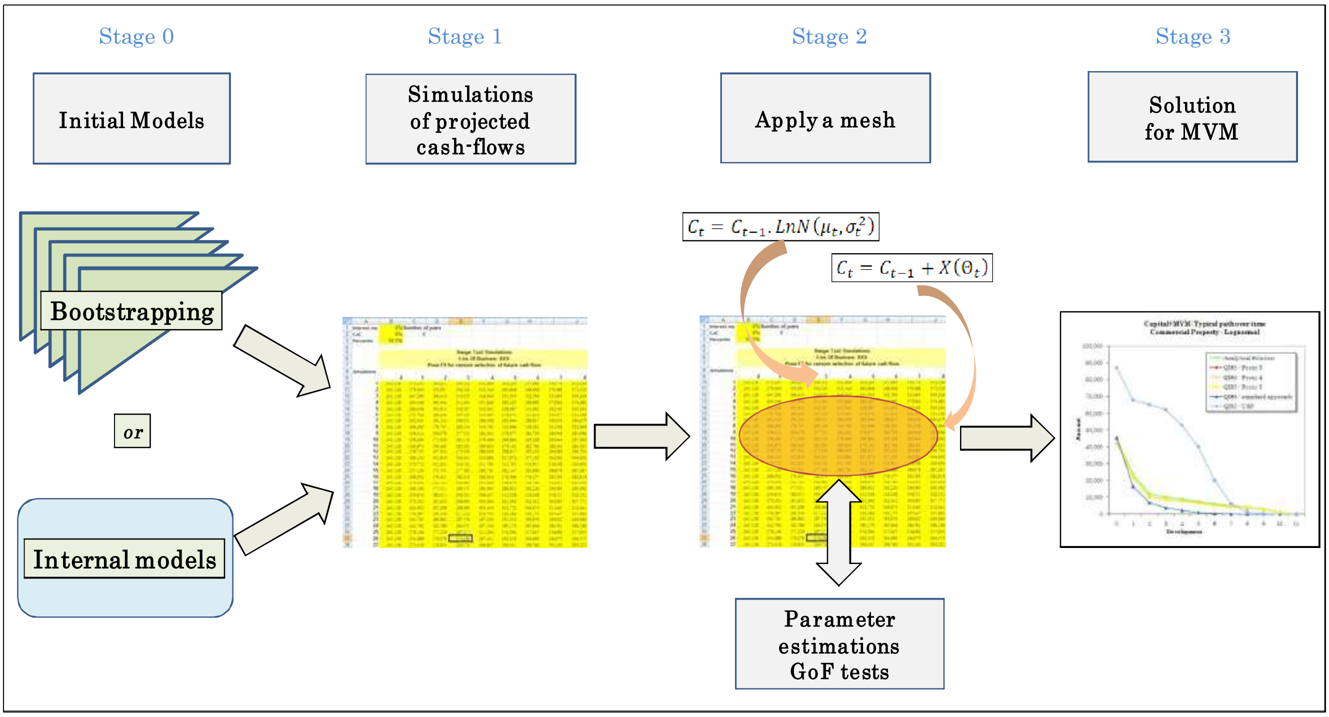


FIG. 3.1 – Schematics of the main application processes

– Stage 1 is the inputs data required as a basis of the mesh-fitting of Stage 2. They are described in §3.1.2 and obtained through the initial losses development triangles from which some initial modeling is carried out, as depicted in Stage 0, in order to derive projected simulations of the claims distributions until run-off.
– Stage 2 is where a mesh is fitted to the projected simulations of cash-flows from Stage 1, as explained in §3.2. The mesh structure is either of the two analytical models structure developed and presented in §2.2.6 in the previous Chapter and it is supposed to capture the whole projected underlying distributions of cash-flows into one analytical claims development structure. In order to fit the cash-flows to the mesh,

parameters need to be estimated. The fit will be measured through Goodness Of Fit tests (GoF) as will further be described in §3.2.2.
– Stage 3 will present the results of the MVM estimation. This will be covered in §3.3.

Reference will be made to these four stages in the following sections.

3.1 Overall model

3.1.1 Limitations

This paragraph sets out the scope of the applicative study conducted in this Chapter.
As mentioned in Chapter 2, §2.2.6, only the non-life reserve risk of any bound business is considered, keeping in mind though that it is a large part of most non-life insurance total Solvency Capital Requirements. Expanding on the non-life risk module exposed in Appendix F – "SCR – Standard Formula", the non-life insurance risk is divided into the reserve risk and the premium risk. Reserve risk relates to the liabilities covering insurance policies contracted in the historical years and is sometimes referred to as the risk of the claims reserves deterioration. Premium risk deals with future risks to which the insurer is bound, some of which are already recognized as liabilities in the form of premium reserves (the provision for unearned premium and unexpired risks for instance). Policies expected to be written during the risk period and covered by the corresponding expected premium income are also part of the premium risk. The premium risk is where the exposure to the catastrophe risk lies. However, given its highly specific nature, the catastrophe risk is often treated separately as a third component of the non-life insurance risk.

Now as opposed to a "real" MVM calculation that would be part of a full QIS5 submission (cf. §1.2.4 – "CPs 42 / 71 and QIS5 technical specifications"), whether through the Standard Formula or through an internal model, and whether through a proxy or a "full calculation" (cf. §2.1 – "QIS5 proxies"), only the gross claims deterioration of a single non-life line of business is taken into consideration. More precisely, the following sets the exact scope of the study, conducted within a Solvency II banner.

In terms of risks *not* captured within the capital used as an input to the MVM:
– the unavoidable market risk is left aside[32];
– the study is conducted on a gross of reinsurance basis only, which further simplifies into not having to capture the credit risk with respect to reinsurance contracts and special purposes vehicles;
– the Operational Risk is not captured.

Moreover, the following assumptions are used:
– as only a single line of business is considered, dependencies between LoBs are outside the scope of this paper;
– the Catastrophe risk is beyond the scope of the study
– a constant cost-of-capital rate is assumed, set at 6%;
– a deterministic discount rate yield curve is used, as provided by CEIOPS.

3.1.2 Inputs data – simulated cash-flows of claims payments

From an initial claims loss development triangle, it is in theory relatively straightforward to derive simulations for the completed triangle, which translates into the projection of simulated cash-flows that are needed in Stage 1. There are a number of stochastic models used to derive stochastic reserves: Mack, Projected Case Estimates, lognormal, GLM, Munich Chain Ladder models or the Bootstrap procedure to name a few.

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[32] However, QIS5 Technical Specifications [3] (TP.5.18) state that: "*With respect to market risk only the unavoidable market risk should be taken into account in the risk margin. Undertakings should follow a practicable approach when they assess the unavoidable market risk. It only needs to be taken into account where it is significant. For non-life insurance obligations and short-term and mid-term life insurance obligations the unavoidable market risk can be considered to be nil.*"

In this project, the following has been conducted:
– a full and detailed analysis from Stage 0 to Stage 3 was carried out from two initial real triangles, where Bootstrapping has been carried out (Stage 0)
– outputs of various lines of business from an internal model (i.e. Stage 1) have been used in order to comment on the wider picture, from data readily available internally. The underlying modeling of the internal model is not discussed here. Stage 2 has been conducted but the results are not presented in full, for the sake of keeping this reasonably concise and avoiding overloading the Appendices. However, we comment on the final results of Stage 3.

3.1.2.1 Triangles

The triangles used and bootstrapped come from publicly available information from the FSA returns[33] as at the year-ended 2009. The following set of data was thus collected and used:
– Paid claims triangle
– Incurred claims triangle
– Earned premiums

The two lines of business under study, namely Commercial Property (CProp) and Employer's Liability (EL) have been chosen in order to capture potentially different results and behaviors, as in a short-tailed and in a long-tailed class of business respectively. Both come from a medium size, relatively well-diversified non-life insurance company.

The set of triangles used are presented in Appendix C; however they have been slightly anonymized for confidentiality purposes.

3.1.2.2 Bootstrap

Bootstrapping is a simulation-based approach. It is a method for producing sampled distributions for statistical quantities of interest by generating pseudo samples, which are obtained by randomly drawing, with replacement, from observed data. In simple terms, bootstrapping is a re-sampling procedure and all the pseudo samples generated by bootstrapping are subsets of the observed sample or identical to the observed sample. The algorithm is widely used in general insurance claims reserving to obtain the estimation error of the reserves estimates and further obtain their predictive distributions when the procedure is used together with an underlying reserving model correctly calibrated to simulate the process error.

As mentioned earlier in this document, a Bootstrap on a claims process as formulated by T. Mack and D. Murphy (cf. [16],[17]) was performed, using E&Y internal software WinRange® and 10,000 simulations. The theory underlying this is detailed in Appendix D. From a given simulation obtained by resampling the triangle of residuals, a full completed triangle (i.e. the upper + the lower triangles) is built to derive a corresponding reserve amount. And in this process, the sum of each cell of the incremental diagonal of the lower triangle gives the future expected claims payment cash-flow in that given realization. As a result of this, simulated projected cash-flows of future claims payments are obtained for Stage 2.

It should be born in mind that the mesh acting as a supra-model as we defined it earlier heavily relies on how the parameters (i.e. the development factors λ_j, the Mack parameters σ_j which denote variability and the initial expected loss ratio (IELR) I_i^* when using the Bornhuetter-Ferguson Model) are estimated in the bootstrapping procedure, with a direct knock-on effect on the mesh-fitting – and ultimately on the MVM calculation results.

[33] The FSA Returns are regulatory forms sent annually to the Financial Services Authority (FSA) and prepared for each regulated operating insurance company in the UK. The FSA returns comprise detailed financial information on solvency, investments, business mix, claims and premiums, etc. and are publicly available.

First, the choice of the underlying reserving model can potentially provide different sets of outputs. The most commonly used models are based on resampling the Pearson residuals, which in turns relies on the "Over-Dispersed Poisson (ODP) distribution" to model the traditional link ratio method.

Then, in addition, if in theory the future development factors are unknown and are estimated by minimizing the least-squares from the full original triangle of historical data, it should be noted that in practice, a number of judgmental assumptions are added by the actuary into the analysis. These can be either supported by the knowledge of the underlying book of business (such as speeding up of claims settlement trends or management actions, or a large one-off claim that is not believed to reflect the book history). In order to achieve this, individual factors can be excluded from the calculation of the volume average for instance, or the volume average could take a volume average of a user-defined selected number of accident years, or a tail factor could be added, etc. Furthermore, the analysis could be performed on either paid or incurred claims data.
With no prior knowledge on the underlying business, my general approach here has been to work on a mix of paid and incurred claims data and to use a chain ladder ("CL") method on the short-tailed LoB, while on the long-tailed LoB using a chain ladder ("CL") on 2008 and prior accident years and a Bornhuetter-Ferguson ("BF") method for the 2009 accident year. And where appropriate, some unusual development factors have been excluded.

Consequently, one should keep in mind that the MVM calculations results are to some extent subject to how the bootstrapping was performed, and this includes:
– model selection
– judgmental assumptions.

The number of simulations carried out could also have an effect on the final result, albeit probably not significant over 5,000 simulations.
Some sensitivity testing could further be conducted to measure these choices and/or interventions; however this is out of scope of this study.

3.1.2.3 Internal model

The following lines of business are being tested: Casualty, Professional Indemnity (PI), Energy, Auto and Property. Most of the outputs comprise 10,000 simulations. As mentioned above, their origin is not fully known except that they mostly come from medium-sized companies. I am not aware of the underlying stochastic techniques that were used to derive each of those. In addition, these would be subject to confidentiality requirements. As mentioned previously, these readily available outputs will only be used to provide some wider comments when comparing the results against classes.

In each case, the outputs are exported onto Excel (via Access as an intermediate processing tool), and become the inputs of Stage 2 as described in FIG. 3.1 and detailed in the following.

3.2 Mesh-fitting

The aim is to define a statistical distribution matching the projected cash-flows as simulated under the bootstrap method, in order to describe those more concisely and in a tractable manner. In order to validate the approach, the distribution must be compared to the bootstrap results. This can be done with the following steps:
– Get a sample of projected cash-flows big enough to closely approximate the population of possible claims payments paths. Here we chose to take 10,000 elements.
– Calculate the distribution parameters under our two analysed structures giving the best fit with the sample.

Having thus obtained the distribution parameters, a Goodness of Fit test will describe how well the distribution fits the set of simulated projected cash-flows.
I used "R" for the GoF study. More precisely, I used RExcel, which allows access to "R" from within Excel while enabling the VBA platform.

3.2.1 Fitting an analytical mesh to simulated cash-flows – parameter estimations

3.2.1.1 Model fit

In this context, "fit" refers to the ability of a model to reproduce the data. Here, the simulated cash-flows are the observed data that we want to describe through an analytical mesh. To do so, some model fitting needs to be done, and in general this requires carrying out the following steps:

– Select the model or function to which the data is to be fitted: a family of distributions is usually considered and a priori thought to represent the data adequately. In our case, the lognormal and normal distributions are considered.
– Estimate parameters: once the model has been selected, the next step is to estimate the unknown parameters in the function under consideration; there are a number of ways to accomplish this, the two major methods of parameter estimation are the maximum likelihood and the least squares.
– Evaluate the quality of fit: this can be performed graphically, by plotting the theoretical "pdf" (Probability Density Function) curve against the histogram of the empirical frequencies, or by producing a Quantile-Quantile (QQ) plot (cf. Appendix E for more details)
– Perform Goodness of Fit statistical tests: these tests measure the compatibility of a random sample with a theoretical probability distribution function. They show how well the selected distribution fits to the data. They are a form of hypothesis testing where the null and alternative hypotheses are:

H_0: Sample data come from the stated distribution
H_1: Sample data do not come from the stated distribution

3.2.1.2 Parameter estimation

In many fields involving statistics, a number of parameter estimation methods for probability distribution functions are used. For instance, the following methods are common:

– moments,
– probability weighted moments,
– L-moments,
– least squares (on the original or linearized data),
– weighted least squares,
– maximum likelihood,
– minimum cross entropy,
– Bayesian estimation.

This document will not discuss the merits nor compare the performances of these methods. The maximum likelihood method will be selected and described, as it is quite a standard and easily implementable approach to most parameters estimation problems.

Bearing in mind that the two models ultimately considered here to describe the cumulative claims payments C_t being the lognormal and the normal distributions, a similar procedure to address the parameter estimation for both models will be explained. Indeed, if Y is a lognormally distributed random variable such that $Y \sim LnN(\mu, \sigma^2)$, then $X = ln(Y)$ is a normally distributed random variable with $X \sim N(\mu, \sigma^2)$. As a result, the location parameter is equal to the mean of the logarithm of the data points, and the scale parameter is equal to the standard deviation of the logarithm of the data points. Thus, the lognormal distribution does not have to be dealt with as a separate distribution. By taking the logarithm of the data points, the techniques developed for the normal distribution can be used to estimate the parameters of the lognormal distribution.

3.2.1.2.1 Log-likelihood parameters estimation

The probability density function of the normal distribution for $X \sim N(\mu, \sigma^2)$ is:

$$f_X(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad x \in \mathbb{R} \tag{3-1}$$

In general, having a sample $x = (x_1, x_2, \dots, x_N)$ from a normal $N(\mu, \sigma^2)$ population, the standard approach to approximate values of parameters (μ, σ^2) is the maximum likelihood method, which requires maximization of the likelihood (or equivalently, log-likelihood) function:

$$\mathcal{L}(\mu, \sigma^2) = \prod_{i=1}^{N} f_{X_i}(\mu, \sigma^2) \tag{3-2}$$

Taking the log:

$$ln\mathcal{L}(\mu, \sigma^2) = \sum_{i=1}^{N} ln\left(f_{X_i}(\mu, \sigma^2)\right) \tag{3-3}$$

Replacing each $f_{X_i}(\mu, \sigma^2)$ by their corresponding formulation from (3-1):

$$ln\mathcal{L}(\mu, \sigma^2) = -\frac{N}{2} ln(2\pi) - N\, ln(\sigma) - \frac{1}{2\sigma^2} \sum_{i=1}^{N} (x_i - \mu)^2 \tag{3-4}$$

Taking the derivatives with respect to μ and σ:

$$\frac{\partial ln\mathcal{L}(\mu, \sigma^2)}{\partial \mu} = \frac{1}{2\sigma^2} \sum_{i=1}^{N} 2(x_i - \mu) \tag{3-5}$$

$$\frac{\partial ln\mathcal{L}(\mu, \sigma^2)}{\partial \sigma} = -\frac{N}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{N} (x_i - \mu)^2 \tag{3-6}$$

and setting them to zero (in order to maximize $ln\mathcal{L}(\mu, \sigma^2)$) we get the resulting system of first order conditions:

$$\frac{\partial ln\mathcal{L}(\mu, \sigma^2)}{\partial \mu} = \sum_{i=1}^{N} (x_i - \mu) = 0 \tag{3-7}$$

$$\frac{\partial ln\mathcal{L}(\mu, \sigma^2)}{\partial \sigma^2} = -N + \frac{1}{\sigma^2} \sum_{i=1}^{N} (x_i - \mu)^2 = 0 \tag{3-8}$$

whose solution yields the maximum likelihood estimates for sample $x = (x_1, x_2, \dots, x_N)$ from a normal $N(\mu, \sigma^2)$ population:

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} x_i = \bar{x} \tag{3-9}$$

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \hat{\mu})^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2 \tag{3-10}$$

In practice, the unbiased estimator $s^2 = \frac{N}{N-1}\hat{\sigma}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$ is used instead of $\hat{\sigma}^2$, allowing for the Bessel correction, however the difference between $s^2$and $\hat{\sigma}^2$ becomes negligibly small for large samples.

The maximum likelihood estimators for sample $y = (y_1, y_2, \dots, y_N)$ from a lognormal $lnN(\mu, \sigma^2)$ population:

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} ln(y_i) \tag{3-11}$$

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{N} (ln(y_i) - \hat{\mu})^2 \tag{3-12}$$

3.2.1.2.2 Cumulated means and (squared) standard deviations $(M_{1:t}, S_{1:t})$ estimators

Here, we will be looking backwards at the past data, and for the purposes of estimating the unconditional (μ_k, σ_k^2) ultimately, we will *not* consider conditional distributions, unlike what we do when solving the regimes.
Coming back to the two claims structures under consideration, these will now be written as follows:

– lognormal:

$$C_t = C_{t-1}.LnN(\mu_t, \sigma_t^2) = C_0.LnN\left(\sum_{k=1}^{t} \mu_k, \sum_{k=1}^{t} \sigma_k^2\right) = C_0.LnN\left(\sum_{k=1}^{t} \mu_k, \sum_{k=1}^{t} {\sigma_k}^2 + 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} \rho_{i,j}\sigma_i\sigma_j\right)$$
$$= C_0.LnN(M_{1:t}, S_{1:t})$$

– normal:

$$C_t = C_{t-1} + N(\mu_t, \sigma_t^2) = N\left(\sum_{k=1}^{t} \mu_k, \sum_{k=1}^{t} {\sigma_k}^2 + 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} \rho_{i,j}\sigma_i\sigma_j\right) + C_0$$
$$= N(M_{1:t}, S_{1:t}) + C_0$$

as formulated in equations (A.1- 6) and (A.2- 17) respectively, it can be seen that the parameters that should first be estimated are the cumulated mean and (squared) standard deviations $\left(\sum_{k=1}^{t} \mu_k, \sum_{k=1}^{t} {\sigma_k}^2 + 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} \rho_{i,j}\sigma_i\sigma_j\right)$ of the cumulated claims developments from time equals 0 to t.

The previous results can be transposed as follows, where:

$$\begin{pmatrix} C_0^{(1)} & \cdots & C_t^{(1)} & \cdots & C_{n-1}^{(1)} \\ \vdots & & \vdots & & \vdots \\ \vdots & \cdots & C_t^{(i)} & \cdots & \vdots \\ \vdots & & \vdots & & \vdots \\ C_0^{(N)} & \cdots & C_t^{(N)} & \cdots & C_{n-1}^{(N)} \end{pmatrix} \tag{3-13}$$

denotes the set of projected cash-flows (note that ${C_0}^{(i)} = C_0$ for each simulation i, relating to the cumulative payments to date, at the valuation date) from time equals 0 to t and for each simulation i and where $N = 10{,}000$ is the number of simulations obtained in Stage 1 (cf. FIG. 3.1).

– Lognormal structure: each projection period $t \in [\![1, \dots, n-1]\!]$ has the sample $y_t = ({y_t}^{(1)}, {y_t}^{(2)}, \dots, {y_t}^{(i)}, \dots, {y_t}^{(N)})$ to fit, with each ${y_t}^{(i)}$ being the cumulated development factor from origin (time $t = 0$) for simulation i, or more formally: ${y_t}^{(i)} = \frac{{C_t}^{(i)}}{C_0}$.

  The maximum likelihood estimators for these samples are thus given by:

$$\widehat{\sum_{k=1}^{t} \mu_k} = \widehat{M_{1:t}} = \frac{1}{N}\sum_{i=1}^{N} ln\left(\frac{{C_t}^{(i)}}{C_0}\right) \tag{3-14}$$

$$\left(\sum_{k=1}^{t} {\sigma_k}^2 + 2\widehat{\sum_{i=1}^{t-1}\sum_{j=i+1}^{t}} \rho_{i,j}\sigma_i\sigma_j\right) = \widehat{S_{1:t}} = \frac{1}{N}\sum_{i=1}^{N}\left(ln\left(\frac{{C_t}^{(i)}}{C_0}\right) - \widehat{\sum_{k=1}^{t}} \mu_k\right)^2 \tag{3-15}$$

– Normal structure: the sample for each projection period $t \in [\![1, \dots, n-1]\!]$ is now $x_t = ({x_t}^{(1)}, {x_t}^{(2)}, \dots, {x_t}^{(i)}, \dots, {x_t}^{(N)})$ with each ${x_t}^{(i)}$ being the cumulated increments since origin (time $t = 0$) for each simulation i, or more formally: ${x_t}^{(i)} = {C_t}^{(i)} - C_0$.

The maximum likelihood estimators for these samples are thus given by:

$$\widehat{\sum_{k=1}^{t}\mu_k} = \widehat{M_{1:t}} = \frac{1}{N}\sum_{i=1}^{N}\left(C_t{}^{(i)} - C_0\right) \tag{3-16}$$

$$\left(\widehat{\sum_{k=1}^{t}\sigma_k{}^2 + 2\sum_{\iota=1}^{t-1}\sum_{\jmath=\iota+1}^{t}\rho_{\iota,\jmath}\sigma_\iota\sigma_\jmath}\right) = \widehat{S_{1:t}} = \frac{1}{N}\sum_{i=1}^{N}\left(\left(C_t{}^{(i)} - C_0\right) - \widehat{\sum_{k=1}^{t}\mu_k}\right)^2 \tag{3-17}$$

It is worth remembering that the cumulated payments $C_t{}^{(i)}$ are on a discounted basis, as described in (2-14).

3.2.1.2.3 Dependency measurement

The two analytical models assume a dependency structure between development factors under the lognormal model and between increments under the additive model, as mentioned in §2.2.6.2.1 and §2.2.6.2.2. In order to include this, we will measure the appropriate resulting correlations from the cash-flows within all projection periods.

Using the notations introduced above, the following need to be considered:

– under the lognormal model, for each $t, u \in [\![1, \dots, n-1]\!]$ correlation between the samples $y_t{}'$ and $y_u{}'$ will be measured, where $y_t{}' = (y_t{}'^{(1)}, y_t{}'^{(2)}, \dots, y_t{}'^{(i)}, \dots, y_t{}'^{(N)})$, with each $y_t{}'^{(i)}$ being the year-to-year development factor for simulation i, or more formally: $y_t{}'^{(i)} = \frac{C_t{}^{(i)}}{C_{t-1}{}^{(i)}}$

Note that we have $y_t{}^{(i)} = \prod_{k=1}^{t} y_k{}'^{(i)} = \frac{C_t{}^{(i)}}{C_0}$.

– under the normal model, for each $t, u \in [\![1, \dots, n-1]\!]$ correlation between the samples $x_t{}'$ and $x_u{}'$ will be measured, where $x_t{}' = (x_t{}'^{(1)}, x_t{}'^{(2)}, \dots, x_t{}'^{(i)}, \dots, x_t{}'^{(N)})$, with each $x_t{}'^{(i)}$ being the year-to-year increments for simulation i, or more formally: $x_t{}'^{(i)} = C_t{}^{(i)} - C_{t-1}{}^{(i)}$

Similarly, note that we have $x_t{}^{(i)} = \sum_{k=1}^{t} x_k{}'^{(i)} = C_t{}^{(i)} - C_0$.

Several correlation measures can be applied.
The most familiar measure is the Pearson product-moment correlation coefficient, or "Pearson's correlation." It is obtained by dividing the covariance of the two variables by the product of their standard deviations. The population correlation coefficient $\rho_{y_t',y_u'}$ between the two random variables $y_t{}'$ (or $x_t{}'$) and y_u' (or x_u') with expected values μ_t and μ_u and standard deviations σ_t and σ_u is defined as:

$$\rho_{y_t',y_u'} = corr(y_t', y_u') = \frac{cov(y_t', y_u')}{\sigma_t\sigma_u} = \frac{E[(y_t' - \mu_t)(y_u' - \mu_u)]}{\sigma_t\sigma_u}$$

where cov means covariance, and $corr$ a widely used alternative notation for Pearson's correlation.
The Pearson correlation is defined only if both of the standard deviations are finite and both of them are nonzero, which is the case in our worked examples.
When dealing with population samples, the sample correlation coefficient can be written as:

$$r_{y_t',y_u'} = \frac{\sum_{i=1}^{N}\left(y_t'^{(i)} - \overline{y_t'}\right)\left(y_u'^{(i)} - \overline{y_u'}\right)}{(N-1).\,s_t s_u}$$

where $\overline{y_t'}$ and $\overline{y_u'}$ are the sample means of y_t' and y_u', s_t and s_u are the sample standard deviations of y_t' and y_u'.

Rank correlation coefficients are other types of dependence measures. Spearman's rank correlation coefficient and Kendall's rank correlation coefficient (τ) for instance measure the extent to which, as one variable increases, the other variable tends to increase, without requiring that increase to be represented by a linear relationship. These two measures have not been used here.

Finally, the following correlation matrix is obtained for the lognormal model:

$$\boldsymbol{\Lambda}_{\mathbf{1},\boldsymbol{n}} = \begin{pmatrix} {\sigma_{y'_1}}^2 & \cdots\cdots\cdots & & \rho_{y'_1,y'_n}\sigma_{y'_1}\sigma_{y'_n} \\ & \ddots & \rho_{y'_t,y'_u}\sigma_{y'_t}\sigma_{y'_u} & \\ \vdots & & & \vdots \\ & & \ddots & \\ \rho_{y'_1,y'_n}\sigma_{y'_1}\sigma_{y'_n} & & & {\sigma_{y'_n}}^2 \end{pmatrix}$$

A similar correlation matrix can be obtained for the normal model, where the y'_t are replaced by the $x_t{}'$.

3.2.1.2.4 Parameter estimation of (μ_t, σ_t^2): year-to-year view

Each time-step mean will simply be estimated recursively as follows:

$$\widehat{\mu_t} = \widehat{\sum_{k=1}^{t} \mu_k} - \sum_{k=1}^{t-1} \widehat{\mu_k} \tag{3-18}$$

or:

$$\widehat{\mu_t} = \widehat{M_{1:t}} - \sum_{k=1}^{t-1} \widehat{\mu_k} \tag{3-19}$$

For the year-to-year standard deviation, we can use (A.3- 22) (with $u = 1$ and $v = t - 1$, for $t > 1$):

$$S_{1:t} - S_{1:t-1} = \left(\sum_{k=1}^{t} {\sigma_k}^2 + 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} \rho_{i,j}\sigma_i\sigma_j\right) - \left(\sum_{k=1}^{t-1} {\sigma_k}^2 + 2\sum_{i=1}^{t-2}\sum_{j=i+1}^{t-1} \rho_{i,j}\sigma_i\sigma_j\right) = {\sigma_t}^2 + 2\sum_{j=1}^{t-1} \rho_{t,j}\sigma_t\sigma_j$$

from which we can derive:

$${\sigma_t}^2 + 2\sigma_t \sum_{j=1}^{t-1} \rho_{t,j}\sigma_j + (S_{1:t-1} - S_{1:t}) = 0 \tag{3-20}$$

This is a second-order equation in σ_t that we solve as follows:

With

$$\Delta_t = \left(\sum_{j=1}^{t-1} \rho_{t,j}\sigma_j\right)^2 + (S_{1:t} - S_{1:t-1}) \tag{3-21}$$

The (positive) root is then:

$$\sigma_t = -\left(\sum_{j=1}^{t-1} \rho_{t,j}\sigma_j\right) + \sqrt{\Delta_t} \tag{3-22}$$

We can then estimate the year-to-year standard deviation recursively as follows:

$$\widehat{\sigma_1} = \widehat{S_{1:1}} \tag{3-23}$$

and then, for $2 \leq t \leq n$:

$$\widehat{\sigma_t} = -\left(\sum_{j=1}^{t-1} \rho_{t,j}\widehat{\sigma_j}\right) + \sqrt{\left(\sum_{j=1}^{t-1} \rho_{t,j}\widehat{\sigma_j}\right)^2 + \left(\widehat{S_{1:t}} - \widehat{S_{1:t-1}}\right)} \tag{3-24}$$

This is independent of the model structure.

For this to work it can be seen that we need the following to hold:

$$\left(\sum_{j=1}^{t-1} \rho_{t,j} \widehat{\sigma_j}\right) \leq \sqrt{\left(\sum_{j=1}^{t-1} \rho_{t,j} \widehat{\sigma_j}\right)^2 + \left(\widehat{S_{1:t}} - \widehat{S_{1:t-1}}\right)} \tag{3-25}$$

which requires that:

$$\widehat{S_{1:t-1}} \geq \widehat{S_{1:t}} \tag{3-26}$$

Note that in the case of total independency between time periods, this previous condition would come down to the following one:

$$\widehat{\sum_{k=1}^{t} \sigma_k^2} \geq \sum_{k=1}^{t-1} \widehat{\sigma_k}^2 \tag{3-27}$$

In both cases, the underlying condition is that the volatilities of the cumulated payments increase as we move further into the future, in other words, C_t is more volatile than C_{t-1} for $t \in [\![1, \ldots, n-1]\!]$.
If this is true by construction in the analytical models since:

– Lognormal model: $C_t = C_0.LnN\left(\sum_{k=1}^{t} \mu_k, \sum_{k=1}^{t} \sigma_k{}^2 + 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} \rho_{i,j}\sigma_i\sigma_j\right)$

$$var(C_t) = C_0{}^2.\left(e^{\sum_{k=1}^{t}\sigma_k{}^2 + 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t}\rho_{i,j}\sigma_i\sigma_j} - 1\right).e^{2\sum_{k=1}^{t}\mu_k + \sum_{k=1}^{t}\sigma_k{}^2 + 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t}\rho_{i,j}\sigma_i\sigma_j} \quad \forall t \in [\![1, \ldots, n-1]\!] \Rightarrow var(C_t) \geq var(C_{t-1}) \tag{3-28}$$

– Normal model: $C_t = N\left(\sum_{k=1}^{t} \mu_k, \sum_{k=1}^{t} \sigma_k{}^2 + 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} \rho_{i,j}\sigma_i\sigma_j\right) + C_0$

$$var(C_t) = \sum_{k=1}^{t} \sigma_k{}^2 + 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} \rho_{i,j}\sigma_i\sigma_j \quad \forall t \in [\![1, \ldots, n-1]\!] \Rightarrow var(C_t) \geq var(C_{t-1}) \tag{3-29}$$

(both variances are increasing functions of time t if all correlations factors are positive, which is the case in our applications)
it is however not established that the series of projected cash-flows to which the meshes are fitted should fulfill that condition. Using the matrix notation introduced in (3-13), this comes down to verifying that the set of simulations $\left(C_t^{(1)}, \ldots, C_t^{(i)}, \ldots, C_t^{(N)}\right)^T$ is more volatile than $\left(C_{t-1}^{(1)}, \ldots, C_{t-1}^{(i)}, \ldots, C_{t-1}^{(N)}\right)^T$ $\forall t \in [\![1, \ldots, n-1]\!]$ – where $\boldsymbol{x}^T$ denotes the transpose of vector $\boldsymbol{x}$.

Nevertheless, it is generally the case that Bootstrapping or internal models inherently capture the "funnel of uncertainty" mentioned earlier in this document, by generating increasingly more volatile outputs as we go further in the claims projections development. In most cases, there is a statistical model underlying the simulations leading to the stochastic reserves determination and their run-off over time, as in the T. Mack and D. Murphy's formulation variation of the Bootstrapping used here and described in Appendix D.

3.2.2 Goodness of fit and best mesh selection

Goodness of fit assesses whether a given distribution is suited to a data-set. As a first step, one can conduct a visual examination of the fitted curve by drawing its empirical histogram together with the theoretical probability density function curve. As a second step, a QQ-plot (where "Q" stands for "quantile") can compare the two probability distributions by plotting their percentiles against each other. If the two distributions being compared are similar, the points in the QQ-plot will approximately lie on the line $y = x$. Beyond those, some quantitative techniques should be examined. The general procedure consists of defining a test statistic which is some function of the data measuring the distance between the theoretical candidate probability distribution function (the hypothesis) and the data, and then calculating the probability of obtaining data which have a still larger value of this test statistic than the value observed, assuming the hypothesis is true. This probability is called the confidence level. More can be found in Appendix E.

The following tests and their underlying measures of fit were used.
– Chi-square test (on binned data);
– Kolmogorov–Smirnov test (for continuous distributions);

Each of these GoF tests statistics provide a p-value, which is the probability of obtaining a test statistic at least as extreme as the one that was actually observed, assuming that the null hypothesis is true. In other words, the p-value, which directly depends on a given sample, attempts to provide a measure of the strength of the results of a test, in contrast to a simple reject or do not reject. If the null hypothesis is true and the chance of random variation is the only reason for sample differences, then the p-value is a quantitative measure to feed into the decision making process as evidence.
A widely accepted interpretation of the p-value found in many scientific papers can be as follows:

p-value	Interpretation
$p < 0.01$	very strong evidence against H_0
$0.01 \leq p < 0.05$	moderate evidence against H_0
$0.05 \leq p < 0.10$	suggestive evidence against H_0
$0.10 \leq p$	little or no real evidence against H_0

When comparing the fit of a data set to several theoretical candidate functions, a natural ranking approach could be to select the model that results in the highest p-value. However, this might not be so straightforward in this exercise where we are mostly interested in the tail, which is hard to assess via GoF statistics.

3.2.3 Mesh-fitting results

This section summarizes the results of the parameters estimations, the Goodness of Fit and the tests for independence throughout the projection period for the two lines of business under consideration, each under the two analytical model fits. This closes Step 2 of the process described in FIG. 3.1.

More precisely, the four combinations are covered in the following order:
– Lognormal model for CProp;
– Lognormal model for EL;
– Normal model for CProp;
– Normal model for EL;

3.2.3.1 Lognormal model

Line of Business	Commercial Property										
Model	Lognormal										
Time t	1	2	3	4	5	6	7	8	9	10	11
Parameters estimates (cumulated)											
Mean $\widehat{M_{1:t}}$	9.727%	12.608%	13.454%	13.886%	14.242%	14.444%	14.511%	14.530%	14.532%	14.533%	14.533%
Standard deviation $\sqrt{\widehat{S_{1:t}}}$	1.577%	1.991%	2.095%	2.174%	2.256%	2.341%	2.408%	2.464%	2.515%	2.559%	2.574%
Parameters estimates (incremental)											
Mean $\widehat{\mu_t}$	9.727%	2.881%	0.846%	0.432%	0.356%	0.202%	0.067%	0.018%	0.003%	0.001%	0.000%
Standard deviation $\widehat{\sigma_t}$	1.577%	0.815%	0.373%	0.326%	0.289%	0.250%	0.202%	0.167%	0.151%	0.115%	0.037%
Goodness of fit tests											
Chi-Square											
Test statistic (X)	11.241	19.235	16.115	17.720	13.243	14.838	11.436	13.545	15.466	18.696	23.210
p-value	0.591	0.116	0.243	0.168	0.429	0.318	0.574	0.407	0.279	0.133	0.039
Kolmogorov-Smirnov											
Test statistic (D)	0.007	0.009	0.008	0.009	0.008	0.007	0.006	0.006	0.006	0.007	0.008
p-value	0.634	0.373	0.557	0.436	0.490	0.709	0.888	0.891	0.807	0.715	0.619

Table 3-1 – Mesh-fitting results: (A) Lognormal model (B) Commercial Property

Line of Business	Employer Liability																		
Model	Lognormal																		
Time t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Parameters estimates (cumulated)																			
Mean $\widehat{M_{1:t}}$	15.321%	28.330%	38.304%	44.988%	48.702%	50.632%	51.863%	52.657%	53.172%	53.560%	53.861%	54.117%	54.301%	54.482%	54.671%	54.888%	55.085%	55.140%	55.140%
Standard deviation $\sqrt{\widehat{S_{1:t}}}$	1.127%	1.813%	2.441%	2.962%	3.299%	3.524%	3.699%	3.824%	3.931%	4.040%	4.147%	4.270%	4.393%	4.510%	4.628%	4.765%	4.913%	5.035%	5.085%
Parameters estimates (incremental)																			
Mean $\widehat{\mu_t}$	15.321%	13.009%	9.974%	6.685%	3.714%	1.930%	1.231%	0.794%	0.515%	0.387%	0.301%	0.256%	0.184%	0.181%	0.189%	0.216%	0.197%	0.055%	0.000%
Standard deviation $\widehat{\sigma_t}$	1.127%	1.103%	1.019%	0.875%	0.672%	0.536%	0.462%	0.379%	0.353%	0.343%	0.331%	0.339%	0.327%	0.292%	0.279%	0.289%	0.295%	0.258%	0.102%
Goodness of fit tests																			
Chi-Square																			
Test statistic (X)	10.673	12.702	13.623	12.480	21.073	11.977	18.584	18.039	11.962	16.520	17.617	9.330	12.219	10.671	9.786	14.221	15.846	21.003	22.457
p-value	0.638	0.471	0.401	0.489	0.071	0.530	0.137	0.156	0.531	0.222	0.173	0.407	0.201	0.299	0.368	0.115	0.070	0.013	0.008
Kolmogorov-Smirnov																			
Test statistic (D)	0.006	0.006	0.008	0.007	0.006	0.006	0.007	0.007	0.006	0.008	0.008	0.010	0.009	0.010	0.010	0.011	0.013	0.013	0.014
p-value	0.876	0.802	0.540	0.659	0.918	0.814	0.791	0.699	0.807	0.590	0.503	0.243	0.332	0.251	0.303	0.174	0.075	0.079	0.036

Table 3-2 – Mesh-fitting results: (A) Lognormal model (B) Employer's Liability

The following depicts the qualitative GoF tests obtained for $t = 1$ only, the full run-off results can be found in Appendix B. The graph on the left shows the empirical histogram of the (log-) cumulated development factors while the graph on the right shows their respective QQ-plot.

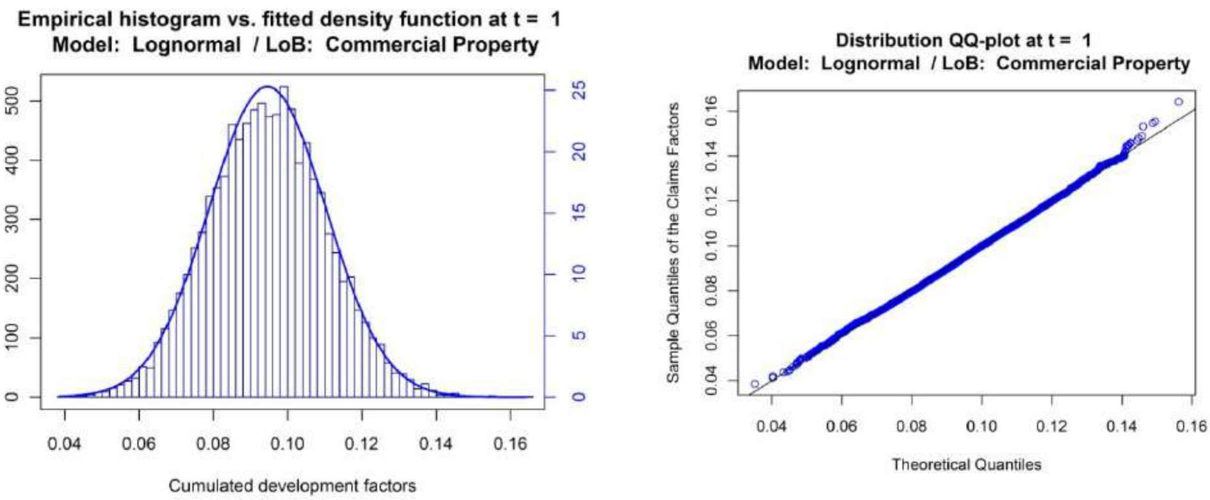

FIG. 3.2 – Mesh-fitting results: (A) Lognormal model (B) Commercial Property

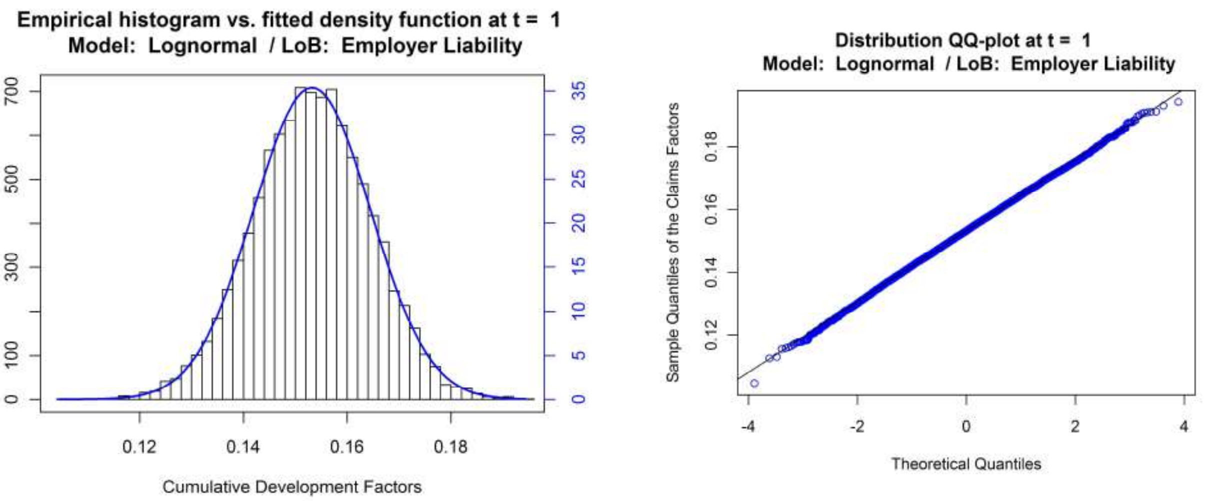

FIG. 3.3 – Mesh-fitting results: (A) Lognormal model (B) Employer's Liability

The fit on CProp passes both the Chi-Square and the Kolmogorov-Smirnov tests on all projection periods, with the exception of the last year (at time t = 11) where the result of the Chi-Square test at the 95% confidence interval would lead to rejecting the null hypothesis of normality of the development factors. However, the K-S test would lead to accepting it.
Looking at the graphs now listed in Appendix B, it can be seen that from time t = 1 to 11, the "Empirical vs. fitted density" is relatively good, with no noticeable deterioration as we move further away from origin. The QQ-plots also show a very good fit overall, with some deviations in the tail of the distributions.

The fit on EL passes both the Chi-Square and the Kolmogorov-Smirnov tests on all projection periods, with the exception of the last two years, where at time t = 18 and t = 19, the result of the Chi-Square test at the 95% confidence interval would lead to rejecting the null hypothesis of normality of the development factors. The K-S test would only reject the fit at time t =19.
The graphs seem to support these quantitative results. Even if some deviations can be seen in the tail of the distributions, nothing seems to suggest that the fit should be rejected.

The correlation matrices are presented in Appendix B.3.

3.2.3.2 Normal model

Line of Business	Commercial Property										
Model	Normal										
Time t	1	2	3	4	5	6	7	8	9	10	11
Parameters estimates (cumulated)											
Mean $\widehat{M_{1:t}}$	86,494	113,813	121,979	126,182	129,669	131,663	132,333	132,527	132,564	132,585	132,589
Standard deviation $\sqrt{\widehat{S_{1:t}}}$	14,705	19,123	20,283	21,146	22,016	22,894	23,561	24,114	24,625	25,054	25,200
Parameters estimates (incremental)											
Mean $\widehat{\mu_t}$	86,494	27,318	8,167	4,203	3,486	1,995	670	193	37	21	4
Standard deviation $\widehat{\sigma_t}$	14,705	7,984	3,639	3,184	2,840	2,456	1,986	1,642	1,484	1,133	363
Goodness of fit tests											
Chi-Square											
Test statistic (X)	20.229	22.224	27.060	24.403	24.426	22.439	27.995	24.796	25.153	28.488	13.414
p-value	0.163	0.102	0.028	0.059	0.058	0.097	0.022	0.053	0.048	0.019	0.570
Kolmogorov-Smirnov											
Test statistic (D)	0.009	0.012	0.011	0.012	0.012	0.010	0.009	0.009	0.011	0.011	0.011
p-value	0.361	0.105	0.145	0.128	0.113	0.278	0.357	0.415	0.207	0.215	0.166

Table 3-3 – Mesh-fitting results: (A) Normal model (B) Commercial Property

Line of Business	Employer Liability																		
Model	Lognormal																		
Time t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Parameters estimates (cumulated)																			
Mean $\widehat{M_{1:t}}$	15.321%	28.330%	38.304%	44.988%	48.702%	50.632%	51.863%	52.657%	53.172%	53.560%	53.861%	54.117%	54.301%	54.482%	54.671%	54.888%	55.085%	55.140%	55.140%
Standard deviation $\sqrt{\widehat{S_{1:t}}}$	1.127%	1.813%	2.441%	2.962%	3.299%	3.524%	3.699%	3.824%	3.931%	4.040%	4.147%	4.270%	4.393%	4.510%	4.628%	4.765%	4.913%	5.035%	5.085%
Parameters estimates (incremental)																			
Mean $\widehat{\mu_t}$	15.321%	13.009%	9.974%	6.685%	3.714%	1.930%	1.231%	0.794%	0.515%	0.387%	0.301%	0.256%	0.184%	0.181%	0.189%	0.216%	0.197%	0.055%	0.000%
Standard deviation $\widehat{\sigma_t}$	1.127%	1.103%	1.019%	0.875%	0.672%	0.536%	0.462%	0.379%	0.353%	0.343%	0.331%	0.339%	0.327%	0.292%	0.279%	0.289%	0.295%	0.258%	0.102%
Goodness of fit tests																			
Chi-Square																			
Test statistic (X)	10.673	12.702	13.623	12.480	21.073	11.977	18.584	18.039	11.962	16.520	17.617	9.330	12.219	10.671	9.786	14.221	15.846	21.003	22.457
p-value	0.638	0.471	0.401	0.489	0.071	0.530	0.137	0.156	0.531	0.222	0.173	0.407	0.201	0.299	0.368	0.115	0.070	0.013	0.008
Kolmogorov-Smirnov																			
Test statistic (D)	0.006	0.006	0.008	0.007	0.006	0.006	0.007	0.007	0.006	0.008	0.008	0.010	0.009	0.010	0.010	0.011	0.013	0.013	0.014
p-value	0.876	0.802	0.540	0.659	0.918	0.814	0.791	0.699	0.807	0.590	0.503	0.243	0.332	0.251	0.303	0.174	0.075	0.079	0.036

Table 3-4 – Mesh-fitting results: (A) Normal model (B) Employer's Liability

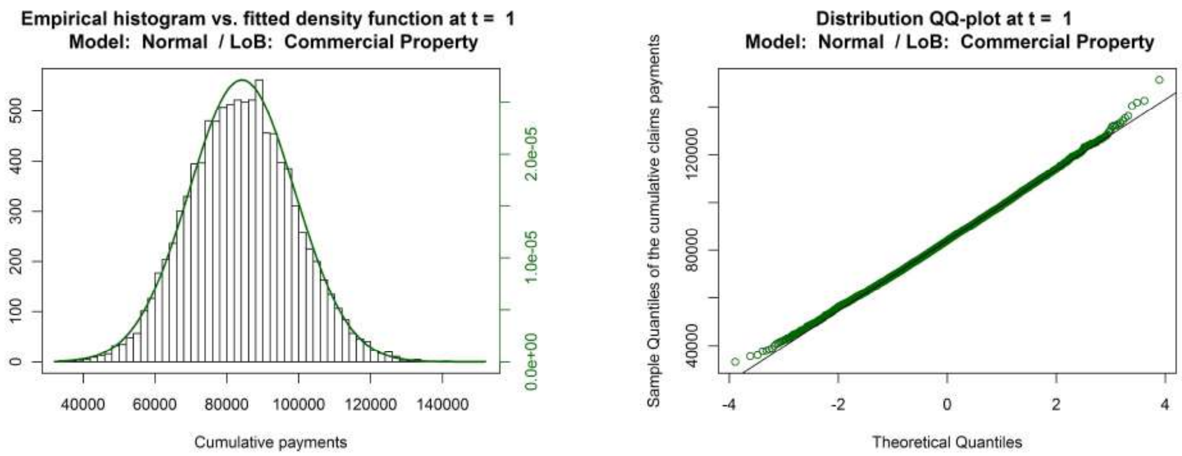

FIG. 3.4 – Mesh-fitting results: (A) Normal model (B) Commercial Property

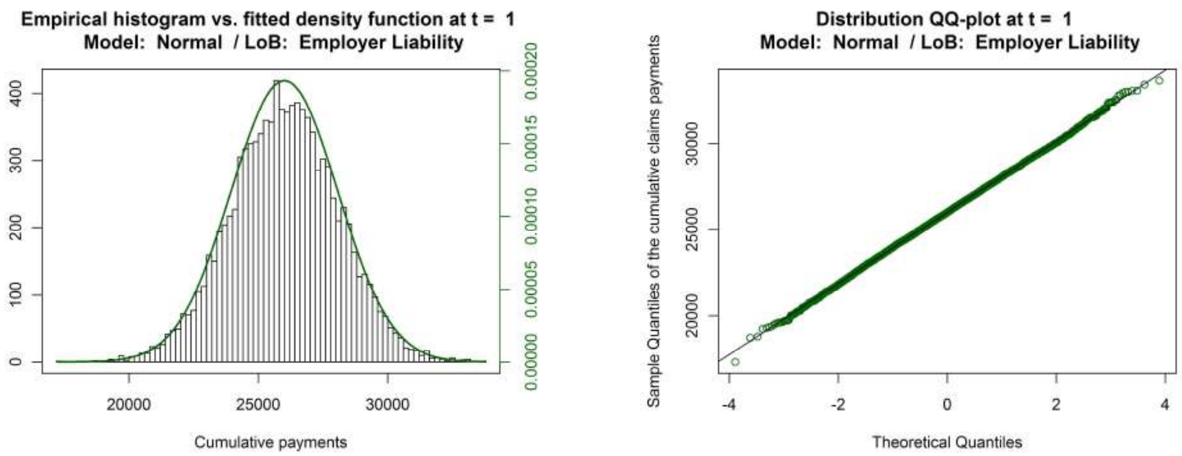

FIG. 3.5 – Mesh-fitting results: (A) Normal model (B) Employer's Liability

On CProp, the Chi-Square test at the 95% confidence interval would lead to rejecting the fit for normality at times $t = 3, 7, 9, 10$, whereas it passes the K-S test on all projection periods. It can be noted, however, that the Chi-Square tests in general rely on how data is binned, with no known optimal algorithm to define those bins and their number. The K-S p-values look reasonable, and as such, there is no strong evidence to reject the fit.
The graphs also seem to indicate a good fit, albeit with more deviations in the tails than their lognormal counterparts.

On EL, there seems to be some evidence to reject the fit from time $t = 9$ onwards, as it does not pass the K-S, however, it would pass under the Chi-Square test this time, at times $t = 11, 12, 13, 14$. The graphs clearly suggest a poorer fit than the lognormal model, with increasing deviations from normality in the tail as we move closer to the horizon.

Comparing the fits on the lognormal and normal models, the p-values on the lognormal model seem to indicate that the lognormal fit is better than the normal one.

The correlation factors results are fairly similar to those described in the lognormal results above.

3.3 MVM Results

3.3.1 Practical MVM through the analytical solutions

The main aim of this study has been to build a theoretical analytical model as an answer to the inherent circularity issue lying in the MVM formulation. The theoretical solutions have then been applied on two real case studies.

The next paragraphs will present the analytical MVM results and at the same time compare them against the QIS5 proxies,along a similar order as the GoF section above, namely:
– Lognormal model for CProp;
– Lognormal model for EL;
– Normal model for CProp;
– Normal model for EL.

3.3.2 Quantifying proxies materiality

It first needs to be explained what exactly lies behind the various results being compared.

– Analytical solution: whether using the lognormal or normal model, the analytical solution is the one solved for $({Cap_t}^{1C},\ {MVM_t}^{C})$ as fully described in §2.2.6, calibrated with the parameters presented in the previous paragraph.

– Unstressed MVM: this gives the analytical result for $({Cap_t}^{1A},\ {MVM_t}^{A})$ where the capital ignores the movement of the MVM under a stress scenario over a 1-year horizon. This will provide a quantification of the materiality of this currently used simplification.

– QIS5 – proxy 3 / QIS5 – proxy 4 / QIS5 – proxy 5: these elements are meant to quantify the proxies suggested by QIS5 as formulated in §2.2.6.2.3, on the MVM approximation in isolation. This means that the capital base these are using is the one determined under the analytical solution, in order to strip out the effect of the approximations also applied when determining capital requirement under the Standard Approach in QIS5 (cf. below). As a result, the volatility taken into account (in the capital calculation, and hence, in the MVM number) is estimated by the mesh parameterisation, which in turn finds its roots in the input cash-flows.

– QIS5 – standard approach: the Standard Formula provides a simplified way of assessing the capital amount at time $t = 0$. Under the scope of this study, the capital amount is made of non-life reserve risk capital requirement for a single line of business only. The technical specifications[34] provide the following calculations steps, using the same notations as used throughout this document:

$$NL_{res(t|t)} = Cap_t^{QIS5} = \left(\frac{e^{\Phi^{-1}[99.5\%].\sqrt{\ln(\sigma^2+1)}}}{\sqrt{\sigma^2+1}} - 1\right).E(R_t|\mathcal{F}_t) \qquad \textbf{(3-30)}$$

where σ is the standard deviation for the reserve risk (per reserve unit) provided in the Technical Specifications[35]. It comes out as 11% on both Commercial Property and Employer's Liability lines of business, taken from "Fire and other damage" and "Third-party liability" calibrations respectively, using the QIS5 class mapping. The standard deviation is assumed to be constant throughout the whole run-off period.

With the capital defined as such, the MVM is further approximated using Proxy 3, as described in equations (2-43) and (2-44). It is worth reminding here that Cap_t^{QIS5} ignores the circularity issue.
This will give an idea of where the capital determined through the Standard Formula approach stands in comparison to an exact solution. It will not, however, provide a direct quantification of the proxy adopted on the MVM calculation.

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[34] QIS5 Technical Specifications SCR.9.14 (cf. [3])
[35] QIS5 Technical Specifications SCR.9.29 (cf. [3])

– QIS5 – USP[36]: in an extension to the previous, the undertaking can use its own specific parameters (Undertaking Specific Parameters) as a measure of the standard deviation for the reserve risk and use these in the Standard Formula. At the undertaking level, the purpose is to contribute to a more risk-sensitive capital requirement and allow a better assessment of the underwriting risk that undertakings are exposed to. At the European level, encouraging companies to calculate these USP will help revising the calibration of the corresponding market parameters prescribed under the Standard Approach. A credibility element is attached to the final volatility to use, as follows:

$$\sigma_{(res,lob)} = c.\sigma_{(U,res,lob)} + (1-c).\sigma_{(M,res,lob)}$$

where c is the credibility factor, $\sigma_{(U,res,lob)}$ is the undertaking-specific estimate of the standard deviation for reserve risk, for a given line of business, and $\sigma_{(M,res,lob)}$ is the standard parameter provided in the Technical Specifications and $\sigma_{(res,lob)}$ is the resulting USP parameter to be used in the Standard Formula calculations. The credibility factor depends on the length of available historical data. In the case of our two examples, CProp and EL, where 10 years and 18 years of triangle data respectively are available, full credibility can be granted (i.e $c = 1$).

The Technical Specifications further suggest three methods, briefly described as follows:

- Method 1 assumes a constant proportionality relationship between the "variance of the best estimate for claims outstanding in one year plus the incremental claims paid over the one year by LoB" and the "current best estimate for claims outstanding". The information required in addition to the standard paid claims triangles is an "as at" view for each past calendar year of the BEL and how it runs-off after a year (i.e. the split between the new BEL and incremental paid claims for the following calendar year), as the constant of proportionality is estimated on this historic view. Without any actual reserving history on the two examples chosen in this case study, and without going into the heavy task of re-reserving prior years triangles, this method will not be considered here.
- Methods 2 and 3 are based on the Merz and Wüthrich method[37] with the calculation of the mean squared error of prediction (MSEP) of the claims development result over the one year. The resulting USP standard deviation is computed as follows:

$$\sigma_{(U,res,lob)} = \frac{\sqrt{MSEP}}{BEL}$$

where BEL is the actual BEL reported by the undertaking for Method 2, and what the best estimate of the reserves would be using the Chain-Ladder approach for Method 3.

The results of these calculations using the Helper Tabs[38] are summarized below:

	Commercial Property			Employer Liability		
Reserve Risk	Method 1	Method 2	Method 3	Method 1	Method 2	Method 3
Final sigma $\sigma_{(u,res,lob)}$		13.91%	16.36%		10.00%	7.88%
N (lob)		10	10		18	18
Standard gross factor $\sigma_{(M,prem,lob)}$		11%			11%	
Credibility factor c		100%	100%		100%	100%
USP		13.91%	16.36%		10.00%	7.88%

Table 3-5 – USP volatility results

The results being presented show a "typical path" throughout the whole run-off of the book where the same proxies structures are used throughout the projection period. As mentioned previously, the way in which the liabilities will evolve over time is random, which has the effect that all the calculations leading to the exact solutions use expected values conditioned to the available information at the time in the projection. This has implications on the path-dependent structures only (i.e. the lognormal structure in this case), where for a given selected path, there will be a different capital and MVM amounts. By "typical path", we mean a selected simulation that is not too atypical and that could be considered to be average from the set of simulations under study.

[36] QIS5 Technical Specifications SCR.10.6 (cf. [3])

[37] "Modelling The Claims Development Result For Solvency Purposes" by Michael Merz and Mario V Wüthrich, Casualty Actuarial Society E-Forum, Fall 2008

[38] H_Risk_Margin_201000906.xls (as at 23.09.2010)

The general layout of the results is as follows. For each given model and line of business, the first table presents the numerical results accompanied by those of intermediate steps of the analytical solutions, for the capital and the MVM amounts throughout the projection period (11 years for Commercial Property and 18 years for Employer's Liabilities, which relate to the number of accident years of the respective claims triangles). Then, the next three graphs show a comparison of the proxies described above on a typical path for (i) the MVM, (ii) the Capital and (iii) the MVM + Capital. The latter element (MVM + Capital) is actually the most important and meaningful from an economic point of view. Indeed, the mutual relationship between MVM and Capital holds in the way each of the two components move against each other. If MVM is larger, then the Capital should be smaller since the MVM can be used to fund future capital requirements, as MVM drops to zero over time and that release from the technical provisions is an offset to capital. As the QIS5 Proxies 3, 4 and 5 use the same Capital amounts as the analytical solutions, the Capital curve in (ii) only compares the analytical solution to the Standard Approach ones, be it using Standard Factors or USP. Nevertheless, we have added the reserves values to it in order to get a graphical feel as to how the Capital amounts compare to them in terms of absolute amounts and evolutions. Finally, when moving to the normal models, however, we removed the comparison to the Standard Approaches, as the results will be identical to those presented under the lognormal models. In addition, given the underlying lognormal assumptions of the QIS5 calculations for the reserve risk as seen in Equation (3-30), it makes more sense to compare them alongside the analytical lognormal model. Instead, and in order to link the two analytical solutions together, the analytical lognormal curves have been added to the normal models graphs.

Then we present some high-level results of comparisons against classes of business. The underlying data comes from internal models outputs and have not been further analyzed as was previously explained. The aim here is to get a wider picture of the QIS5 proxy materiality.

We comment on these results in the last paragraph of this section.

3.3.3 Lognormal model

The values for C_t are that of a typical selected path. The expected reserves conditioned to the information available at the time in the projection are shown for illustrative purposes. They do not directly enter the MVM results. Note that the $\widetilde{\mu_t}, \widetilde{\sigma_t}, \widetilde{M_t}$ and $\widetilde{S_t}$ are the values obtained from the selected path ω_t.

3.3.3.1 Commercial Property

Line of Business	Commercial Property											
Model	Lognormal											
t	0	1	2	3	4	5	6	7	8	9	10	11
C(t)	845,550	907,118	927,305	935,805	941,521	944,865	947,544	947,179	945,987	946,616	946,718	946,605
μ(t)		9.73%	2.88%	0.85%	0.43%	0.36%	0.20%	0.07%	0.02%	0.00%	0.00%	0.00%
σ(t)		1.58%	0.82%	0.37%	0.33%	0.29%	0.25%	0.20%	0.17%	0.15%	0.11%	0.04%
M(t)		14.53%	4.81%	1.92%	1.08%	0.65%	0.29%	0.09%	0.02%	0.00%	0.00%	0.00%
S(t)		0.07%	0.03%	0.02%	0.01%	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%
Selected path ω(t)		7.03%	2.20%	0.91%	0.61%	0.35%	0.28%	-0.04%	-0.13%	0.07%	0.01%	-0.01%
μ(t) ~		9.73%	2.44%	0.74%	0.41%	0.39%	0.21%	0.11%	0.02%	-0.02%	0.04%	0.00%
σ(t) ~		1.58%	0.77%	0.36%	0.31%	0.26%	0.21%	0.17%	0.13%	0.11%	0.06%	0.02%
M(t) ~		14.06%	4.33%	1.89%	1.16%	0.75%	0.36%	0.15%	0.03%	0.02%	0.04%	0.00%
S(t) ~		0.07%	0.03%	0.02%	0.01%	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%
Y(t)	0.8685	0.9575	0.9812	0.9884	0.9925	0.9964	0.9985	0.9996	0.9998	0.9996	1.0000	1.0000
F(t)	0.0448	0.0198	0.0090	0.0077	0.0065	0.0052	0.0041	0.0033	0.0028	0.0014	0.0004	0.0000
W(t)	0.0982	0.0592	0.0402	0.0313	0.0237	0.0173	0.0121	0.0079	0.0047	0.0019	0.0004	0.0000
[W(t)-W(t+1)]/Y(t)]	0.0450	0.0198	0.0090	0.0077	0.0065	0.0052	0.0042	0.0033	0.0028	0.0014	0.0004	0.0000
Capital(t)	38,023	17,976	8,391	7,197	6,097	4,949	3,933	3,080	2,620	1,371	422	0
MVM(t)	5,737	3,363	2,279	1,778	1,349	983	687	450	265	108	25	0
R(t) | C(t)	127,978	40,310	17,808	10,935	7,107	3,433	1,412	333	160	345	8	0
Implied Percentile	65%	68%	75%	73%	71%	69%	67%	64%	60%	58%	56%	

Table 3-6 – MVM results: (A) Lognormal model (B) Commercial Property

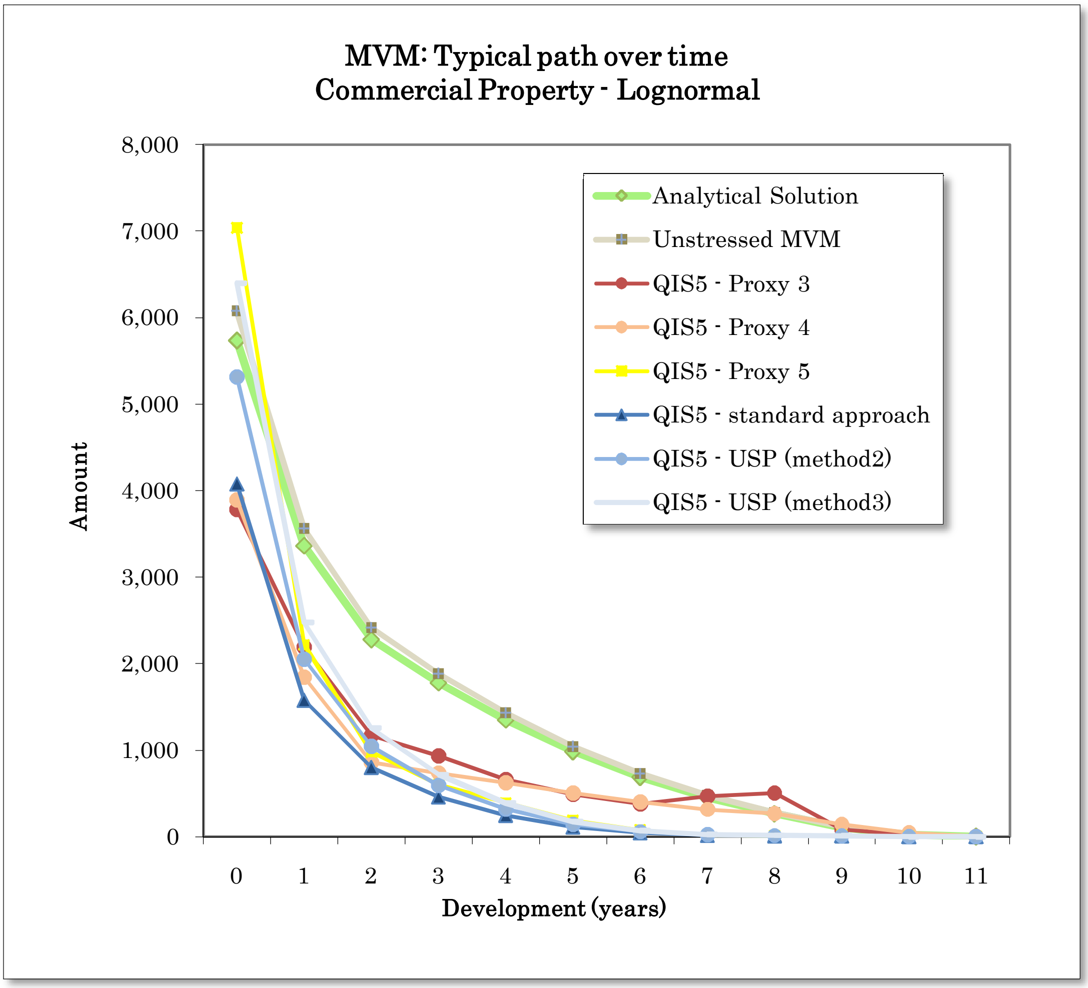


FIG. 3.6 – MVM: (A) Lognormal model (B) Commercial Property

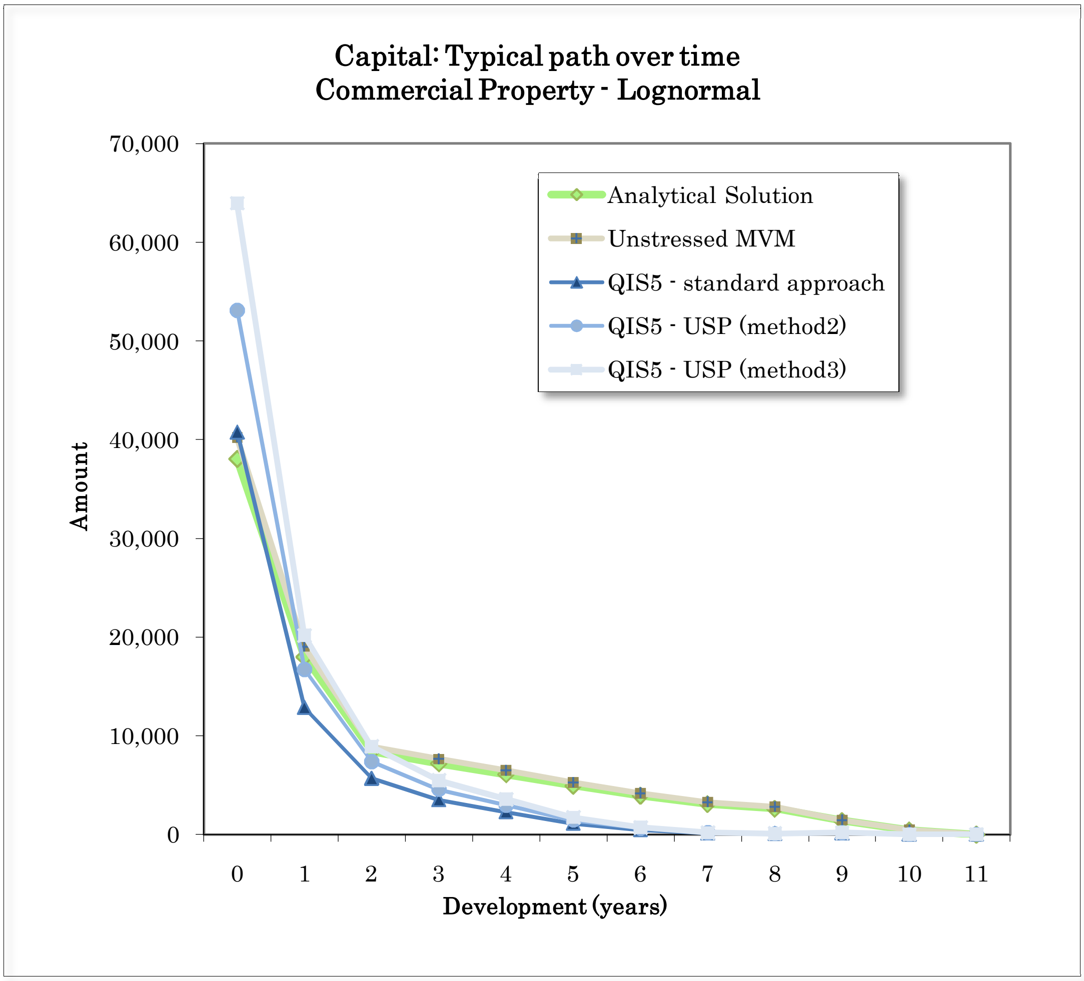


FIG. 3.7 – Capital: (A) Lognormal model (B) Commercial Property

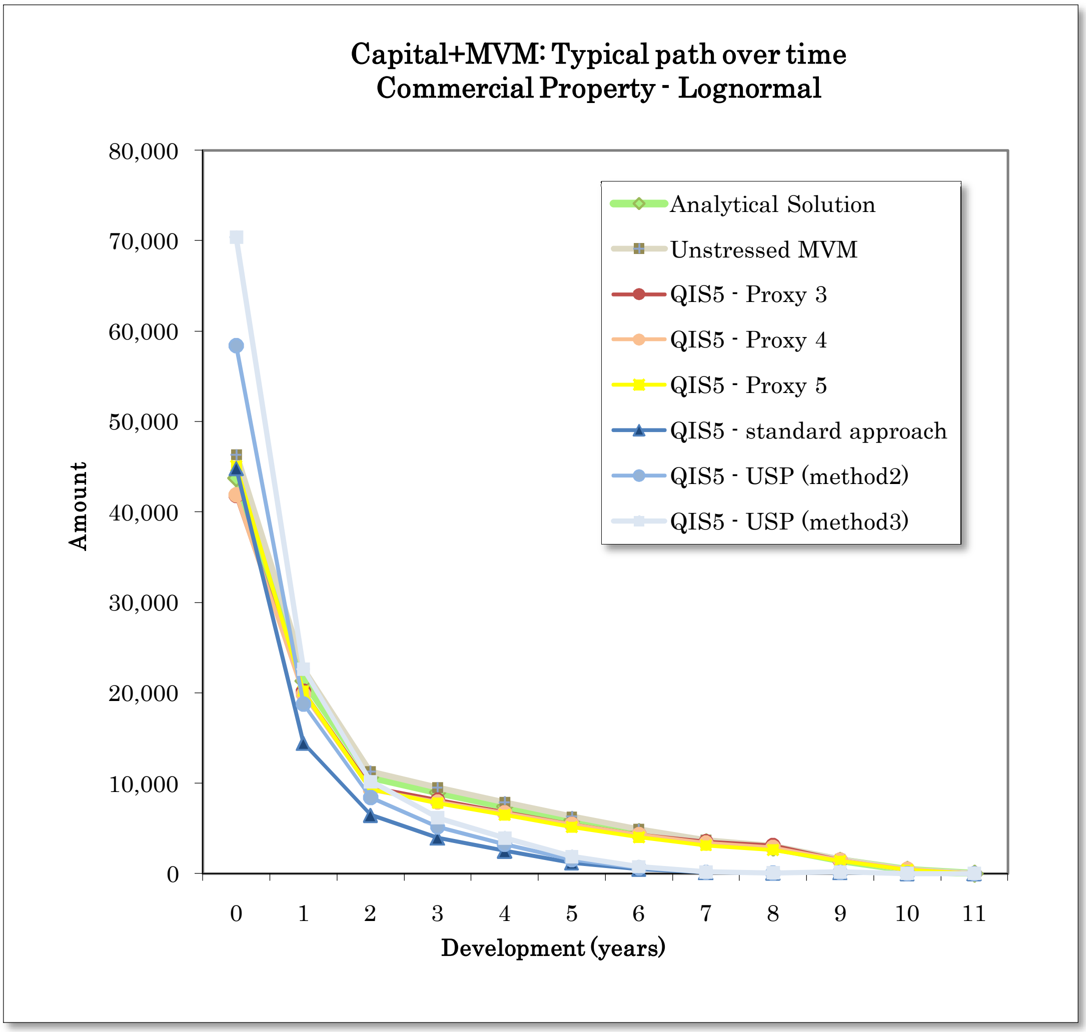


FIG. 3.8 – Capital + MVM: (A) Lognormal model (B) Commercial Property

3.3.3.2 Employer's Liability

Line of Business	Employer Liability									
Model	Lognormal									
t	0	1	2	3	4	5	6	7	8	9
C(t)	156,994	187,361	215,020	240,279	257,151	266,280	273,689	277,650	278,053	279,325
μ(t)		15.321%	13.009%	9.974%	6.685%	3.714%	1.930%	1.231%	0.794%	0.515%
σ(t)		1.127%	1.103%	1.019%	0.875%	0.672%	0.536%	0.462%	0.379%	0.353%
M(t)		55.140%	39.819%	26.810%	16.836%	10.151%	6.438%	4.508%	3.277%	2.483%
S(t)		0.259%	0.209%	0.163%	0.129%	0.106%	0.091%	0.078%	0.068%	0.059%
Selected path ω(t)		17.683%	13.770%	11.107%	6.786%	3.489%	2.744%	1.437%	0.145%	0.456%
μ(t) ~		15.321%	13.753%	10.621%	7.312%	4.009%	2.092%	1.461%	0.932%	0.408%
σ(t) ~		1.127%	1.044%	0.909%	0.758%	0.604%	0.497%	0.427%	0.350%	0.318%
M(t) ~		58.102%	42.781%	29.028%	18.407%	11.095%	7.086%	4.994%	3.533%	2.601%
S(t) ~		0.259%	0.209%	0.163%	0.129%	0.106%	0.091%	0.078%	0.068%	0.059%
Y(t)	0.5586	0.6513	0.7474	0.8313	0.8945	0.9312	0.9509	0.9650	0.9740	0.9781
F(t)	0.0493	0.0391	0.0297	0.0222	0.0164	0.0130	0.0109	0.0088	0.0079	0.0074
W(t)	0.1956	0.1677	0.1420	0.1197	0.1011	0.0863	0.0742	0.0637	0.0552	0.0475
[W(t)-W(t+1)]/Y(t)]	0.0498	0.0395	0.0299	0.0224	0.0165	0.0130	0.0110	0.0088	0.0079	0.0074
Capital(t)	7,822	7,397	6,425	5,374	4,251	3,473	3,002	2,454	2,209	2,064
MVM(t)	3,298	2,895	2,451	2,076	1,744	1,481	1,281	1,100	946	813
R(t) | C(t)	124,052	100,330	72,654	48,747	30,326	19,682	14,128	10,081	7,410	6,266
Implied percentile	85%	84%	83%	83%	84%	85%	85%	87%	85%	83%

t	10	11	12	13	14	15	16	17	18	19
C(t)	279,804	280,225	281,696	282,983	283,265	284,054	284,465	285,298	285,441	285,344
μ(t)	0.0039	0.301%	0.256%	0.184%	0.181%	0.189%	0.216%	0.197%	0.055%	0.000%
M(t)	1.967%	1.580%	1.279%	1.023%	0.839%	0.657%	0.468%	0.252%	0.055%	0.000%
S(t)	0.051%	0.042%	0.034%	0.025%	0.018%	0.012%	0.008%	0.004%	0.001%	0.000%
Selected path ω(t)	0.171%	0.150%	0.524%	0.456%	0.099%	0.278%	0.145%	0.292%	0.050%	-0.034%
μ(t) ~	0.316%	0.186%	0.162%	0.230%	0.306%	0.238%	0.338%	0.238%	0.168%	0.010%
σ(t) ~	0.297%	0.273%	0.263%	0.232%	0.199%	0.179%	0.173%	0.163%	0.132%	0.045%
M(t) ~	2.193%	1.877%	1.691%	1.529%	1.298%	0.992%	0.754%	0.416%	0.178%	0.010%
S(t) ~	0.051%	0.042%	0.034%	0.025%	0.018%	0.012%	0.008%	0.004%	0.001%	0.000%
Y(t)	0.9812	0.9831	0.9847	0.9870	0.9901	0.9925	0.9958	0.9982	0.9999	1.0000
F(t)	0.0067	0.0065	0.0057	0.0049	0.0044	0.0042	0.0040	0.0032	0.0011	0.0000
W(t)	0.0402	0.0336	0.0272	0.0216	0.0168	0.0125	0.0083	0.0043	0.0011	0.0000
[W(t)-W(t+1)]/Y(t)]	0.0067	0.0065	0.0057	0.0049	0.0044	0.0042	0.0040	0.0032	0.0011	0.0000
Capital(t)	1,889	1,818	1,608	1,386	1,240	1,199	1,129	919	315	0
MVM(t)	689	575	468	372	288	214	142	74	19	0
R(t) | C(t)	5,362	4,827	4,376	3,724	2,841	2,160	1,192	510	28	0
Implied percentile	82%	78%	76%	75%	72%	67%	62%	58%	56%	

Table 3-7 – MVM results: (A) Lognormal model (B) Employer Liability

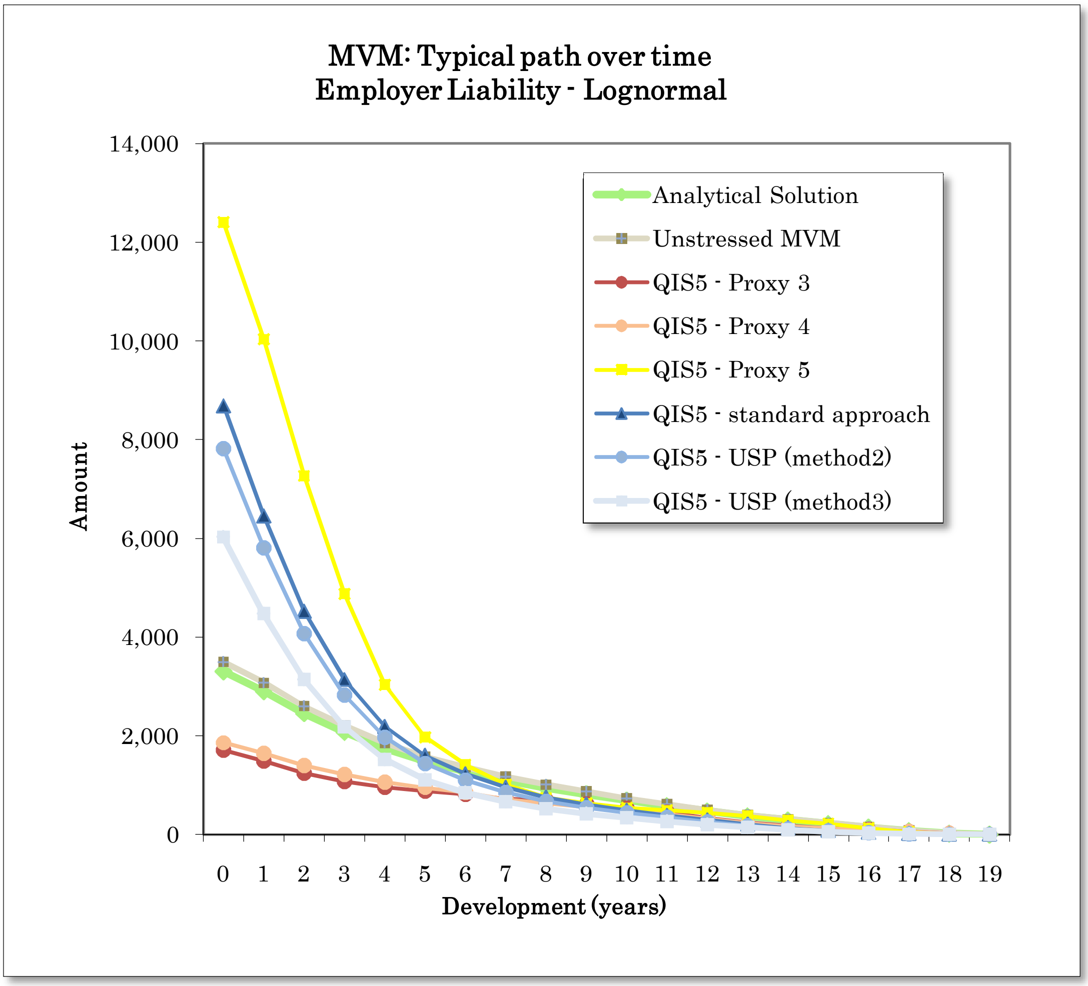


FIG. 3.9 – MVM: (A) Lognormal model (B) Employer Liability

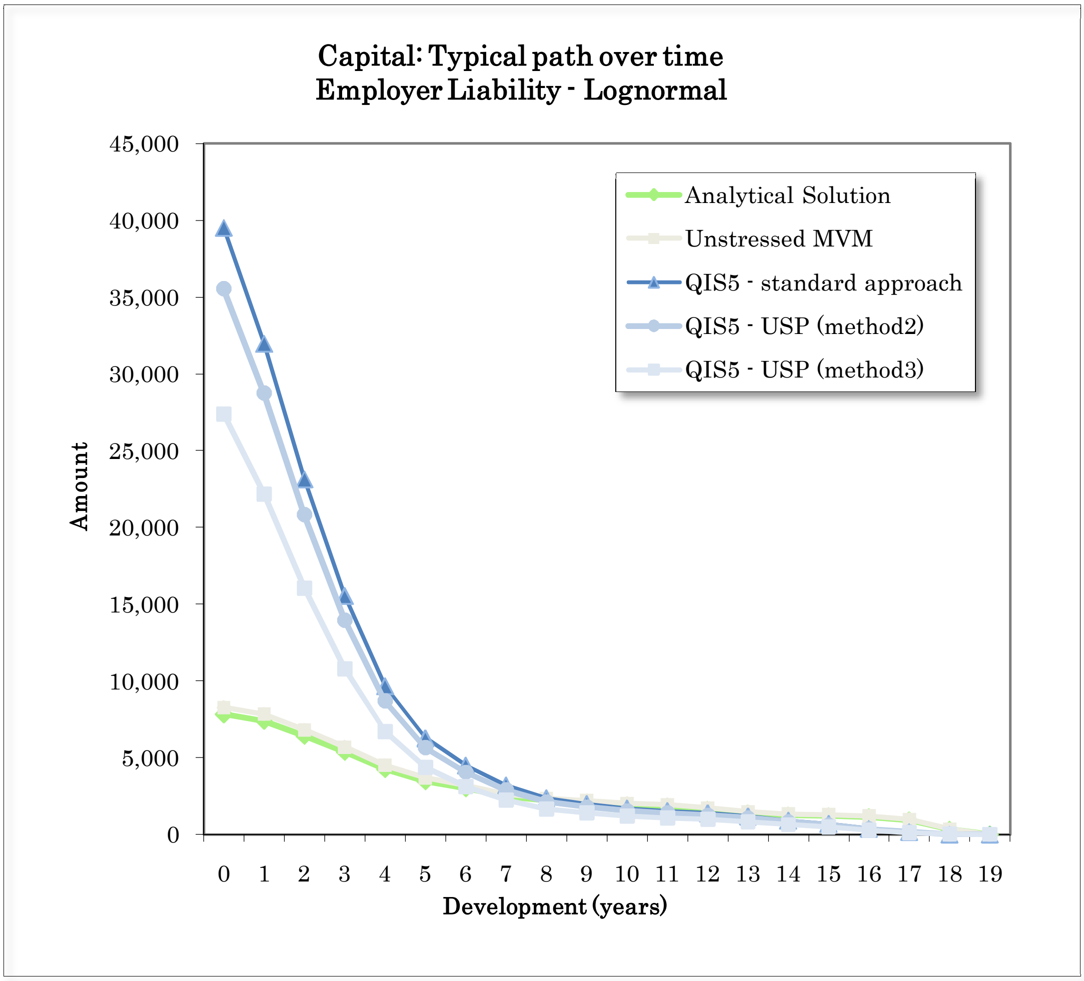


FIG. 3.10 – Capital: (A) Lognormal model (B) Employer Liability

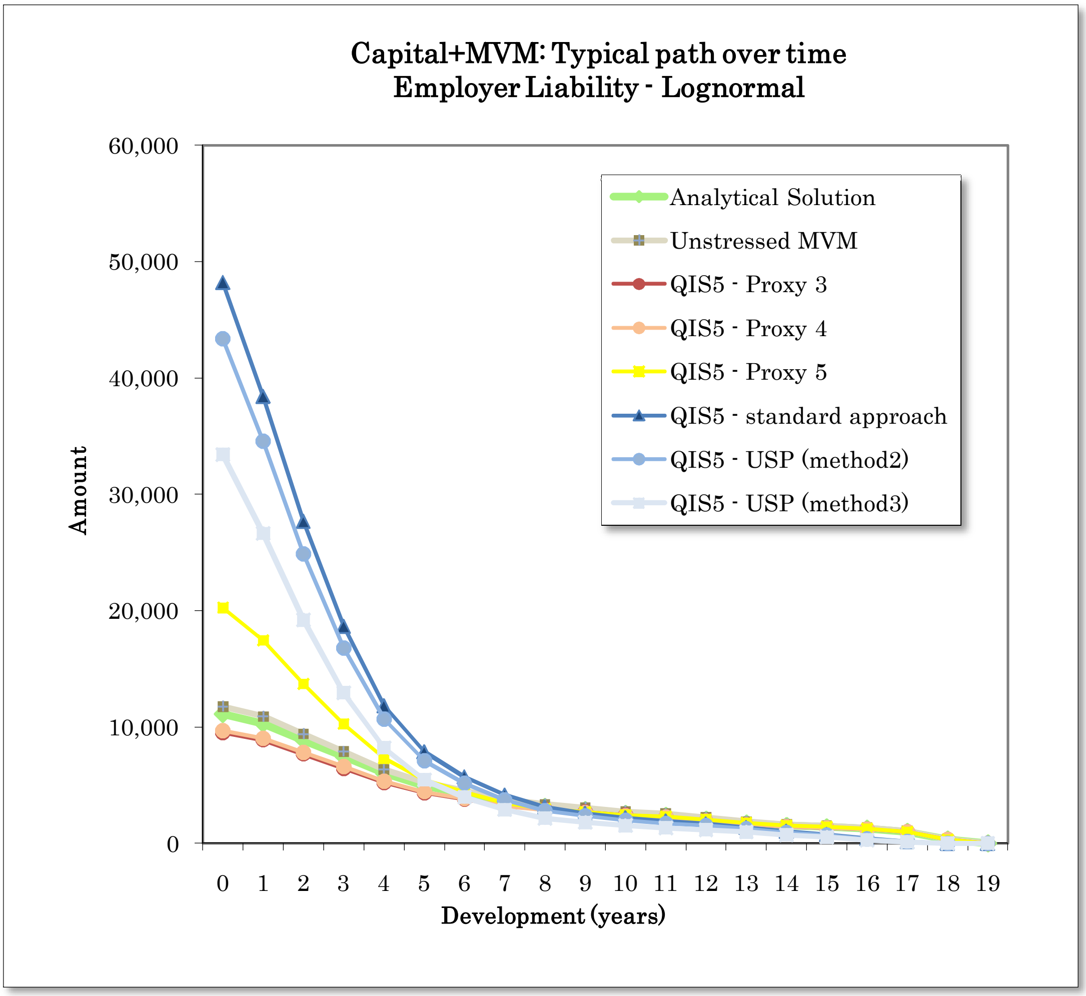


FIG. 3.11 – Capital + MVM: (A) Lognormal model (B) Employer Liability

3.3.4 Normal model

The values for are that of a selected path, although they have no further implications on the results since the chosen structure for Capital and MVM is non path-dependent, as is described in the Appendices. They are only shown for the illustration.

3.3.4.1 Commercial Property

Line of Business	Commercial Property											
Model	Normal											
t	0	1	2	3	4	5	6	7	8	9	10	11
C(t)	845,550	907,118	927,305	935,805	941,521	944,865	947,544	947,179	945,987	946,616	946,718	946,605
μ(t)		86,494	27,318	8,167	4,203	3,486	1,995	670	193	37	21	4
σ(t)		14,705	7,984	3,639	3,184	2,840	2,456	1,986	1,642	1,484	1,133	363
Selected path ω(t)		61,568	20,187	8,501	5,716	3,344	2,678	- 364	- 1,192	629	101	- 112
μ(t) ~		86,494	22,381	6,876	3,828	3,666	2,019	1,078	173	- 181	341	8
σ(t) ~		14,705	7,434	3,522	3,030	2,575	2,099	1,668	1,310	1,118	585	180
Capital(t)	35,734	18,064	8,558	7,364	6,257	5,100	4,053	3,184	2,716	1,422	438	0
MVM(t)	5,573	3,429	2,346	1,832	1,390	1,015	709	466	275	112	26	0
R(t) | C(t)	126,683	40,189	17,809	10,933	7,105	3,439	1,420	341	168	349	8	0
Implied Percentile	65%	68%	75%	73%	71%	69%	66%	64%	60%	58%	56%	

Table 3-8 – MVM results: (A) Normal model (B) Commercial Property

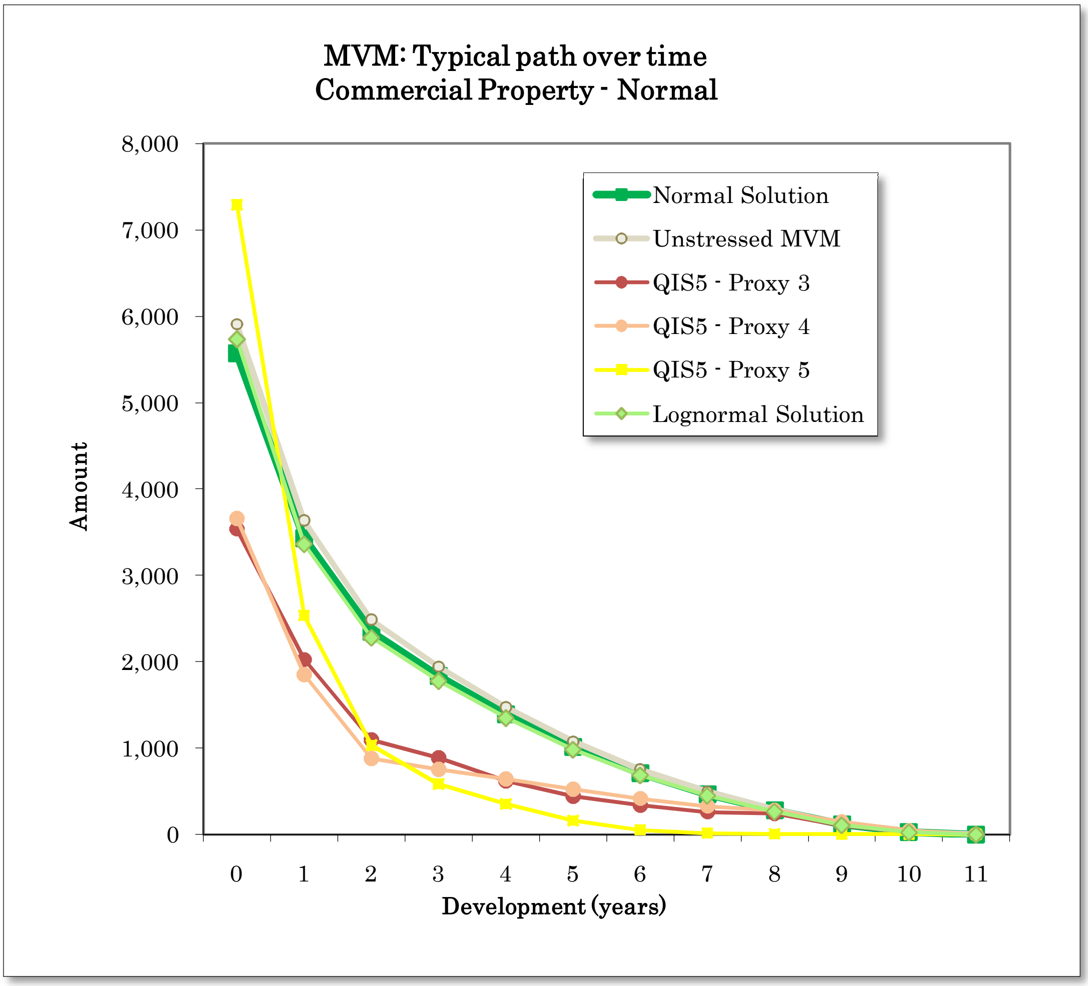


FIG. 3.12 – MVM: (A) Normal model (B) Commercial Property

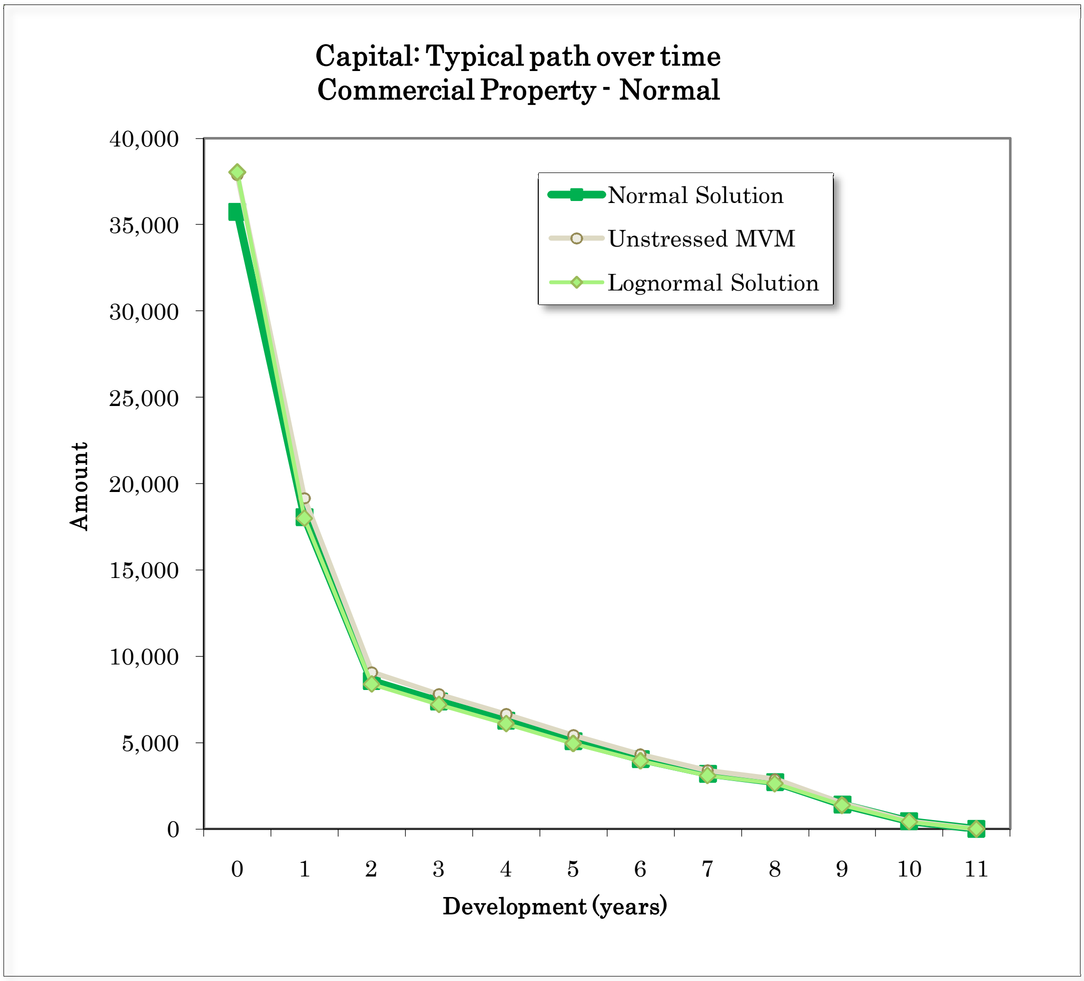


FIG. 3.13 – Capital: (A) Normal model (B) Commercial Property

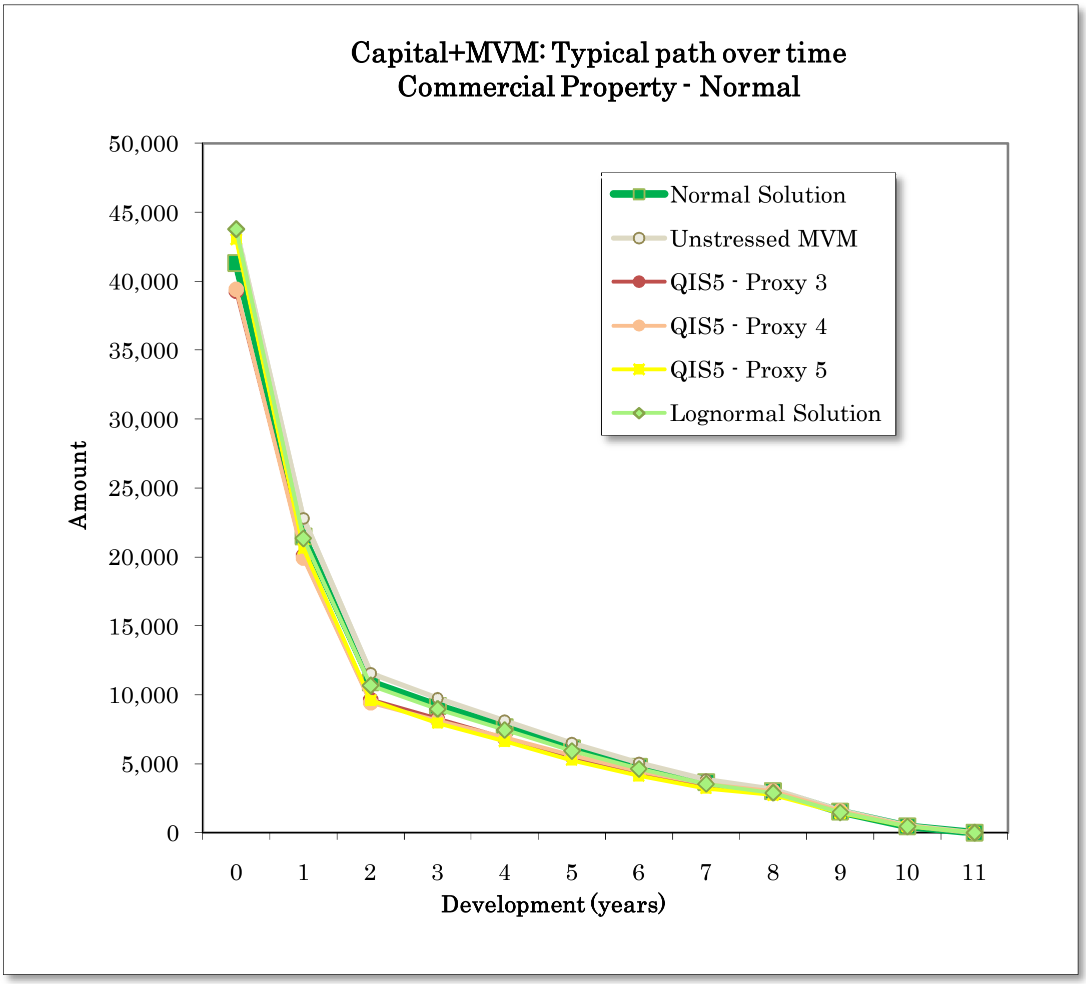


FIG. 3.14 – Capital + MVM: (A) Normal model (B) Commercial Property

3.3.4.2 Employer's Liability

Line of Business	Employer Liability									
Model	Normal									
t	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
C(t)	156,994	187,361	215,020	240,279	257,151	266,280	273,689	277,650	278,053	279,325
μ(t)		26,005	25,445	21,893	15,958	9,345	5,001	3,245	2,117	1,386
σ(t)		2,062	2,407	2,549	2,369	1,852	1,465	1,262	1,036	962
Selected path ω(t)		30,366	27,659	25,259	16,872	9,129	7,409	3,961	404	1,272
μ(t) ~		26,005	27,616	24,088	18,224	10,521	5,626	4,033	2,604	1,136
σ(t) ~		2,062	2,177	2,094	1,866	1,544	1,297	1,130	932	850
Capital(t)	5,012	5,290	5,088	4,534	3,753	3,151	2,745	2,265	2,066	1,942
MVM(t)	2,809	2,508	2,191	1,886	1,614	1,388	1,199	1,035	899	775
R(t) | C(t)	115,851	89,846	64,401	42,508	26,550	17,204	12,203	8,958	6,841	5,455
Implied Percentile	91%	85%	80%	79%	81%	83%	83%	84%	83%	80%

t	10	11	12	13	14	15	16	17	18	19
C(t)	279,804	280,225	281,696	282,983	283,265	284,054	284,465	285,298	285,441	285,344
μ(t)	1,049	821	704	513	505	527	606	557	167	6
σ(t)	935	905	929	897	807	775	810	831	722	285
Selected path ω(t)	479	421	1,471	1,287	281	790	411	833	143	-97
μ(t) ~	885	520	454	657	877	679	967	682	489	29
σ(t) ~	799	736	711	628	542	487	473	447	366	124
Capital(t)	1,787	1,728	1,527	1,317	1,184	1,150	1,087	890	302	0
MVM(t)	658	551	447	356	277	206	137	72	18	0
R(t) | C(t)	4,406	3,585	2,881	2,368	1,863	1,336	730	173	6	0
Implied Percentile	77%	72%	69%	67%	64%	60%	57%	54%	53%	

Table 3-9 – MVM results: (A) Normal model (B) Employer's Liability

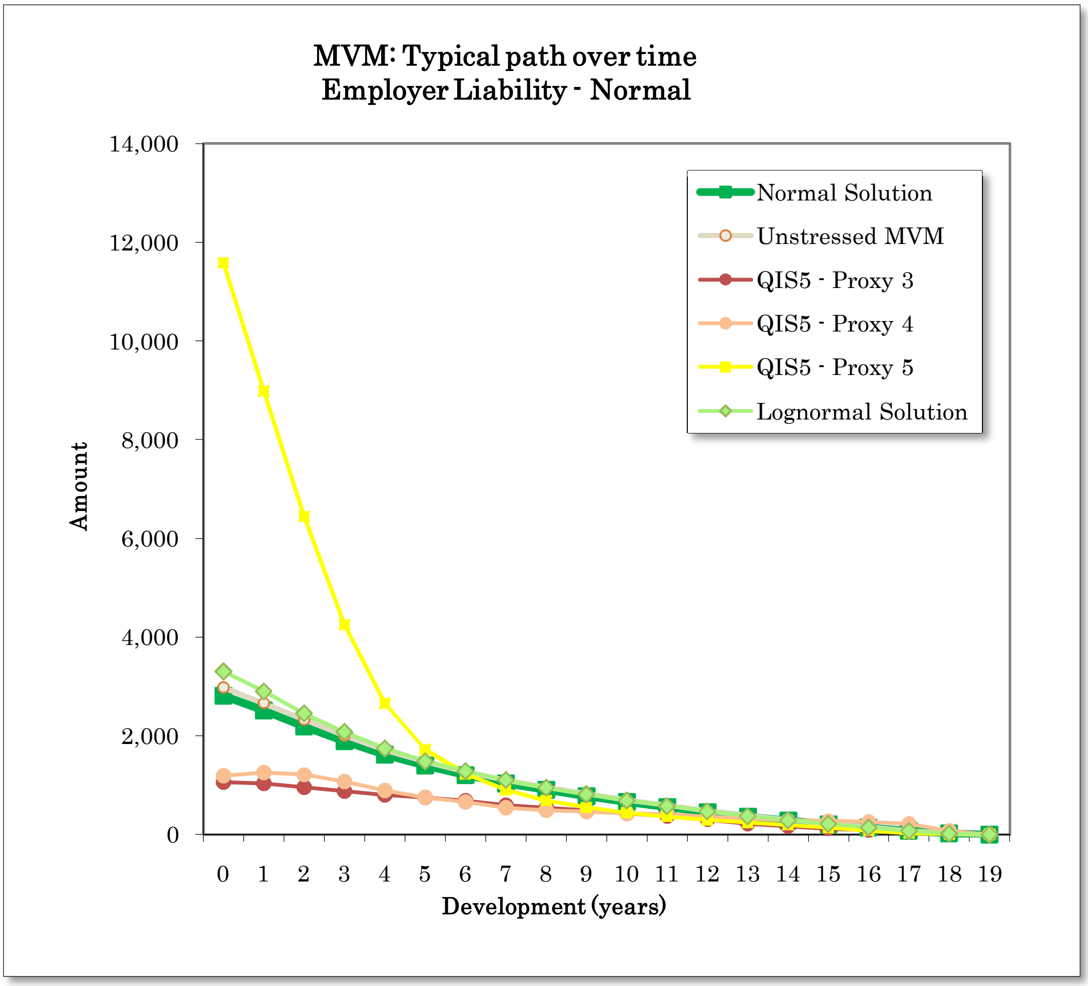


FIG. 3.15 – MVM: (A) Normal model (B) Employer's Liability

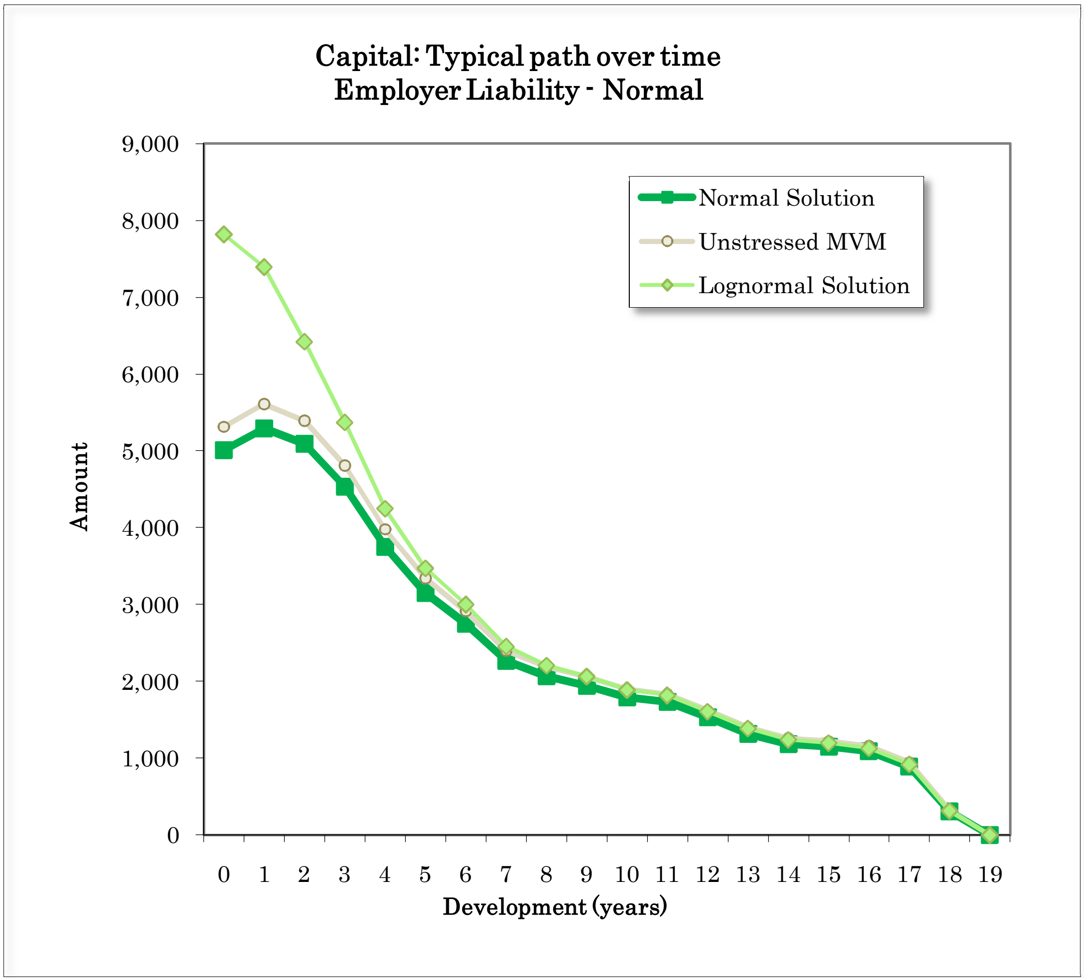


FIG. 3.16 – Capital: (A) Normal model (B) Employer's Liability

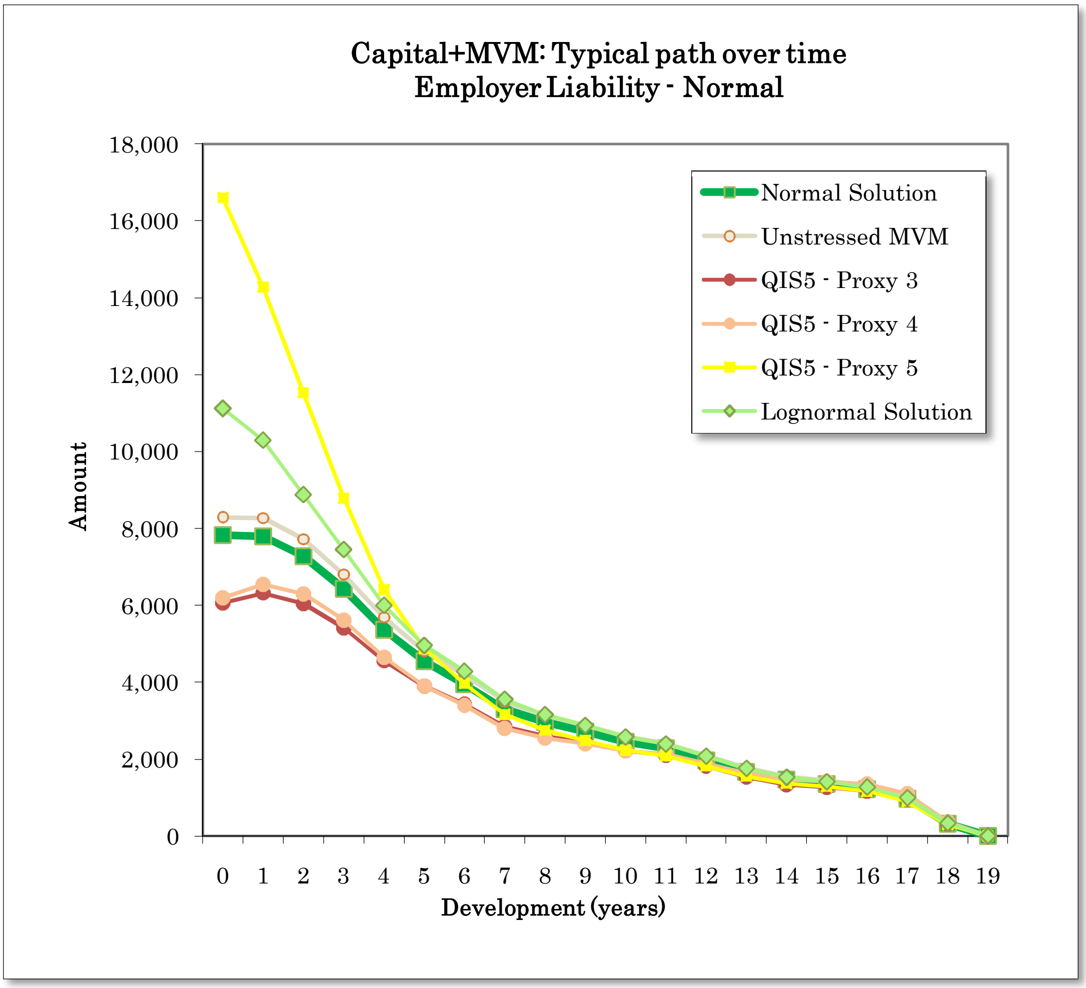


FIG. 3.17 – Capital + MVM: (A) Normal model (B) Employer's Liability

3.3.5 Other results: comparing against class

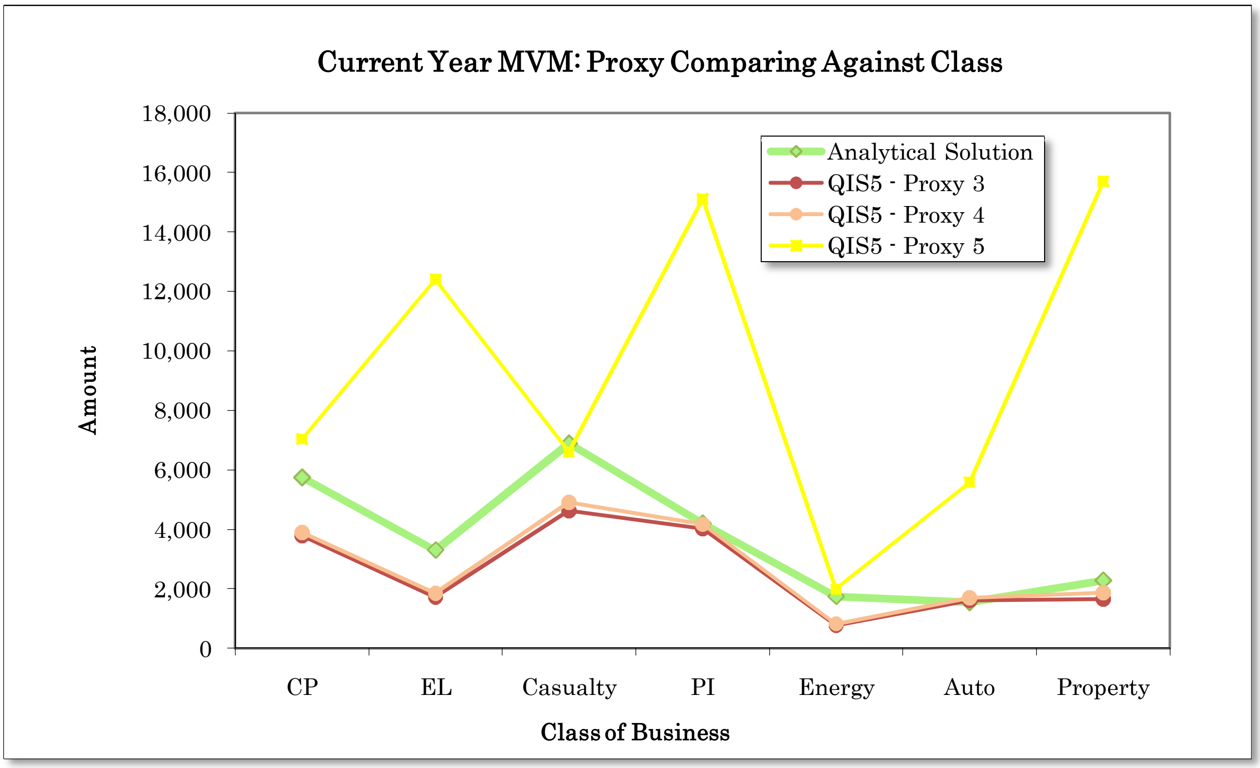


FIG. 3.18 – Current-year MVM for several classes

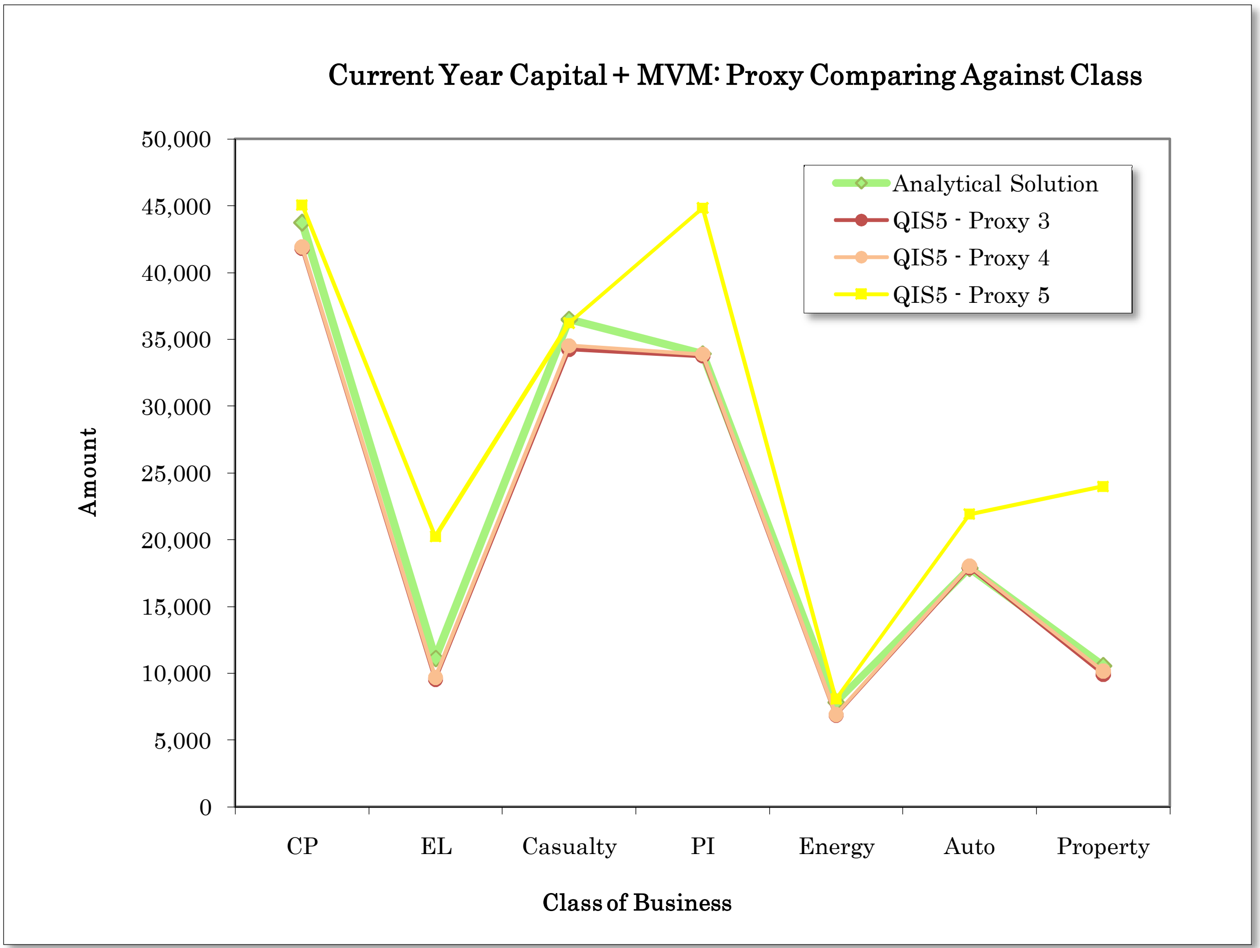


FIG. 3.19 – Current-year Capital + MVM for several classes

3.3.6 Comments on results

3.3.6.1 General comments

3.3.6.1.1 Model validation

The initial objective set in this thesis was to tackle the circularity issue that is part of the MVM formulation. The theoretical developments achieved this goal by solving the problem analytically within a predefined framework and limiting oneself to the reserve risk. Now, as for the practical applications of these theoretical developments, it can be said that the results provided proved to be relevant in the sense that (i) the GoF tests proved to show a relatively good fit and (ii) the MVM results obtained through the analytical solutions seem to have many of the theoretical and intuitive desired features. Indeed, the following observations can be made with respect to the risk margin values and behaviours:

- it has a decreasing pattern over time with:
  - at the end of the projection when all the liabilities run-off;
  - a smoother decrease on the longer tail class than on the short-tail class as it takes longer to run-off the whole portfolio;
- it is a higher proportion of the capital amount at the same time period, on the longer tail class than on the short tail as the MVM is funding more Cost of Capital;
- it is *not* a constant ratio of the reserves, as one would not expect the underlying risks to be measured as a constant proportion of the absolute reserves amounts;
- it would show a higher level per unit reserves if there was more parameter and process error as captured within the mesh for the same type of risk.

3.3.6.1.2 Comparisons

Proxies 3 and 4 seem to systematically underestimate the MVM amount throughout the whole run-off period, while Proxy 5 seems to overestimate it, especially over the first year(s) depending on the class of business, suggesting that the calibration factors were set to be quite punitive in this scenario.
There also appears to be some cases where the Capital and the MVM curves are not monotonically decreasing functions of time. Intuitively it could be argued that if it takes time before any useful information emerges to change the reserves, then it is only when that information emerges that capital is needed.
In all cases, the Capital amounts, and hence, the MVM amounts become null at the end of each of the projections, which is when all the liabilities have run-off. It will be interesting to compare the different speeds of convergence.
It can also be seen that there are cross-over between methods at different points in time, varying between models and lines of business.

Another interesting feature to note is that the lognormal and normal models roughly give the same numerical solutions, which is quite surprising given the difference in the level of complexity of the solving process that these two structures have. As such, the following comparisons will mainly focus on comparing proxies against the lognormal model.

In general, emphasis will be placed on results at time $t = 0$ as technically, only the current year MVM requires to be calculated for Solvency II purposes. In addition, as we move closer towards the horizon, the more uncertain the calculations are; as such, extra care will need to be put on interpreting and comparing the results, with some values getting close to or equal to zero, any related ratio will skyrocket in absolute amounts.

We will also comment on where the MVM stands in terms of percentiles of the 1-year reserves.

Unless otherwise stated, the following comments and comparisons relate to the MVM. Also, when not mentioned, "analytical solution" will refer to the lognormal solution.

3.3.6.2 Comparing the lognormal and the normal models

Quite surprisingly, if we first look at the MVM results only, the normal and the lognormal curves are nearly superimposed (cf. FIG. 3.12 and FIG. 3.15). On Commercial Property, the lognormal solution provides a slightly higher MVM than the normal one on the first year, and getting lower onwards, lying in the range $[-3.6\% ; +2.9\%]$ against each other, if we look at the "lognormal-to-normal" MVM ratio, which is fairly close. On Employer's Liabilities, the lognormal solution is systematically above the normal one during all years with a "lognormal-to-normal" MVM ratio within the $[+3.5\% ; +17.4\%]$ range, with a wider gap in the first years.

In terms of capital amounts now, it can be seen that the solutions are fairly similar after the first year on CProp (cf. FIG. 3.13) with slightly lower capital from the lognormal solution than the normal one. Excluding the first year on CProp, the lognormal capital is between $[-3.6\% ; -0.5\%]$ lower, with the first year being 6.4% higher. On EL, it takes several years for the two curves to nearly align, the two capital curves starting quite far apart in the first years, before converging into being almost superimposed. The normal model in this particular case is indeed featuring an increasing volatility in absolute terms on the amounts paid out each year as we move from the first to the second and from the second to the third years. These increments then become less volatile as we get into the following years until run-off (with the exception of a couple of years as we approach the time-horizon). However, the coefficients of variation still increase over the years throughout the projection, which is in line with the lognormal model, suggesting a slower decrease in the volatility of the expected incremental payments amounts (or of the year-to-year development factors for the lognormal model) than the decrease in their absolute amount.
However, if we compare the capital formulations under both models, it is clear that the "normal" capital at time t is a function of the sole standard deviation over the next year (i.e at time $t + 1$), while the "lognormal" capital captures volatility until ultimate. This can thus make the "normal" capital less smooth and more sensitive to fluctuations than what the "lognormal" model can feature.

Despite the first years differences in capital amounts, it seems that some mitigation appears when deriving the “normal” MVM results from the “normal” capital, with the two “normal” and “lognormal” MVMs being fairly close throughout the projection periods.

One explanation that could be put forward to explain the similarities in patterns and values between the outcomes of the two models could be that both of them are modelisation of the same inputs cash-flows, and as such, should reflect the same underlying risks. That being said, cautiousness should be put in any generalisation of that sort, as it might be the case that the features and behaviours obtained here are only due to the specific characteristics of the data chosen for the case studies in this thesis, and that other classes and/or triangles from other companies would not give the same MVM results.

Further testing should be performed to prove or disprove this, as some highly volatile risks and potentially not as smooth risks might not be captured well under the normal model. More data should be used, taking for example a more comprehensive set of sample data, such as other claims development triangles on CProp and EL, as well as for other lines of business, for this result to be conclusive. As a matter of fact, the results obtained on some of the simulations “borrowed” from the internal models outputs for some classes would tend to disprove the generalisation. Indeed, while both the normal and lognormal models provide results that show a level of similarity within less than 2% difference for Auto and PI, the gap can be quite significant for Energy and Casualty. Some more rigorous sensitivity testing might be required to understand what the drivers of any discrepancies between the two models results are. This is out of the scope of this thesis.

Furthermore, it should be remembered that the Goodness of Fit tests carried out on the normal models did not perform as well as the lognormal models, leading to some mitigation on the reliability that can be put on how well these results would reflect the underlying data being tested.

If it could be proved, though, that these results could be generalized on some classes of business, their application would be quite interesting and straightforward in practice, as the normal model is fairly simple to calculate, whereas the lognormal one had more complexity in its analytical solving.

3.3.6.3 Measuring unstressed MVM simplification

As directly seen in the equations, the simplified approximation that assumes that the change in the MVM in a distressed scenario is immaterial in comparison to the total MVL and as such could be ignored in the capital calculations could be quantified in terms of loading applied on top of the “exact solution”. In the case of the normal structure, the loading is constant throughout the run-off period, and is exactly 6% (i.e the (constant) Cost Of Capital rate) higher than the normal solution allowing for the MVM to be stressed under a 1 in 200 event. There is also the same relationship between the two corresponding Capital amounts. All four graphs for the normal structure show this relationship between the two curves. That same relationship is less straightforward in theory in the case of the lognormal structure, however, the numerical applications confirm that decreasing link as we move further from the time horizon n, where the unstressed MVM (and Capital) lie in the range of [5.6%-6.5%] higher than their respective “exact solution(s)”. For Commercial Property, the range in the relationship for the MVM lie in [5.9%-6.1%]. For Employer’s Liabilities, the [6.0%-6.4%].

In any cases, the CoC rate is a close cap to the materiality of what the difference should be, and the further away we are from the run-off of all liabilities, the smaller that difference is. As a consequence, these results overall suggest that the initial simplified assumption made in Solvency II to overcome the inherent MVM circularity is reasonable.

3.3.6.4 Comparing QIS5 proxies

The implicit assumption made when calculating the QIS5 proxies is that a similar proxy approach persists throughout the projection. More specifically for each year until run-off, the same reserve run-off approach sets the QIS5 – Proxy 3 MVMs determination, similarly the same duration approach is used when

determining QIS5 – Proxy 4 MVMs (albeit using updated durations at each time of the projection) and lastly, the same constant percentage of BEL is applied to derive QIS5 – Proxy 5 MVMs for each class of business.
Table 3-10 shows the main assumptions used in the various lines of business: relevant durations (at time $t = 0$) and percentage of BEL.

In terms of directional results, the main comparison results are fairly similar at time $t = 0$, across the two main classes under study, albeit with wider impacts on EL. More will be said about the comparisons against other classes in §3.3.6.6 below.

Sticking to CProp and EL and to time $t = 0$ for now, the following comments can be made:

– Proxies 3 and 4, both reasonably close, seem to underestimate the MVM amounts: they are a respective ratio of 66% and 68% of the lognormal solution (63% and 66% under the normal solution) for CProp and a respective 52% and 56% for EL (38% and 42% under the normal solution))
– Proxy 5 seems to overestimate the MVM result: the Proxy 5 results come up to 123% for CProp (131% under the normal solution) and 376% (and 412% for the normal solution) for EL. It should first be noted, though, that the analytical results obtained on EL show unusually low age-to-age volatilities as opposed to what one would expect on that type of class of business. However, nothing in the actual initial data would lead to suggest that more volatility would be required, so this could be just a specificity of the data chosen. As such, any comparison to the proxies on that class wherever a calibration is used based on market data should be read with this in mind.
  In addition, as Proxy 5 parameterisation is the result of a calibration over several CProp or EL books of business, it might be that the feature obtained here is only due to the specific characteristics of the data used in this thesis, rather than a general conclusion for Property and Damage or Third Party Liability. Data from more books would be required for this result to be conclusive. It is possible that if all possible classes EL or Property classes were averaged, the proxy would not overstate. On the other hand, from a regulatory perspective it might be right to have a factor that is prudent to encourage undertakings to use the other proxy approaches.

The implicit assumption under QIS5 – Proxy 3 is that the Capital requirement decreasing pattern in the MVM calculation is directly proportional to the reserve pattern. The assumption behind QIS5 – Proxy 5 is that the current and future Cost of Capital in the MVM calculation is independent of where we are in the run-off period but only depends on the absolute level of BEL at the time of calculation. These results would then firstly suggest that approximating the future capital requirements as a proportion of the opening capital by the ratio of the projected reserves to the opening reserves (i.e. the "Proportional" method in QIS5 – Proxy 5) is an understatement of what the actual future capital requirements could be, and secondly, that the calibration of the ratios of the BEL suggested under QIS5 – Proxy 5 is too punitive for those two classes, although this might only be due to the specifics of the data used here.

The closeness between Proxies 3 and 4 is not surprising since the duration is in itself an indicator of how the cash-flows, and hence the reserves run-off over time.
This can also be proved theoretically: indeed, rewriting the two MVM formulations at time $t = 0$ as seen in equations (2-42) and (2-45) (but only with slightly less formal notations):

$$MVM_{0|0}^{QIS5-Proxy3} = cE\left(Cap_0^{1c}|\mathcal{F}_0\right)\sum_{t=0}^{n-1}\frac{E(R_t|\mathcal{F}_0)}{E(R_0|\mathcal{F}_0)} \qquad \textbf{(3-31)}$$

$$MVM_{0}^{QIS5-Proxy4} = cE\left(Cap_0^{1c}|\mathcal{F}_0\right).Dur_{mod}(0) \qquad \textbf{(3-32)}$$

and expressing the duration as the cash-weighted average time in the time series of the projected paid claims, we have:

$$Dur(0) = \frac{\sum_{t=1}^{n} t.E(C_t - C_{t-1}|\mathcal{F}_0)}{\sum_{t=1}^{n} E(C_t - C_{t-1}|\mathcal{F}_0)} \qquad \textbf{(3-33)}$$

with $\sum_{t=1}^{n} E(C_t - C_{t-1}|\mathcal{F}_0) = E(C_n|\mathcal{F}_0) - C_0 = E(R_0|\mathcal{F}_0)$

and after rewriting $\sum_{t=1}^{n} t.E(C_t - C_{t-1}|\mathcal{F}_0) = n.E(C_n|\mathcal{F}_0) - \sum_{t=0}^{n-1} E(C_t|\mathcal{F}_0)$ (with one of each term cancelling out with the following term), which can further be written as $\sum_{t=1}^{n} t.E(C_t - C_{t-1}|\mathcal{F}_0) = \sum_{t=0}^{n-1} E(C_n - C_t|\mathcal{F}_0)$, giving $\sum_{t=1}^{n} t.E(C_t - C_{t-1}|\mathcal{F}_0) = \sum_{t=0}^{n-1} E(R_t|\mathcal{F}_0)$, and finally:

$$Dur(0) = \frac{\sum_{t=0}^{n-1} E(R_t|\mathcal{F}_0)}{E(R_0|\mathcal{F}_0)} = \sum_{t=0}^{n-1} \frac{E(R_t|\mathcal{F}_0)}{E(R_0|\mathcal{F}_0)} \tag{3-34}$$

which brings the two proxies as both coming down to performing a "proportional" run-off of the reserves method.
One varying assumption made when implementing these two solutions, that could explain why the numerical solutions differ slightly, is the fact that the duration is computed directly from the "raw" cash-flows (step 1 of FIG. 3.1) whereas the reserve pattern under QIS5 – Proxy 3 uses the "expected" one, and as such, results from the application of the mesh on those specified cash-flows (i.e. step 3 of FIG. 3.1).

If we now move along the subsequent projected time-steps, it can be seen that QIS5 – Proxy 5 curve has the highest convergence speed downwards. On CProp, it gets below the exact solution after the first year by reverting to being 66% of the lognormal solution (given that the "normal" and "lognormal" MVM are fairly superimposed after the first year on this class, the comments from now on will only discuss the "lognormal" solutions). This ratio as a proportion of the exact solution is continuously decreasing over time. It becomes less than 5% after the seventh year. Compared to the other two proxies, it can be seen that QIS5 – Proxy 5 curve even becomes lower than what QIS5 – Proxies 3 and 4 would suggest after the third year.
On EL, this pattern is a bit longer to achieve: it will take another six years for the QIS5 – Proxy 5 curve to cross the exact solution, where at time $t = 7$, the same indicative ratio becomes 92%. It will then roughly follow the other two proxies after the following year.

All in all, these would suggest that QIS5 – Proxy 5 is too rough an approximation, overstating the MVM in the first few years and underestimating it in the subsequent years. As mentioned above, the implicit assumption in the MVM calculation is a sole dependency on the absolute level of BEL, with no regards to how much longer this BEL will be held and how it will run-off. Also, that a same class of business in two different companies will roughly have the same claims profile, and hence, the same risk will be attached to them. If we were to imagine an extreme configuration on a long-tail with a large size company A on the one hand holding an amount $BEL(X)$ after 10 years of run-off, and a small size company B on the other hand holding the same amount $BEL(X)$, only after the first year into run-off, then, according to the proxy, the two companies would be holding the same amount of MVM on top of their identical BEL. However, it is clear that the uncertainty associated with $BEL(X)$ for Company A is significantly smaller than that of Company B as more is known about the claims development and more stability surround those reserves. In addition, the lines of business segmentation in general and in particular the one as defined in QIS5 are so broad that different companies might write completely different books with different risk characteristics and tails. Similarly within Europe, exactly the same class may behave completely differently due to different legal environments, physical environments, policy terms, etc.

QIS5 – Proxies 3 and 4 turn out to be similar in theory. In practice, duration can often be derived from payments history, while determining how the reserves run-off is subject to a projection analysis. However, it is likely that the proportional method requiring the use of these reserves run-off patterns will nevertheless be more popular, as these will be ready at hand, being directly linked to the reserving exercises. With the exception of a handful of years near the horizon, both proxies systematically underestimate the MVM, the effect being stronger on EL with the proxies being most of the time less than half of the exact value.

3.3.6.5 Comparing QIS5 Standard Formula approaches

3.3.6.5.1 QIS5 Standard Formula – standard approach

The capital amount seems to be overstated (by 7% on CProp and 505% on EL – with the comment expressed above on the unusually low level of volatility in the results) during the first year, while this pattern reverts on the following years where the QIS5 Standard Formula – standard approach capital

becomes less than the exact formulas (at time $t = 1$ onwards on CProp and at time $t = 9$ on EL), sometimes quite significantly as we move further towards the horizon. For example, it becomes less than 50% on CProp after 3 years in the run-off.
One thing to note is that the ratio "QIS5 Standard Formula – standard approach capital"/"Exact solution" is a steadily decreasing function of time.

On the MVM amounts, however, the proxy is a 29% underestimation of the analytical estimate on CProp, and a 264% overestimation on EL.

Looking at the Capital + MVM amount, however, the proxy is relatively close on CProp during the first year (102% of the analytical solution) while it clearly underestimates on the following years. The underlying assumptions that the volatility should remain constant and the same across all time periods and should similarly still be applied to reserves that become smaller and smaller and less volatile as more information is known, could explain why the ratios drop as significantly.

3.3.6.5.2 QIS5 Standard Formula – USP

Not surprisingly, the USPs systematically give higher MVMs and Capital requirements on CProp, while giving lower results on EL. Indeed, as can be seen on Table 3-5, the reserve risk standard deviations calculated as per methods 2 and 3 come up to 13.91% and 16.36% respectively on CProp, as opposed to a standard gross market factor of 11.00%. On the other hand, the two USP methods generate 10.00% and 7.88% on EL respectively, which similarly compares to a 11.00% standard factor. This is due to the fact that the Capital is a sole function of the current time reserves and of the standard deviation only, and the resulting MVM depends on the Capital and the reserve run-off, hence the one-way spread between the standard approach and the USPs curves throughout the projection.

It has been voiced that USP calibrations were set in a too prudential manner. Consequently, whenever this would be the case, the sheer extent of the gap over and above the QIS5 standard approach suggests that few companies would voluntarily adopt the USP approach.

The USP show a similar decreasing pattern as the standard approach.
As such, fairly similar conclusions seem to hold here as for the standard approach, albeit with the USP leading to much more prudence in the first year if we look at the total Capital + MVM amounts.
On CProp, looking at the MVM amounts only and at the first year, the analytical solution lies between the results of the two USP methods, which in turn show more prudence than the standard approach itself. However, this could be seen as a mitigating effect when deriving the MVM amount (using the "Proportional Approach" of Proxy 3) from the Capital amounts, as these clearly show more prudence than the analytical solution. It is the opposite on EL, where both USP methods show lower capital than the standard approach.

3.3.6.6 Comparing against lines of business

The main study was conducted for the Commercial Property and Employer's Liabilities classes of business, the selection of which was with the aim of comparing the behaviour of a short-tail class to that of a longer-tailed one. The durations for these two classes are 1.7 years and 4.0 years respectively, as determined directly from the cash-flows (cf. Table 3-10). Looking at the analytical solutions, it can be seen, as we would expect, that the speed of convergence towards zero is higher on the shorter-tail line. The MVM runs-off at 59% of its initial value in the first projection year on CProp, vs. 88% on EL. The "half-life" – which we could define as being the period of time it takes the MVM to decrease by half – is somewhere between the first and the second year for CProp, and near the fifth years on EL, which respectively represent 18% and 26% of the time until full run-off for both lines of business. It further reaches the third of its value around the third year for CProp, as opposed to the eighth year on EL (which is about 30% vs. 42% along the time-horizon, respectively). This suggests that the longer the timeframe over which the liabilities persist, the higher uncertainty we have over how those liabilities will evolve until they become fully extinct, and as such the longer is the need to hold a loading on top of Best Estimate to take this uncertainty into account.

The following table shows the results as a time $t = 0$ on seven business lines, under the lognormal structure.

Classes of business	CP	EL	Casualty	PI	Energy	Auto	Property
Capital + MVM							
Analytical Solution	43,760	11,120	36,500	33,923	7,822	17,874	10,559
QIS5 - Proxy 3	41,804	9,529	34,256	33,751	6,852	17,934	9,941
QIS5 - Proxy 4	41,917	9,677	34,522	33,896	6,902	18,022	10,153
QIS5 - Proxy 5	45,062	20,227	36,220	44,844	8,069	21,919	23,990
MVM							
Analytical Solution	5,737	3,298	6,878	4,190	1,741	1,539	2,273
QIS5 - Proxy 3	3,781	1,707	4,634	4,018	772	1,599	1,654
QIS5 - Proxy 4	3,894	1,855	4,900	4,164	822	1,687	1,866
QIS5 - Proxy 5	7,039	12,405	6,598	15,111	1,988	5,584	15,704
Capital	38,023	7,822	29,622	29,733	6,081	16,335	8,286
Reserves at t = 0	127,978	124,052	65,981	151,113	13,255	139,589	285,525
Proxy 5 % (% of BEL)	5.5%	10.0%	10.0%	10.0%	15.0%	4.0%	5.5%
Duration	1.7	4.0	2.8	2.3	2.3	1.7	3.8
MVM as % of Capital	15.1%	42.2%	23.2%	14.1%	28.6%	9.4%	27.4%
MVM as % of reserves	4.5%	2.7%	10.4%	2.8%	13.1%	1.1%	0.8%

Table 3-10 – Capital and MVM results by class of business

On the other classes, it seems that Proxy 5 almost systematically overestimates the MVM and Capital + MVM amounts, the only exception being Casualty. This is quite significantly so on PI and Property.

Proxies 3 and 4 are relatively fine on PI and Auto and tend to underestimate the analytical solution on the other classes.

The last two lines of the table provide a comparison of the MVM amount as estimated by the analytical solution as a ratio of Capital and reserves respectively. No clear pattern seems to emerge however when trying to link these to the durations, which seems consistent with the fact that the key drivers of the lognormal analytical solution are the following year volatility and the remaining cumulated volatility until ultimate. PI and Energy have comparable durations, and yet the latter seems to require a higher loading as a proportion of reserves and capital.

3.3.6.7 Comparing against the percentile approach

The results presented on each summary table (Table 3-6, Table 3-7, Table 3-8 and Table 3-9) show the percentiles (of the 1-year reserves deterioration) which correspond to the sum of the MVM and BEL. Not surprisingly, it can be seen that these percentiles levels vary along the projections, with no set trend, although somewhat decreasing. It can also be noted that the lognormal and normal analytical estimates give fairly similar results.

This would tend to suggest that the percentile would need a regular and varying calibration that would be function of where the book is inside its run-off in order to prescribe a relevant percentile.

Chapter 4 Conclusions

4.1 Summary of the chosen approach

This thesis examined the Market Value Margin in its general conceptual definition within the Solvency II framework for a non-life insurer and went through the various methods to calculate it, under the Cost of Capital approach. The simplified approaches suggested in QIS5 have the merit of being straightforward to apply and of requiring little extra data. Among the issues that these simplifications aim at by-passing are the future capital requirements estimations and the self-referential relationship between these capital requirements and the current and future MVMs.

The aim of this thesis has been to build a claims structure model that would overcome the inherent circularity issue in the MVM formulation, and to apply this theoretical model as a mesh to fit some projected data. As a by-product of this work, we have been able to compare and quantify the QIS5 MVM approximations in two case studies, limited to the reserve risk and to a line of business taken in isolation (namely, the short-tail Commercial Property and the long-tail Employer's Liability).
This was achieved in the following manner:

- First, two theoretical claims processes are assumed and two analytical solutions to the MVM self-referential formulation are derived. This has involved some theoretical developments and a backwards solving from the point in time when all liabilities cease to exist.
- Second, these two theoretical structures are used and applied as meshes on simulated projected cash-flows obtained from real initial triangles, in order to further describe these outputs and derive the relevant MVM estimates, from the current time until the whole run-off of the portfolio. Goodness of Fit tests are carried out once the relevant parameters to the model are estimated from the cash-flows.
- Then, similarly, the QIS5 proxies are calculated from the same initial inputs throughout the projection periods and compared to the estimates derived from the analytical solutions.

4.2 Theoretical model validation

This study successfully presented a theoretical solution within a specified framework to the Cost of Capital Market Value Margin formulation. It was also demonstrated that these theoretical developments could be used in practice and implemented on two case studies. Indeed, the results obtained through the analytical solutions have many of the desired features one would expect the MVM to have, among which a decreasing pattern over time while dropping to 0 at the end of the time horizon, a higher level per unit reserve as we include more parameter error and a slower decrease as it takes longer to run-off the liabilities.

4.3 Proxies comparisons

It should be borne in mind that some of the following conclusions do not claim to be generalized in their application to all lines of business and to all portfolios within each line of business, as the results might be due to the specific characteristics of the data chosen in the case studies.

First, as a generic comment that is independent of the class of business, the "Unstressed MVM" analytical solution, defined as the Expected Cost of Capital Risk Margin, with capital capturing the expected BEL deterioration only (i.e this is the simplified solution that by-passes the circularity issue) gives a higher result than the exact solution, with a spread roughly capped by the Cost of Capital rate. Working backwards, this is mainly due to the fact that the MVM replaces the "simplified" capital at time $t = n - 1$ with a reducing effect at each previous projection year.

Then, overall, as a common comment on both analysed classes, it seems that Proxy 5 (i.e MVM calculated as a set percentage of the BEL) is quite punitive on the first year by being 23% above the analytical

solution on CProp and most notably 376% above on EL (although our results on this class suggest a smaller volatility than the ones that can found for similar classes). Also, the results of the study suggest that proxies 3 and 4 (i.e the “proportional” and the “duration” approach respectively) applied to the analytical capital numbers, while both leading to fairly similar results, tend to underestimate the analytical solutions for the MVM amounts. They do however come up with similar results to the QIS5 standard approach on the first year on the short-tail class, while diverging in subsequent periods by releasing margins less quickly than the standard approach. This is less true for the longer-tail class, where proxies 3 and 4 are between 20-30% below the standard approaches during the first year. The USP proxies show varying results on the two lines: on CProp, the analytical estimate lies between the two USP curves in the first year, with both giving higher results than the standard approach; all three then fall below the analytical estimate past the first year. On EL, they are both below the standard approach curve and above the analytical estimate curve in the first year, while decreasing at about the same rate as the standard approach, bringing the three curves below that of the analytical one after the third year.
Of noticeable interest was the fact that the results were reasonably close between the normal and the lognormal models.

Chapter 5 Limits and further possible extensions

This section summarizes the limitations surrounding this study, where the results are not so straightforward to interpret and generalize and hence caution must be applied when reading the conclusions. Alternatively, we will also discuss the potential extensions that could be added to the approach and assumptions used here.

First of all, the study was conducted with a wish to fit the pre-defined framework of Solvency II – QIS5 specifications by working around the constraints and/or suggestions imposed to date by CEIOPS. As such, it is within these that we have added further limitations to the scope of the study, without challenging the general structure and assumptions that define the MVM.

The following lists the main constraints that have been carried over in this thesis:
- a constant cost-of-capital rate is assumed, set at 6%;
- a deterministic discount rate yield curve is used, as provided by CEIOPS;
- only a single line of business is considered in isolation, and as such dependencies between lines are outside the scope of this paper;
- the study is limited to the reserve risk component only;
- the Catastrophe risk is beyond the scope of the study;
- the unavoidable market risk is left aside;
- the study is conducted on a gross of reinsurance basis only, which further simplifies into not having to capture the counterparty risk with respect to reinsurance contracts and special purposes vehicles;
- the Operational Risk is not captured.

Should any of these elements be included in the study, the results might be different, possibly materially so. However, one of the aims of this thesis has been to compare several calculations of the risk margin by keeping the analysis as simple as possible in order to avoid potentially complex interactions that would make the results even more complex to interpret.

Secondly, as mentioned previously, some of the conclusions should only be seen in light of the data chosen and modelled in the study. The analysis could be extended to a greater, more comprehensive set of sample data such as different lines of business, different type of business that would map to the same line of business and indeed different company data within each type of business.

Then, there also were inherent limitations to how the projected cash-flows were generated, which had an impact on the final MVM calculations. The simulations used were obtained by performing a bootstrap as formulated by T. Mack and D. Murphy. Using the Pearson residuals or Over Dispersed Poisson variations might lead to different volatilities, and hence different MVM estimates. The application of the Bootstrap approach involved imposing some judgement upon the raw statistical model and it is possible that even with the same data, other practitioners may have chosen different parameters and may thus have obtained different results.

Among the further possible extensions to the model, other than including what has been excluded as described above, the following could be envisaged:
- including New Future Business;
- including diversifications between lines of business;
- including other types of dependency structures between time periods. For instance, investigate more complex copulas;
- investigating other distributions and claims structures. For example, a linear structure could overtake the simple multiplicative or additive structures considered here in isolation. Also, a Mack-type structure could be envisaged, adding a square-root relationship to the year-to-year cash-flows structure. Note that catastrophe risk can often have a very skew distribution – much more skew than would be possible from a Lognormal distribution. In order to try to replicate cash-flows from these kind of risks with a

theoretical model, it is likely that different distributions would need to be used, at least for earlier development years.

Chapter 6 Appendices

Appendix A. Proofs

This section details the results obtained in §2.2.6 – Analytical models.

A.1 Lognormal model

A.1.1 Model

The model being considered here assumes that a company is exposed to claims with the following known process:

$$\boxed{C_t = C_{t-1}.LnN\left(\widetilde{\mu_t}, \widetilde{\sigma_t^2}\right)} \quad \text{(A.1- 1)}$$

where the logarithm of the "development factors" are assumed to be a multivariate normal distribution which is the same as imposing a Gaussian copula across the lognormal distributions different development periods.

The following developments will be using the definitions and notations and work introduced in Appendix A.3 to describe the dependency structure. In particular, we will use:

$$\widetilde{M}_{u:v} = \sum_{k=u}^{v} \widetilde{\mu_k} \quad \text{(A.1- 2)}$$

$$\tilde{S}_{u:v} = \alpha_{(v-u+1)}{}^{T}\tilde{\Lambda}_{u,v}\alpha_{(v-u+1)} = \sum_{k=u}^{v} \sigma_k{}^2 + 2\sum_{i=u}^{v-1}\sum_{j=i+1}^{v} \rho_{i,j}\sigma_i\sigma_j - \Delta_{1,v} \quad \text{(A.1- 3)}$$

with the convention that $\tilde{S}_{v:v} = \tilde{\sigma}_v{}^2$ and with all the components defined and described in 'Appendix A.3.3.3 - Conditional distributions'.

Note that the notation $\tilde{X}$ is introduced to further allow for the fact that when adding a dependency structure to the model between all time periods, and when conditioning on past information (i.e $X(\theta_u)|X(\theta_1),\dots,X(\theta_i),\dots,X(\theta_{u-1})$ the mean and the variance are altered, as opposed to what would be obtained if only the current information was known. Refer to Appendix A.3.3.3 for more detail.

We get the following properties:

$$E(C_t|\mathcal{F}_{t-1}) = C_{t-1}.e^{\widetilde{\mu_t}+\frac{1}{2}\widetilde{\sigma_t^2}} \quad \text{(A.1- 4)}$$

and

$$Var(C_t|\mathcal{F}_{t-1}) = C_{t-1}^2.e^{2.\tilde{\mu}_t+\tilde{\sigma}_t^2}.\left(e^{\tilde{\sigma}_t^2} - 1\right) \quad \text{(A.1- 5)}$$

This can further be extended to give:

$$C_t = C_0.\prod_{k=1}^{t} LnN(\tilde{\mu}_k, \tilde{\sigma}_k^2) = C_0.\prod_{k=1}^{t} e^{N(\tilde{\mu}_k,\tilde{\sigma}_k^2)} = C_0.e^{\sum_{k=1}^{t} N(\tilde{\mu}_k,\tilde{\sigma}_k^2)}$$

$$\Rightarrow C_t = C_0.e^{N(\widetilde{M}_{1:t},\tilde{S}_{1:t})} = C_0.LnN\left(\widetilde{M}_{1:t}, \tilde{S}_{1:t}\right) \quad \text{(A.1- 6)}$$

(by setting $u = 1$ and $v = t$ in equation (A.3- 17))

The various formulas exposed in the general structure in §2.2.6.1 can then be transposed to this lognormal scenario as follows:

– The reserves R_t as at time t now become:

$$R_t = (C_n|\mathcal{F}_t) - C_t = C_t.\left(LnN\left(\widetilde{M}_{t+1:n}, \tilde{S}_{t+1:n}\right) - 1\right) \tag{A.1- 7}$$

Thus:

$$E(R_t|\mathcal{F}_t) = C_t.\left(e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}} - 1\right) \tag{A.1- 8}$$

$$Var(R_t|\mathcal{F}_t) = C_t{}^2.e^{2\widetilde{M}_{t+1:n}+\tilde{S}_{t+1:n}}.\left(e^{\tilde{S}_{t+1:n}} - 1\right) \tag{A.1- 9}$$

– The future losses:

$$L_t = R_t - E(R_t) = C_t.\left(LnN\left(\widetilde{M}_{t+1:n}, \tilde{S}_{t+1:n}\right) - e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}}\right) \tag{A.1- 10}$$

$$E(L_t|\mathcal{F}_t) = 0 \tag{A.1- 11}$$

$$Var(L_t|\mathcal{F}_t) = Var(R_t|\mathcal{F}_t) \tag{A.1- 12}$$

– The 1-year loss deterioration:

$$L_t^1 = E(C_n|\mathcal{F}_{t+1}) - E(C_n|\mathcal{F}_t)$$

$$L_t^1 = C_{t+1}.e^{\widetilde{M}_{t+2:n}+\frac{1}{2}\tilde{S}_{t+2:n}} - C_t.e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}}$$

$$L_t^1 = C_t.\left(LnN(\tilde{\mu}_{t+1}, \tilde{\sigma}_{t+1}^2).e^{\widetilde{M}_{t+2:n}+\frac{1}{2}\tilde{S}_{t+2:n}} - e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}}\right) \tag{A.1- 13}$$

– The capital: as noted in §2.2.6.1, the formulas for the capital requirements exposed below do not include the risk margin deterioration yet, but are rather shown to underline their general properties and for ease of comparison between the two time horizons. Formulas (A.1- 16) and onwards will avoid the shortcuts with the goal of finding an exact solution for $Cap_t{}^{1C}$ and $MVM_t{}^{C}$.

– On an *ultimate horizon* basis, $Cap_t = 99.5\%[L_t]$, which is now transposed into:

$$Cap_t = C_t.\left(e^{\widetilde{M}_{t+1:n}+\sqrt{\tilde{S}_{t+1:n}}.\phi} - e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}}\right)$$

$$Cap_t = C_t.e^{\widetilde{M}_{t+1:n}}\left(e^{\sqrt{\tilde{S}_{t+1:n}}.\phi} - e^{\frac{1}{2}\tilde{S}_{t+1:n}}\right) \tag{A.1- 14}$$

with the notation $\phi = \Phi^{-1}(99.5\%)$ where $\Phi^{-1}(p)$ denotes the quantile function of the standard normal distribution of order p, i.e. $\Phi^{-1}(p) = x$ such that $\Phi(x) = p$.

– On a *1-year horizon* basis, $Cap_t{}^1 = 99.5\%\left[L_t{}^1\right]$, transposed into:

$$Cap_t{}^1 = C_t.\left(e^{\tilde{\mu}_{t+1}+\tilde{\sigma}_{t+1}\phi}.e^{\widetilde{M}_{t+2:n}+\frac{1}{2}\tilde{S}_{t+2:n}} - e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}}\right)$$

$$Cap_t{}^1 = C_t.e^{\widetilde{M}_{t+1:n}}\left(e^{\frac{1}{2}\tilde{S}_{t+2:n}+\tilde{\sigma}_{t+1}\phi} - e^{\frac{1}{2}\tilde{S}_{t+1:n}}\right) \tag{A.1- 15}$$

which is put in a form directly comparable with Cap_t in (A.1- 14) above.
Unsurprisingly, if the losses mature after one year then these different approaches give the same capital requirements.

A.1.2 System to solve

The actual capital we are after is $Cap_t{}^{1c}$ as a solution of the system in (2-23).
Under the LogNormal assumptions, our iterative regime for $Cap_t{}^{1c}$ – and consequently $MVM_t{}^{c}$ from (2-28) becomes:

$$\left\{\begin{aligned} Cap_t{}^{1c} &= \frac{1}{1+c}\left(99.5\%\left[c\sum_{i=t+1}^{n-1} E\left(Cap_i{}^{1c}|\mathcal{F}_{t+1}\right) + C_{t+1}e^{\widetilde{M}_{t+2:n}+\frac{1}{2}\tilde{S}_{t+2:n}}|\mathcal{F}_t\right]\right. \\ &\qquad \left. - c\sum_{i=t+1}^{n-1} E\left(Cap_i{}^{1c}|\mathcal{F}_t\right) - C_t e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}}\right) \qquad \text{(A.1- 16)} \\ &\text{with} \\ Cap_{n-1}{}^{1c} &= \frac{1}{1+c}\left(C_{n-1}e^{\widetilde{M}_n}\left(e^{\tilde{\sigma}_n\phi} - e^{\frac{1}{2}\tilde{\sigma}_n^2}\right)\right) \qquad \text{(A.1- 17)} \end{aligned}\right.$$

with the following notations:
– 99.5%[…] to denote the percentile amount, i.e. the VaR, at the 99.5th level;
–$\phi = \Phi^{-1}[99.5\%]$ and Φ^{-1} being the percentile function of the standard normal cumulative distribution function;
– conditioning on $\mathcal{F}_t$ (with the notation $|\mathcal{F}_t$) means we are conditioning on all past information up to time t.

A.1.3 Solution

Let us define X_t as follows:

$$Cap_t{}^{1c} = C_t X_t \qquad \text{(A.1- 18)}$$

where $X_t$is independent of C_t.

We get the following:

$$\begin{aligned} C_t X_t &= \frac{1}{1+c}\left(\%\left[c\sum_{i=t+1}^{n-1} E(C_i X_i|\mathcal{F}_{t+1}) + C_{t+1}e^{\widetilde{M}_{t+2:n}+\frac{1}{2}\tilde{S}_{t+2:n}}|\mathcal{F}_t\right]\right. \\ &\qquad \left. - c\sum_{i=t+1}^{n-1} E(C_i X_i|\mathcal{F}_t) - C_t e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}}\right) \end{aligned}$$

$$\begin{aligned} C_t X_t &= \frac{1}{1+c}\left(\%\left[c\sum_{i=t+1}^{n-1} X_i E(C_i|\mathcal{F}_{t+1}) + C_{t+1}e^{\widetilde{M}_{t+2:n}+\frac{1}{2}\tilde{S}_{t+2:n}}|\mathcal{F}_t\right]\right. \\ &\qquad \left. - c\sum_{i=t+1}^{n-1} X_i E(C_i|\mathcal{F}_t) - C_t e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}}\right) \end{aligned} \qquad \text{(A.1- 19)}$$

With

$E(C_i|\mathcal{F}_{t+1}) = C_{t+1}e^{\widetilde{M}_{t+2:i}+\frac{1}{2}\tilde{S}_{t+2:i}}$

we get the following intermediate building blocks:

$\sum_{i=t+1}^{n-1} X_i E(C_i|\mathcal{F}_{t+1}) = \sum_{i=t+1}^{n-1} X_i C_{t+1} e^{\widetilde{M}_{t+2:i}+\frac{1}{2}\tilde{S}_{t+2:i}}$

$c\sum_{i=t+1}^{n-1} X_i E(C_i|\mathcal{F}_{t+1}) + C_{t+1}e^{\widetilde{M}_{t+2:n}+\frac{1}{2}\tilde{S}_{t+2:n}} = C_{t+1}e^{\widetilde{M}_{t+2:n}+\frac{1}{2}\tilde{S}_{t+2:n}}\left(c\sum_{i=t+1}^{n-1} X_i e^{-\widetilde{M}_{i+1:n}-\frac{1}{2}\tilde{S}_{i+1:n}} + 1\right)$

We can now consider the following:

$\% \left[c \sum_{i=t+1}^{n-1} X_i E(C_i | \mathcal{F}_{t+1}) + C_{t+1} e^{\widetilde{M}_{t+2:n} + \frac{1}{2}\tilde{S}_{t+2:n}} | \mathcal{F}_t \right]$

$= \% \left[C_{t+1} e^{\widetilde{M}_{t+2:n} + \frac{1}{2}\tilde{S}_{t+2:n}} \left(c \sum_{i=t+1}^{n-1} X_i e^{-\widetilde{M}_{i+1:n} - \frac{1}{2}\tilde{S}_{i+1:n}} + 1 \right) | \mathcal{F}_t \right]$

$= \%[C_{t+1} | \mathcal{F}_t]. e^{\widetilde{M}_{t+2:n} + \frac{1}{2}\tilde{S}_{t+2:n}} \left(c \sum_{i=t+1}^{n-1} X_i e^{-\widetilde{M}_{i+1:n} - \frac{1}{2}\tilde{S}_{i+1:n}} + 1 \right)$

$= C_t e^{\tilde{\mu}_{t+1} + \tilde{\sigma}_{t+1}\phi} e^{\widetilde{M}_{t+2:n} + \frac{1}{2}\tilde{S}_{t+2:n}} \left(c \sum_{i=t+1}^{n-1} X_i e^{-\widetilde{M}_{i+1:n} - \frac{1}{2}\tilde{S}_{i+1:n}} + 1 \right)$

Similarly– second part of the numerator,

$\sum_{i=t+1}^{n-1} X_i E(C_i | \mathcal{F}_t) = \sum_{i=t+1}^{n-1} X_i C_t e^{\widetilde{M}_{t+1:i} + \frac{1}{2}\tilde{S}_{t+1:i}}$

$c \sum_{i=t+1}^{n-1} X_i E(C_i | \mathcal{F}_t) + C_t e^{\widetilde{M}_{t+1:n} + \frac{1}{2}\tilde{S}_{t+1:n}} = C_t e^{\widetilde{M}_{t+1:n} + \frac{1}{2}\tilde{S}_{t+1:n}} \left(c \sum_{i=t+1}^{n-1} X_i e^{-\widetilde{M}_{i+1:n} - \frac{1}{2}\tilde{S}_{i+1:n}} + 1 \right)$

Bringing these together we get

$$C_t X_t = \frac{1}{1+c} \left(C_t e^{\tilde{\mu}_{t+1} + \tilde{\sigma}_{t+1}\phi} e^{\widetilde{M}_{t+2:n} + \frac{1}{2}\tilde{S}_{t+2:n}} \left(c \sum_{i=t+1}^{n-1} X_i e^{-\widetilde{M}_{i+1:n} - \frac{1}{2}\tilde{S}_{i+1:n}} + 1 \right) - C_t e^{\widetilde{M}_{t+1:n} + \frac{1}{2}\tilde{S}_{t+1:n}} \left(c \sum_{i=t+1}^{n-1} X_i e^{-\widetilde{M}_{i+1:n} - \frac{1}{2}\tilde{S}_{i+1:n}} + 1 \right) \right)$$

which further simplifies into:

$$X_t = \frac{1}{1+c} \left(e^{\tilde{\mu}_{t+1} + \tilde{\sigma}_{t+1}\phi} e^{\widetilde{M}_{t+2:n} + \frac{1}{2}\tilde{S}_{t+2:n}} \left(c \sum_{i=t+1}^{n-1} X_i e^{-\widetilde{M}_{i+1:n} - \frac{1}{2}\tilde{S}_{i+1:n}} + 1 \right) - e^{\widetilde{M}_{t+1:n} + \frac{1}{2}\tilde{S}_{t+1:n}} \left(c \sum_{i=t+1}^{n-1} X_i e^{-\widetilde{M}_{i+1:n} - \frac{1}{2}\tilde{S}_{i+1:n}} + 1 \right) \right)$$

If we now define

$$Y_i = e^{-\widetilde{M}_{i+1:n} - \frac{1}{2}\tilde{S}_{i+1:n}} \tag{A.1- 20}$$

we get:

$$X_t = \frac{1}{1+c} \left(e^{\tilde{\mu}_{t+1} + \tilde{\sigma}_{t+1}\phi} e^{\widetilde{M}_{t+2:n} + \frac{1}{2}\tilde{S}_{t+2:n}} \left(c \sum_{i=t+1}^{n-1} X_i Y_i + 1 \right) - e^{\widetilde{M}_{t+1:n} + \frac{1}{2}\tilde{S}_{t+1:n}} \left(c \sum_{i=t+1}^{n-1} X_i Y_i + 1 \right) \right) \tag{A.1- 21}$$

and if we also set the following:

$$A_t = \frac{e^{\tilde{\mu}_{t+1} + \tilde{\sigma}_{t+1}\phi} e^{\widetilde{M}_{t+2:n} + \frac{1}{2}\tilde{S}_{t+2:n}}}{1+c} = \frac{e^{\tilde{\sigma}_{t+1}\phi} e^{\widetilde{M}_{t+1:n} + \frac{1}{2}\tilde{S}_{t+2:n}}}{1+c} \tag{A.1- 22}$$

$$B_t = \frac{e^{\widetilde{M}_{t+1:n} + \frac{1}{2}\tilde{S}_{t+1:n}}}{1+c} \tag{A.1- 23}$$

This gives us

$$X_t = A_t\left(c\sum_{i=t+1}^{n-1} X_iY_i + 1\right) - B_t\left(c\sum_{i=t+1}^{n-1} X_iY_i + 1\right)$$

which implies

$$X_t = c(A_t - B_t)\sum_{i=t+1}^{n-1} X_iY_i + A_t - B_t \quad \text{(A.1- 24)}$$

If we now set the following:

$$F_t = A_t - B_t \quad \text{(A.1- 25)}$$

then we have the following equation:

$$X_t = F_t\left(c\sum_{i=t+1}^{n-1} X_iY_i + 1\right) \quad \text{(A.1- 26)}$$

with the constraint that $X_n = 0$.
Let us now let

$$W_t = \sum_{i=t}^{n-1} X_iY_i \quad \text{(A.1- 27)}$$

with $W_n = 0$, clearly, we get:

$$X_t = \frac{W_t - W_{t+1}}{Y_t} \quad \text{(A.1- 28)}$$

Thus,

$$W_t - W_{t+1} = cF_tY_tW_{t+1} + F_tY_t \quad \text{(A.1- 29)}$$

implying

$$W_t = (cF_tY_t + 1)W_{t+1} + F_tY_t \quad \text{(A.1- 30)}$$

The resultant regime can thus be written as:

$$Cap_t{}^{1c} = C_t\frac{W_t - W_{t+1}}{Y_t}$$

with

$$W_t = (1 + cF_tY_t)W_{t+1} + F_tY_t$$

$$Y_i = e^{-\widetilde{M}_{t+1:n} - \frac{1}{2}\tilde{S}_{t+1:n}}$$

$$F_t = \frac{1}{1+c}\, e^{\widetilde{M}_{t+1:n}}\left(e^{\tilde{\sigma}_{t+1}\phi}e^{\frac{1}{2}\tilde{S}_{t+2:n}} - e^{\frac{1}{2}\tilde{S}_{t+1:n}}\right)$$

$$\widetilde{M}_{t:n} = \sum_{k=t}^{n} \tilde{\mu}_k \quad \text{(A.1- 31)}$$

$$\tilde{S}_{t:n} = \sum_{k=t}^{n} \sigma_k{}^2 + 2\sum_{i=t}^{n-1}\sum_{j=i+1}^{n} \rho_{i,j}\sigma_i\sigma_j - \Delta_{1,v}$$

where $\Delta_{1,v}$ and $\tilde{\mu}_k$ are defined in Appendix A.3 and capture the past dependencies on which the current and future information are conditioned to.

$$\phi = \Phi^{-1}(99.5\%)$$

Similarly:

$$MVM_t{}^C = c\sum_{i=t}^{n-1} E\left(Cap_i{}^{1c}|\mathcal{F}_t\right)$$

$$MVM_t{}^C = c\,Cap_t{}^{1c} + c\sum_{i=t+1}^{n-1} E(C_i|\mathcal{F}_t)\left(\frac{W_i - W_{i+1}}{Y_i}\right)$$

$$MVM_t{}^C = c\,Cap_t{}^{1c} + c\sum_{i=t+1}^{n-1} C_t e^{\tilde{M}_{t+1:i}+\frac{1}{2}\tilde{S}_{t+1:i}}\,(W_i - W_{i+1})e^{\tilde{M}_{i+1:n}+\frac{1}{2}\tilde{S}_{i+1:n}}$$

$$MVM_t{}^C = c\,Cap_t{}^{1c} + c\,C_t e^{\tilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}}\sum_{i=t+1}^{n-1}(W_i - W_{i+1})$$

$$MVM_t{}^C = c\,Cap_t{}^{1c} + c\,C_t e^{\tilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}}(W_{t+1} - W_n)$$

$$\boxed{MVM_t{}^C = c\,Cap_t{}^{1c} + c\,C_t e^{\tilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}}W_{t+1}} \tag{A.1- 32}$$

with $W_n = 0$.

A.1.4 Solving simplified solution

The $\left(Cap_t^{1A}, MVM_t^A\right)$ system formulation presented in "Table 2-1 – Capital and Market Value Margins formulations" and defined as the Expected Cost of Capital Risk Margin, with capital capturing the expected BEL deterioration only, can be solved in a relatively straightforward manner. This, alongside our "exact solutions" will provide us with a measure of the simplifying assumption made to ignore the change in the MVM over a 1-year horizon.

$$MVM_t^A = c\sum_{i=t}^{n-1} E\left[Cap_i^{1A}|\mathcal{F}_t\right] \tag{A.1- 33}$$

$$MVM_t^A = c\sum_{i=t}^{n-1} E[99.5\%[L_t^1]|\mathcal{F}_t] = c\sum_{i=t}^{n-1} E[99.5\%[E(C_n|\mathcal{F}_{i+1}) - E(C_n|\mathcal{F}_i)|\mathcal{F}_i]|\mathcal{F}_t] \tag{A.1- 34}$$

$$MVM_t^A = c\sum_{i=t}^{n-1} E[99.5\%[E(C_n|\mathcal{F}_{i+1})|\mathcal{F}_i]|\mathcal{F}_t] - E[E(C_n|\mathcal{F}_i)|\mathcal{F}_t] \tag{A.1- 35}$$

$$MVM_t^A = c\sum_{i=t}^{n-1} [E[99.5\%[E(C_n|\mathcal{F}_{i+1})|\mathcal{F}_i]|\mathcal{F}_t] - E(C_n|\mathcal{F}_t)] \tag{A.1- 36}$$

$$MVM_t^A = c\sum_{i=t}^{n-1} E[99.5\%[E(C_n|\mathcal{F}_{i+1})|\mathcal{F}_i]|\mathcal{F}_t] - c(n-t).E(C_n|\mathcal{F}_t) \tag{A.1- 37}$$

$$MVM_t^A = c\sum_{i=t}^{n-1} E\left[99.5\%\left[C_i e^{\tilde{\mu}_{i+1}+\tilde{\sigma}_{i+1}\mathrm{N}(0,1)} e^{\tilde{M}_{i+2:n}+\frac{1}{2}\tilde{S}_{i+2:n}}\right]|\mathcal{F}_t\right] - c(n-t).E(C_n|\mathcal{F}_t) \tag{A.1- 38}$$

$$MVM_t^A = c\sum_{i=t}^{n-1} E\left[C_i e^{\tilde{\mu}_{i+1}+\tilde{\sigma}_{i+1}\phi} e^{\widetilde{M}_{i+2:n}+\frac{1}{2}\tilde{S}_{i+2:n}} | \mathcal{F}_t\right] - c(n-t).E(C_n|\mathcal{F}_t) \quad \text{(A.1- 39)}$$

$$MVM_t^A = c\sum_{i=t}^{n-1} C_t e^{\widetilde{M}_{t+1:i}+\frac{1}{2}\tilde{S}_{t+1:i}} e^{\tilde{\mu}_{i+1}+\tilde{\sigma}_{i+1}\phi} e^{\widetilde{M}_{i+2:n}+\frac{1}{2}\tilde{S}_{i+2:n}} - c(n-t).C_t e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}} \quad \text{(A.1- 40)}$$

$$\boxed{MVM_t^A = c.C_t e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}} \left(\sum_{i=t}^{n-1} e^{\tilde{\sigma}_{i+1}\phi - \frac{1}{2}\left(\tilde{\sigma}_{i+1}^2 + 2\sum_{j=i+2}^{n} \rho_{i+1,j}\sigma_{i+1}\sigma_j\right) - \Delta_{1,u}} - (n-t)\right) \quad \text{for } t \in [\![0, n-1]\!]} \quad \text{(A.1- 41)}$$

$$Cap_t^{1A} = 99.5\%\left[C_{t+1} e^{\widetilde{M}_{t+2:n}+\frac{1}{2}\tilde{S}_{t+2:n}} | \mathcal{F}_t\right] - C_t e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}} \quad \text{(A.1- 42)}$$

$$Cap_t^{1A} = C_t.e^{\tilde{\mu}_{t+1}+\tilde{\sigma}_{t+1}\phi} e^{\widetilde{M}_{t+2:n}+\frac{1}{2}\tilde{S}_{t+2:n}} - C_t e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}} \quad \text{(A.1- 43)}$$

$$\boxed{Cap_t^{1A} = C_t e^{\widetilde{M}_{t+1:n}+\frac{1}{2}\tilde{S}_{t+1:n}} \left(e^{\tilde{\sigma}_{t+1}\phi - \frac{1}{2}\tilde{\sigma}_{t+1}^2} - 1\right)} \quad \text{(A.1- 44)}$$

A.2 The additive model

A.2.1 Model

The model being considered now assumes that a company is exposed to claims with the following known process:

$$\boxed{C_t = C_{t-1} + X(\Theta_t)} \tag{A.2- 1}$$

Where the $X(\Theta_t)$ are correlated, closed under addition (i.e. if $X(\Theta_k)$ and $X(\Theta_l)$ for $k, t \geq 1$ follow a given statistical law X, then $X(\Theta_k) + X(\Theta_l)$ will follow the same statistical law X), with Θ_t being a vector parameter. This is equivalent to assuming that the increments are distributed along the law of $X(\Theta_t)$ and correlated between different development periods.

The various formulas exposed in the general structure in the section above can then be transposed to this additive structure as follows:

– The reserves are defined as:

$$R_t = (C_n|\mathcal{F}_t) - C_t = C_t + \sum_{k=t+1}^{n} X(\Theta_k) - C_t = \sum_{k=t+1}^{n} X(\Theta_k) \tag{A.2- 2}$$

Unlike with the multiplicative assumption on development factors, under the additive model structure, the reserves at time t are independent of C_t.

Thus, the following properties hold:

$$E(R_t|\mathcal{F}_t) = E(R_t) = \sum_{k=t+1}^{n} E[X(\Theta_k)] \tag{A.2- 3}$$

$$Var(R_t|\mathcal{F}_t) = Var(R_t) = Var\left(\sum_{k=t+1}^{n} X(\Theta_k)\right) \tag{A.2- 4}$$

– The future losses become:

$$L_t = R_t - E(R_t) = \sum_{k=t+1}^{n} X(\Theta_k) - \sum_{k=t+1}^{n} E[X(\Theta_k)] \tag{A.2- 5}$$

with the following properties:

$$E(L_t|\mathcal{F}_t) = 0 \tag{A.2- 6}$$

$$Var(L_t|\mathcal{F}_t) = Var(R_t|\mathcal{F}_t) = Var\left(\sum_{k=t+1}^{n} X(\Theta_k)\right) \tag{A.2- 7}$$

– The 1-year loss deterioration becomes:

$$L_t^1 = E(C_n|\mathcal{F}_{t+1}) - E(C_n|\mathcal{F}_t)$$

$$L_t^1 = E\left[C_{t+1} + \sum_{k=t+2}^{n} X(\Theta_k)\,|\mathcal{F}_t\right] - E\left[C_t + \sum_{k=t+1}^{n} X(\Theta_k)|\mathcal{F}_t\right]$$

$$L_t^1 = C_t + X(\Theta_{t+1}) + \sum_{k=t+2}^{n} E[X(\Theta_k)] - C_t - \sum_{k=t+1}^{n} E[X(\Theta_k)]$$

$$L_t^1 = X(\Theta_{t+1}) - E[X(\Theta_{t+1})] \tag{A.2- 8}$$

Unlike the lognormal structure, the 1-year loss deterioration on the additive model only depends on how the claims payments evolve over the next year (i.e. time $t+1$), instead of how they move until ultimate. This has the effect that the dependencies between different time periods are captured within the model to the extent that the volatilities will be altered by past dependencies, and also, in how the year-to-year increments are estimated from the cumulated paid claims from origin (cf. §3.2.1.2.4).

– The capital (not including the MVM deterioration at this stage, as already mentioned above)

- On an *ultimate horizon* basis, $Cap_t = 99.5\%[L_t]$ is transposed into:

$$Cap_t = 99.5\%\left[\sum_{k=t+1}^{n} X(\Theta_k)\right] - \sum_{k=t+1}^{n} E[X(\Theta_k)] \tag{A.2- 9}$$

- On a *1-year horizon* basis, $Cap_t{}^1 = 99.5\%\left[L_t{}^1\right]$ is transposed into:

$$Cap_t{}^1 = 99.5\%[X(\Theta_{t+1})] - E[X(\Theta_{t+1})] \tag{A.2- 10}$$

which is in a form directly comparable with Cap_t in (A.2- 9) above.

As with the multiplicative structure, if the losses mature after one year then these different approaches give the same capital requirements.

A.2.2 System to solve

We need to solve $Cap_t{}^{1c}$ as a solution of the system described in equations (2-28).
Under the additive structure assumptions, our iterative regime for $Cap_t{}^{1c}$ – and consequently $MVM_t{}^C$ becomes:

$$Cap_t{}^{1c} = \frac{1}{1+c}\left(F_{x(\Theta_{t+1})}{}^{-1}(99.5\%, \Theta_{t+1}) - E[X(\Theta_{t+1})]\right) \tag{A.2- 11}$$

with $F_{x(\Theta_t)}{}^{-1}(99.5\%, \Theta_t)$ being the VaR at the 99.5th percentile of the distribution for $x(\Theta_t)$ of cumulative distribution function $F_{x(\Theta_t)}$.
Indeed,

$$\begin{aligned} Cap_t{}^{1c} = \frac{1}{1+c}\Bigg(\%\Bigg[c\sum_{i=t+1}^{n-1} E\left[Cap_i^{1c}|\mathcal{F}_{t+1}\right] + C_{t+1} + \sum_{k=t+2}^{n} E[X(\Theta_k)]\,|\mathcal{F}_t\Bigg] \\ - c\sum_{i=t+1}^{n-1} E\left[Cap_i^{1c}|\mathcal{F}_t\right] - C_t - \sum_{k=t+1}^{n} E[X(\Theta_k)]\Bigg) \end{aligned}$$

$$\begin{aligned} Cap_t{}^{1c} = \frac{1}{1+c}\Bigg(c\sum_{i=t+1}^{n-1} E\left[Cap_i^{1c}\right] + C_t + F_{x(\Theta_{t+1})}{}^{-1}(99.5\%, \Theta_{t+1}) + \sum_{k=t+2}^{n} E[X(\Theta_k)] \\ - c\sum_{i=t+1}^{n-1} E\left[Cap_i^{1c}\right] - C_t - \sum_{k=t+1}^{n} E[X(\Theta_k)]\Bigg) \end{aligned}$$

$$Cap_t{}^{1c} = \frac{1}{1+c}\left(F_{x(\Theta_{t+1})}{}^{-1}(99.5\%, \Theta_{t+1}) - E[X(\Theta_{t+1})]\right)$$

Here, for practical reasons, we introduced a further, strong simplification, by assuming that the capital amounts will not be path-dependent, meaning that the capital at time t under the additive assumption is independent of C_t (in other words, $\mathcal{F}_t$ does not provide additional information to determine the required capital at time t), hence the fact that $\sum_{k=t+2}^{n} E[X(\Theta_k)]\,|\mathcal{F}_t = \sum_{k=t+2}^{n} E[X(\Theta_k)]$ which is no longer random, and under the same reasoning, $E\left(Cap_i{}^{1c}|\mathcal{F}_{t+1}\right) = E\left(Cap_i{}^{1c}\right)$ for $i \geq t+1$ is no longer random either. This simplifies the regime into equation (A.2- 11). This choice was made because the focus is on the lognormal model.

A.2.3 Solution

Under the additive structure, and given the parameters of the additive distribution, the solutions for $Cap_t{}^{1c}$ and $MVM_t{}^{C}$ can simply be written as:

$$\boxed{Cap_t{}^{1c} = \frac{1}{1+c}\left({F_{x(\Theta_{t+1})}}^{-1}(99.5\%, \Theta_{t+1}) - E[X(\Theta_{t+1})]\right)} \quad \text{(A.2- 12)}$$

And:

$$MVM_t{}^{C} = c\sum_{i=t}^{n-1} E\left(Cap_i{}^{1c}|\mathcal{F}_t\right) = c\sum_{i=t}^{n-1} E\left(Cap_i{}^{1c}\right)$$

$$\boxed{MVM_t{}^{C} = \frac{c}{1+c}\sum_{i=t}^{n-1}\left({F_{x(\Theta_{i+1})}}^{-1}(99.5\%, \Theta_{i+1}) - E[X(\Theta_{i+1})]\right)} \quad \text{(A.2- 13)}$$

An interesting feature is that the circularity theoretically collapses under the additive structure, where $Cap_t{}^{1c}$ at time t does not depend on future $MVM_{t+i}{}^{C}$.

In addition, as already mentioned above, the capital requirement only depends on the following year paid claims volatility, unlike the lognormal model which relies on how the claims structure behaves until ultimate.

We can now easily transpose those results to the Normal distributions.

A.2.3.1 Normal assumption

$\boxed{X(\Theta_t) = N(\tilde{\mu}_t, \tilde{\sigma}_t^2)}$ with $X(\Theta_t)$ defined as in (A.2- 1) above.

The claims payment model becomes:

$$\boxed{C_t = C_{t-1} + N(\tilde{\mu}_t, \tilde{\sigma}_t^2)} \quad \text{(A.2- 14)}$$

We get the following properties:

$$E(C_t|\mathcal{F}_{t-1}) = C_{t-1} + \tilde{\mu}_t \quad \text{(A.2- 15)}$$

$$Var(C_t|\mathcal{F}_{t-1}) = \tilde{\sigma}_t^2 \quad \text{(A.2- 16)}$$

This can further be extended to give:

$$C_t = \sum_{k=1}^{t} N(\tilde{\mu}_t, \tilde{\sigma}_t^2) + C_0 \quad \text{(A.2- 17)}$$

We then have the following properties:

$E[X(\Theta_t)] = E[N(\tilde{\mu}_t, \tilde{\sigma}_t^2)] = \tilde{\mu}_t$
$Var[X(\Theta_t)] = Var[N(\tilde{\mu}_t, \tilde{\sigma}_t^2)] = \tilde{\sigma}_t^2$
$C_t = \sum_{k=1}^{t} N(\tilde{\mu}_k, \tilde{\sigma}_k^2) + C_0 = N\left(\sum_{k=1}^{t} \tilde{\mu}_k\,, \tilde{S}_{1:t}\right) + C_0$
$R_t = \sum_{k=t+1}^{n} N(\tilde{\mu}_k, \tilde{\sigma}_k^2) = N\left(\sum_{k=t+1}^{n} \tilde{\mu}_k\,, \tilde{S}_{t+1:n}\right)$

$E(R_t|\mathcal{F}_t) = E\left[N\left(\sum_{k=t+1}^{n} \tilde{\mu}_k , \tilde{S}_{t+1:n}\right)|\mathcal{F}_t\right] = \sum_{k=t+1}^{n} \tilde{\mu}_k$
$Var(R_t|\mathcal{F}_t) = Var\left[N\left(\sum_{k=t+1}^{n} \tilde{\mu}_k , \tilde{S}_{t+1:n}\right)\right] = \tilde{S}_{t+1:n}$
$L_t = R_t - E(R_t) = N\left(\sum_{k=t+1}^{n} \tilde{\mu}_k , \tilde{S}_{t+1:n}\right) - \sum_{k=t+1}^{n} \tilde{\mu}_k$
$L_t^1 = E(C_n|\mathcal{F}_{t+1}) - E(C_n|\mathcal{F}_t) = C_{t+1} + \sum_{k=t+2}^{n} \tilde{\mu}_k - \sum_{k=t+1}^{n} \tilde{\mu}_k - C_t = C_t + N(\tilde{\mu}_{t+1}, \tilde{\sigma}_{t+1}^2) - \tilde{\mu}_{t+1} - C_t \quad \Rightarrow L_t^1 = N(\tilde{\mu}_{t+1}, \tilde{\sigma}_{t+1}^2) - \tilde{\mu}_{t+1}$

$Cap_t = 99.5\%[L_t] = 99.5\%\left[N\left(\sum_{k=t+1}^{n} \tilde{\mu}_k , \tilde{S}_{t+1:n}\right)\right] - \sum_{k=t+1}^{n} \tilde{\mu}_k$
$= \sum_{k=t+1}^{n} \tilde{\mu}_k + \phi\sqrt{\tilde{S}_{t+1:n}} - \sum_{k=t+1}^{n} \tilde{\mu}_k$
$\Rightarrow Cap_t = \phi\sqrt{\tilde{S}_{t+1:n}}$

To which we can compare:
$Cap_t{}^1 = 99.5\%\left[L_t{}^1\right] = 99.5\%[N(\tilde{\mu}_{t+1}, \tilde{\sigma}_{t+1}^2)] - \tilde{\mu}_{t+1} = \tilde{\mu}_{t+1} + \phi\tilde{\sigma}_{t+1} - \tilde{\mu}_{t+1}$
$\boxed{\Rightarrow Cap_t{}^1 = \phi\tilde{\sigma}_{t+1}}$

with $\phi = \Phi^{-1}[99.5\%]$ and Φ^{-1} being the percentile function of the standard normal cumulative distribution function.

We now have:
$F_{x(\Theta_{t+1})}{}^{-1}(99.5\%, \Theta_{t+1}) - E[X(\Theta_{t+1})] = F_{N(\tilde{\mu}_{t+1}, \tilde{\sigma}_{t+1}^2)}{}^{-1}(99.5\%, \tilde{\mu}_{t+1}, \tilde{\sigma}_{t+1}^2) - \tilde{\mu}_{t+1} = \tilde{\mu}_{t+1} + \phi\tilde{\sigma}_{t+1} - \tilde{\mu}_{t+1} = \phi\tilde{\sigma}_{t+1}$

We can thus simplify the resultant regime as:

$$Cap_t^{1c} = \frac{\phi\tilde{\sigma}_{t+1}}{1+c}$$

with

$$\phi = \Phi^{-1}[99.5\%] = 2.576$$

and finally,

$$MVM_t{}^C = \frac{c\phi}{1+c}\sum_{i=t}^{n-1} \tilde{\sigma}_{i+1} \qquad \text{(A.2- 18)}$$

Another interesting feature that is best reflected under the normal assumption compared to the lognormal is that the $MVM_t{}^C$ moves from being root sum squares to a linear function of the future standard deviations.

A.2.4 Solving simplified solution for the Normal model

Similarly to the lognormal model, we will solve the $\left(Cap_t^{1A}, MVM_t^A\right)$ system formulation to provide us with a measure of the simplifying assumption made to ignore the change in the MVM over a 1-year horizon.

$$Cap_t{}^{1A} = 99.5\%\left[N\left(\tilde{\mu}_{t+1}, \tilde{\sigma}_{t+1}^2\right) - \tilde{\mu}_{t+1}|\mathcal{F}_t\right] \qquad \text{(A.2- 19)}$$

$$\boxed{Cap_t^{1A} = \phi\tilde{\sigma}_{t+1}} \qquad \text{(A.2- 20)}$$

$$MVM_t^A = c\sum_{i=t}^{n-1} E\left[Cap_i^{1A}|\mathcal{F}_t\right] \qquad \text{(A.2- 21)}$$

$$MVM_t^A = c\sum_{i=t}^{n-1} E[99.5\%[L_t^1]|\mathcal{F}_t] = c\sum_{i=t}^{n-1} E\left[99.5\%\left[N\left(\tilde{\mu}_{i+1}, \tilde{\sigma}_{i+1}^2\right) - \tilde{\mu}_{i+1}|\mathcal{F}_i\right]|\mathcal{F}_t\right] \qquad \text{(A.2- 22)}$$

$$\boxed{MVM_t^A = c\phi \sum_{i=t}^{n-1} \tilde{\sigma}_{i+1}} \tag{A.2- 23}$$

We can note that we have:

$$Cap_t{}^{1c} = \frac{Cap_t{}^{1_A}}{1+c} \tag{A.2- 24}$$

and similarly:

$$MVM_t^C = \frac{MVM_t^A}{1+c} \tag{A.2- 25}$$

at each time $t \leq n-1$, which denote that the capital and MVM allowing for the MVM to be stressed over a 1-year horizon is less than the same values ignoring this possibility.

A.3 Dependencies

A.3.1 General notations and definitions

Variance-Covariance matrix

If we consider a vector $\boldsymbol{X}$ of random n variables X_i ($i \in [\![1, \dots, n]\!]$) each with finite variance, using the notation:

$$\boldsymbol{X} = (X_1, \dots, X_i, \dots, X_n)^T$$

then the covariance matrix $\boldsymbol{\Lambda}$ is the matrix whose (i, j) entry is the covariance:

$$\Lambda_{i,j} = \mathrm{cov}(X_i, X_j) = \mathrm{E}\left((X_i - \mu_i)(X_j - \mu_j)\right)$$

where $\mu_i = E(X_i)$ for each $i \in [\![1, \dots, n]\!]$.

We have:

$$\boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{E}[(\boldsymbol{X_1} - \boldsymbol{\mu_1})^2] & \dots\dots\dots & & \boldsymbol{\Lambda_{i,n}} \\ & \ddots & \boldsymbol{\Lambda_{i,j}} & \\ \vdots & & & \vdots \\ & & \ddots & \\ \boldsymbol{\Lambda_{n,1}} & & \boldsymbol{E}[(\boldsymbol{X_n} - \boldsymbol{\mu_n})^2] \end{pmatrix} \quad \text{(A.3- 1)}$$

which can also be written as:

$$\boldsymbol{\Lambda} = \boldsymbol{E}\left[(\boldsymbol{X} - \boldsymbol{E}(\boldsymbol{X}))(\boldsymbol{X} - \boldsymbol{E}(\boldsymbol{X}))^{\boldsymbol{T}}\right] \quad \text{(A.3- 2)}$$

Correlation matrix

Using the Pearson's product-moment correlation coefficient defined as:

$$\rho_{i,j} = \frac{\mathrm{cov}(X_i, X_j)}{\sigma_{X_i}\sigma_{X_j}}$$

where σ_{X_i}, σ_{X_j} are the standard deviations of random variables X_i and X_j.
The correlation matrix $\mathbf{P}$ is then defined as:

$$\mathbf{P} = \begin{pmatrix} \boldsymbol{1} & & \dots & & \boldsymbol{\rho_{1,n}} \\ & \ddots & & & \\ & & & \boldsymbol{\rho_{i,j}} & \\ & & \boldsymbol{1} & & \\ & & & \ddots & \\ \boldsymbol{\rho_{n,1}} & & \dots & & \boldsymbol{1} \end{pmatrix} \quad \text{(A.3- 3)}$$

We can see that $\boldsymbol{\Lambda}$ can also be expressed in terms of the correlation factors:

$$\boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{\sigma_{X_1}}^2 & \dots\dots\dots & & \boldsymbol{\rho_{1,n}\sigma_{X_1}\sigma_{X_n}} \\ & \ddots & \boldsymbol{\rho_{i,j}\sigma_{X_i}\sigma_{X_j}} & \\ \vdots & & & \vdots \\ & & \ddots & \\ \boldsymbol{\rho_{1,n}\sigma_{X_1}\sigma_{X_n}} & & & \boldsymbol{\sigma_{X_n}}^2 \end{pmatrix} \quad \text{(A.3- 4)}$$

A.3.2 Gaussian vectors

$\boldsymbol{X}$ is a standard Gaussian (or normal) vector defined as follows:

$$\boldsymbol{X} \sim \boldsymbol{N_n}(\boldsymbol{0}, \boldsymbol{I_n}) \Leftrightarrow X_1, \dots, X_i, \dots, X_n \text{ are i.i.d } N(0,1)$$

with: $\boldsymbol{E}(\boldsymbol{X}) = (\boldsymbol{0}, \dots, \boldsymbol{0})$ vector of expected value of $\boldsymbol{X}$ and $\boldsymbol{V}(\boldsymbol{X}) = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} = \boldsymbol{I_n}$ variance matrix of $\boldsymbol{X}$.

More generally, $\boldsymbol{Y}$ is Gaussian if there exist $\boldsymbol{X}$ standard Gaussian, $\boldsymbol{A}$ and $\boldsymbol{B}$ such that: $\boldsymbol{Y} = \boldsymbol{AX} + \boldsymbol{B}$ with $\boldsymbol{Y}$, $\boldsymbol{A}$, $\boldsymbol{X}$ and $\boldsymbol{B}$ of sizes $(p,1), (p,n), (n,1)$ and $(p,1)$ respectively.

This is also referred to as multivariate normal distributions or multivariate Gaussian distributions which is a generalization of the one-dimensional (univariate) normal distribution to higher dimensions.

We have:

$$\boldsymbol{E}(\boldsymbol{Y}) = \boldsymbol{E}(\boldsymbol{AX} + \boldsymbol{B}) = \boldsymbol{AE}(\boldsymbol{X}) + \boldsymbol{B} = \boldsymbol{B} \tag{A.3- 5}$$

$$\boldsymbol{V}(\boldsymbol{Y}) = \boldsymbol{AV}(\boldsymbol{X})\boldsymbol{A^T} = \boldsymbol{AA^T} \tag{A.3- 6}$$

We can further use the following notation:

$$\boldsymbol{Y} \sim \boldsymbol{N_p}(\boldsymbol{B}, \boldsymbol{AA^T}) \tag{A.3- 7}$$

The following property holds (closeness under linear transform property):
If $Y = (Y_1, \dots, Y_i, \dots, Y_p)$ is Gaussian, then $\sum_{i=1}^{p} \alpha_i Y_i$ is Gaussian as well, and more precisely we have:

$$\boldsymbol{Y} \sim \boldsymbol{N_p}(\boldsymbol{\mu}, \boldsymbol{\Lambda}) \text{ then } \sum_{i=1}^{p} \alpha_i Y_i \sim N(\alpha^T \mu, \alpha^T \Lambda \alpha) \tag{A.3- 8}$$

A.3.3 Applications

A.3.3.1 Variance of a sum of Gaussian random vectors

Using the definitions and notations introduced above, we will further introduce another set of notations for the purposes of writing the dependency structure of the lognormal and normal models under study in this thesis. This dependency structure is equivalent to using Gaussian Copulas applied to multivariate Gaussian variables in normal or log-space.

If we let:

$$\boldsymbol{X_{u,v}} = \left(X(\theta_u), \dots, X(\theta_i), \dots, X(\theta_v)\right)^T \tag{A.3- 9}$$

with $1 \le u \le i \le v \le n$ ($\boldsymbol{X_{u,v}}$ is a column vector of dimension $v - u + 1$) and each $X(\theta_i) \sim N(\mu_i, \sigma_i^2)$ for each $1 \le u \le i \le v \le n$,
the variance-covariance matrix is as follows:

$$\boldsymbol{\Lambda_{u,v}} = \begin{pmatrix} \boldsymbol{\sigma_{X(\theta_u)}}^2 & \cdots\cdots\cdots & & \boldsymbol{\rho_{u,v}\sigma_{X(\theta_u)}\sigma_{X(\theta_v)}} \\ & \ddots & \boldsymbol{\rho_{i,j}\sigma_{X(\theta_i)}\sigma_{X(\theta_j)}} & \\ & \vdots & & \vdots \\ & & \ddots & \\ \boldsymbol{\rho_{u,v}\sigma_{X(\theta_v)}\sigma_{X(\theta_u)}} & & & \boldsymbol{\sigma_{X(\theta_v)}}^2 \end{pmatrix} \tag{A.3- 10}$$

or

(A.3- 11)

$$\boldsymbol{\Lambda}_{u,v} = \begin{pmatrix} \sigma_u{}^2 & \dots\dots\dots & & \rho_{u,v}\sigma_u\sigma_v \\ & \ddots & \rho_{i,j}\sigma_i\sigma_j & \\ \vdots & & & \vdots \\ & & \ddots & \\ \rho_{u,v}\sigma_u\sigma_v & & & \sigma_v{}^2 \end{pmatrix}$$

with $\rho_{i,j}$ correlation factors between $X(\theta_i)$ and $X(\theta_j)$.
$\boldsymbol{\Lambda}_{u,v}$ is symmetric, non-negative definite and of dimension $(v-u+1)\times(v-u+1)$.
Note that $\boldsymbol{\Lambda}_{u,v}$ is a sub-matrix of $\boldsymbol{\Lambda}_{1,n}$.

We can further introduce the following notation:

$$\boldsymbol{\mu}_{u,v} = (\mu_u, \dots, \mu_i, \dots, \mu_v)^T \qquad \text{(A.3- 12)}$$

Using property (A.3- 8) we can write: $\sum_{k=u}^{v} X(\Theta_k) \sim N\left(\sum_{k=u}^{v} E[X(\Theta_k)], \boldsymbol{\alpha}_{(v-u+1)}{}^T \boldsymbol{\Lambda}_{u,v} \boldsymbol{\alpha}_{(v-u+1)}\right)$
with $\boldsymbol{\alpha}_{(v-u+1)} = (1, \dots, 1, \dots, 1)^T$ of dimension $(v-u+1)$.

The variance can also be decomposed into the following sum (using the fact that $\boldsymbol{\Lambda}_{u,v}$ is symmetric):

$$\boldsymbol{\alpha}_{(v-u+1)}{}^T \boldsymbol{\Lambda}_{u,v} \boldsymbol{\alpha}_{(v-u+1)} = \sum_{k=u}^{v} \boldsymbol{\sigma}_{X(\theta_k)}{}^2 + 2\sum_{i=u}^{v-1}\sum_{j=i+1}^{v} \boldsymbol{\rho}_{i,j}\boldsymbol{\sigma}_{X(\theta_i)}\boldsymbol{\sigma}_{X(\theta_j)} \qquad \text{(A.3- 13)}$$

or

$$\boldsymbol{\alpha}_{(v-u+1)}{}^T \boldsymbol{\Lambda}_{u,v} \boldsymbol{\alpha}_{(v-u+1)} = \sum_{k=u}^{v} \sigma_k{}^2 + 2\sum_{i=u}^{v-1}\sum_{j=i+1}^{v} \rho_{i,j}\sigma_i\sigma_j \qquad \text{(A.3- 14)}$$

Finally, we can write:

$$\boldsymbol{E}\left(\sum_{k=u}^{v} \boldsymbol{N}(\boldsymbol{\mu}_k, \boldsymbol{\sigma}_k^2)\right) = \sum_{k=u}^{v} \boldsymbol{E}\left(\boldsymbol{N}(\boldsymbol{\mu}_k, \boldsymbol{\sigma}_k^2)\right) = \sum_{k=u}^{v} \boldsymbol{\mu}_k \qquad \text{(A.3- 15)}$$

$$\boldsymbol{Var}\left(\sum_{k=u}^{v} \boldsymbol{N}(\boldsymbol{\mu}_k, \boldsymbol{\sigma}_k^2)\right) = \boldsymbol{\alpha}_{(v-u+1)}{}^T \boldsymbol{\Lambda}_{u,v} \boldsymbol{\alpha}_{(v-u+1)} = \sum_{k=u}^{v} \boldsymbol{\sigma}_k{}^2 + 2\sum_{i=u}^{v-1}\sum_{j=i+1}^{v} \boldsymbol{\rho}_{i,j}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j \qquad \text{(A.3- 16)}$$

and so:

$$\boxed{\sum_{k=u}^{v} \boldsymbol{N}(\boldsymbol{\mu}_k, \boldsymbol{\sigma}_k^2) \sim \boldsymbol{N}\left(\sum_{k=u}^{v} \boldsymbol{\mu}_k, \sum_{k=u}^{v} \boldsymbol{\sigma}_k{}^2 + 2\sum_{i=u}^{v-1}\sum_{j=i+1}^{v} \boldsymbol{\rho}_{i,j}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\right)} \qquad \text{(A.3- 17)}$$

To simplify matters, let us make the following substitutions:

$$\boldsymbol{M}_{u:v} = \sum_{k=u}^{v} \boldsymbol{\mu}_k \qquad \text{(A.3- 18)}$$

$$\boldsymbol{S}_{u:v} = \boldsymbol{\alpha}_{(v-u+1)}{}^T \boldsymbol{\Lambda}_{u,v} \boldsymbol{\alpha}_{(v-u+1)} = \sum_{k=u}^{v} \boldsymbol{\sigma}_k{}^2 + 2\sum_{i=u}^{v-1}\sum_{j=i+1}^{v} \boldsymbol{\rho}_{i,j}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j \qquad \text{(A.3- 19)}$$

in which, case (A.3- 17) simplifies into:

(A.3- 20)

$$\boxed{\sum_{k=u}^{v} N(\mu_k, \sigma_k^2) \sim N(\boldsymbol{M}_{u:v}, \boldsymbol{S}_{u:v})}$$

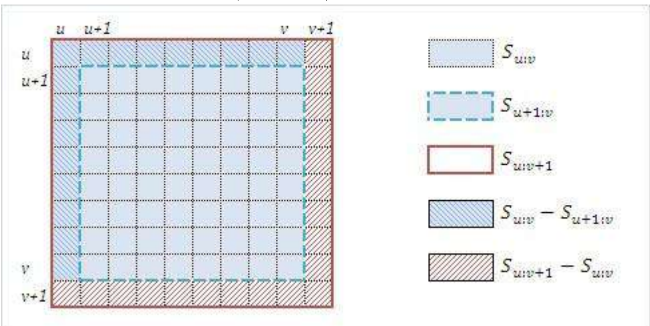
A.3.3.2 Other properties

The parameter estimation section (cf. 3.2.1.2) makes useful use of the following properties:

$$\begin{aligned}\boldsymbol{S}_{u:v} - \boldsymbol{S}_{u+1:v} &= \left(\sum_{k=u}^{v} \boldsymbol{\sigma}_k{}^2 + 2\sum_{i=u}^{v-1}\sum_{j=i+1}^{v} \boldsymbol{\rho}_{i,j}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\right) - \left(\sum_{k=u+1}^{v} \boldsymbol{\sigma}_k{}^2 + 2\sum_{i=u+1}^{v-1}\sum_{j=i+1}^{v} \boldsymbol{\rho}_{i,j}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\right) \\ &= \boldsymbol{\sigma}_u{}^2 + 2\sum_{j=u+1}^{v} \boldsymbol{\rho}_{u,j}\boldsymbol{\sigma}_u\boldsymbol{\sigma}_j\end{aligned} \qquad \text{(A.3- 21)}$$

$$\begin{aligned}\boldsymbol{S}_{u:v+1} - \boldsymbol{S}_{u:v} &= \left(\sum_{k=u}^{v+1} \boldsymbol{\sigma}_k{}^2 + 2\sum_{i=u}^{v}\sum_{j=i+1}^{v+1} \boldsymbol{\rho}_{i,j}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\right) - \left(\sum_{k=u}^{v} \boldsymbol{\sigma}_k{}^2 + 2\sum_{i=u}^{v-1}\sum_{j=i+1}^{v} \boldsymbol{\rho}_{i,j}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\right) \\ &= \boldsymbol{\sigma}_{v+1}{}^2 + 2\sum_{j=u}^{v} \boldsymbol{\rho}_{v+1,j}\boldsymbol{\sigma}_{v+1}\boldsymbol{\sigma}_j\end{aligned} \qquad \text{(A.3- 22)}$$

This can be proven using the matrix notations or looking at the following summation table for the variance-covariance matrix indices defined in (A.3- 11).



$$S_{u:v+1} = \boldsymbol{\alpha}_{(v-u+2)}{}^{T}\boldsymbol{\Lambda}_{u,v+1}\boldsymbol{\alpha}_{(v-u+2)} = \left(\boldsymbol{\alpha}_{(v+1-u,0)} + \beta_{v-u+2}\right)^{T}\boldsymbol{\Lambda}_{u,v+1}\left(\boldsymbol{\alpha}_{(v+1-u,0)} + \beta_{v-u+2}\right)$$

where $\boldsymbol{\alpha}_{(v+1-u,0)} = (1,1,\dots,1,0)$ and $\beta_{v-u+2} = (0,0,\dots,0,1)_{v-u+2}$ are orthogonal vectors of dimension $v-u+2$.

Developing this, we get:

$$S_{u:v+1} = \boldsymbol{\alpha}_{(v+1-u,0)}{}^{T}\boldsymbol{\Lambda}_{u,v+1}\boldsymbol{\alpha}_{(v+1-u,0)} + \beta_{v-u+2}{}^{T}\boldsymbol{\Lambda}_{u,v+1}\beta_{v-u+2} + \boldsymbol{\alpha}_{(v+1-u,0)}{}^{T}\boldsymbol{\Lambda}_{u,v+1}\beta_{v-u+2} + \beta_{v-u+2}{}^{T}\boldsymbol{\Lambda}_{u,v+1}\boldsymbol{\alpha}_{(v+1-u,0)}$$

It is then relatively straightforward that we have:

First term

$$\boldsymbol{\alpha}_{(v+1-u,0)}{}^{T}\boldsymbol{\Lambda}_{u,v+1}\boldsymbol{\alpha}_{(v+1-u,0)} = \boldsymbol{\alpha}_{(v+1-u)}{}^{T}\boldsymbol{\Lambda}_{u,v}\boldsymbol{\alpha}_{(v+1-u)} = S_{u:v}$$

Second term:

$$\beta_{v-u+2}{}^{T}\boldsymbol{\Lambda}_{u,v+1}\beta_{v-u+2} = \boldsymbol{\sigma}_{v+1}{}^{2}$$

Third and fourth terms:

$$\boldsymbol{\alpha}_{(v+1-u,0)}{}^{T}\boldsymbol{\Lambda}_{u,v+1}\beta_{v-u+2} = \beta_{v-u+2}{}^{T}\boldsymbol{\Lambda}_{u,v+1}\boldsymbol{\alpha}_{(v+1-u,0)}$$

as $\boldsymbol{\Lambda}_{u,v+1}$ is definite positive and

$$\left(\boldsymbol{\alpha}_{(v+1-u,0)}{}^{T}\boldsymbol{\Lambda}_{u,v+1}\beta_{v-u+2}\right)^{T} = \beta_{v-u+2}{}^{T}\boldsymbol{\Lambda}_{u,v+1}{}^{T}\boldsymbol{\alpha}_{(v+1-u,0)} = \beta_{v-u+2}{}^{T}\boldsymbol{\Lambda}_{u,v+1}\boldsymbol{\alpha}_{(v+1-u,0)}$$

$$\boldsymbol{\alpha}_{(v+1-u,0)}{}^{T}\boldsymbol{\Lambda}_{u,v+1}\beta_{v-u+2} = \beta_{v-u+2}{}^{T}\boldsymbol{\Lambda}_{u,v+1}\boldsymbol{\alpha}_{(v+1-u,0)} = \sum_{j=u}^{v}\boldsymbol{\rho}_{v+1,j}\boldsymbol{\sigma}_{v+1}\boldsymbol{\sigma}_{j}$$

Grouping these together, we get (A.3- 22).

A similar reasoning can be used to prove (A.3- 21).

A.3.3.3 Conditional distributions

Conditional distributions for multivariate normal distributions have the general following properties[39] (temporarily introducing slightly different notations here):

If $\boldsymbol{\mu}$ and $\boldsymbol{\Lambda}$ are the mean vector and the covariance matrix respectively, and are partitioned as follows:

$$\boldsymbol{\mu} = \begin{pmatrix}\boldsymbol{\mu}_1\\ \boldsymbol{\mu}_2\end{pmatrix} \text{ with sizes } \begin{pmatrix} q \times 1 \\ (n-q)\times 1\end{pmatrix}$$

$$\boldsymbol{\Lambda} = \begin{pmatrix}\boldsymbol{\Lambda}_{1,1} & \boldsymbol{\Lambda}_{1,2}\\ \boldsymbol{\Lambda}_{2,1} & \boldsymbol{\Lambda}_{2,2}\end{pmatrix} \text{ with sizes } \begin{pmatrix} q\times q & q\times(n-q)\\ (n-q)\times q & (n-q)\times(n-q)\end{pmatrix}$$

then the distribution of $\boldsymbol{X}_1$ conditional on $\boldsymbol{X}_2 = \boldsymbol{a}$ is multivariate normal $\boldsymbol{X}_1|\boldsymbol{X}_2 = \boldsymbol{a} \sim \boldsymbol{N}(\widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{\Lambda}})$ where:

$$\widetilde{\boldsymbol{\mu}} = \boldsymbol{\mu}_1 + \boldsymbol{\Lambda}_{1,2}\boldsymbol{\Lambda}_{2,2}{}^{-1}(\boldsymbol{a} - \boldsymbol{\mu}_2)$$

and the covariance matrix:

$$\widetilde{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda}_{1,1} - \boldsymbol{\Lambda}_{1,2}\boldsymbol{\Lambda}_{2,2}{}^{-1}\boldsymbol{\Lambda}_{2,1}$$

This matrix is the Schur complement of $\boldsymbol{\Lambda}_{2,2}$ in $\boldsymbol{\Lambda}$.

Note that knowing that $\boldsymbol{X}_2 = \boldsymbol{a}$ alters the variance, though the new variance does not depend on the specific value of $\boldsymbol{a}$; perhaps more surprisingly, the mean is shifted by $\boldsymbol{\Lambda}_{1,2}\boldsymbol{\Lambda}_{2,2}{}^{-1}(\boldsymbol{a} - \boldsymbol{\mu}_2)$; in comparison, when in the situation of not knowing the value of $\boldsymbol{a}$, $\boldsymbol{X}_1$ would have distribution $\boldsymbol{N}(\boldsymbol{\mu}_1, \boldsymbol{\Lambda}_{1,1})$.

We will now apply this to our study, using our existing notations and introducing some more.

First, let's consider the following partition:

$$\boldsymbol{\Lambda}_{1,v} = \begin{pmatrix}\boldsymbol{\Lambda}_{1,u-1} & \boldsymbol{\Sigma}_{u,v}{}^{\mathrm{T}}\\ \boldsymbol{\Sigma}_{u,v} & \boldsymbol{\Lambda}_{u,v}\end{pmatrix} = \begin{pmatrix}\sigma_1{}^2 & \cdots\cdots\cdots & & \rho_{1,v}\sigma_1\sigma_v\\ & \ddots & \rho_{i,j}\sigma_i\sigma_j & \\ \vdots & & & \vdots\\ & & \ddots & \\ \rho_{1,v}\sigma_1\sigma_v & & & \sigma_v{}^2\end{pmatrix} \qquad \text{(A.3- 23)}$$

$$\text{with sizes } \begin{pmatrix}(u-1)\times(u-1) & (u-1)\times(v-u+1)\\ (v-u+1)\times(u-1) & (v-u+1)\times(v-u+1)\end{pmatrix}$$

If we condition $\boldsymbol{X}_{u,v}$ on the path: $\boldsymbol{X}_{1,u-1} = (\omega_1, \dots, \omega_i, \dots, \omega_{u-1})^T = \boldsymbol{\Omega}_{u-1}$, then we can write:

$$\boldsymbol{X}_{u,v}|\boldsymbol{X}_{1,u-1} = \boldsymbol{\Omega}_{u-1} \sim \boldsymbol{N}(\widetilde{\boldsymbol{\mu}}_{u,v}, \widetilde{\boldsymbol{\Lambda}}_{u,v}) \qquad \text{(A.3- 24)}$$

where:

$$\widetilde{\boldsymbol{\mu}}_{u,v} = \boldsymbol{\mu}_{u,v} + \boldsymbol{\Sigma}_{u,v}\boldsymbol{\Lambda}_{1,u-1}{}^{-1}(\boldsymbol{\Omega}_{u-1} - \boldsymbol{\mu}_{1,u-1}) \qquad \text{(A.3- 25)}$$

and

$$\widetilde{\boldsymbol{\Lambda}}_{u,v} = \boldsymbol{\Lambda}_{u,v} - \boldsymbol{\Sigma}_{u,v}\boldsymbol{\Lambda}_{1,u-1}{}^{-1}\boldsymbol{\Sigma}_{u,v}{}^{\mathrm{T}} \qquad \text{(A.3- 26)}$$

[39] http://en.wikipedia.org/wiki/Multivariate_normal_distribution#Conditional_distributions

We can now transpose property (A.3- 20) into the conditional distribution of the sum:

$$\boxed{\sum_{k=u}^{v} N(\widetilde{\boldsymbol{\mu}}_k, \widetilde{\boldsymbol{\sigma}}_k^2)\,|\boldsymbol{X}_{1,u-1} = (\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_i, \dots, \boldsymbol{\omega}_{u-1})^T = \boldsymbol{\Omega}_{u-1} \sim N(\widetilde{\boldsymbol{M}}_{u,v}, \widetilde{\boldsymbol{S}}_{u,v})} \tag{A.3- 27}$$

with the following substitutions:

$$\widetilde{\boldsymbol{M}}_{u,v} = \sum_{k=u}^{v} \widetilde{\boldsymbol{\mu}}_k \tag{A.3- 28}$$

Where $\tilde{\mu}_k$ is each of the coordinate of the vector $\widetilde{\boldsymbol{\mu}}_{u,v}$ as expressed in (A.3- 25).

And:

$$\tilde{\mathbf{S}}_{u,v} = \boldsymbol{\alpha}_{(v-u+1)}{}^T \widetilde{\boldsymbol{\Lambda}}_{u,v} \boldsymbol{\alpha}_{(v-u+1)} = \sum_{k=u}^{v} \boldsymbol{\sigma}_k{}^2 + 2\sum_{i=u}^{v-1}\sum_{j=i+1}^{v} \boldsymbol{\rho}_{i,j}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j - \boldsymbol{\Delta}_{1,v} \tag{A.3- 29}$$

where $\boldsymbol{\Delta}_{1,v} = \boldsymbol{f}\left(\rho_{i,j}\sigma_i\sigma_j, 1 \le u \le i, \boldsymbol{j} \le v\right) = \boldsymbol{\alpha}_{(v-u+1)}{}^T \left(\boldsymbol{\Sigma}_{u,v}\boldsymbol{\Lambda}_{1,u-1}{}^{-1}\boldsymbol{\Sigma}_{u,v}{}^{\mathbf{T}}\right)\boldsymbol{\alpha}_{(v-u+1)}$

Note that we have:

$$\tilde{\mathbf{S}}_{u,v} = \boldsymbol{S}_{u:v} - \boldsymbol{\Delta}_{1,v} \tag{A.3- 30}$$

The $(\tilde{\mu}_u, \tilde{\sigma}_u^2)$ can be obtained by using the results for $(\widetilde{M}_{u,u}, \tilde{S}_{u,u})$ respectively (i.e setting $u = v$) .

$\boldsymbol{\Lambda}_{1,u} = \begin{pmatrix} \boldsymbol{\Lambda}_{1,u-1} & \boldsymbol{\Sigma}_{u,u}{}^{\mathbf{T}} \\ \boldsymbol{\Sigma}_{u,u} & \boldsymbol{\Lambda}_{u,u} \end{pmatrix}$ with only the following simplifications:

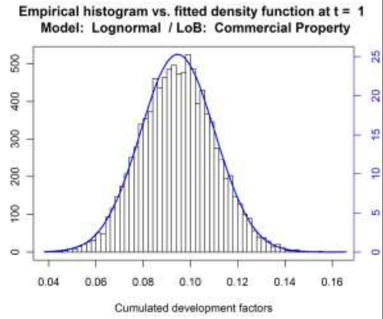
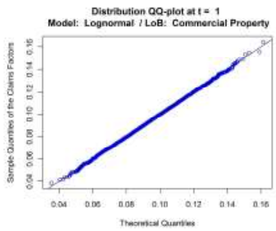
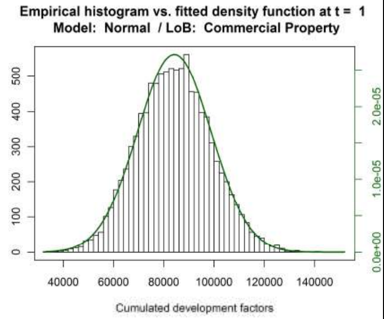
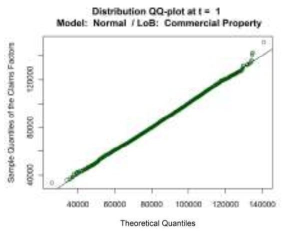
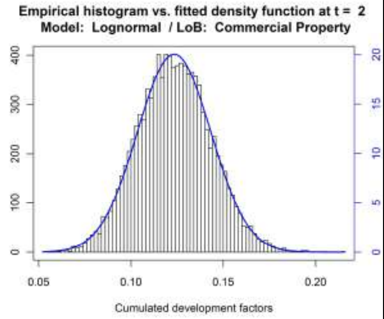
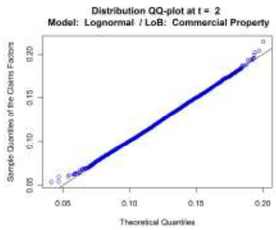
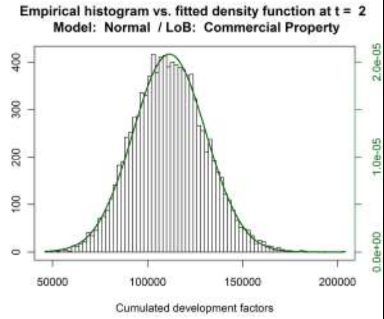
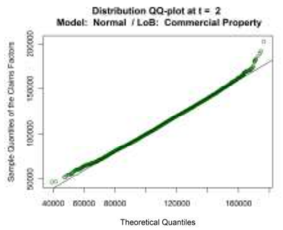
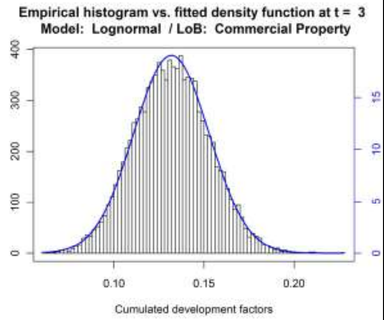
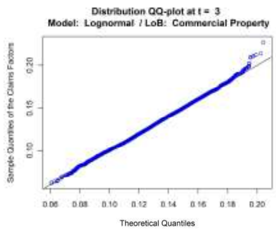
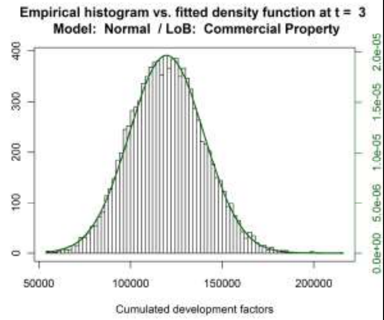
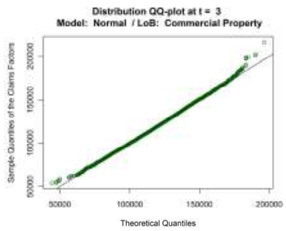
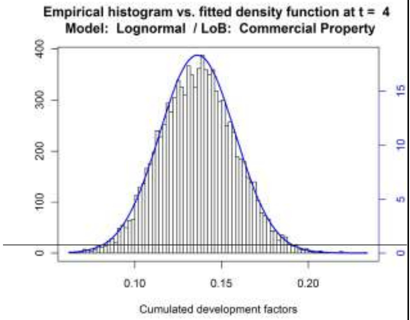
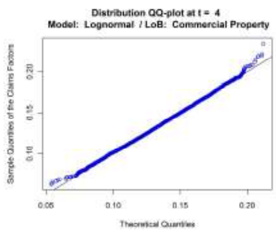
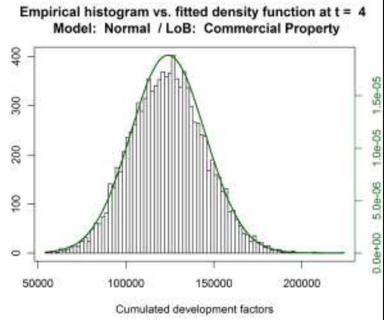
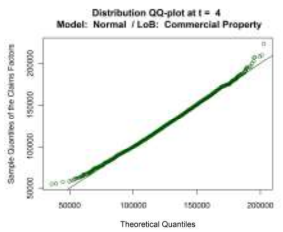
$\Lambda_{u,u} = \sigma_u^2$ is now a scalar and $\boldsymbol{\Sigma}_{u,u} = (\rho_{1,u}\sigma_1\sigma_u, \dots, \rho_{i,u}\sigma_i\sigma_u, \dots, \rho_{u-1,u}\sigma_{u-1}\sigma_u)$ is a $(u-1)$ dimension vector. Their determination still implies inverting the covariance matrix of the past information (i.e $\boldsymbol{\Lambda}_{1,u-1}$ which captures the dependency structure from $t = 1$ to $t = u - 1$).

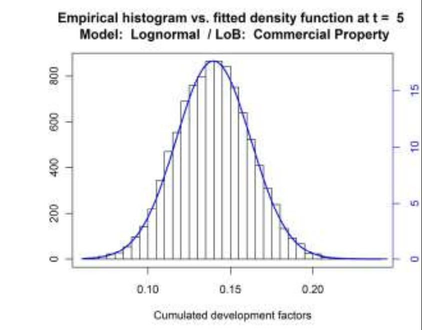
There are no simple ways of further expressing $\widetilde{\boldsymbol{M}}_{u:v}$ and $\tilde{\boldsymbol{S}}_{u:v}$ or even $(\widetilde{\boldsymbol{\mu}}_u, \widetilde{\boldsymbol{\sigma}}_u^2)$ analytically.

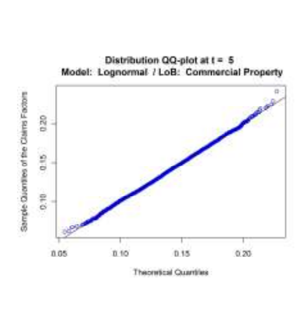
Appendix B. Detailed results

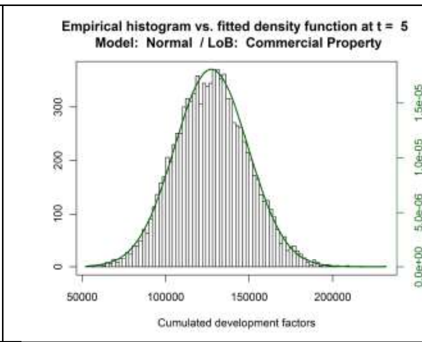
B.1 GoF tests results

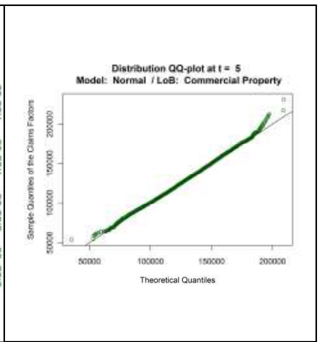
B.1.1 GoF for Commercial Property

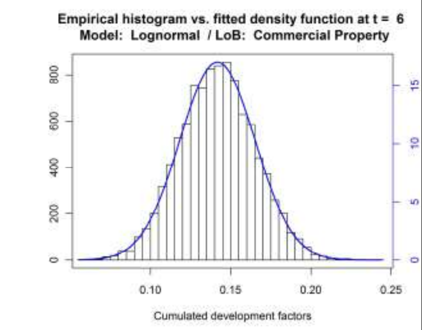
Lognormal model		Normal model	
Empirical vs. fitted density	QQ-plot	Empirical vs. fitted density	QQ-plot
			
			
			
			

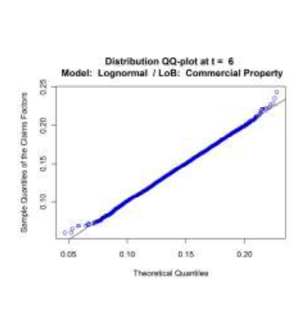

Empirical histogram vs. fitted density function at t = 5
Model: Lognormal / LoB: Commercial Property
Cumulated development factors

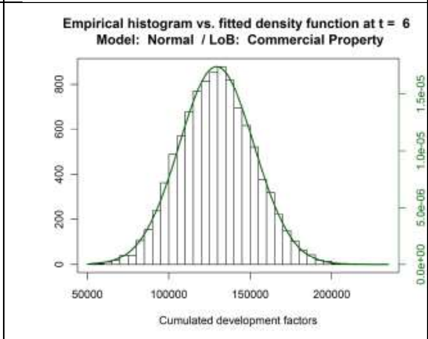

Distribution QQ-plot at t = 5
Model: Lognormal / LoB: Commercial Property
Sample Quantiles of the Claims Factors
Theoretical Quantiles

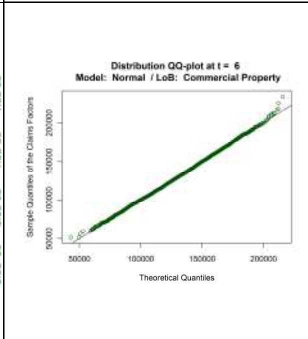

Empirical histogram vs. fitted density function at t = 5
Model: Normal / LoB: Commercial Property
Cumulated development factors

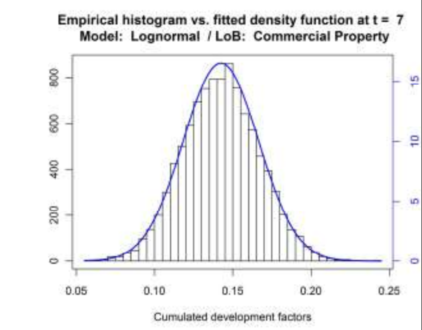

Distribution QQ-plot at t = 5
Model: Normal / LoB: Commercial Property
Sample Quantiles of the Claims Factors
Theoretical Quantiles

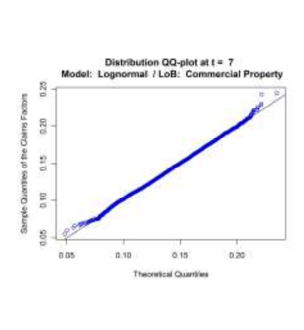

Empirical histogram vs. fitted density function at t = 6
Model: Lognormal / LoB: Commercial Property
Cumulated development factors

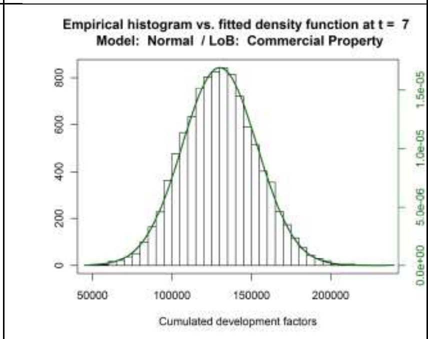

Distribution QQ-plot at t = 6
Model: Lognormal / LoB: Commercial Property
Sample Quantiles of the Claims Factors
Theoretical Quantiles

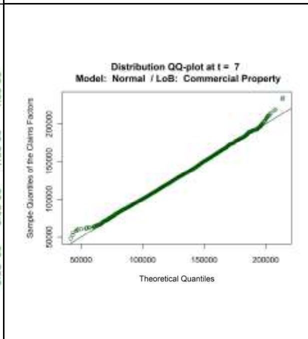

Empirical histogram vs. fitted density function at t = 6
Model: Normal / LoB: Commercial Property
Cumulated development factors

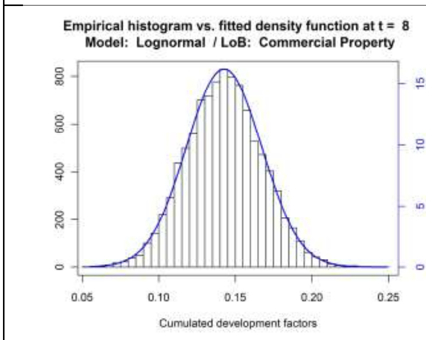

Distribution QQ-plot at t = 6
Model: Normal / LoB: Commercial Property
Sample Quantiles of the Claims Factors
Theoretical Quantiles

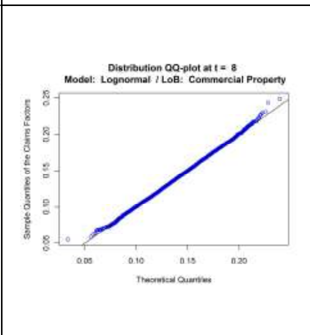

Empirical histogram vs. fitted density function at t = 7
Model: Lognormal / LoB: Commercial Property
Cumulated development factors

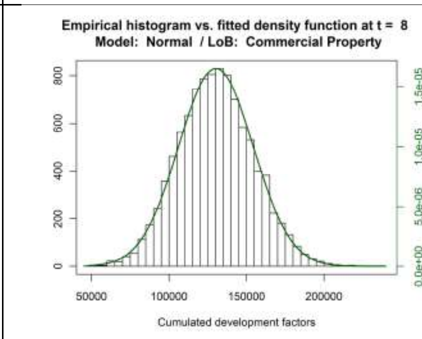

Distribution QQ-plot at t = 7
Model: Lognormal / LoB: Commercial Property
Sample Quantiles of the Claims Factors
Theoretical Quantiles

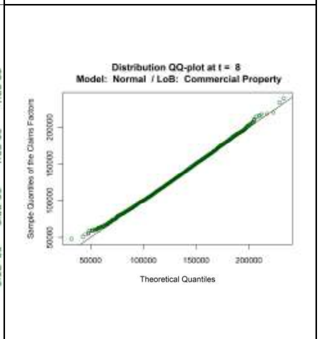

Empirical histogram vs. fitted density function at t = 7
Model: Normal / LoB: Commercial Property
Cumulated development factors

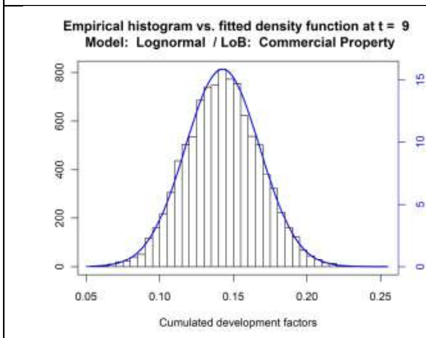

Distribution QQ-plot at t = 7
Model: Normal / LoB: Commercial Property
Sample Quantiles of the Claims Factors
Theoretical Quantiles

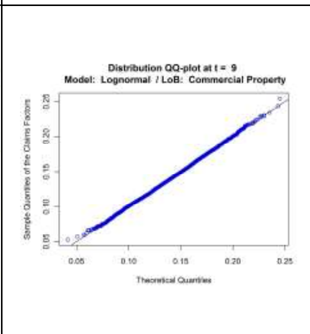

Empirical histogram vs. fitted density function at t = 8
Model: Lognormal / LoB: Commercial Property
Cumulated development factors

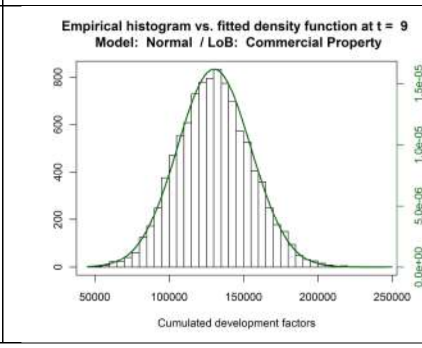

Distribution QQ-plot at t = 8
Model: Lognormal / LoB: Commercial Property
Sample Quantiles of the Claims Factors
Theoretical Quantiles

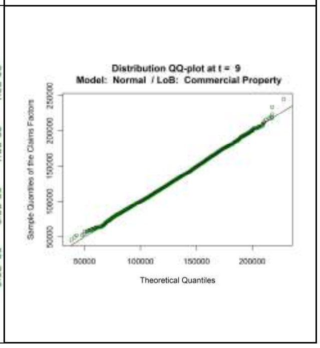

Empirical histogram vs. fitted density function at t = 8
Model: Normal / LoB: Commercial Property
Cumulated development factors

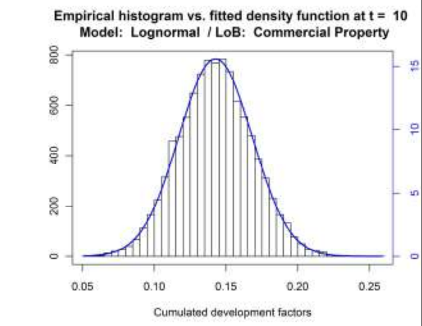
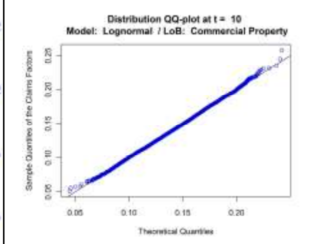
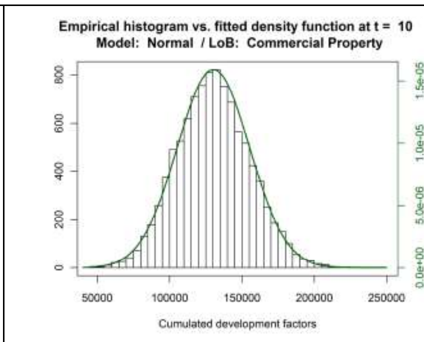
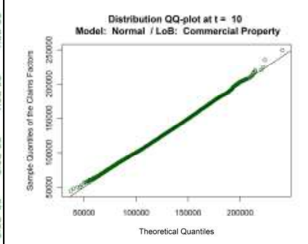
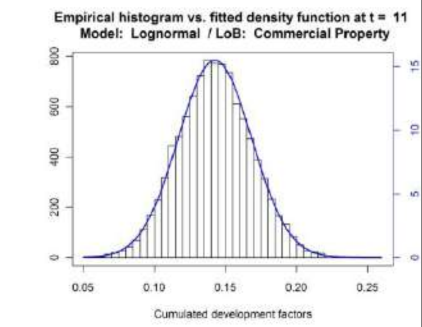
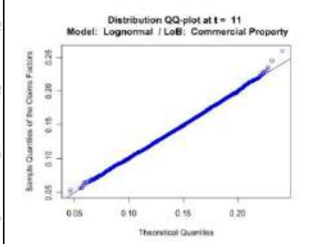
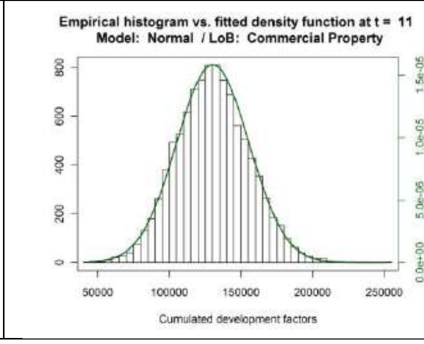
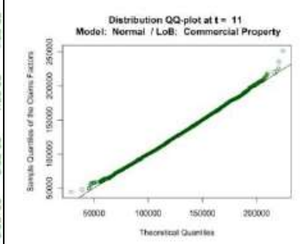

Distribution QQ-plot at t = 8
Model: Normal / LoB: Commercial Property
Sample Quantiles of the Claims Factors
Theoretical Quantiles


Empirical histogram vs. fitted density function at t = 9
Model: Lognormal / LoB: Commercial Property
Cumulated development factors

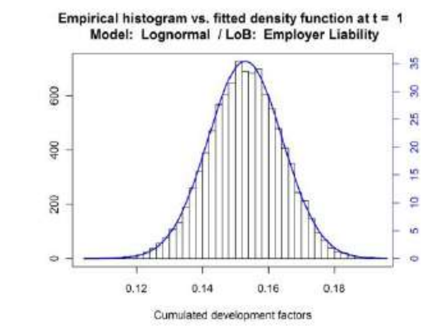
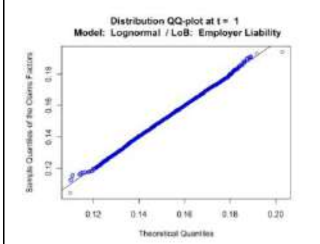
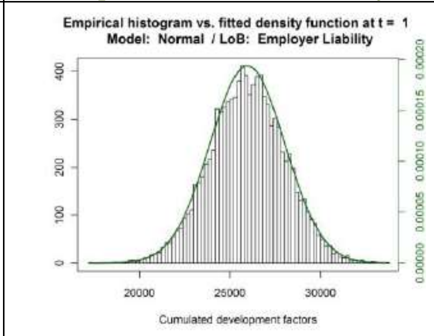
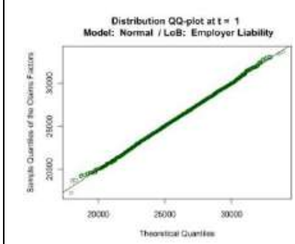
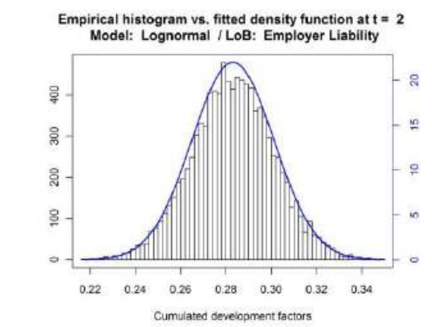
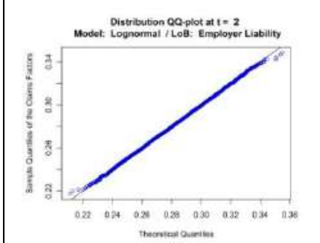
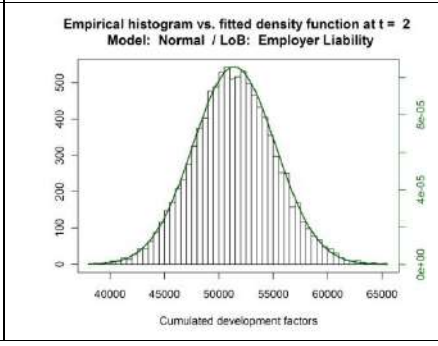
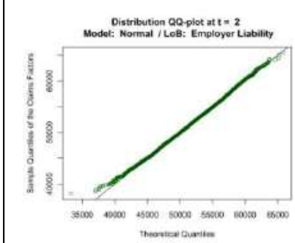

Distribution QQ-plot at t = 9
Model: Lognormal / LoB: Commercial Property
Sample Quantiles of the Claims Factors
Theoretical Quantiles

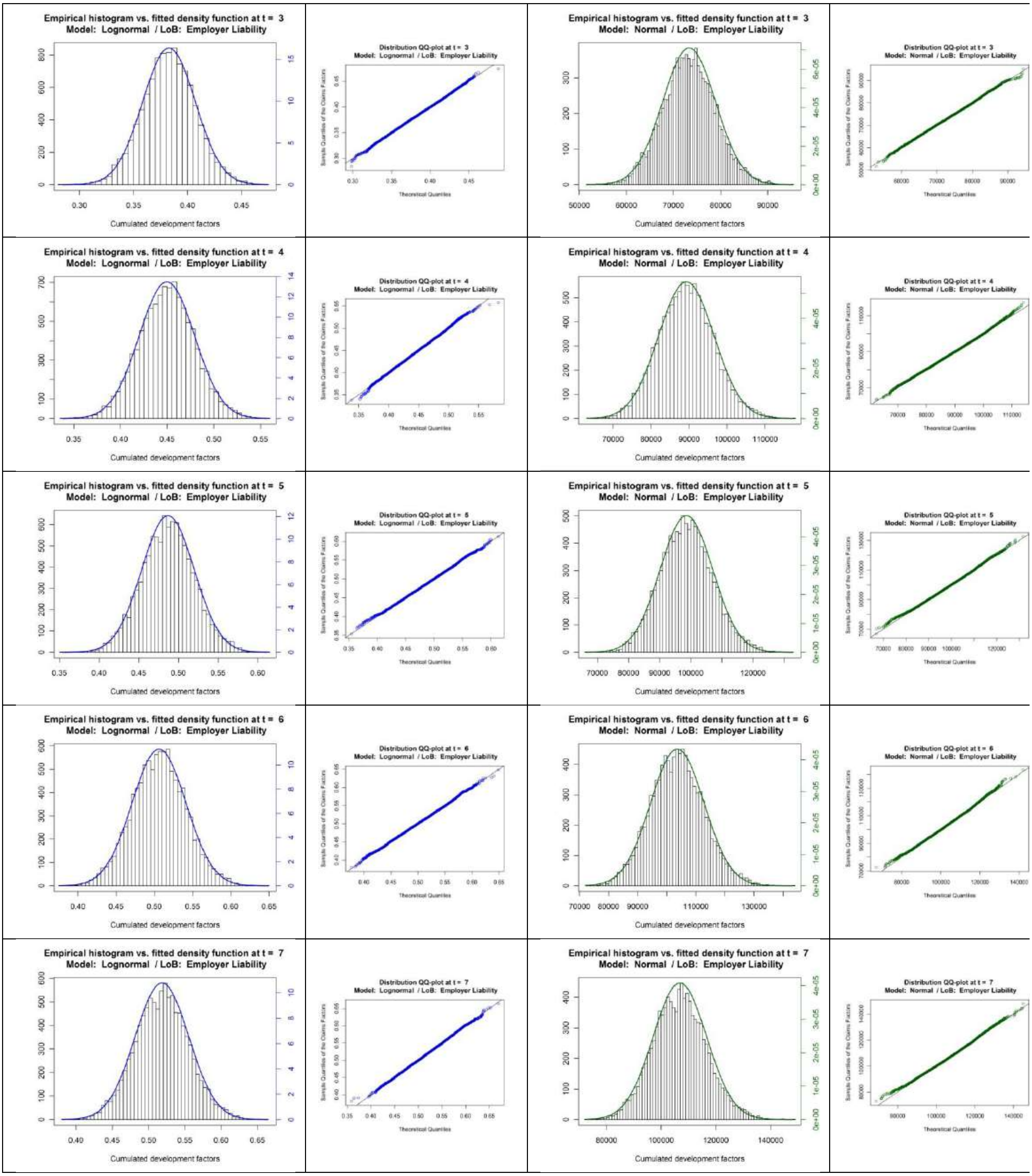

Empirical histogram vs. fitted density function at t = 9
Model: Normal / LoB: Commercial Property
Cumulated development factors

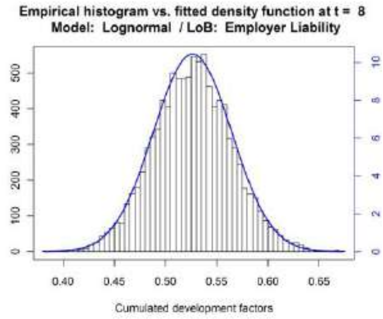

Distribution QQ-plot at t = 9
Model: Normal / LoB: Commercial Property
Sample Quantiles of the Claims Factors
Theoretical Quantiles

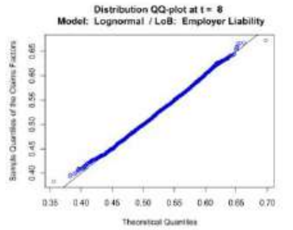
			
			

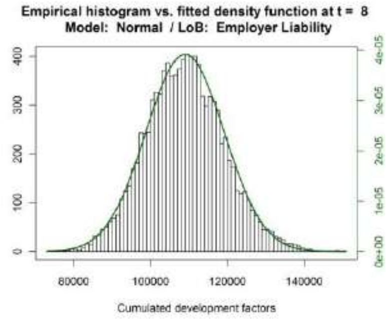
B.1.2 GoF for Employer's Liability

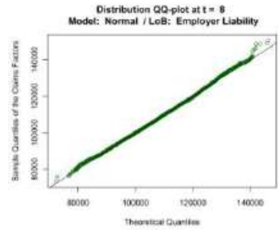
Lognormal model		Normal model	
Empirical vs. fitted density	QQ-plot	Empirical vs. fitted density	QQ-plot
			
			

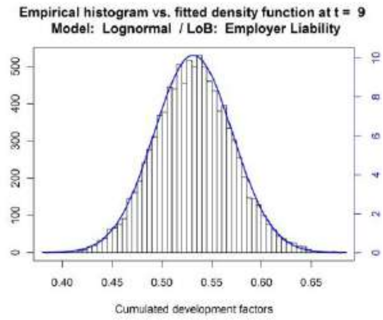

Empirical histogram vs. fitted density function at t = 3
Model: Lognormal / LoB: Employer Liability
Cumulated development factors
Distribution QQ-plot at t = 3
Model: Lognormal / LoB: Employer Liability
Theoretical Quantiles
Sample Quantiles of the Claims Factors
Empirical histogram vs. fitted density function at t = 3
Model: Normal / LoB: Employer Liability
Cumulated development factors
Distribution QQ-plot at t = 3
Model: Normal / LoB: Employer Liability
Theoretical Quantiles
Sample Quantiles of the Claims Factors
Empirical histogram vs. fitted density function at t = 4
Model: Lognormal / LoB: Employer Liability
Cumulated development factors
Distribution QQ-plot at t = 4
Model: Lognormal / LoB: Employer Liability
Theoretical Quantiles
Sample Quantiles of the Claims Factors
Empirical histogram vs. fitted density function at t = 4
Model: Normal / LoB: Employer Liability
Cumulated development factors
Distribution QQ-plot at t = 4
Model: Normal / LoB: Employer Liability
Theoretical Quantiles
Sample Quantiles of the Claims Factors
Empirical histogram vs. fitted density function at t = 5
Model: Lognormal / LoB: Employer Liability
Cumulated development factors
Distribution QQ-plot at t = 5
Model: Lognormal / LoB: Employer Liability
Theoretical Quantiles
Sample Quantiles of the Claims Factors
Empirical histogram vs. fitted density function at t = 5
Model: Normal / LoB: Employer Liability
Cumulated development factors
Distribution QQ-plot at t = 5
Model: Normal / LoB: Employer Liability
Theoretical Quantiles
Sample Quantiles of the Claims Factors
Empirical histogram vs. fitted density function at t = 6
Model: Lognormal / LoB: Employer Liability
Cumulated development factors
Distribution QQ-plot at t = 6
Model: Lognormal / LoB: Employer Liability
Theoretical Quantiles
Sample Quantiles of the Claims Factors
Empirical histogram vs. fitted density function at t = 6
Model: Normal / LoB: Employer Liability
Cumulated development factors
Distribution QQ-plot at t = 6
Model: Normal / LoB: Employer Liability
Theoretical Quantiles
Sample Quantiles of the Claims Factors
Empirical histogram vs. fitted density function at t = 7
Model: Lognormal / LoB: Employer Liability
Cumulated development factors
Distribution QQ-plot at t = 7
Model: Lognormal / LoB: Employer Liability
Theoretical Quantiles
Sample Quantiles of the Claims Factors
Empirical histogram vs. fitted density function at t = 7
Model: Normal / LoB: Employer Liability
Cumulated development factors
Distribution QQ-plot at t = 7
Model: Normal / LoB: Employer Liability
Theoretical Quantiles
Sample Quantiles of the Claims Factors

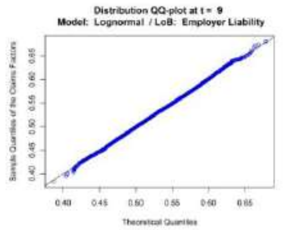

Empirical histogram vs. fitted density function at t = 8
Model: Lognormal / LoB: Employer Liability
Cumulated development factors

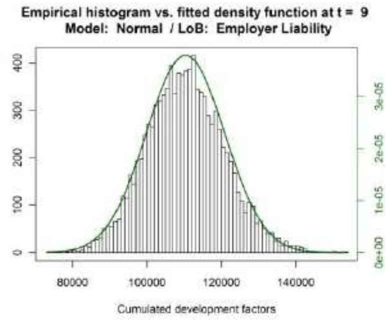

Distribution QQ-plot at t = 8
Model: Lognormal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

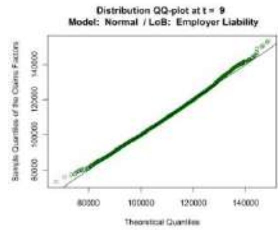

Empirical histogram vs. fitted density function at t = 8
Model: Normal / LoB: Employer Liability
Cumulated development factors

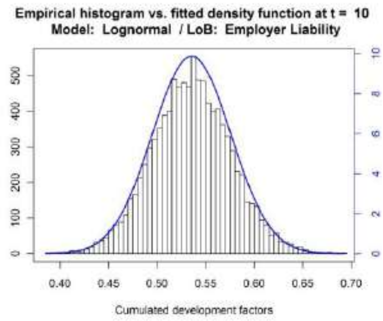

Distribution QQ-plot at t = 8
Model: Normal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles


Empirical histogram vs. fitted density function at t = 9
Model: Lognormal / LoB: Employer Liability
Cumulated development factors

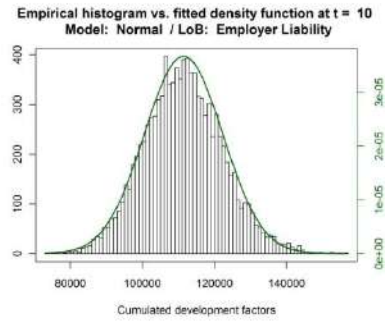

Distribution QQ-plot at t = 9
Model: Lognormal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

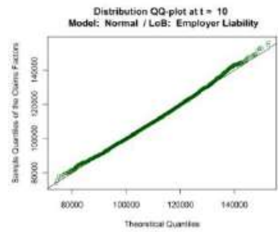

Empirical histogram vs. fitted density function at t = 9
Model: Normal / LoB: Employer Liability
Cumulated development factors

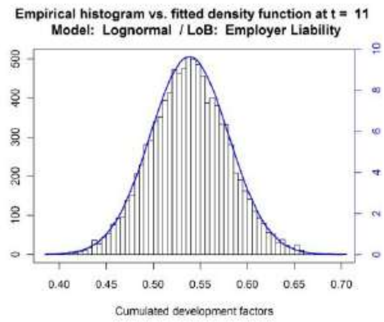

Distribution QQ-plot at t = 9
Model: Normal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles


Empirical histogram vs. fitted density function at t = 10
Model: Lognormal / LoB: Employer Liability
Cumulated development factors

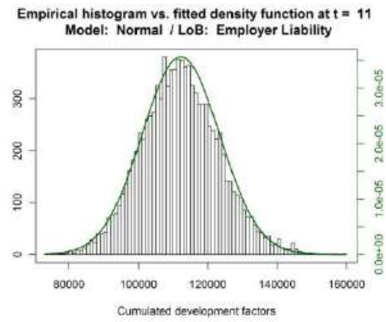

Distribution QQ-plot at t = 10
Model: Lognormal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

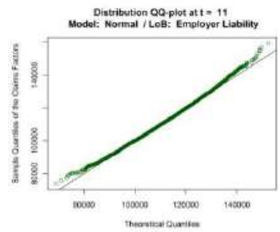

Empirical histogram vs. fitted density function at t = 10
Model: Normal / LoB: Employer Liability
Cumulated development factors

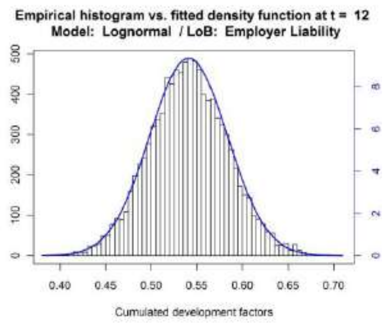

Distribution QQ-plot at t = 10
Model: Normal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

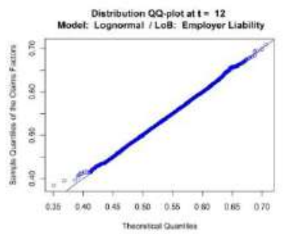

Empirical histogram vs. fitted density function at t = 11
Model: Lognormal / LoB: Employer Liability
Cumulated development factors

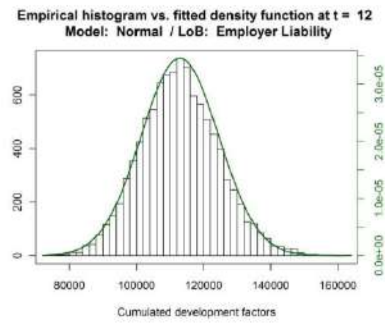

Distribution QQ-plot at t = 11
Model: Lognormal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

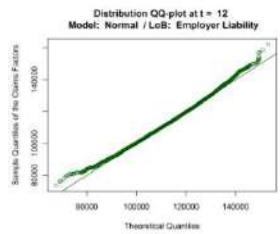

Empirical histogram vs. fitted density function at t = 11
Model: Normal / LoB: Employer Liability
Cumulated development factors

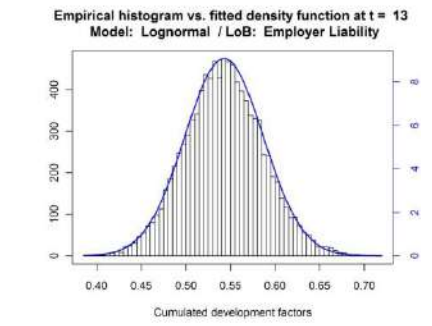

Distribution QQ-plot at t = 11
Model: Normal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

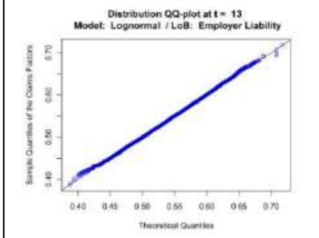

Empirical histogram vs. fitted density function at t = 12
Model: Lognormal / LoB: Employer Liability
Cumulated development factors

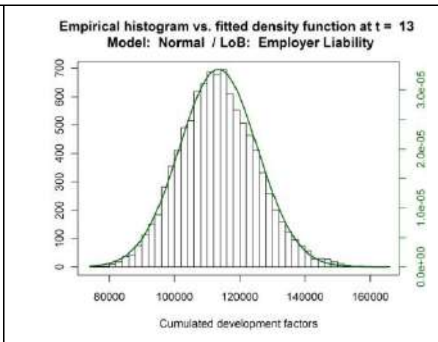

Distribution QQ-plot at t = 12
Model: Lognormal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

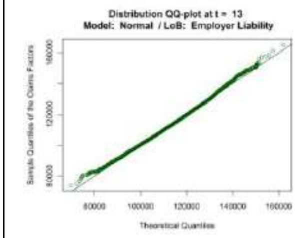

Empirical histogram vs. fitted density function at t = 12
Model: Normal / LoB: Employer Liability
Cumulated development factors

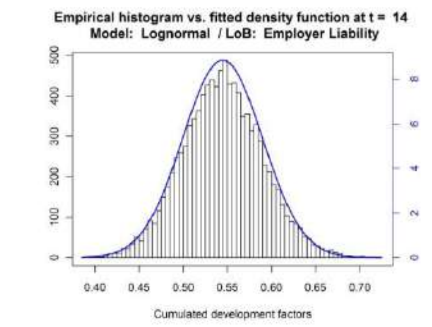

Distribution QQ-plot at t = 12
Model: Normal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

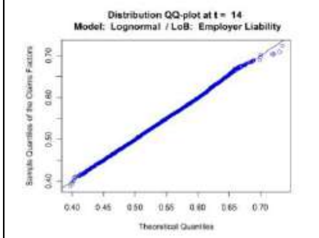

Empirical histogram vs. fitted density function at t = 13
Model: Lognormal / LoB: Employer Liability
Cumulated development factors

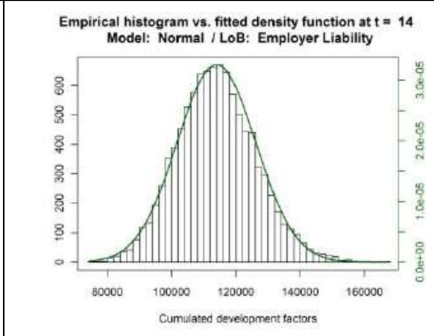

Distribution QQ-plot at t = 13
Model: Lognormal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

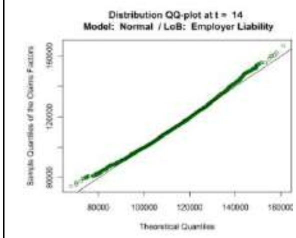

Empirical histogram vs. fitted density function at t = 13
Model: Normal / LoB: Employer Liability
Cumulated development factors

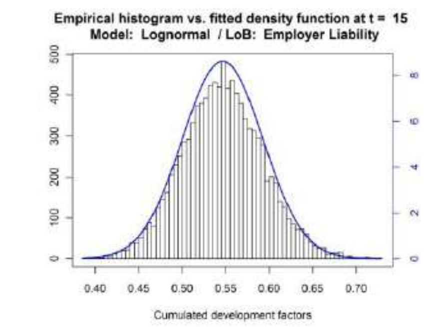

Distribution QQ-plot at t = 13
Model: Normal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

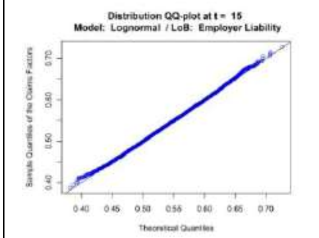

Empirical histogram vs. fitted density function at t = 14
Model: Lognormal / LoB: Employer Liability
Cumulated development factors

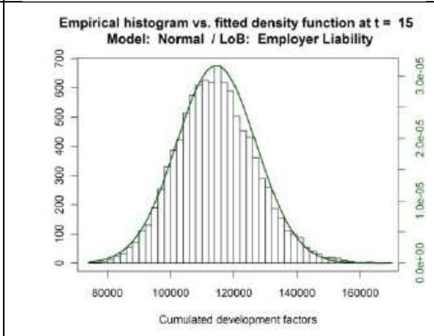

Distribution QQ-plot at t = 14
Model: Lognormal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

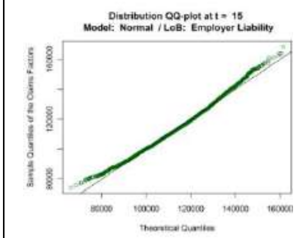

Empirical histogram vs. fitted density function at t = 14
Model: Normal / LoB: Employer Liability
Cumulated development factors

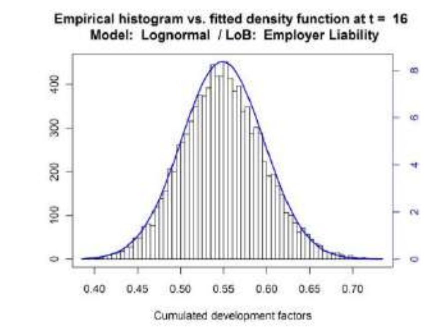

Distribution QQ-plot at t = 14
Model: Normal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

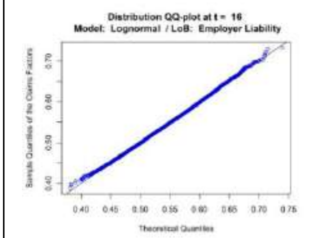

Empirical histogram vs. fitted density function at t = 15
Model: Lognormal / LoB: Employer Liability
Cumulated development factors

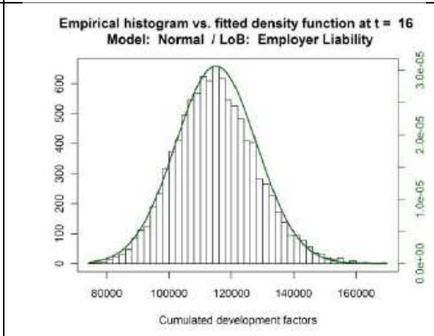

Distribution QQ-plot at t = 15
Model: Lognormal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

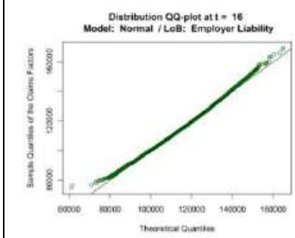

Empirical histogram vs. fitted density function at t = 15
Model: Normal / LoB: Employer Liability
Cumulated development factors

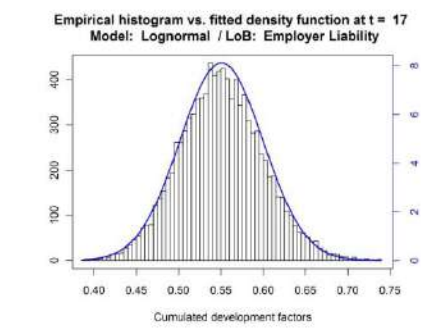

Distribution QQ-plot at t = 15
Model: Normal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

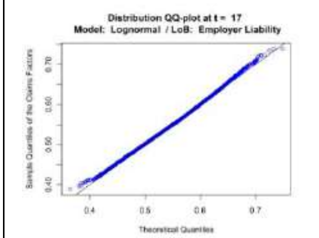

Empirical histogram vs. fitted density function at t = 16
Model: Lognormal / LoB: Employer Liability
Cumulated development factors

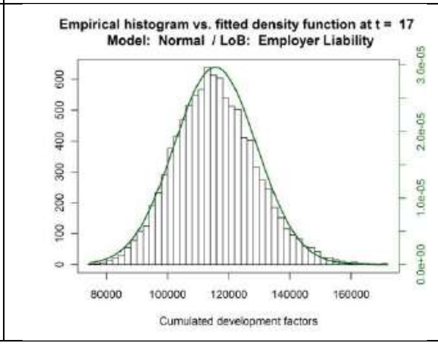

Distribution QQ-plot at t = 16
Model: Lognormal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

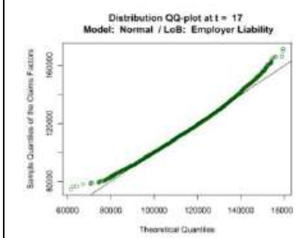

Empirical histogram vs. fitted density function at t = 16
Model: Normal / LoB: Employer Liability
Cumulated development factors

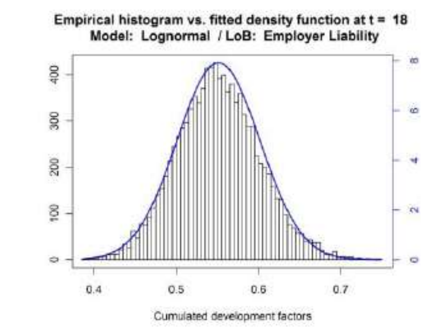
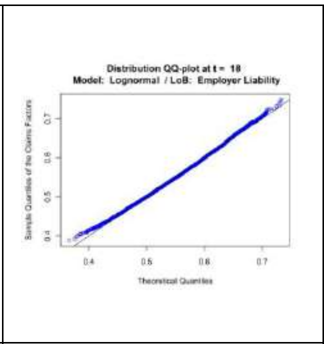
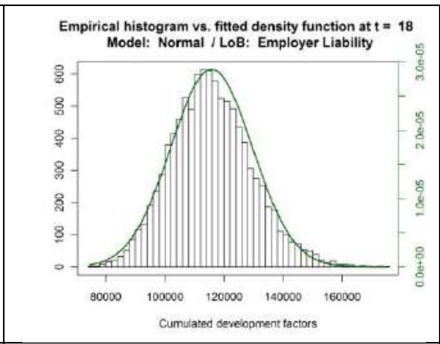
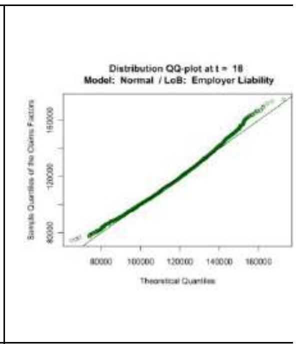
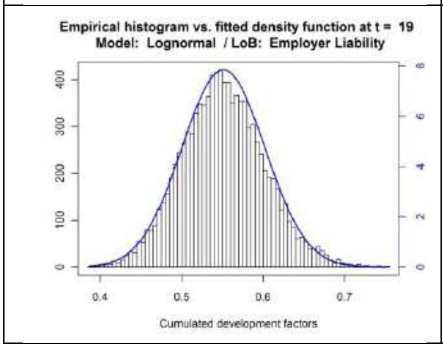
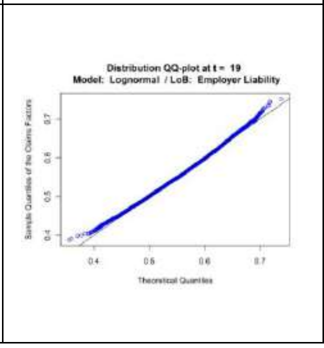
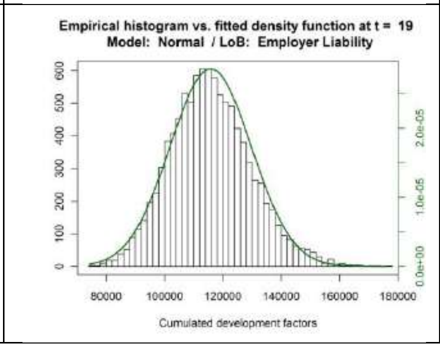
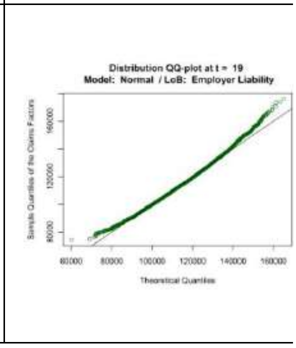

Distribution QQ-plot at t = 16
Model: Normal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles


Empirical histogram vs. fitted density function at t = 17
Model: Lognormal / LoB: Employer Liability
Cumulated development factors


Distribution QQ-plot at t = 17
Model: Lognormal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles


Empirical histogram vs. fitted density function at t = 17
Model: Normal / LoB: Employer Liability
Cumulated development factors


Distribution QQ-plot at t = 17
Model: Normal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles


Empirical histogram vs. fitted density function at t = 18
Model: Lognormal / LoB: Employer Liability
Cumulated development factors

Distribution QQ-plot at t = 18
Model: Lognormal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

Empirical histogram vs. fitted density function at t = 18
Model: Normal / LoB: Employer Liability
Cumulated development factors

Distribution QQ-plot at t = 18
Model: Normal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

Empirical histogram vs. fitted density function at t = 19
Model: Lognormal / LoB: Employer Liability
Cumulated development factors

Distribution QQ-plot at t = 19
Model: Lognormal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

Empirical histogram vs. fitted density function at t = 19
Model: Normal / LoB: Employer Liability
Cumulated development factors

Distribution QQ-plot at t = 19
Model: Normal / LoB: Employer Liability
Sample Quantiles of the Claims Factors
Theoretical Quantiles

B.2 MVM results

B.2.1 Commercial Property

Commercial Property - Lognormal

Capital+MVM: Typical path over time

Capital + MVM	0	1	2	3	4	5	6	7	8	9	10	11
Analytical Solution	43,760	21,339	10,670	8,975	7,447	5,932	4,620	3,530	2,885	1,478	448	0
Unstressed MVM	46,307	22,598	11,310	9,519	7,899	6,293	4,900	3,744	3,060	1,568	475	0
QIS5 - Proxy 3	41,804	20,167	9,564	8,133	6,756	5,441	4,311	3,550	3,123	1,455	448	0
QIS5 - Proxy 4	41,917	19,817	9,251	7,934	6,722	5,456	4,336	3,395	2,889	1,511	466	0
QIS5 - Proxy 5	45,062	20,193	9,371	7,798	6,488	5,138	4,011	3,098	2,629	1,390	423	0
QIS5 - standard approach	44,835	14,414	6,471	3,940	2,511	1,203	493	123	61	117	2	0
QIS5 - USP (method2)	58,401	18,775	8,429	5,132	3,270	1,567	643	160	79	152	3	0
QIS5 - USP (method3)	70,361	22,620	10,155	6,184	3,940	1,888	774	192	96	183	4	0

MVM: Typical path over time

MVM	0	1	2	3	4	5	6	7	8	9	10	11
Analytical Solution	5,737	3,363	2,279	1,778	1,349	983	687	450	265	108	25	0
Unstressed MVM	6,077	3,565	2,417	1,886	1,431	1,043	728	477	281	114	27	0
QIS5 - Proxy 3	3,781	2,191	1,173	936	659	492	378	470	503	84	25	0
QIS5 - Proxy 4	3,894	1,841	859	737	624	507	403	315	268	140	43	0
QIS5 - Proxy 5	7,039	2,217	979	601	391	189	78	18	9	19	0	0
QIS5 - standard approach	4,077	1,576	800	458	247	110	44	16	10	7	0	0
QIS5 - USP (method2)	5,311	2,053	1,041	596	322	143	57	21	13	9	0	0
QIS5 - USP (method3)	6,399	2,473	1,255	718	388	173	69	26	16	11	0	0

Capital: Typical path over time

Capital	0	1	2	3	4	5	6	7	8	9	10	11
Analytical Solution	38,023	17,976	8,391	7,197	6,097	4,949	3,933	3,080	2,620	1,371	422	0
Unstressed MVM	40,230	19,033	8,894	7,632	6,468	5,250	4,172	3,267	2,780	1,454	448	0
QIS5 - Proxy 3	38,023	17,976	8,391	7,197	6,097	4,949	3,933	3,080	2,620	1,371	422	0
QIS5 - Proxy 4	38,023	17,976	8,391	7,197	6,097	4,949	3,933	3,080	2,620	1,371	422	0
QIS5 - Proxy 5	38,023	17,976	8,391	7,197	6,097	4,949	3,933	3,080	2,620	1,371	422	0
QIS5 - standard approach	40,758	12,838	5,672	3,483	2,263	1,093	450	106	51	110	2	0
QIS5 - USP (method2)	53,090	16,722	7,388	4,536	2,948	1,424	586	138	67	143	3	0
QIS5 - USP (method3)	63,963	20,147	8,901	5,465	3,552	1,716	706	167	80	172	4	0

Reserves	127,978	40,310	17,808	10,935	7,107	3,433	1,412	333	160	345	8	0

Table 6-1 – MVM and Capital results (A) Lognormal model (B) Commercial Property

Commercial Property - Normal

Capital+MVM: Typical path over time

Capital + MVM	0	1	2	3	4	5	6	7	8	9	10	11
Normal Solution	41,308	21,494	10,904	9,196	7,647	6,114	4,761	3,650	2,991	1,534	464	0
Unstressed MVM	43,786	22,783	11,558	9,748	8,106	6,481	5,047	3,869	3,171	1,626	492	0
QIS5 - Proxy 3	39,271	20,089	9,652	8,247	6,878	5,539	4,387	3,443	2,954	1,520	464	0
QIS5 - Proxy 4	39,394	19,915	9,435	8,118	6,898	5,622	4,468	3,510	2,995	1,568	483	0
QIS5 - Proxy 5	43,027	20,600	9,591	7,948	6,609	5,260	4,104	3,198	2,720	1,423	438	0
Lognormal Solution	43,760	21,339	10,670	8,975	7,447	5,932	4,620	3,530	2,885	1,478	448	0

MVM: Typical path over time

MVM	0	1	2	3	4	5	6	7	8	9	10	11
Normal Solution	5,573	3,429	2,346	1,832	1,390	1,015	709	466	275	112	26	0
Unstressed MVM	5,908	3,635	2,486	1,942	1,474	1,076	751	494	291	118	28	0
QIS5 - Proxy 3	3,536	2,024	1,094	883	621	439	334	259	238	98	26	0
QIS5 - Proxy 4	3,660	1,850	877	754	641	522	415	326	278	146	45	0
QIS5 - Proxy 5	7,292	2,535	1,033	584	352	161	51	14	3	1	0	0
Lognormal Solution	5,737	3,363	2,279	1,778	1,349	983	687	450	265	108	25	0

Capital: Typical path over time

Capital	0	1	2	3	4	5	6	7	8	9	10	11
Normal Solution	35,734	18,064	8,558	7,364	6,257	5,100	4,053	3,184	2,716	1,422	438	0
Unstressed MVM	37,879	19,148	9,072	7,806	6,632	5,406	4,296	3,375	2,879	1,507	464	0
QIS5 - Proxy 3	35,734	18,064	8,558	7,364	6,257	5,100	4,053	3,184	2,716	1,422	438	0
QIS5 - Proxy 4	35,734	18,064	8,558	7,364	6,257	5,100	4,053	3,184	2,716	1,422	438	0
QIS5 - Proxy 5	35,734	18,064	8,558	7,364	6,257	5,100	4,053	3,184	2,716	1,422	438	0
Lognormal Solution	38,023	17,976	8,391	7,197	6,097	4,949	3,933	3,080	2,620	1,371	422	0

Reserves	126,683	40,189	17,809	10,933	7,105	3,439	1,420	341	168	349	8	0

Table 6-2 – MVM and Capital results (A) Normal model (B) Commercial Property

B.2.2 Employer's Liability

Employer Liability - Lognormal

Capital+MVM: Typical path over time

Capital + MVM	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Analytical Solution	11,120	10,292	8,876	7,449	5,995	4,954	4,283	3,554	3,154	2,877	2,577	2,393	2,076	1,758	1,528	1,413	1,270	993	334	0
Unstressed MVM	11,757	10,896	9,400	7,890	6,351	5,253	4,544	3,773	3,350	3,059	2,741	2,547	2,210	1,871	1,625	1,503	1,350	1,053	354	0
QIS5 - Proxy 3	9,529	8,878	7,669	6,449	5,204	4,346	3,808	3,171	2,900	2,682	2,416	2,261	1,935	1,620	1,416	1,329	1,227	977	334	0
QIS5 - Proxy 4	9,677	9,038	7,819	6,587	5,309	4,419	3,851	3,159	2,835	2,620	2,361	2,230	1,932	1,619	1,407	1,322	1,213	976	334	0
QIS5 - Proxy 5	20,227	17,430	13,690	10,248	7,284	5,441	4,415	3,462	2,950	2,691	2,425	2,301	2,046	1,758	1,524	1,415	1,248	970	318	0
QIS5 - standard approach	48,200	38,407	27,660	18,662	11,844	7,863	5,722	4,160	3,108	2,601	2,191	1,916	1,680	1,388	1,034	763	413	173	9	0
QIS5 - USP (method2)	43,355	34,546	24,880	16,786	10,654	7,072	5,147	3,742	2,796	2,339	1,970	1,723	1,511	1,248	930	686	371	155	8	0
QIS5 - USP (method3)	33,419	26,629	19,178	12,939	8,212	5,451	3,967	2,884	2,155	1,803	1,519	1,329	1,165	962	717	529	286	120	7	0

MVM: Typical path over time

MVM	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Analytical Solution	3,298	2,895	2,451	2,076	1,744	1,481	1,281	1,100	946	813	689	575	468	372	288	214	142	74	19	0
Unstressed MVM	3,495	3,070	2,600	2,203	1,851	1,573	1,361	1,170	1,005	865	732	612	497	395	306	227	150	79	20	0
QIS5 - Proxy 3	1,707	1,481	1,244	1,075	952	873	806	716	691	618	528	443	327	234	176	130	98	58	19	0
QIS5 - Proxy 4	1,855	1,641	1,394	1,213	1,058	946	849	705	626	556	473	412	324	233	167	123	85	57	19	0
QIS5 - Proxy 5	12,405	10,033	7,265	4,875	3,033	1,968	1,413	1,008	741	627	536	483	438	372	284	216	119	51	3	0
QIS5 - standard approach	8,692	6,454	4,521	3,137	2,186	1,594	1,223	949	748	605	483	379	286	202	130	75	33	10	0	0
QIS5 - USP (method2)	7,819	5,805	4,067	2,822	1,967	1,434	1,100	854	673	544	434	341	257	181	117	67	30	9	0	0
QIS5 - USP (method3)	6,027	4,475	3,135	2,175	1,516	1,105	848	658	519	420	335	263	198	140	90	52	23	7	0	0

Capital: Typical path over time

Capital	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Analytical Solution	7,822	7,397	6,425	5,374	4,251	3,473	3,002	2,454	2,209	2,064	1,889	1,818	1,608	1,386	1,240	1,199	1,129	919	315	0
Unstressed MVM	8,262	7,826	6,799	5,688	4,500	3,680	3,183	2,604	2,344	2,193	2,009	1,936	1,712	1,475	1,319	1,276	1,199	975	334	0
QIS5 - Proxy 3	7,822	7,397	6,425	5,374	4,251	3,473	3,002	2,454	2,209	2,064	1,889	1,818	1,608	1,386	1,240	1,199	1,129	919	315	0
QIS5 - Proxy 4	7,822	7,397	6,425	5,374	4,251	3,473	3,002	2,454	2,209	2,064	1,889	1,818	1,608	1,386	1,240	1,199	1,129	919	315	0
QIS5 - Proxy 5	7,822	7,397	6,425	5,374	4,251	3,473	3,002	2,454	2,209	2,064	1,889	1,818	1,608	1,386	1,240	1,199	1,129	919	315	0
QIS5 - standard approach	39,507	31,953	23,139	15,525	9,658	6,268	4,499	3,210	2,360	1,996	1,708	1,537	1,394	1,186	905	688	380	162	9	0
QIS5 - USP (method2)	35,536	28,741	20,813	13,964	8,687	5,638	4,047	2,888	2,123	1,795	1,536	1,383	1,254	1,067	814	619	341	146	8	0
QIS5 - USP (method3)	27,392	22,154	16,043	10,764	6,696	4,346	3,120	2,226	1,636	1,384	1,184	1,066	966	822	627	477	263	113	6	0

Reserves	124,052	100,330	72,654	48,747	30,326	19,682	14,128	10,081	7,410	6,266	5,362	4,827	4,376	3,724	2,841	2,160	1,192	510	28	0

Table 6-3 – MVM and Capital results (A) Lognormal model (B) Employer's Liability

Employer Liability - Normal

Capital+MVM: Typical path over time

Capital + MVM	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Normal Solution	7,821	7,798	7,279	6,420	5,366	4,540	3,944	3,299	2,965	2,717	2,446	2,279	1,974	1,673	1,461	1,356	1,224	961	321	0
Unstressed MVM	8,290	8,266	7,716	6,805	5,688	4,812	4,181	3,497	3,143	2,880	2,592	2,416	2,092	1,773	1,548	1,437	1,298	1,019	340	0
QIS5 - Proxy 3	6,069	6,319	6,043	5,412	4,555	3,899	3,430	2,850	2,603	2,429	2,210	2,102	1,824	1,533	1,340	1,266	1,169	945	321	0
QIS5 - Proxy 4	6,201	6,545	6,295	5,610	4,643	3,899	3,396	2,802	2,556	2,402	2,211	2,138	1,889	1,629	1,465	1,423	1,345	1,101	374	0
QIS5 - Proxy 5	16,597	14,275	11,528	8,785	6,408	4,872	3,965	3,160	2,750	2,487	2,228	2,086	1,815	1,554	1,370	1,284	1,160	907	303	0
Lognormal Solution	11,120	10,292	8,876	7,449	5,995	4,954	4,283	3,554	3,154	2,877	2,577	2,393	2,076	1,758	1,528	1,413	1,270	993	334	0

MVM: Typical path over time

MVM	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Normal Solution	2,809	2,508	2,191	1,886	1,614	1,388	1,199	1,035	899	775	658	551	447	356	277	206	137	72	18	0
Unstressed MVM	2,978	2,659	2,322	1,999	1,710	1,472	1,271	1,097	953	821	698	584	474	377	293	218	145	76	19	0
QIS5 - Proxy 3	1,057	1,029	955	877	802	747	686	586	537	487	422	374	298	216	157	116	81	55	18	0
QIS5 - Proxy 4	1,189	1,255	1,207	1,076	890	748	651	537	490	461	424	410	362	312	281	273	258	211	72	0
QIS5 - Proxy 5	11,585	8,985	6,440	4,251	2,655	1,720	1,220	896	684	546	441	359	288	237	186	134	73	17	1	0
Lognormal Solution	3,298	2,895	2,451	2,076	1,744	1,481	1,281	1,100	946	813	689	575	468	372	288	214	142	74	19	0

Capital: Typical path over time

Capital	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Normal Solution	5,012	5,290	5,088	4,534	3,753	3,151	2,745	2,265	2,066	1,942	1,787	1,728	1,527	1,317	1,184	1,150	1,087	890	302	0
Unstressed MVM	5,313	5,607	5,393	4,806	3,978	3,341	2,909	2,401	2,190	2,058	1,895	1,832	1,618	1,396	1,255	1,219	1,153	943	321	0
QIS5 - Proxy 3	5,012	5,290	5,088	4,534	3,753	3,151	2,745	2,265	2,066	1,942	1,787	1,728	1,527	1,317	1,184	1,150	1,087	890	302	0
QIS5 - Proxy 4	5,012	5,290	5,088	4,534	3,753	3,151	2,745	2,265	2,066	1,942	1,787	1,728	1,527	1,317	1,184	1,150	1,087	890	302	0
QIS5 - Proxy 5	5,012	5,290	5,088	4,534	3,753	3,151	2,745	2,265	2,066	1,942	1,787	1,728	1,527	1,317	1,184	1,150	1,087	890	302	0
Lognormal Solution	7,822	7,397	6,425	5,374	4,251	3,473	3,002	2,454	2,209	2,064	1,889	1,818	1,608	1,386	1,240	1,199	1,129	919	315	0

Reserves	115,851	89,846	64,401	42,508	26,550	17,204	12,203	8,958	6,841	5,455	4,406	3,585	2,881	2,368	1,863	1,336	730	173	6	0

Table 6-4 – MVM and Capital results (A) Normal model (B) Employer's Liability

B.3 Correlation results

B.3.1 Commercial Property

Correlation matrix of the (log-) year-to-year development factors - Lognormal model - Commercial Property

t	1	2	3	4	5	6	7	8	9	10	11
1	100%	32%	14%	10%	10%	10%	7%	5%	4%	3%	3%
2	32%	100%	20%	13%	13%	18%	14%	11%	9%	8%	8%
3	14%	20%	100%	28%	27%	30%	32%	27%	24%	26%	25%
4	10%	13%	28%	100%	36%	35%	37%	41%	37%	36%	39%
5	10%	13%	27%	36%	100%	43%	36%	39%	45%	45%	46%
6	10%	18%	30%	35%	43%	100%	45%	41%	49%	51%	49%
7	7%	14%	32%	37%	36%	45%	100%	52%	43%	44%	45%
8	5%	11%	27%	41%	39%	41%	52%	100%	56%	54%	63%
9	4%	9%	24%	37%	45%	49%	43%	56%	100%	84%	80%
10	3%	8%	26%	36%	45%	51%	44%	54%	84%	100%	82%
11	3%	8%	25%	39%	46%	49%	45%	63%	80%	82%	100%

Correlation matrix of the year-to-year increments - Normal model - Commercial Property

t	1	2	3	4	5	6	7	8	9	10	11
1	100%	36%	18%	12%	12%	12%	8%	6%	4%	4%	3%
2	36%	100%	23%	15%	15%	19%	14%	12%	9%	8%	8%
3	18%	23%	100%	29%	29%	31%	32%	28%	25%	26%	26%
4	12%	15%	29%	100%	37%	36%	38%	41%	37%	36%	40%
5	12%	15%	29%	37%	100%	44%	36%	39%	45%	45%	47%
6	12%	19%	31%	36%	44%	100%	46%	41%	49%	51%	49%
7	8%	14%	32%	38%	36%	46%	100%	52%	43%	44%	45%
8	6%	12%	28%	41%	39%	41%	52%	100%	56%	54%	63%
9	4%	9%	25%	37%	45%	49%	43%	56%	100%	85%	80%
10	4%	8%	26%	36%	45%	51%	44%	54%	85%	100%	82%
11	3%	8%	26%	40%	47%	49%	45%	63%	80%	82%	100%

B.3.2 Employer's Liability

Correlation matrix of the (log-) year-to-year development factors - Lognormal model - Employer Liability

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	100%	32%	30%	27%	22%	23%	18%	12%	10%	10%	10%	11%	10%	11%	10%	13%	12%	10%	9%
2	32%	100%	42%	38%	28%	22%	19%	14%	10%	9%	8%	10%	8%	9%	9%	11%	12%	9%	8%
3	30%	42%	100%	44%	34%	22%	18%	16%	12%	10%	8%	8%	6%	6%	8%	9%	11%	7%	6%
4	27%	38%	44%	100%	38%	27%	22%	19%	14%	14%	9%	9%	9%	9%	8%	10%	11%	8%	6%
5	22%	28%	34%	38%	100%	29%	25%	21%	17%	17%	15%	14%	12%	10%	11%	13%	15%	10%	10%
6	23%	22%	22%	27%	29%	100%	31%	27%	24%	24%	23%	23%	19%	18%	18%	19%	20%	17%	17%
7	18%	19%	18%	22%	25%	31%	100%	32%	29%	29%	28%	30%	27%	24%	22%	23%	23%	21%	20%
8	12%	14%	16%	19%	21%	27%	32%	100%	39%	36%	35%	34%	33%	31%	26%	25%	25%	21%	22%
9	10%	10%	12%	14%	17%	24%	29%	39%	100%	42%	38%	38%	36%	34%	33%	27%	25%	23%	23%
10	10%	9%	10%	14%	17%	24%	29%	36%	42%	100%	51%	48%	45%	45%	45%	41%	35%	31%	33%
11	10%	8%	8%	9%	15%	23%	28%	35%	38%	51%	100%	57%	54%	52%	51%	51%	47%	40%	45%
12	11%	10%	8%	9%	14%	23%	30%	34%	38%	48%	57%	100%	66%	61%	60%	61%	61%	58%	58%
13	10%	8%	6%	9%	12%	19%	27%	33%	36%	45%	54%	66%	100%	69%	66%	65%	64%	62%	62%
14	11%	9%	6%	9%	10%	18%	24%	31%	34%	45%	52%	61%	69%	100%	72%	68%	64%	59%	62%
15	10%	9%	8%	8%	11%	18%	22%	26%	33%	45%	51%	60%	66%	72%	100%	76%	69%	64%	67%
16	13%	11%	9%	10%	13%	19%	23%	25%	27%	41%	51%	61%	65%	68%	76%	100%	81%	73%	78%
17	12%	12%	11%	11%	15%	20%	23%	25%	25%	35%	47%	61%	64%	64%	69%	81%	100%	85%	87%
18	10%	9%	7%	8%	10%	17%	21%	21%	23%	31%	40%	58%	62%	59%	64%	73%	85%	100%	83%
19	9%	8%	6%	6%	10%	17%	20%	22%	23%	33%	45%	58%	62%	62%	67%	78%	87%	83%	100%

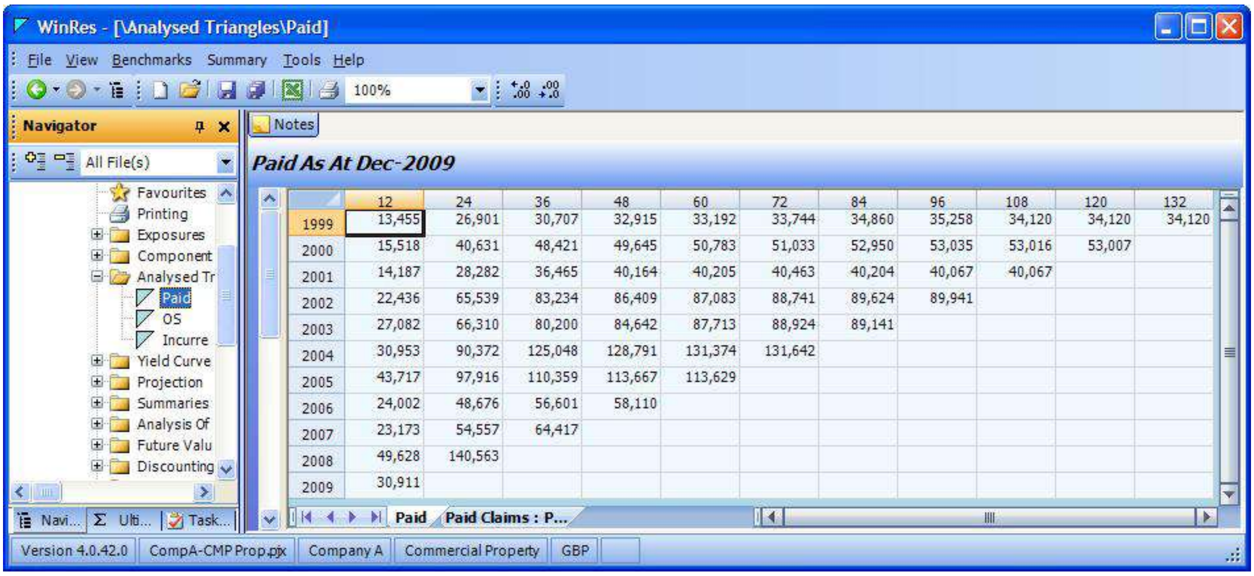
Correlation matrix of the year-to-year increments - Normal model - Employer Liability

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	100%	43%	40%	36%	31%	29%	24%	17%	14%	13%	12%	13%	12%	12%	12%	15%	14%	10%	9%
2	43%	100%	54%	50%	39%	31%	27%	21%	15%	13%	11%	13%	10%	12%	12%	15%	15%	11%	9%
3	40%	54%	100%	56%	46%	33%	27%	23%	17%	14%	12%	12%	9%	10%	11%	13%	14%	9%	8%
4	36%	50%	56%	100%	49%	37%	30%	26%	20%	18%	13%	12%	11%	12%	12%	14%	15%	11%	8%
5	31%	39%	46%	49%	100%	38%	33%	28%	22%	21%	18%	17%	14%	13%	14%	16%	18%	12%	11%
6	29%	31%	33%	37%	38%	100%	37%	33%	28%	27%	26%	26%	22%	21%	20%	22%	23%	19%	18%
7	24%	27%	27%	30%	33%	37%	100%	36%	32%	32%	30%	32%	29%	26%	25%	26%	26%	22%	21%
8	17%	21%	23%	26%	28%	33%	36%	100%	41%	38%	37%	36%	35%	33%	29%	28%	28%	22%	23%
9	14%	15%	17%	20%	22%	28%	32%	41%	100%	45%	40%	40%	38%	36%	34%	30%	27%	24%	24%
10	13%	13%	14%	18%	21%	27%	32%	38%	45%	100%	53%	50%	47%	46%	46%	43%	37%	32%	34%
11	12%	11%	12%	13%	18%	26%	30%	37%	40%	53%	100%	58%	56%	53%	52%	52%	48%	41%	45%
12	13%	13%	12%	12%	17%	26%	32%	36%	40%	50%	58%	100%	67%	63%	62%	62%	62%	59%	59%
13	12%	10%	9%	11%	14%	22%	29%	35%	38%	47%	56%	67%	100%	70%	67%	66%	66%	63%	63%
14	12%	12%	10%	12%	13%	21%	26%	33%	36%	46%	53%	63%	70%	100%	73%	69%	65%	60%	63%
15	12%	12%	11%	12%	14%	20%	25%	29%	34%	46%	52%	62%	67%	73%	100%	77%	70%	65%	68%
16	15%	15%	13%	14%	16%	22%	26%	28%	30%	43%	52%	62%	66%	69%	77%	100%	82%	74%	79%
17	14%	15%	14%	15%	18%	23%	26%	28%	27%	37%	48%	62%	66%	65%	70%	82%	100%	85%	87%
18	10%	11%	9%	11%	12%	19%	22%	22%	24%	32%	41%	59%	63%	60%	65%	74%	85%	100%	84%
19	9%	9%	8%	8%	11%	18%	21%	23%	24%	34%	45%	59%	63%	63%	68%	79%	87%	84%	100%

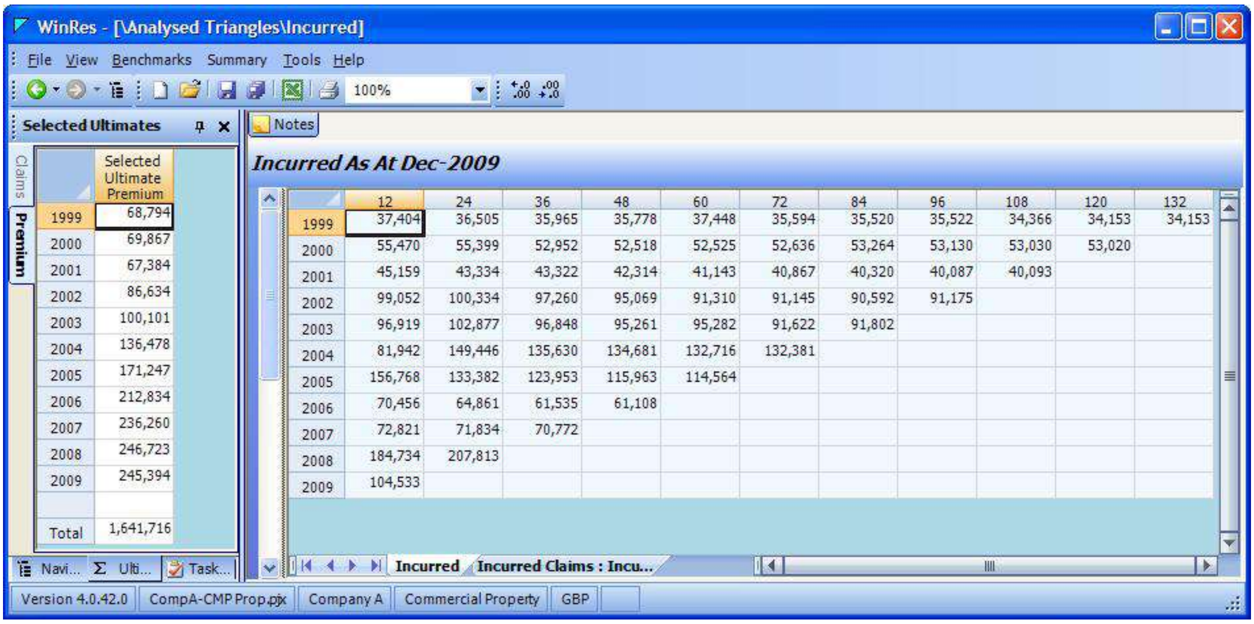
Appendix C. Data used

C.1 Data triangles and premiums

C.1.1 Commercial Property



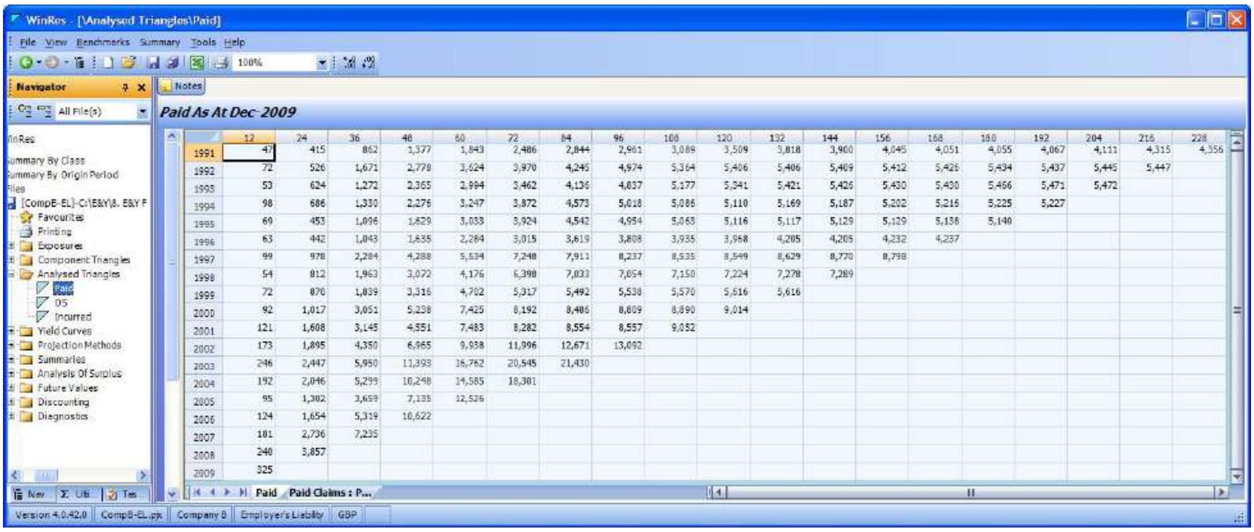
	12	24	36	48	60	72	84	96	108	120	132
1999	13,455	26,901	30,707	32,915	33,192	33,744	34,860	35,258	34,120	34,120	34,120
2000	15,518	40,631	48,421	49,645	50,783	51,033	52,950	53,035	53,016	53,007	
2001	14,187	28,282	36,465	40,164	40,205	40,463	40,204	40,067	40,067		
2002	22,436	65,539	83,234	86,409	87,083	88,741	89,624	89,941			
2003	27,082	66,310	80,200	84,642	87,713	88,924	89,141				
2004	30,953	90,372	125,048	128,791	131,374	131,642					
2005	43,717	97,916	110,359	113,667	113,629						
2006	24,002	48,676	56,601	58,110							
2007	23,173	54,557	64,417								
2008	49,628	140,563									
2009	30,911										



	Selected Ultimate Premium
1999	68,794
2000	69,867
2001	67,384
2002	86,634
2003	100,101
2004	136,478
2005	171,247
2006	212,834
2007	236,260
2008	246,723
2009	245,394
Total	1,641,716

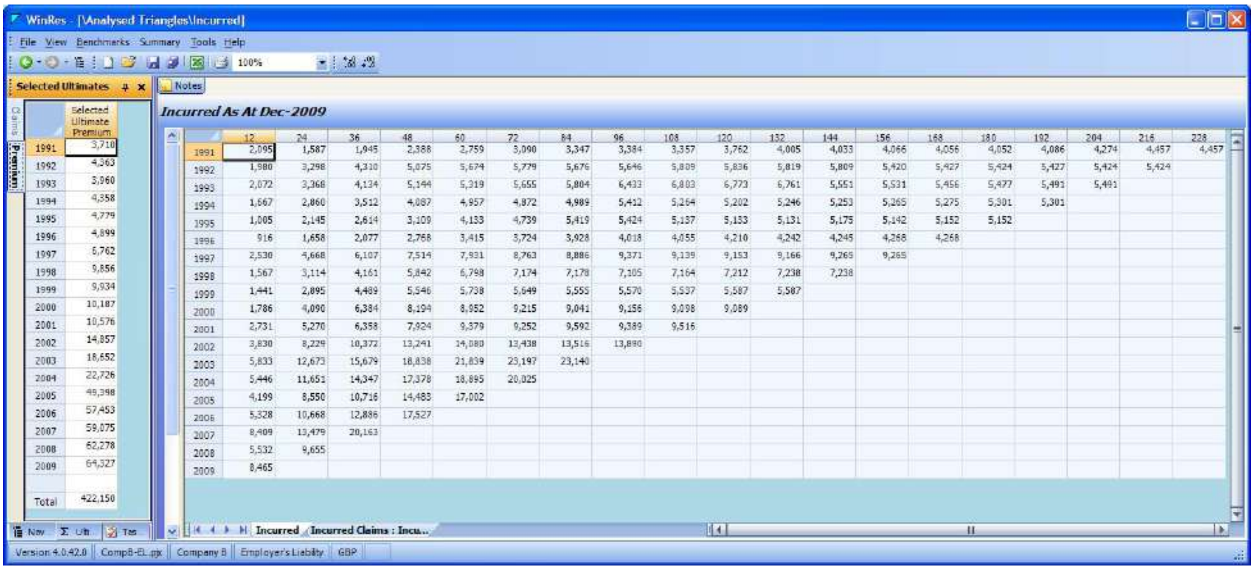
	12	24	36	48	60	72	84	96	108	120	132
1999	37,404	36,505	35,965	35,778	37,448	35,594	35,520	35,522	34,366	34,153	34,153
2000	55,470	55,399	52,952	52,518	52,525	52,636	53,264	53,130	53,030	53,020	
2001	45,159	43,334	43,322	42,314	41,143	40,867	40,320	40,087	40,093		
2002	99,052	100,334	97,260	95,069	91,310	91,145	90,592	91,175			
2003	96,919	102,877	96,848	95,261	95,282	91,622	91,802				
2004	81,942	149,446	135,630	134,681	132,716	132,381					
2005	156,768	133,382	123,953	115,963	114,564						
2006	70,456	64,861	61,535	61,108							
2007	72,821	71,834	70,772								
2008	184,734	207,813									
2009	104,533										

C.1.2 Employer's Liability



Paid As At Dec-2009

	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228
1991	47	415	862	1,377	1,843	2,486	2,844	2,961	3,089	3,509	3,818	3,900	4,045	4,051	4,055	4,067	4,111	4,315	4,356
1992	72	526	1,671	2,778	3,624	3,970	4,245	4,974	5,364	5,406	5,406	5,409	5,412	5,426	5,434	5,437	5,445	5,447	
1993	53	624	1,272	2,365	2,994	3,462	4,136	4,837	5,177	5,341	5,421	5,426	5,430	5,430	5,466	5,471	5,472		
1994	98	686	1,330	2,276	3,247	3,872	4,573	5,018	5,086	5,110	5,169	5,187	5,202	5,216	5,225	5,227			
1995	69	453	1,096	1,629	3,033	3,924	4,542	4,954	5,063	5,116	5,117	5,129	5,129	5,138	5,140				
1996	63	442	1,043	1,635	2,284	3,015	3,619	3,808	3,935	3,968	4,205	4,205	4,232	4,237					
1997	99	978	2,284	4,288	5,534	7,248	7,911	8,237	8,535	8,549	8,629	8,770	8,798						
1998	54	812	1,963	3,072	4,176	6,398	7,033	7,054	7,150	7,224	7,278	7,289							
1999	72	870	1,839	3,316	4,702	5,317	5,492	5,538	5,570	5,616	5,616								
2000	92	1,017	3,051	5,238	7,425	8,192	8,486	8,809	8,890	9,014									
2001	121	1,608	3,145	4,551	7,483	8,282	8,554	8,557	9,052										
2002	173	1,895	4,350	6,965	9,938	11,996	12,671	13,092											
2003	246	2,447	5,950	11,393	16,762	20,545	21,430												
2004	192	2,046	5,299	10,248	14,585	18,301													
2005	95	1,302	3,659	7,135	12,526														
2006	124	1,654	5,319	10,622															
2007	181	2,736	7,235																
2008	240	3,857																	
2009	325																		



Selected Ultimates

	Selected Ultimate Premium
1991	3,710
1992	4,363
1993	3,960
1994	4,358
1995	4,779
1996	4,899
1997	6,762
1998	9,856
1999	9,934
2000	10,187
2001	10,576
2002	14,857
2003	18,652
2004	22,726
2005	49,398
2006	57,453
2007	59,075
2008	62,278
2009	64,327
Total	422,150

Incurred As At Dec-2009

	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228
1991	2,095	1,587	1,945	2,388	2,759	3,090	3,347	3,384	3,357	3,762	4,005	4,033	4,066	4,056	4,052	4,086	4,274	4,457	4,457
1992	1,980	3,298	4,310	5,075	5,674	5,779	5,676	5,646	5,809	5,836	5,819	5,809	5,420	5,427	5,424	5,427	5,424	5,424	
1993	2,072	3,368	4,134	5,144	5,319	5,655	5,804	6,433	6,803	6,773	6,761	5,551	5,531	5,456	5,477	5,491	5,491		
1994	1,667	2,860	3,512	4,087	4,957	4,872	4,989	5,412	5,264	5,202	5,246	5,253	5,265	5,275	5,301	5,301			
1995	1,005	2,145	2,614	3,109	4,133	4,739	5,419	5,424	5,137	5,133	5,131	5,175	5,142	5,152	5,152				
1996	916	1,658	2,077	2,768	3,415	3,724	3,928	4,018	4,055	4,210	4,242	4,245	4,268	4,268					
1997	2,530	4,668	6,107	7,514	7,931	8,763	8,886	9,371	9,139	9,153	9,166	9,265	9,265						
1998	1,567	3,114	4,161	5,842	6,798	7,174	7,178	7,105	7,164	7,212	7,238	7,238							
1999	1,441	2,895	4,489	5,546	5,738	5,649	5,555	5,570	5,537	5,587	5,587								
2000	1,786	4,090	6,384	8,194	8,952	9,215	9,041	9,156	9,098	9,089									
2001	2,731	5,270	6,358	7,924	9,379	9,252	9,592	9,389	9,516										
2002	3,830	8,229	10,372	13,241	14,080	13,438	13,516	13,890											
2003	5,833	12,673	15,679	18,838	21,839	23,197	23,140												
2004	5,446	11,651	14,347	17,378	18,895	20,025													
2005	4,199	8,550	10,716	14,483	17,002														
2006	5,328	10,668	12,886	17,527															
2007	8,409	13,479	20,163																
2008	5,532	9,655																	
2009	8,465																		

C.2 Line of business description

C.2.2 Commercial Property

Commercial property insurance covers an organization or individuals' business premises (buildings) or property, against any loss or damage to property caused by theft, accident or some other means. Residential buildings, commercial buildings and moveable property are examples of properties that can be covered by the policy. The important characteristic of this class from a reserving point of view is that the run-off claims will be relatively brief. Thus, within about two years of the accident year end i.e. the year when the claim occurred, one would expect the great majority of the outstanding claims to have been settled. Hence, property insurance is classified as a short-tail class, where the delay between a claim occurring and it being settled is short.

C.2.2 Employer's Liability

Employers' liability (EL) insurance insures employers against the costs of compensation for those who are injured or made ill at work through the fault of their employer. It provides greater security to firms against costs which could otherwise result in financial difficulty and to employees that resources will be available for compensation even where firms have become insolvent. It supports the right of employees injured through their employer's negligence to be fairly compensated - the principle of 'access to justice'; and the responsibility of employers to fund the costs of their negligence - the principle of 'polluter pays'. It is compulsory for employers to have EL insurance.

EL is one of five main risk groups under liability insurance (alongside public liability, product liability and professional indemnity). Liability insurance is considered to be a long-tail class, where the delay between the occurrence of a claim and it being settled is long.

Appendix D. Bootstrap theory

In order to derive the characteristics of the distribution of the insurance reserves, essentially percentiles or Value at Risk (VaR), the underlying law of these reserves needs to be estimated. In this respect, Bootstrapping is a very popular approach in general insurance stochastic claims reserving, for its relative simplicity and flexibility.

The Bootstrap method introduced in Efron (1979) is a very general resampling procedure for estimating the distributions of statistics based on independent observations. It is used to estimate, in a consistent way, the variability of a parameter. This resampling method replaces theoretical deductions in statistical analysis by Monte-Carlo type simulations obtained by repeatedly resampling the "original" data and making inferences from the resamples.

Its use in actuarial sciences becomes widely spread, with pricing in particular and technical reserves estimations. This can be explained by the whole range of applications of the method: confidence intervals, hypothesis testing, generalized linear models, etc. The Bootstrap technique has been commonly used in the recent years to analyse claims reserves variability and obtain prediction errors for different claims-reserving methods, namely, the chain ladder technique and methods based on generalized linear models.

Actuarial literature around the Bootstrap technique is quite substantial (see for instance [18],[19] and [20]).

In this thesis, in order to obtain the projected simulated cash-flows I used the internal software of my company which performs Bootstrapping of the claims development model as formulated by T. Mack and D. Murphy ([16],[17]).

Below is a brief presentation of the methodology and of its practical implementation.

D.1 Method

The basis is to generate new samples by drawing random samples with replacement from a unique initial sample. At root is the idea that if the original sample is a good approximation of the population, the Bootstrap method will provide a good approximation of the sampling distribution of the population.
Applied to non-life claims, Bootstrapping is sampling with replacement from the observed normalized errors in the historical loss development data. The sampling produces new "realisations" of historical development data that have the same statistical characteristics as the actual data.

D.1.1 Chain Ladder Model: Formulation

For a claims triangle made up of aggregate claims, where i denotes origin period and j denotes development period, T. Mack's and D. Murphy's formulation gives:

$$\boxed{C_{i,j} = \lambda_j C_{i,j-1} + \sigma_j \sqrt{C_{i,j-1}}\varepsilon_{i,j}} \qquad \text{(D. 1)}$$

where:
– $C_{i,j}$ denotes the cumulated claims payments for the i^{th} origin period and at the j^{th} development period;
– λ_j denotes the development factor at the j^{th} development period;
– σ_j denotes the Mack Parameter which determines variability;
– the errors $\varepsilon_{i,j}$ are independent and identically distributed (iid) with mean 0 and variance 1;
The method for estimation of σ_j is described below.

Note that the C_t referred to in the analytical models under study in this thesis and defined as the cumulated payments at time t relate to the $C_{i,j}$ as follows:

$$C_t = \sum_{i+j=n+1+t} C_{i,j}$$

(the C_t are the sum across all accident periods and for the calendar year t, i.e the t^{th} diagonal of the completed triangle, where n is the number of accident years history (and incidentally also the number of projection years until run-off))

D.1.2 Chain Ladder Model: Justification of Volume-Weighted Chain Ladder Estimates

The values of the development factors λ_j in Equation (D. 1) are given by the least-squares solution, i.e. by minimizing the expression

$$\sum_i \left\{ \frac{\left(C_{i,j} - \lambda_j C_{i,j-1}\right)^2}{C_{i,j-1}} \right\}$$

consistent with the normal model. Differentiating this with respect to λ_j, we obtain

$$\frac{d}{d\lambda_j} \sum_i \left\{ \frac{\left(C_{i,j} - \lambda_j C_{i,j-1}\right)^2}{C_{i,j-1}} \right\} = \frac{d}{d\lambda_j} \sum_i \left\{ \frac{{C_{i,j}}^2}{C_{i,j-1}} - 2\lambda_j C_{i,j} + {\lambda_j}^2 C_{i,j-1} \right\}$$

Setting the resulting expression equal to zero,

$$-2 \sum_i C_{i,j} + 2\hat{\lambda}_j \sum_i C_{i,j-1} = 0$$

Solving for the estimator $\hat{\lambda}_j$, we obtain the least-squares estimate:

$$\hat{\lambda}_j = \frac{\sum_i C_{i,j}}{\sum_i C_{i,j-1}} \tag{D. 2}$$

The Mack Parameters σ_j are estimated by the expression

$$\hat{\sigma}_j^2 = \frac{1}{n-1} \sum_i \frac{\left(C_{i,j} - \hat{\lambda}_j C_{i,j-1}\right)^2}{C_{i,j-1}} \tag{D. 3}$$

giving the unbiased estimator for σ_j.

D.1.3 Chain Ladder Model: Parameter Error Estimation

The parameter error in the development factors λ_j can be quantified through the use of "bootstrapping" techniques. The approach used is as follows.
A triangle of residuals is constructed according to the formula

$$\varepsilon_{i,j} = \frac{C_{i,j} - \hat{\lambda}_j C_{i,j-1}}{\hat{\sigma}_j \sqrt{C_{i,j-1}}} \tag{D. 4}$$

which is a rearrangement of (D. 1).
This triangle of residuals can be seen as a sample from the true distribution of the parameter error term. By resampling this triangle with replacement, i.e. allowing individual values from the original triangle to appear more than once in the new sample triangle, a new "sample" from the same distribution can be

obtained. This process gives the bootstrap residuals ε^*, which can be used to calculate pseudo data as below:

$$\boldsymbol{C^*_{i,j} = \hat{\lambda}_j C_{i,j-1} + \hat{\sigma}_j \sqrt{C_{i,j-1}} \varepsilon^*_{i,j}} \tag{D. 5}$$

The pseudo data triangle thus obtained is a simulation of another claims triangle sampled from the same stochastic process. Pseudo development factors λ^*_j can then be computed using Equation (D. 2), with the numerator replaced by the sum of pseudo claims and the denominator by the relevant sum of actual claims:

$$\boldsymbol{\lambda^*_j = \frac{\sum_i C^*_{i,j}}{\sum_i C_{i,j-1}}} \tag{D. 6}$$

As λ^*_j is a known function of $C^*_{i,j}$, the pseudo development factors can be thought of as a sample from the true distribution of development factors. The expected value of λ^*_j can be shown to be the original estimator for λ_j, as follows:

$E\left[\lambda^*_j | C_{i,j-1}\right] = \frac{E\left[\sum_i C^*_{i,j} | C_{i,j-1}\right]}{\sum_i C_{i,j-1}} = \frac{E\left[\sum_i\left(\hat{\lambda}_j C_{i,j-1} + \hat{\sigma}_j \sqrt{C_{i,j-1}} \varepsilon^*_{i,j}\right) | C_{i,j-1}\right]}{\sum_i C_{i,j-1}}$ from Equation (D. 5)

$$E\left[\lambda^*_j | C_{i,j-1}\right] = \frac{\sum_i\left(E\left[\hat{\lambda}_j C_{i,j-1} + \hat{\sigma}_j \sqrt{C_{i,j-1}} \varepsilon^*_{i,j} | C_{i,j-1}\right]\right)}{\sum_i C_{i,j-1}} = \frac{\sum_i\left(E\left[\hat{\lambda}_j C_{i,j-1} | C_{i,j-1}\right] + E\left[\hat{\sigma}_j \sqrt{C_{i,j-1}} \varepsilon^*_{i,j} | C_{i,j-1}\right]\right)}{\sum_i C_{i,j-1}}$$

$$= \frac{\sum_i\left(E\left[\hat{\lambda}_j C_{i,j-1} | C_{i,j-1}\right] + E\left[\hat{\sigma}_j \sqrt{C_{i,j-1}} | C_{i,j-1}\right] E\left[\varepsilon^*_{i,j} | C_{i,j-1}\right]\right)}{\sum_i C_{i,j-1}} = \frac{\sum_i\left(E\left[\hat{\lambda}_j C_{i,j-1} | C_{i,j-1}\right]\right)}{\sum_i C_{i,j-1}}$$

as $\varepsilon^*_{i,j}$ are iid and $E\left[\varepsilon^*_{i,j}\right] = 0$

$\Rightarrow E\left[\lambda^*_j | C_{i,j-1}\right] = \frac{\hat{\lambda}_j \sum_i C_{i,j-1}}{\sum_i C_{i,j-1}} = \hat{\lambda}_j$

The values of λ^*_j obtained using this method can be used to approximate a sample of possible grossing up factors (GUF) G^*_j, where:

$$\boldsymbol{G_j = \prod_{k=j+1}^{N} \lambda_k} \tag{D. 7}$$

is the factor required to take the aggregate claims $C_{i,j}$ at development period j to ultimate. N is the number of development periods to ultimate. These GUFs can then be used to infer a sample of ultimate claims for origin period i, U^*_i.

D.1.4 Bornhuetter-Ferguson Model: Formulation

For given cumulative paid claims $C_{i,j}$, ultimate claims under the Bornhuetter-Ferguson (BF) method are given by

$$\boldsymbol{U_i = C_{i,j} + I_i P_i \left(1 - \frac{1}{G_j}\right)} \tag{D. 8}$$

where:

– U_i denotes ultimate claims for origin period i
– I_i denotes the initial expected loss ratio (IELR) for origin period i
– P_i denotes the ultimate premiums for origin period i
– G_j denotes the GUF from development period j to ultimate, derived from the chain ladder model.

In the work that follows, a normal distribution is assumed for the IELR I_i.

D.1.5 Bornhuetter-Ferguson Model: Estimation

The model assumes that the selected IELR for the latest origin period, I_c, is the average of historical ultimate loss ratios (ULRs) adjusted for premium rate changes. Thus, by taking an average of the selected

ULRs over all previous origin periods in the triangle, adjusting each to the current year's premium rates, we can calculate an expected IELR, I'_c:

$$\boldsymbol{I'_c = \frac{\sum_i (I_i r_i)}{n}} \tag{D. 9}$$

where:

– r_i denotes the premium rate index for origin period i, adjusted so that $r_i = 1$ for the latest origin period
– n denotes the number of origin periods over which the ULRs are averaged.

From the selected future development profile and the selected ultimate premiums and claims in the Chain Ladder Model, we also have an implied IELR from Equation (D. 8), $I"_c$:

$$\boldsymbol{I"_c = \frac{\dot{U}_c - C_{c1}}{\dot{P}_c\left(1 - \dot{G}_1^{-1}\right)}} \tag{D. 10}$$

where:

– the subscript c indicates values for the current origin period
– $\dot{U}_c$ denotes the selected ultimate claims for the current origin period
– $\dot{P}_c$ denotes the selected ultimate premiums for the current origin period
– $\dot{G}_1$ denotes the GUF from development period 1 to ultimate, derived from the selected development profile.

However, in practice, $I'_c \neq I"_c$. The model assumes that the reason for this difference is that the premium rates used to adjust historical ULRs are slightly inaccurate. The premium rates are therefore adjusted by the ratio of the implied to expected IELRs, so that $I'^{new}_c = I"_c = \mu_I$, i.e. the implied IELRs are equal to the expected. This gives new premium rates r'_i, where:

$$\boldsymbol{r'_i = \frac{I"_c}{I'_c} r_i} \tag{D. 11}$$

D.1.6 Bornhuetter-Ferguson Model: Parameter Error Estimation

From the work in the previous section, we have a mean IELR calculated from historical ULRs adjusted by the new premium rate indices r'_i. However, as this mean is based on a finite number of observations, i.e. the ULRs for n origin periods, it is subject to parameter error. In order to quantify this error, we assume that I_c is normally distributed with mean μ_I and a standard deviation calculated from the rate-adjusted ULRs as below:

$$\boldsymbol{s = \sqrt{\frac{\sum_i (I_i r'_i)^2}{n-1} - \mu_I^2 \frac{n}{n-1}}} \tag{D. 12}$$

IELRs for the latest origin period can therefore be simulated from the following normal distribution:

$$\boldsymbol{I_c^* \sim N\left(\mu_I, \frac{s^2}{n}\right)}$$

IELRs for each origin period where the BF method is being employed can then be calculated from these simulated IELRs by adjusting by the rate indices r'_i:

$$\boldsymbol{I_i^* = \frac{I_c^*}{r'_i}} \tag{D. 13}$$

Ultimate claims U_i^* can then be calculated for these origin periods using Equation (D. 8).

D.1.7 Process Error

D.1.7.1 Chain Ladder Model

At each stage of the Chain Ladder model, we simulate $C_{i,j}$ given the simulated value for $C_{i,j-1}$:

$$\boldsymbol{C_{i,j} | C_{i,j-1} \sim \Gamma\left(\left(\frac{\lambda^*}{\sigma_j}\right)^2 C_{i,j-1}, \frac{\lambda^*}{\sigma_j{}^2}\right)}$$

The mean of this gamma distribution gives the mean value of $C_{i,j}|C_{i,j-1} = \lambda^* C_{i,j-1}$, and the variance of the distribution gives the appropriate variance, $\sigma_j{}^2 C_{i,j-1}$.
Using the law of total variance (Equation (D. 14)), we can combine the process variance occurring at each stage in the chain ladder model to calculate the total variance at the ultimate level.

$$\boldsymbol{Var\left(C_{i,j}\right) = Var\left(E\left[C_{i,j}|C_{i,j-1}\right]\right) + E\left(Var\left[C_{i,j}|C_{i,j-1}\right]\right)} \tag{D. 14}$$

Substituting in the mean and variance of the gamma distribution above:

$$\boldsymbol{Var\left(C_{i,j}\right) = Var\left(\lambda_j^* C_{i,j-1}\right) + E\left(\sigma_j{}^2 C_{i,j-1}\right) = \lambda_j^{*2} Var\left(C_{i,j-1}\right) + \sigma_j{}^2 E\left(C_{i,j-1}\right)} \tag{D. 15}$$

Equation (D. 15) gives the iterative relationship between the variance of $C_{i,j}$ and the known expectation and variance of the previous term in the chain ladder, $C_{i,j-1}$.

D.1.7.2 Bornhuetter-Ferguson Model

Taking the simulated mean ultimate claims, U_i^*, as described above, we divide by the bootstrap GUFs G_j^* to give a pseudo leading diagonal for the triangle, $C_{i,N-i+1}^*$. The mean value of ultimate claims M_i given this leading diagonal and the appropriate GUF is then:

$$\boldsymbol{E\left(C_{i,N}^*|C_{i,N-i+1}^*, G_{N-i+1}^*\right) = C_{i,N-i+1}^* G_{N-i+1}^* = U_i^* = M_i} \tag{D. 16}$$

where N is the number of development periods to ultimate.
The variance at ultimate, V_i, can be derived using Equation (D. 15), as described in the section above.
Values for ultimate claims $C_{i,N}^*$ are then simulated from a lognormal distribution based on this mean and variance:

$$\boldsymbol{C_{i,N}^* \sim logN\left(\mu_{iN}, \sigma_{iN}^2\right)}$$

where $\sigma_{iN}^2 = ln\left(1 + \frac{V_i}{M_i^2}\right), \mu_{iN} = ln(M_i) - \frac{1}{2}\sigma_{iN}^2$

Using ultimate claims values simulated from this distribution, we then chain back from the ultimate to the first development period, calculating $C_{i,j-1}^*|C_{i,j}^*$ by dividing by the appropriate development factors.

D.2 Implementation

Model to use: key in the process is the choice of the model underlying the fitting and reserves determination. As mentioned above, the software can handle the Chain-Ladder as well as Bornhuetter-Ferguson models on selected accident years.

Steps (see graph below):

1. Start with the triangle of observed cumulative loss payments $C_{i,j}$ of claims arising from accident year i and development year j and let us assume that we are in year n and that we know all the past information, i.e., $C_{i,j}$ with $i \in [\![1, \dots, n]\!]$, $j \in [\![1, \dots, n-i+1]\!]$.

2. Estimate the Mack model parameters, $\hat{\lambda}_j = \frac{\sum_i C_{i,j}}{\sum_i C_{i,j-1}}$ and $\hat{\sigma}_j^2 = \frac{1}{n-1}\sum_i \frac{\left(C_{i,j} - \hat{\lambda}_j C_{i,j-1}\right)^2}{C_{i,j-1}}$

3. Compute the residuals $\varepsilon_{i,j} = \frac{C_{i,j} - \hat{\lambda}_j C_{i,j-1}}{\hat{\sigma}_j \sqrt{C_{i,j-1}}}$ to bootstrap

   The Bootstrap technique must be adapted to each situation. For the linear model ("classical" or generalized) it is common to adopt one of two possible ways:

- paired Bootstrap: the resampling is done directly from the observations (difference between the observed and the expected values); and
- residuals Bootstrap: the resampling is applied to the residuals of the model.

Despite the fact that the paired bootstrap is more robust than the residual bootstrap, only the latter could be implemented in the context of the claim reserving, given the dependence between some observations and the parameter estimates.
Next, to define the most adequate residuals for the Bootstrap, it is important to remember two points:
the resampling is based on the hypothesis that the residuals are independent and identically distributed (iid); and (ii) it is indifferent to resample the residuals or the residuals multiplied by a constant, as long as we take that fact into account in the generation of the pseudo data.

Different types of residuals can be chosen. With an expected value of zero and a constant variance, the $\varepsilon_{i,j}$ of the Mack model are considered to be adequate residuals for the Bootstrap (the variance of the paired Bootstrap is not constant, making them less eligible candidates for the resampling that requires iid).

Resample the $\varepsilon_{i,j}$ residuals m times to derive m new triangles of residuals $\varepsilon^*_{i,j}$
With the initial triangle of residuals as a starting point, we sample m times with replacement. Each new triangle is a permutation of the original triangle.

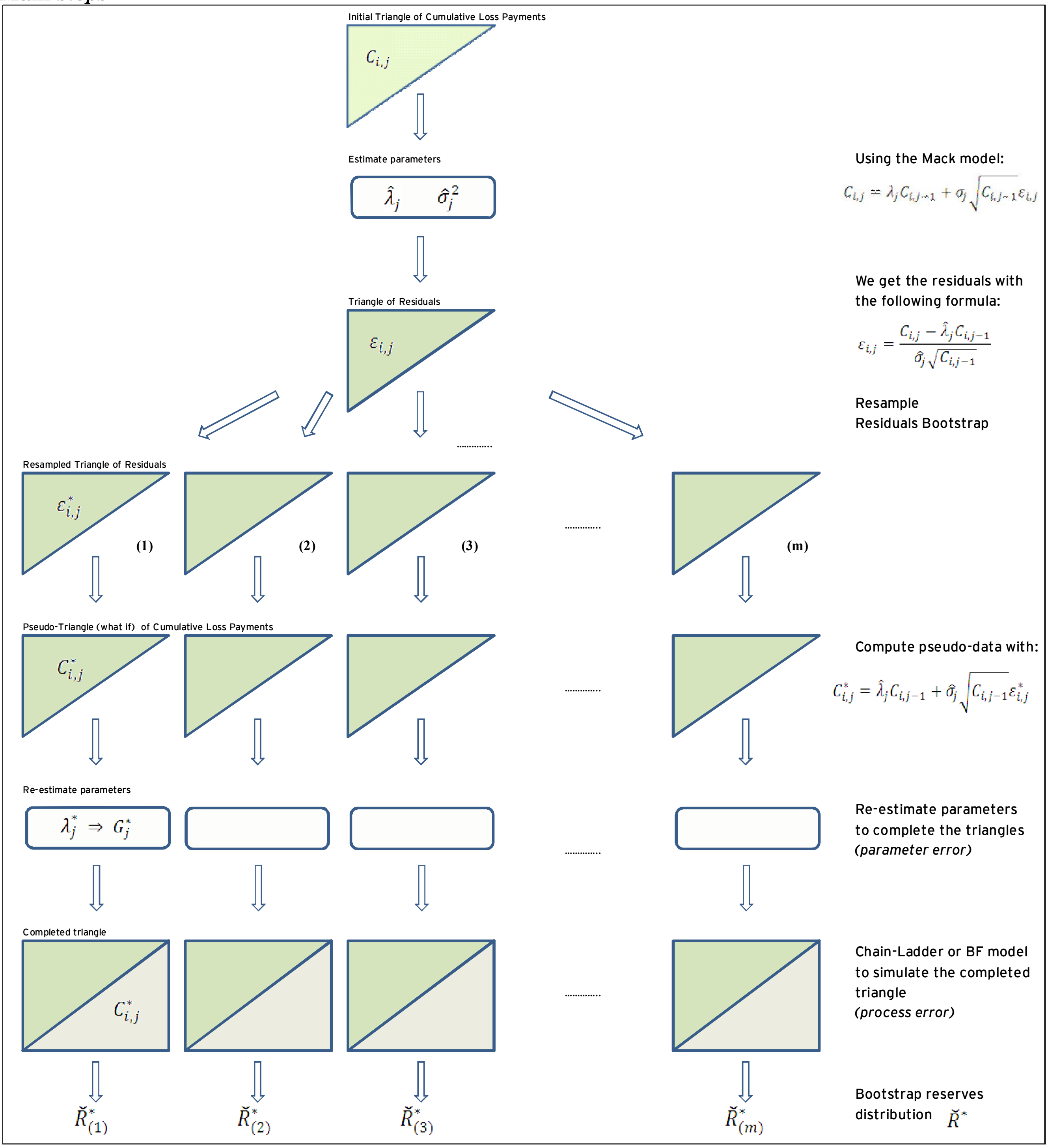
4. Construct the "Pseudo-Triangle" (what if) of Cumulative Loss Payments $C^*_{i,j}$, through the Mack model, using the parameters estimated in step 2: $C^*_{i,j} = \hat{\lambda}_j C_{i,j-1} + \hat{\sigma}_j \sqrt{C_{i,j-1}} \varepsilon^*_{i,j}$

5. Re-estimate the parameters, based on the "Pseudo-Triangles" data $\lambda^*_j = \frac{\sum_i C^*_{i,j}}{\sum_i C_{i,j-1}}$ for the Chain-Ladder model, and $I^*_i = \frac{I^*_c}{r'_i}$ for the Bornhuetter-Ferguson model.

6. Finally, “re-reserve” each simulated triangle, giving us the reserves distribution $\check{R}$, using Chain-Ladder or Bornhuetter-Ferguson.

The graph below summarizes the different implementation steps:

Main steps


Initial Triangle of Cumulative Loss Payments
$C_{i,j}$
Estimate parameters
$\hat{\lambda}_j \quad \hat{\sigma}_j^2$
Using the Mack model:
$C_{i,j} = \lambda_j C_{i,j-1} + \sigma_j \sqrt{C_{i,j-1}}\varepsilon_{i,j}$
We get the residuals with the following formula:
$\varepsilon_{i,j} = \dfrac{C_{i,j} - \hat{\lambda}_j C_{i,j-1}}{\hat{\sigma}_j \sqrt{C_{i,j-1}}}$
Triangle of Residuals
$\varepsilon_{i,j}$
Resample Residuals Bootstrap
Resampled Triangle of Residuals
$\varepsilon^*_{i,j}$
(1)
(2)
(3)
(m)
Pseudo-Triangle (what if) of Cumulative Loss Payments
$C^*_{i,j}$
Compute pseudo-data with:
$C^*_{i,j} = \hat{\lambda}_j C_{i,j-1} + \hat{\sigma}_j \sqrt{C_{i,j-1}}\varepsilon^*_{i,j}$
Re-estimate parameters
$\lambda^*_j \Rightarrow G^*_j$
Re-estimate parameters to complete the triangles *(parameter error)*
Completed triangle
$C^*_{i,j}$
Chain-Ladder or BF model to simulate the completed triangle *(process error)*
$\breve{R}^*_{(1)}$
$\breve{R}^*_{(2)}$
$\breve{R}^*_{(3)}$
$\breve{R}^*_{(m)}$
Bootstrap reserves distribution $\breve{R}^*$

Appendix E. Goodness of fit (GoF) tests theory

A lot of literature can be found on the subject.
This Appendix gives the background of the goodness of fit tests, and is extensively indebted to the websites referred below[40]. More details could be found on W*ikipedia, the free encyclopedia*[41], for instance.

E.1 General

The goodness of fit of a statistical model describes how well it fits a set of observations. Measures of goodness of fit typically summarize the discrepancy between observed values and the values expected under the model in question. Such measures can be used in statistical hypothesis testing, e.g. to test for normality of residuals, to test whether two samples are drawn from identical distributions (see Kolmogorov–Smirnov test), or whether outcome frequencies follow a specified distribution (see Pearson's chi-square test). In the analysis of variance, one of the components into which the variance is partitioned may be a lack-of-fit sum of squares.

E.2 Chi-square

The Chi-Squared test is used to determine if a sample comes from a population with a specific distribution.
This test is applied to binned data, so the value of the test statistic depends on how the data is binned.
Although there is no optimal choice for the number of bins (k), there are several formulas which can be used to calculate this number based on the sample size (N). For example, the following empirical formula can be employed: $k = 1 + \ln(N)$, which is what has been employed.
The data can be grouped into intervals of equal probability or equal width. The first approach is generally more acceptable since it handles peaked data much better. Each bin should contain at least 5 or more data points, so certain adjacent bins sometimes need to be joined together for this condition to be satisfied.

Definition
The Chi-Squared statistic is defined as

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

where O_i is the observed frequency for bin i, and E_i is the expected frequency for bin i calculated by $E_i = F(x_2) - F(x_1)$, where F is the CDF of the probability distribution being tested, and x_1, x_2 are the limits for bin i.

H_0: The data follow the specified distribution.
H_1: The data do not follow the specified distribution.

The hypothesis regarding the distributional form is rejected at the chosen significance level (α) if the test statistic is greater than the critical value defined as ${\chi^2}_{1-\alpha,k-1}$ – meaning the Chi-Squared inverse CDF with $k - 1$ degrees of freedom and a significance level of α.

Characteristics and Limitations
An attractive feature of the chi-square goodness-of-fit test is that it can be applied to any univariate distribution for which the cumulative distribution function can be calculated. The chi-square goodness-of-fit test is applied to binned data (i.e., data put into classes), with the limitation that the value of the chi-square test statistic is dependent on how the data is binned. Another disadvantage of the chi-square test is that it requires a sufficient sample size in order for the chi-square approximation to be valid. This, however, is not an issue in our case.

[40] http://www.mathwave.com/articles/goodness_of_fit.html and http://www.itl.nist.gov/div898/handbook/index.htm
[41] http://en.wikipedia.org/wiki/Goodness_of_fit (as at 29th July 2010)

E.3 Kolmogorov–Smirnov test

This test is used to decide if a sample comes from a hypothesized continuous distribution. It is based on the empirical cumulative distribution function (ECDF). Assume that we have a random sample $x_1, x_2, \ldots, x_n$ from some continuous distribution with CDF $F(x)$. The empirical CDF is denoted by:

$$F_n(x) = \frac{1}{n}[number\ of\ observations\ \leq x]$$

Definition

The Kolmogorov-Smirnov statistic (K) is based on the largest vertical difference between $F(x)$ and $F_n(x)$. It is defined as

$$K_n = \sup_x |F_n(x) - F(x)|$$

H_0: The data follow the specified distribution.
H_1: The data do not follow the specified distribution.

The hypothesis regarding the distributional form is rejected at the chosen significance level (α) if the test statistic K, is greater than the critical value obtained from a table.

Characteristics and Limitations

An attractive feature of this test is that the distribution of the K-S test statistic itself does not depend on the underlying cumulative distribution function being tested. Another advantage is that it is an exact test (the chi-square goodness-of-fit test depends on an adequate sample size for the approximations to be valid). Despite these advantages, the K-S test has several important limitations:

– It only applies to continuous distributions.
– It tends to be more sensitive near the centre of the distribution than at the tails.
– Perhaps the most serious limitation is that the distribution must be fully specified. That is, if location, scale, and shape parameters are estimated from the data, the critical region of the K-S test is no longer valid. It typically must be determined by simulation.

E.4 QQ-plots

In statistics, a QQ-plot ("Q" stands for quantile) is a probability plot, which is a graphical method for comparing two probability distributions by plotting their percentiles against each other. If the two distributions being compared are similar, the points in the QQ-plot will approximately lie on the line y = x. If the distributions are linearly related, the points in the QQ-plot will approximately lie on a line, but not necessarily on the line y = x. QQ-plots can also be used as a graphical means of estimating parameters in a location-scale family of distributions.

The main step in constructing a QQ-plot is calculating or estimating the percentiles to be plotted. If one or both of the axes in a QQ-plot is based on a theoretical distribution with a continuous cumulative distribution function (CDF), all percentiles are uniquely defined and can be obtained by inverting the CDF. If a theoretical probability distribution with a discontinuous CDF is one of the two distributions being compared, some of the percentiles may not be defined, so an interpolated percentile may be plotted. If the QQ-plot is based on data, there are multiple percentile estimators in use. Rules for forming QQ-plots when percentiles must be estimated or interpolated are called plotting positions.
More abstractly, given two cumulative probability distribution functions F and G, with associated quantile functions F^{-1} and G^{-1} (the inverse function of the CDF is the percentile function), the QQ-plot draws the q^{th} percentile of F against the q^{th} percentile of G for a range of values of q. Thus, the QQ-plot is a parametric curve indexed over [0,1] with values in the real plane R^2.

Appendix F. References to the European Directive

F.1 Level 1 text

Extracts from the European Directive (cf.[4]). The Level 1 text was adopted (as a Law) by the European Parliament on 22 April 2009.

Article 76
General provisions

1. Member States shall ensure that insurance and reinsurance undertakings establish technical provisions with respect to all of their insurance and reinsurance obligations towards policy holders and beneficiaries of insurance or reinsurance contracts.
2. The value of technical provisions shall correspond to the current amount insurance and reinsurance undertakings would have to pay if they were to transfer their insurance and reinsurance obligations immediately to another insurance or reinsurance undertaking.
3. The calculation of technical provisions shall make use of and be consistent with information provided by the financial markets and generally available data on underwriting risks (market consistency).
4. Technical provisions shall be calculated in a prudent, reliable and objective manner.
5. Following the principles set out in paragraphs 2, 3 and 4 and taking into account the principles set out in Article 75(1), the calculation of technical provisions shall be carried out in accordance with Articles 77 to 82 and 86.

Article 77
Calculation of technical provisions

1. The value of technical provisions shall be equal to the sum of a best estimate and a risk margin as set out in paragraphs 2 and 3.

2. The best estimate shall correspond to the probability-weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure.

   The calculation of the best estimate shall be based upon up-to-date and credible information and realistic assumptions and be performed using adequate, applicable and relevant actuarial and statistical methods.

   The cash-flow projection used in the calculation of the best estimate shall take account of all the cash in- and out-flows required to settle the insurance and reinsurance obligations over the lifetime thereof.

   The best estimate shall be calculated gross, without deduction of the amounts recoverable from reinsurance contracts and special purpose vehicles. Those amounts shall be calculated separately, in accordance with Article 81.

3. The risk margin shall be such as to ensure that the value of the technical provisions is equivalent to the amount that insurance and reinsurance undertakings would be expected to require in order to take over and meet the insurance and reinsurance obligations.

4. Insurance and reinsurance undertakings shall value the best estimate and the risk margin separately.

However, where future cash flows associated with insurance or reinsurance obligations can be replicated reliably using financial instruments for which a reliable market value is observable, the value of technical provisions associated with those future cash flows shall be determined on the basis of the market value of those financial instruments. In this case, separate calculations of the best estimate and the risk margin shall not be required.

5. Where insurance and reinsurance undertakings value the best estimate and the risk margin separately, the risk margin shall be calculated by determining the cost of providing an amount of eligible own funds equal to the Solvency Capital Requirement necessary to support the insurance and reinsurance obligations over the lifetime thereof.

   The rate used in the determination of the cost of providing that amount of eligible own funds (Cost-of-Capital rate) shall be the same for all insurance and reinsurance undertakings and shall be reviewed periodically.

   The Cost-of-Capital rate used shall be equal to the additional rate, above the relevant risk-free interest rate, that an insurance or reinsurance undertaking would incur holding an amount of eligible own funds, as set out in Section 3, equal to the Solvency Capital Requirement necessary to support insurance and reinsurance obligations over the lifetime of those obligations.

Article 86
Implementing measures

The Commission shall adopt implementing measures laying down the following:

(a) actuarial and statistical methodologies to calculate the best estimate referred to in Article 77(2);

(b) the relevant risk-free interest rate term structure to be used to calculate the best estimate referred to in Article 77(2);

(c) the circumstances in which technical provisions shall be calculated as a whole, or as a sum of a best estimate and a risk margin, and the methods to be used in the case where technical provisions are calculated as a whole;

(d) the methods and assumptions to be used in the calculation of the risk margin including the determination of the amount of eligible own funds necessary to support the insurance and reinsurance obligations and the calibration of the Cost-of-Capital rate;

(e) the lines of business on the basis of which insurance and reinsurance obligations are to be segmented in order to calculate technical provisions;

(f) the standards to be met with respect to ensuring the appropriateness, completeness and accuracy of the data used in the calculation of technical provisions, and the specific circumstances in which it would be appropriate to use approximations, including case-by-case approaches, to calculate the best estimate;

(g) the methodologies to be used when calculating the counterparty default adjustment referred to in Article 81 designed to capture expected losses due to default of the counterparty;

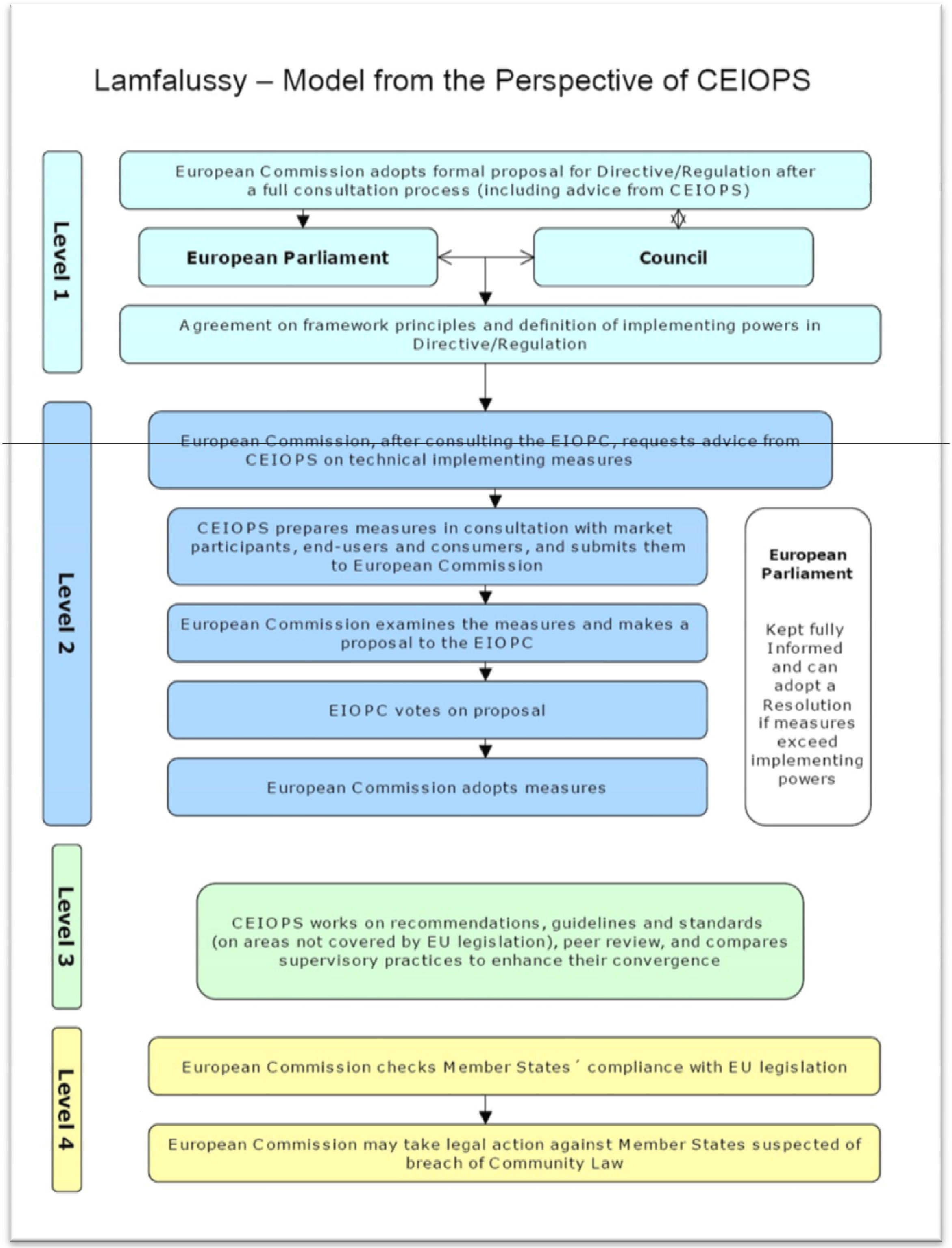
(h) where necessary, simplified methods and techniques to calculate technical provisions, in order to ensure the actuarial and statistical methods referred to in points (a) and (d) are proportionate to the nature, scale and complexity of the risks supported by insurance and reinsurance undertakings including captive insurance and reinsurance undertakings.

Those measures, designed to amend non-essential elements of this Directive by supplementing it, shall be adopted in accordance with the regulatory procedure with scrutiny referred to in Article 301(3).

F.2 Lamfalussy four-level process

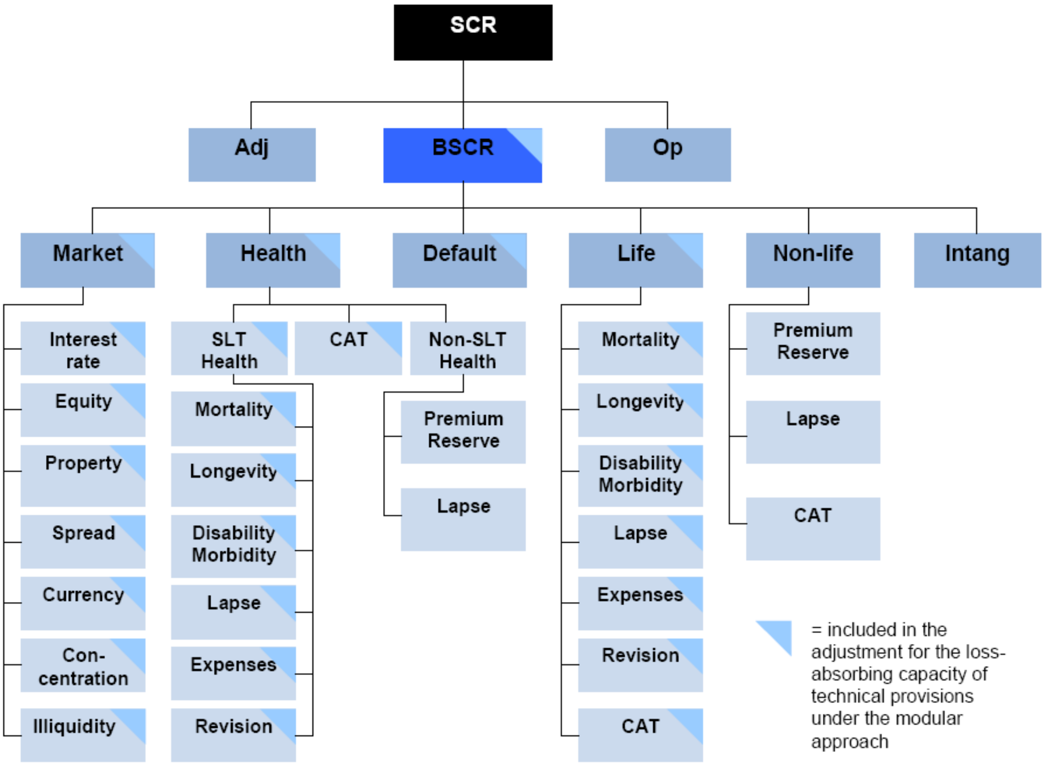
The Lamfalussy Process is an approach to the development of financial service industry regulations used by the European Union. It is composed of four "levels," each focusing on a specific stage of the implementation of legislation.

Source: https://www.ceiops.eu/en/find/index.html?tx_indexedsearch%5Bsword%5D=lamfalussy



F.3 SCR – Standard Formula[42]

The overall structure of the Standard Formula SCR is depicted below:



[42] QIS5 Technical Specification (cf.[3]) (SCR.1.1)

Appendix G. Abbreviations and notations

MVM	Market Value Margin
MVA	Market Value of Assets
MVL	Market consistent Value of Liabilities
BEL	Best Estimate of the Liability
SCR	Solvency Capital Requirement
MCoC	Risk Margin under the Market Cost of Capital approach
CEIOPS	Committee of European Insurance and Occupational Pensions Supervisors
CRO Forum	Chief Risk Officers Forum
CEA	Comité Européen des Assurances
QIS	Quantitative Impact Studies
CP	Consultation Paper
SST	Swiss Solvency Test
Pdf	Probability Density Function
Cdf	Cumulative Density Function
CProp	Commercial Property
EL	Employer's Liabilities
UPR	Unearned Premium Reserves
SST	Swiss Solvency Test
USP	Undertaking Specific Parameters
Main variables	
$c = CoC$	Cost of Capital
n	Time horizon
N	Number of simulations
C_t	Undiscounted cumulative claims payments as at time t
ϕ	$\phi = \Phi^{-1}[99.5\%] = 2.576$, where $\Phi^{-1}(p)$ denotes the quantile function of the standard normal distribution of order p, i.e. $\Phi^{-1}(p) = x$ such that $\Phi(x) = p$.

Chapter 7 References

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