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The impact of the underlying interest rate process
- When calculating the best estimate of liabilities

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Abstract

Solvency II requires a stochastic valuation for most products with guarantees. This is done in order to determine the time value of options and guarantees (TVOG), which is a part of the best estimate of liabilities. One way to determine the TVOG is by projecting a large number of economic scenarios in a financial projection model. This paper aims to explain and examine the impact of using the following underlying interest rate processes: Hull and White (HW); Cox–Ingersoll–Ross (CIR); and Libor Market Model (LMM) when generating the economic scenarios used for the valuation. These three processes are all used in the insurance industry and fulfill the market consistency and risk neutrality required by EIOPA under Solvency II; while HW allows for negative interest rates; the CIR and LMM does not. The differences in their distributions yield different results when the TVOG is determined; which indicates the importance of using the appropriate model as well as understanding it. The TVOG is significantly different for products with a guaranteed rate of 0 % due to the allowance of negative rates in HW. This master thesis is limited to a simple guarantee product and a sample of 80 model points, applying different guaranteed rates to examine how the distributions of the generated scenario impacts the outcome.

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Contents

1 Introduction

The upcoming European regulation Solvency II, as described in full detail in EIOPA's technical documentation (2013), increases the overall effort insurance companies' needs to put into their regulatory reporting; in many areas this will also force companies to perform calculations which haven't be required before. This master thesis will focus on one of these new "areas"; the calculations of the time value of options and guarantees (TVOG). The purpose of the TVOG is to reflect the value of the uncertainty of the obligations the insurance company has taken by promising future guaranteed amounts to their policyholders; guarantees which create an option value due to the asymmetry in the contracts. A typical example is an insurance contract with profit participation. If the insurance company earns an investment return which is in excess of the guaranteed amount, the policyholder will get a discretionary amount in addition to the guaranteed. However, if the investment return is insufficient to cover the guaranteed amount; the insurance company has to cover the loss. This creates the asymmetry which creates the option value, Hull (2006) describes option values in great detail in his book *Options, Futures, and Other Derivatives*.

Frasca and LaSorella (2009) describe one of the most common ways to determine the TVOG in their article *Embedded Value: Practice and Theory*. It is done by projecting a large number of economic scenarios in a financial projection model. This is done by calculating the best estimate of liabilities for each of the economic scenarios and then taking the average of it, this is followed by calculating the best estimate of liabilities under the deterministic certain equivalent scenario, also known as the "central scenario". By subtracting the best estimate of liabilities under this certain equivalent scenario from the averaged amount of the stochastic results, one would get the TVOG.

Many articles, an example is Li and Zhao (2006), have been written discussing the difficulties of replicating market prices; complex models have been introduced to capture the behavior of the market. The complexity of the models increases the overall effort required for model calibrations, so a more complex model might not always be preferable.

There exists a very limited amount of articles (no articles have been found during the duration of this work) which examines the prospective impact of the underlying interest rates models for life insurance contracts, i.e. the valuation of the TVOG. This work will focus on this area and give an answer the question of if the choice of interest rate model impacts the valuation of the best estimate of liability. The work is structured with six district sections as follow; an overview of the properties of the different interest rate models, a short description of the market data used in the calibrations, an analysis of the resulting scenario files once they have been generated, a descriptions of the guaranteed product which has been modelled, the results of the valuation and finally the conclusions of the findings.

2 Interest rate models

There are several different providers of economics scenario generators (ESG) in Europe as analyzed by InsuranceERM (2013); these generators are in many cases based on different underlying interest rate models. The two main requirements for scenarios used to determine the TVOG is that they are market consistent and risk neutral. In short, market consistent refers to reproduce any market price observed in the market; risk neutral refers to being arbitrage free by fulfilling the a martingale test (1=1). The following underlying interest rate processes: Hull and White (HW); Cox–Ingersoll–Ross (CIR); and Libor Market Model (LMM) all fulfill these demands to certain extent. All processes are used in the insurance industry, even though LMM tend to be the most common one based the data collected by InsuranceERM (2013). It's clear from the structure of the interest processes that the distribution will be very different; while HW allows for negative interest rates; the CIR and LMM does not. There are modifications of LMM which allows for negative interest rates, but the process used here does not. The CIR uses a fix volatility term; HW and LMM allows for fluctuation of the volatility term structure. This makes it significantly more difficult to replicate the markets option prices surface using CIR, compared to HW and LMM. There are other versions of CIR where it allows for a fluctuation of the volatility term; thus those have been considered to be out of scope for this work.

2.1 Vasicek model

Both the CIR and the HW models are extensions of the Vasicek model which was introduced by Oldrich Vasicek in 1977 and has an Ornstein-Uhlenbeck process as its base. This is a short rate model following the dynamics given by the stochastic differential equation:

$$dr(t) = a\big(\beta - r(t)\big)dt + \sigma dW(t) \quad [1]$$

Where:

$r(t)$ = Short rate

a = Short rate reversion parameter

β = Long term mean

σ = Volatility

$W(t)$ = Wiener process

The Vasicek model is known as being the first model to capture the mean reversion characteristics. This can be described in the notation above as moving towards the long term mean of β over time, the speed of this reversion is given by the short rate reversion parameter of a. It's shown below that the variance of the random process $r(t)$ is independent of β, but not from a, which is intuitive since β only carries information of the long term mean.

Ito's Lemma states that for any transformed random process f(r(t), t), where f(r, t) is a function, which has continuous derivatives, second in r and first in t, and r(t) is the random process given by the stochastic differential equation [1]:

$$df(r(t),t) = \left[a\big(\beta - r(t)\big) \cdot \frac{\partial f(r(t),t)}{\partial r(t)} + \frac{\partial f(r(t),t)}{\partial t} + \frac{\sigma^2}{2} \cdot \frac{\partial^2 f(r(t),t)}{\partial r(t)^2}\right] dt$$

$$+\sigma \frac{\delta f(r(t),t)}{\delta r(t)} dW(t) \qquad [2]$$

By setting $f(r(t),t)$ to:

$$f(r(t),t) = (r(t) - \beta) \cdot e^{at} \quad [3]$$

We get the following derivatives:

$$\frac{\partial f(r(t),t)}{\partial r(t)} = e^{at} \quad [4], \quad \frac{\partial f(r(t),t)}{\partial t} = a \cdot (r(t) - \beta) \cdot e^{at} \qquad [5], \qquad \frac{\partial^2 f(r(t),t)}{\partial r(t)^2} = 0 \quad [6]$$

Using these results of [4], [5] and [6] in [2] yields:

$$df(r(t),t) = \left[a\big(\beta - r(t)\big) \cdot e^{at} + a \cdot (r(t) - \beta) \cdot e^{at}\right] dt + \sigma e^{at} dW(t) =$$

$$\sigma e^{at} dW(t) \qquad [7]$$

If we now would integrate [7] between 0 and t:

$$(r(t) - \beta) \cdot e^{at} - (r(0) - \beta) = \sigma e^{-at} \int_0^t e^{as} dW(s) \qquad [8]$$

Rewriting [8]:

$$r(t) = r(0) \cdot e^{-at} + \beta(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{as} dW(s) \quad [9]$$

It's now straightforward to calculate the expected value and the variance of the Vasicek model:

$$E[r(t)] = r(0) \cdot e^{-at} + \beta(1 - e^{-at}) \qquad [10]$$

$$Var[r(t)] = \sigma^2 e^{-2at} E\left[\left(\int_0^t e^{as} dW(s)\right)^2\right] = [It\tilde{o}\ isometry] =$$

$$= \sigma^2 e^{-2at} \int_0^t e^{2as} ds = \sigma^2 e^{-2at} \frac{1}{2a}(e^{2at} - 1) = \frac{\sigma^2}{2a}(1 - e^{-2at}) \quad [11]$$

A observation of the results tells us that if $r(0) = \beta$, then the expected value of $r(t)$ is equal to β. As well as that an increase of a would decrease the variance.

2.2 Cox-Ingersoll-Ross model

Cos-Ingersoll-Ross (CIR) in a one factor interest rate model and was introduced by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross in 1985. The model is a modification of the Vasicek model, with the following short rate dynamics given by the stochastic differential equation:

$$dr(t) = a\big(\beta - r(t)\big)dt + \ \sigma\sqrt{r(t)}dW(t) \qquad [12]$$

Where:

- $r(t)$ = Short rate
- a = Short rate reversion parameter
- β = Long term mean
- σ = Volatility
- $W(t)$ = Wiener process

The square root of the short rate guarantees the short rate will never turn negative.

The expected value and variance of $r(t)$ can be determined very similar to the way it was done for the Vasicek model. The only change one would need to do in equation [7] is to replace σ with $\sigma \cdot \sqrt{r(t)}$, it would then yield:

$$df(r(t),t) = \sigma \cdot \sqrt{r(t)} \cdot e^{at}dW(t) \qquad [13]$$

If we now would integrate [13] between 0 and t:

$$(r(t) - \beta) \cdot e^{at} - (r(0) - \beta) = \sigma e^{-at} \int_0^t e^{as}\sqrt{r(s)}dW(s) \qquad [14]$$

Rewriting [14]:

$$r(t) = r(0) \cdot e^{-at} + \beta(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{as}\sqrt{r(s)}dW(s) \qquad [15]$$

One can now calculate the expected value and the variance of the CIR model:

$$E[r(t)] = r(0) \cdot e^{-at} + \beta(1 - e^{-at}) \qquad [16]$$

$$Var[r(t)] = \sigma^2 e^{-2at} E\left[\left(\int_0^t e^{as}\sqrt{r(s)}dW(s)\right)^2\right]$$

$$= [It\tilde{o}\ isometry] = \sigma^2 e^{-2at} \int_0^t e^{2as} E[r(s)]ds = \big[\,use\ [16]\big]$$

$$= \sigma^2 e^{-2at} \int_0^t e^{2as}(r(0) \cdot e^{-as} + \beta(1 - e^{-as}))ds$$

$$= \sigma^2 e^{-2at} \int_0^t \left(r(0) \cdot e^{as} + \beta(e^{2as} - e^{as})\right) ds$$

$$= \sigma^2 e^{-2at} \cdot \left(\frac{r(0)}{a} \cdot (e^{at} - 1)\right) + \sigma^2 e^{-2at} \cdot \frac{\beta}{2a} \cdot (e^{2at} - 1 - 2e^{at} + 2)$$

$$= \frac{\sigma^2 \cdot r(0)}{a} \cdot (e^{-at} - e^{-2at}) + \frac{\sigma^2 \cdot \beta}{2a} \cdot (e^{at} - 1)^2 \qquad [17]$$

The expected value is the same as in the Vasicek model, but the variance has changed due to the square root of $r(t)$ in [12]. In the Vasicek model the variance of the random process was independent of β, this is no longer the case for the CIR-process. The reason for this dependency is related to the dependence of $r(t)$ in the variance, since the two factors becomes more integrated in CIR than in Vasicek.

In order to fit the market value of zero coupon bonds, Brigo and Mercurio (2001) introduced an external deterministic time-dependent shift, $\varphi(t)$ of $r(t)$, such that:

$$r_{ESG}(t) = r(t) + \varphi(t) \qquad [18]$$

It is in fact this $r_{ESG}(t)$ that is used in the ESG. Once the parameters have been calibrated in the ESG with market data, the scenarios can be generated through the short rate dynamics described above.

The scenarios used in this work have been calibrated and generated with Towers Watson's MoSes ESG.

2.3 Hull and White model

There are several different types of Hull and White models (HW) the description here will be limited to the one-factor HW (also known as the extended Vasicek model), the model was introduced by John Hull and Alan White in 1993. The model follows to the short rate dynamics given by the stochastic differential equation:

$$dr(t) = a\left(\vartheta(t) - r(t)\right)dt + \sigma(t)dW(t) \qquad [19]$$

Where:

$r(t)$ = Short rate

a = Short rate reversion parameter

$\vartheta(t)$ = Long term mean

$\sigma(t)$ = Volatility

$W(t)$ = Wiener process

The short rate reversion parameter is adjusting the speed of reversion towards the long term mean; naturally, the long term mean is set by the initial yield curve. For the volatility a piecewise volatility is calibrated in the economic scenario generator (ESG) with the following structure:

$$\sigma(t) = \begin{cases} \sigma_1 & if\ 0 \leq t < 1 \\ \sigma_2 & if\ 1 \leq t < 2 \\ \sigma_3 & if\ 2 \leq t < 3 \\ \sigma_4 & if\ 3 \leq t < 4 \\ \sigma_5 & if\ 4 \leq t < 5 \\ \sigma_7 & if\ 5 \leq t < 7 \\ \sigma_{10} & if\ 7 \leq t < 10 \\ \sigma_{15} & if\ 10 \leq t < 15 \\ \sigma_{20} & if\ 15 \leq t < 20 \\ \sigma_{25} & if\ 20 \leq t < 25 \\ \sigma_{30} & if\ 25 \leq t < 30 \end{cases}$$

Similar to the calculations of the Vasicek and CIR model, one can relatively easy calculate the expected value and the variance of the HW model by applying Ito's lemma to [18]:

$$df(r(t),t) = \left[a\big(\vartheta(t) - r(t)\big) \cdot \frac{\partial f(r(t),t)}{\partial r(t)} + \frac{\partial f(r(t),t)}{\partial t} + \frac{\sigma(t)^2}{2} \cdot \frac{\partial^2 f(r(t),t)}{\partial r(t)^2}\right] dt + \sigma(t)\frac{\partial f(r(t),t)}{\partial r(t)} dW(t) \quad [20]$$

By setting $f(r(t),t)$ to:

$$f(r(t),t) = (r(t) - \vartheta(t)) \cdot e^{at} \qquad [21]$$

We get the following derivatives:

$$\frac{\partial f(r(t),t)}{\partial r(t)} = e^{at} [22], \quad \frac{\partial f(r(t),t)}{\partial t} = a \cdot (r(t) - \vartheta(t)) \cdot e^{at} \quad [23], \quad \frac{\partial^2 f(r(t),t)}{\partial r(t)^2} = 0 \ [24]$$

Using these results of [22], [23] and [24] in [20] yields:

$$df(r(t),t) = \left[a\big(\vartheta(t) - r(t)\big) \cdot e^{at} + a \cdot (r(t) - \vartheta(t)) \cdot e^{at}\right] dt + \sigma e^{at} dW(t) = \sigma(t) e^{at} dW(t) \qquad [25]$$

If we now would integrate [25] between 0 and t:

$$(r(t) - \vartheta(t)) \cdot e^{at} - \big(r(0) - \vartheta(0)\big) = e^{-at} \int_0^t e^{as} \sigma(t) dW(s) \qquad [26]$$

Rewriting [26]:

$$r(t) = r(0) \cdot e^{-at} + \vartheta(0) - \vartheta(t) e^{-at} + e^{-at} \int_0^t e^{as} \sigma(s) dW(s) \quad [27]$$

The expected value is straight forward:

$$E[r(t)] = r(0) \cdot e^{-at} + \vartheta(0) - \vartheta(t)e^{-at} \quad [28]$$

The variance:

$$Var[r(t)] = \sigma^2 e^{-2at} E\left[\left(\int_0^t e^{as}\sigma(s)dW(s)\right)^2\right] = [It\tilde{o}\ isometry] =$$
$$e^{-2at}\int_0^t e^{2as}\sigma(s)^2 ds \qquad [29]$$

A notable observation is that in comparison to CIR, the variance is independent of $\vartheta(t)$ which correspond the β in [17].

Once the parameters have been calibrated in the ESG with market data, the scenarios can be generated through the short rate dynamics described above.

The scenarios used in this work have been calibrated and generated with Towers Watson’s MoSes ESG.

2.4 Libor Market Model

Brigo and Mercurio (2001) explores the Libor Market Model (LMM) as it is very different in comparison to the above mentioned models of Vasicek, HW and CIR; LMM captures the dynamics of the entire yield curve by using building blocks of forward rates. They define the standard lognormal LMM model as:

$$dF_i(t) = \mu_i(F(t), t)dt + F_i(t)\sigma_i(t)dW(t) \qquad [30]$$

Where:

$F_i(t)$ = Forward rates

$\mu(F, t)$ = The drift of forward rate Fi

$\sigma_i(t)$ = Volatility

$W(t)$ = Wiener process

In practice, the standard lognormal LMM fails to capture the volatility smile in derivatives market; however, by adjusting the volatility within the LMM to the SABR volatility model this can be accomplished. Hagan and Lesniewski (2008) derive the SABR-LMM by first defining the SABR volatility model using the following system of stochastic differential equations:

$$dF_i(t) = F_i(t)^\beta \sigma_i(t)dW(t) \qquad [31]$$

$$d\sigma_i(t) = \alpha\sigma_i(t)dZ(t) \qquad [32]$$

Where:

$Z(t)$ = Wiener process with correlation coefficient -1 < p < 1 with $W(t)$.

α = A constant parameter, > 0

β = A constant parameter, 1 > b > 0

This yields the SABR-LMM; which is the one used in the scenario generation;

$$dF_i(t) = \mu_i^{SABR}(F(t), \sigma(t), t)dt + F_i(t)^{\beta}\sigma_i(t) \sum_{j=1}^{N_f} b_{ij}\, dZ_j(t) \quad [33]$$

$$d\sigma_i(t) = \tau_i^{SABR}(F(t), \sigma(t), t)dt + \varphi_i(t) \sum_{j=1}^{N_f} c_{ij}\, dW_j(t) \quad [34]$$

Where:

μ_i^{SABR} = The drift of forward rate Fi in SABR-LMM

τ_i^{SABR} = The drift of the volatility in SABR-LMM

b_{ij} = A time independent parameter

c_{ij} = A time independent parameter

$\varphi_i(t)$ = A time dependent exponential function

The complexity of the structure in comparison to the Vasicek, CIR and HW makes it significantly more difficult to derive the expected value. Hence, the derivation is therefore left out of this work. The reason for this increased complexity is mainly caused by the fact that SABR-LMM isn't a short rate model, but a model of the entire yield curve at every point t. The other models described here only model the short rate at every point t.

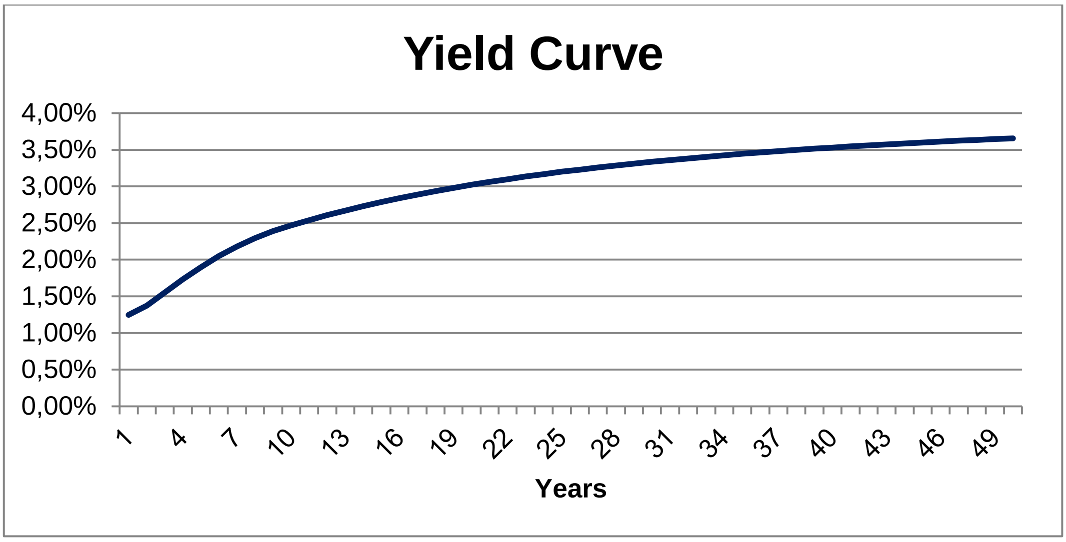
The scenarios used in this work have been calibrated and generated with an under development version of Towers Watson's STAR RN ESG.

3 Market data

The market data used in the scenario files is based on data from Bloomberg as of 2013-06-30.

3.1 Yield Curve

The yield curve has been derived out of Swedish (SEK) swap rates up until the last liquid point; the last liquid point has been defined to be 10 years. After the last liquid point the Smith-Wilson extrapolation method has been applied using a convergence period of 10 years with an ultimate forward rate of 4.2%.



3.2 Swaption volatilities

Swedish (SEK) swaption volatilities are the following.

Volatility Surface (%) - Market														
Option Expiry (Years)	Swap Tenor (Years)													
	1	2	3	4	5	6	7	8	9	10	15	20	25	30
1	30.30%	34.60%	36.05%	34.75%	33.35%	31.55%	30.00%	28.85%	27.60%	26.60%	22.58%	21.34%	20.89%	20.59%
2	32.10%	33.50%	32.70%	31.60%	30.80%	29.80%	29.00%	28.30%	27.50%	26.60%	23.47%	22.55%	22.23%	21.98%
3	33.90%	31.80%	30.40%	29.35%	28.60%	28.00%	27.60%	27.00%	26.60%	26.10%	23.73%	23.02%	22.85%	22.67%
4	31.45%	29.90%	28.70%	27.70%	26.95%	26.60%	26.10%	25.80%	25.60%	25.40%	23.58%	23.19%	23.28%	23.01%
5	28.75%	27.70%	26.80%	26.15%	25.60%	25.30%	25.00%	24.90%	24.80%	24.70%	23.23%	22.98%	23.07%	22.89%
7	25.60%	25.60%	25.20%	24.40%	23.60%	23.70%	23.90%	24.10%	24.30%	24.40%	23.67%	22.99%	22.99%	22.79%
10	23.60%	22.80%	22.50%	22.50%	22.40%	22.40%	22.40%	22.40%	22.40%	22.40%	21.39%	21.39%	20.91%	20.42%
15	21.86%	21.74%	22.02%	22.60%	22.98%	23.17%	23.05%	22.96%	22.90%	22.81%	21.54%	20.71%	19.65%	18.79%
20	22.43%	23.86%	23.75%	23.85%	23.75%	23.75%	23.88%	23.49%	23.23%	22.90%	21.20%	18.98%	17.82%	16.96%
25	23.80%	23.95%	23.94%	24.05%	23.75%	23.55%	23.88%	23.33%	22.88%	22.35%	19.27%	17.25%	16.19%	15.42%
30	22.33%	21.93%	21.54%	21.24%	20.86%	20.48%	20.72%	20.20%	19.78%	19.37%	16.86%	15.32%	14.45%	14.26%

4 Analysis of scenario files

The scenario sets used in this work consists of 3 000 scenarios. For the three different interest rate models described, one set of 3 000 scenarios have been created.

4.1 Risk neutrality and market consistency

The scenario sets must satisfy a number of conditions in order to be risk neutral and market consistent, a few of those question one need to ask are the following;

- Are all investment strategies "in average" equivalent to a risk-free investment?
- Is the yield curve derived from the average of discount factor equal to the market yield curve?
- Do the scenarios replicate swaption prices?
- Do the scenarios replicate index asset-option prices?
- Is the correlation reflected correctly?
- Do the scenarios reflect the markets expectations (e.g. inflation)?

In addition to these conditions other validations and checks are performed to confirm that the scenario set reflects the market (graphical inspections of distributions etc.).

Since only the interest rate models are included in this work and more precisely only the short rate, this will now be limited to look at the average of discount factors, swaption prices and distributions.

4.2 Is the yield curve replicated?

The implied yield curve from the different scenario sets is calculated by averaging the discount factors in the scenario sets. Once the discount factors have been determined the spot rates can be calculated.

$$\beta(t) = E[\beta(t)_i], \; r(t) = \beta(t)^{-\frac{1}{t}} - 1$$

Where;

$\beta(t)_i =$ Discount factor at time t and interation i in a scenario set.

$r(t) =$ The implied spot rate at time t.

Note that this is not the same as taking the average of the spot rates in the scenario set.

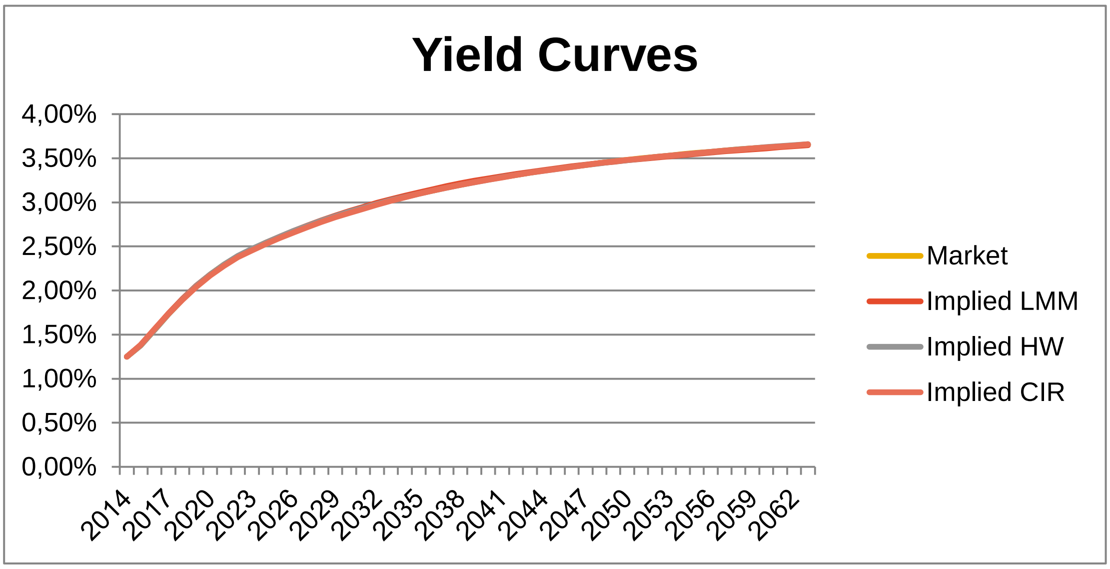


Figure 1: The implied yield curve using the different interest models together with the market yield curve; which was used during the calibration processes. It difficult by a visual inspection to notice any differences; they are all very close to one another.

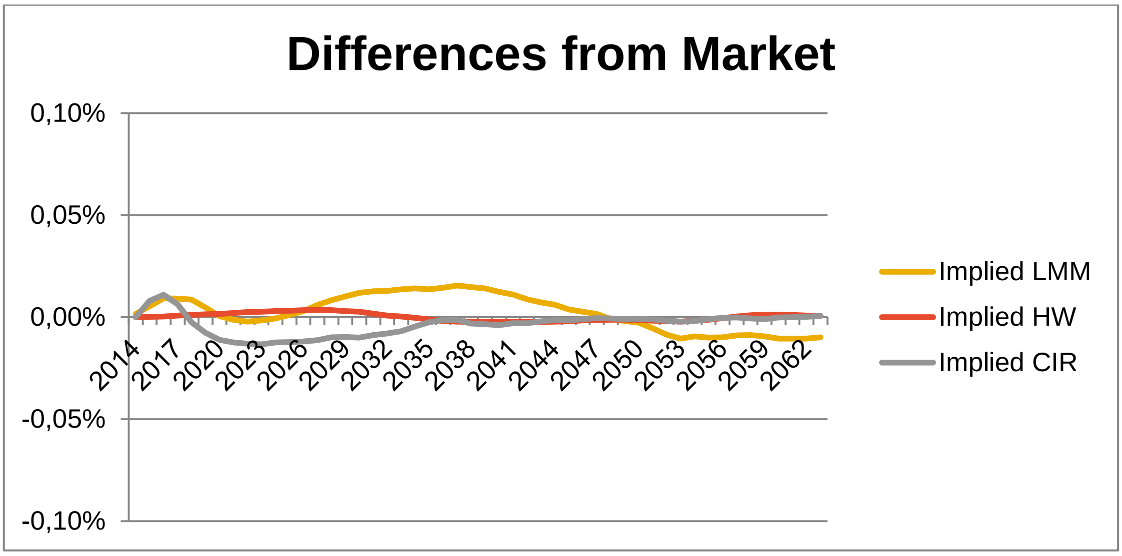


Figure 2: The differences between the implied yield curve and the market yield curve used in the calibrations.

As illustrated in Figure 1 and Figure 2; the differences between the yield curves are no more than 2 bps, which should be considered to be low.

4.3 Are the swaption prices replicated?

The swaption prices are numerically calculated based on the scenario set and these prices are compared to the market prices. As illustrated in Figure 3, Figure 4 and Figure 5; the differences are significant and as expected the CIR-process fails to capture the full volatility surface. The reason for this is mostly due to the fact that the CIR uses a single volatility factor, which makes it impossible for CIR to match the entire volatility surface.

LMM has an absolute total error of 7.5 while HW has an absolute total error of 8.4. The reasoning around why this is would mostly be in regards to the structure of LMM, which is a lot more complicated than the one around HW. This more advanced structure enables the model to capture more of the surface.

Total error (%) - Market price vs Scenario price														
	Swap Tenor (Years)													
Option Expiry (Years)	1	2	3	4	5	6	7	8	9	10	15	20	25	30
1	-10.62%	-30.79%	-39.47%	-41.39%	-41.80%	-40.23%	-37.94%	-35.55%	-31.98%	-28.41%	-7.20%	7.53%	17.40%	24.50%
2	-34.94%	-42.13%	-43.33%	-42.40%	-40.65%	-37.36%	-33.57%	-29.14%	-23.95%	-18.05%	10.27%	28.07%	37.97%	43.46%
3	-46.72%	-46.04%	-44.44%	-41.46%	-37.61%	-32.91%	-27.56%	-21.14%	-15.08%	-8.57%	23.00%	40.48%	47.93%	50.64%
4	-49.89%	-48.45%	-45.17%	-40.14%	-34.08%	-27.58%	-20.03%	-12.72%	-5.70%	1.11%	33.50%	48.42%	52.15%	53.17%
5	-50.05%	-47.82%	-43.19%	-37.03%	-29.23%	-21.07%	-12.43%	-4.41%	3.30%	10.49%	42.74%	55.48%	56.99%	55.32%
7	-48.01%	-46.03%	-40.13%	-30.91%	-19.37%	-9.79%	-0.75%	7.52%	14.81%	21.50%	48.67%	60.50%	58.48%	53.43%
10	-47.41%	-42.70%	-34.62%	-24.12%	-11.41%	1.16%	12.81%	23.06%	31.96%	39.56%	68.40%	70.91%	67.47%	60.85%
15	-51.61%	-45.62%	-35.08%	-22.28%	-8.51%	4.03%	16.12%	26.49%	35.06%	42.33%	65.96%	67.27%	63.05%	56.80%
20	-56.79%	-52.06%	-38.22%	-21.85%	-6.45%	6.65%	17.15%	28.16%	36.93%	44.55%	64.58%	71.23%	65.05%	58.02%
25	-61.80%	-53.10%	-36.26%	-19.16%	-2.92%	10.96%	20.36%	31.64%	40.70%	48.68%	74.30%	76.76%	68.29%	60.33%
30	-62.07%	-49.48%	-27.57%	-6.86%	12.05%	28.71%	39.12%	51.60%	61.37%	69.14%	90.90%	87.25%	76.00%	61.85%

Figure 3: Total error (%) between Market price and Scenario price for CIR.

Total error (%) - Market price vs Scenario price														
	Swap Tenor (Years)													
Option Expiry (Years)	1	2	3	4	5	6	7	8	9	10	15	20	25	30
1	39.64%	8.15%	-6.03%	-10.32%	-12.68%	-12.55%	-11.85%	-11.45%	-9.85%	-8.60%	-1.70%	-2.16%	-4.20%	-5.74%
2	17.00%	3.07%	-1.77%	-4.13%	-6.08%	-6.34%	-6.51%	-6.25%	-5.37%	-3.93%	1.16%	0.08%	-2.05%	-3.56%
3	-0.40%	-0.50%	-1.15%	-1.60%	-2.00%	-2.30%	-2.58%	-1.99%	-2.03%	-1.61%	1.81%	0.53%	-1.84%	-3.43%
4	-2.99%	-2.43%	-1.74%	-0.73%	-0.02%	-0.09%	0.48%	0.31%	-0.19%	-0.64%	1.46%	-0.75%	-3.97%	-5.08%
5	-1.70%	-0.83%	0.42%	1.21%	2.28%	2.39%	2.43%	1.69%	0.98%	0.32%	1.88%	-0.42%	-3.31%	-4.49%
7	-2.89%	-4.10%	-3.10%	-0.82%	1.42%	-0.02%	-1.83%	-3.55%	-5.19%	-6.39%	-7.03%	-6.86%	-8.75%	-9.52%
10	0.39%	2.51%	2.69%	1.67%	1.18%	0.34%	-0.43%	-1.13%	-1.77%	-2.36%	-0.40%	-2.16%	-1.42%	-0.43%
15	6.91%	6.83%	4.95%	1.89%	-0.19%	-1.43%	-1.39%	-1.41%	-1.59%	-1.60%	2.14%	4.48%	8.35%	11.58%
20	-1.29%	-7.04%	-6.98%	-7.66%	-7.59%	-7.88%	-8.63%	-7.53%	-6.87%	-5.89%	-0.32%	8.99%	14.01%	17.69%
25	-10.23%	-10.90%	-11.02%	-11.52%	-10.71%	-10.24%	-11.52%	-9.84%	-8.47%	-6.71%	5.56%	15.41%	20.46%	23.91%
30	-11.16%	-9.82%	-8.43%	-7.40%	-5.97%	-4.48%	-5.59%	-3.50%	-1.73%	0.08%	12.95%	22.44%	28.04%	28.16%

Figure 4: Total error (%) between Market price and Scenario price for HW.

Total error (%) - Market price vs Scenario price														
	Swap Tenor (Years)													
Option Expiry (Years)	1	2	3	4	5	6	7	8	9	10	15	20	25	30
1	8.72%	0.27%	-4.11%	-4.62%	-5.38%	-4.16%	-2.87%	-2.42%	-1.12%	-0.39%	3.97%	1.54%	-2.16%	-5.54%
2	4.18%	0.57%	-0.71%	-1.96%	-3.57%	-3.83%	-4.44%	-5.05%	-5.08%	-4.49%	-2.56%	-5.41%	-9.05%	-12.23%
3	-3.87%	-0.56%	-0.04%	-0.07%	-0.74%	-1.67%	-3.06%	-3.63%	-4.74%	-5.31%	-5.24%	-8.24%	-11.99%	-15.16%
4	-4.35%	-1.86%	-0.10%	1.02%	1.02%	-0.37%	-1.16%	-2.50%	-4.09%	-5.55%	-7.04%	-10.73%	-14.84%	-17.49%
5	-1.34%	1.54%	2.99%	2.94%	2.51%	1.13%	-0.15%	-2.08%	-3.86%	-5.46%	-6.96%	-10.46%	-14.34%	-17.27%
7	6.22%	4.51%	4.14%	4.84%	5.56%	2.56%	-0.51%	-3.25%	-5.68%	-7.48%	-9.91%	-11.29%	-14.65%	-17.73%
10	5.30%	7.24%	6.64%	4.73%	3.37%	1.71%	0.37%	-0.76%	-1.68%	-2.49%	-1.86%	-5.47%	-7.73%	-9.72%
15	7.89%	7.49%	4.88%	1.34%	-1.08%	-2.63%	-3.02%	-3.46%	-4.14%	-4.71%	-3.94%	-5.90%	-6.33%	-6.36%
20	5.38%	-1.59%	-2.39%	-4.00%	-4.70%	-5.75%	-7.30%	-6.92%	-6.92%	-6.69%	-6.80%	-3.49%	-2.38%	-1.73%
25	-1.51%	-3.16%	-4.01%	-5.41%	-5.52%	-6.30%	-9.11%	-8.96%	-9.06%	-8.71%	-2.99%	2.28%	3.92%	4.53%
30	-0.15%	-1.39%	-2.13%	-3.04%	-3.52%	-3.89%	-6.62%	-5.89%	-5.42%	-4.69%	3.08%	8.11%	9.59%	7.06%

Figure 5: Total error (%) between Market price and Scenario price for LMM.

4.4 The distributions

As a result of the differences in the model structure; the distribution are looks very different from each other.

4.4.1 Short rate

The distributions of the short rate are different between the models; both CIR and LMM results in very high short rate towards the end of the projection. HW on the other hand ends up relatively low rates. Another observation is that the HW distribution includes negative values; while LMM and CIR doesn't.

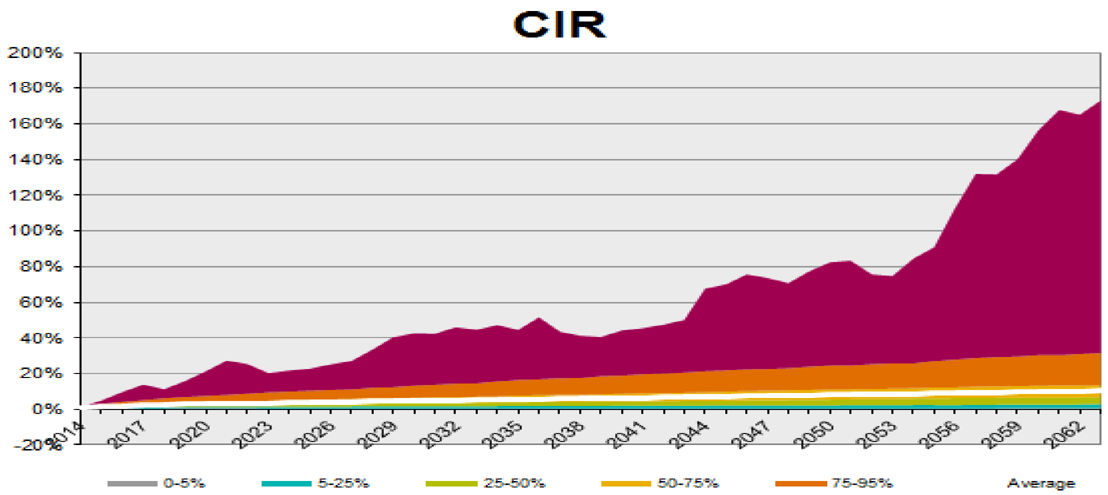


Figure 6: Percentile graph of the short rate for CIR.

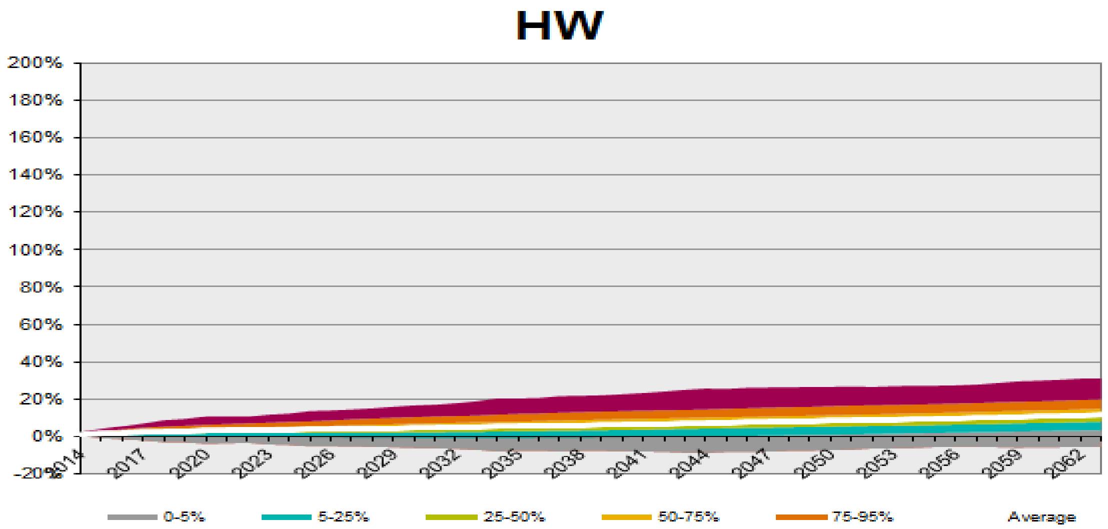


Figure 7: Percentile graph of the short rate for HW.

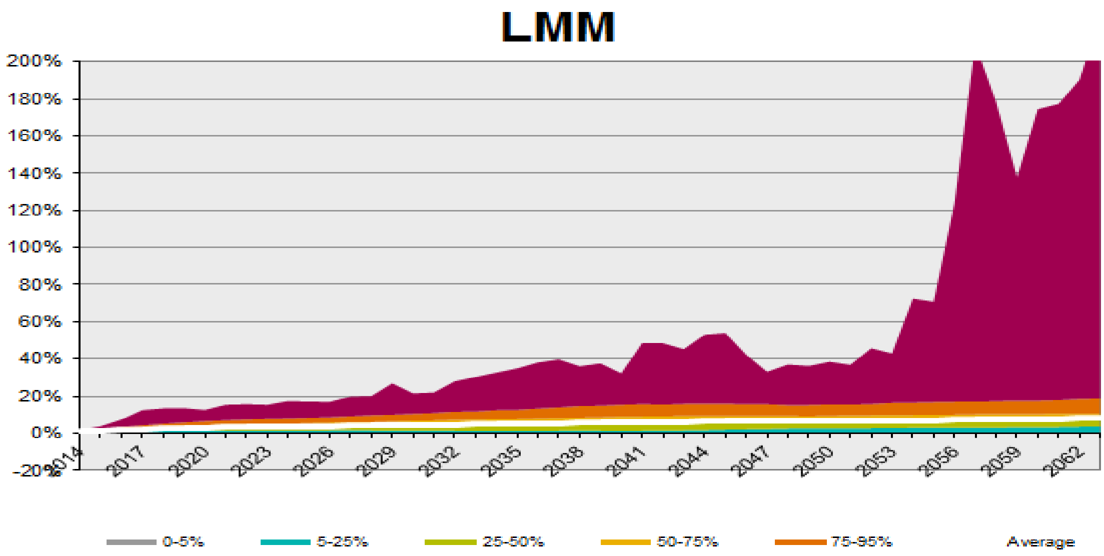


Figure 8: Percentile graph of the short rate for LMM.

4.4.2 Short rate index

The non-negative rates for CIR and LMM prevents them to end up with an index below 1. This is particularly notable for LMM where there is a sharp edge by 1. The HW index has a wider spread than LMM and CIR.

The total number of scenarios is 3 000 per set and there are scenarios with a short rate index above 10 after 30 years for all of these sets, those values are not shown in the figures below.

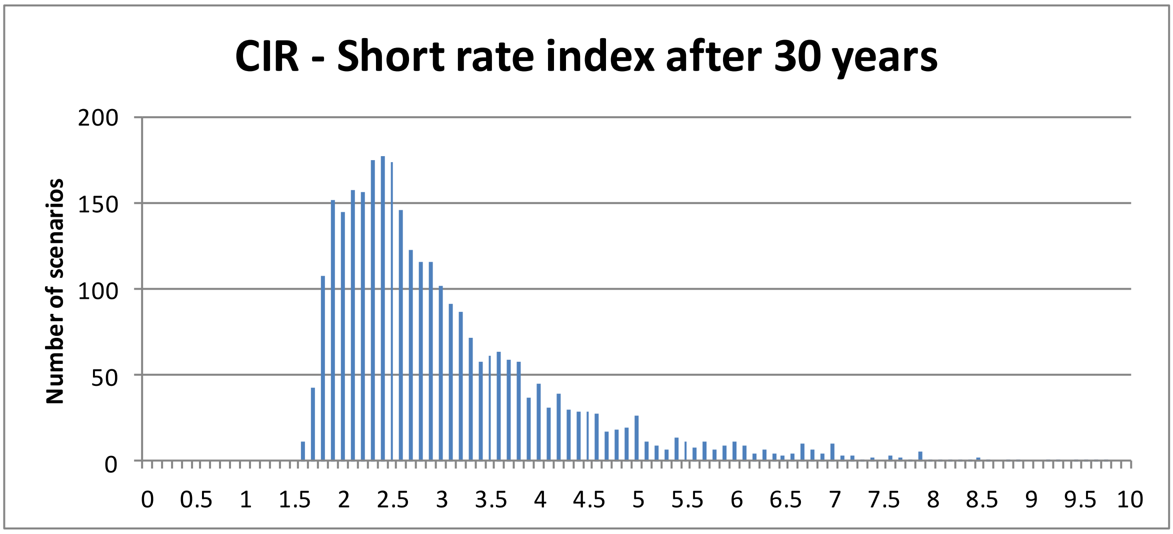


Figure 9: A histogram of the short rate index after 30 years with 3 000 scenarios.

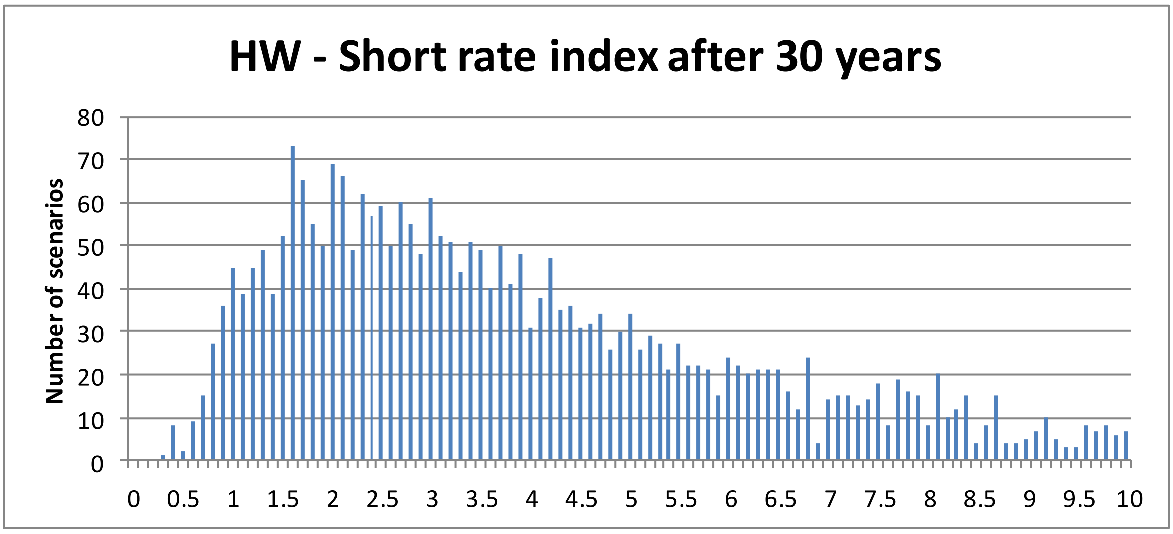


Figure 10: A histogram of the short rate index after 30 years with 3 000 scenarios.

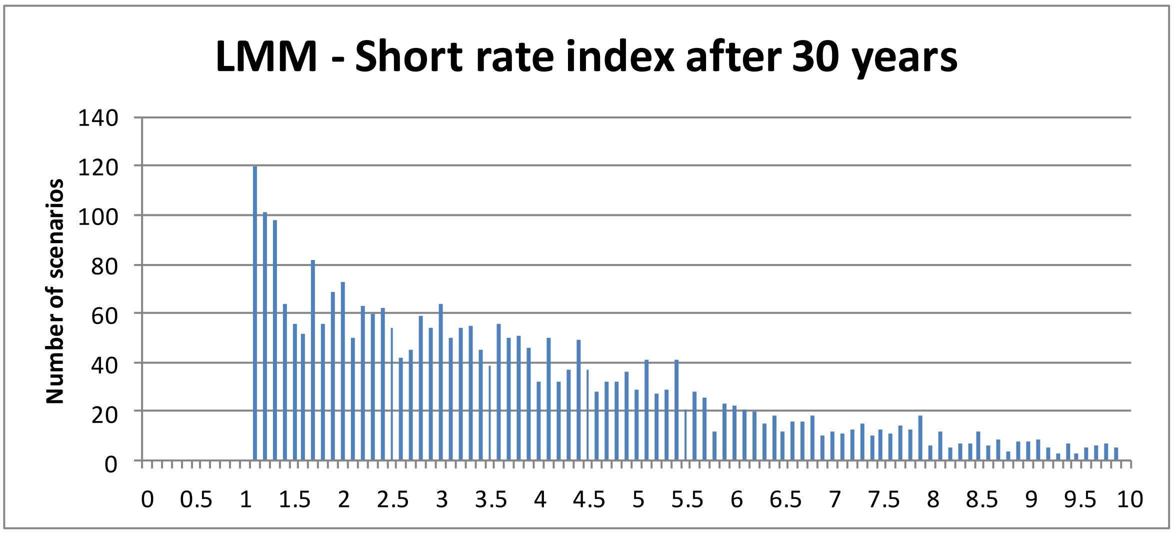


Figure 11: A histogram of the short rate index after 30 years with 3 000 scenarios.

5 The guarantee product

The policyholder will receive a lump sum payment upon retirement and if the policyholder dies before retirement the lump sum payment occurs at the time of death. In both cases the payment will be the maximum of the fund value and the guaranteed value.

If the policyholder surrenders before retirement, he will only receive the fund value.

5.1 Assumptions

For simplification; the assumption is made that all expenses are perfectly matched with premium fees taken directly on the premium paid. It's the net premium paid (after fees) which is added to the guaranteed value and the fund value. In addition to the premium fees, there's also an annual guarantee charge of 0.2% on the fund value.

5.1.1 Guarantee rate: The model will perform three runs with three different net guarantee rates; 0%, 2% and 4%.

5.1.2 For mortality M90 is assumed:

$$\mu_x = \alpha + \beta e^{\ln(10)\gamma(x-f)}$$

Where:

$\alpha = 0.001, \beta = 0.000012, \gamma = 0.044$ and $f = \begin{cases} 6 & for\ females \\ 0 & for\ males \end{cases}$

5.1.3 Paid-up: The model assumes a 2% flat annual paid-up rate.

5.1.4 Surrender: The model assumes a 1% flat annual surrender rate.

5.2 Model points

The portfolio consists of 80 model points; 40 males and 40 females. They are evenly distributed between the ages of 21 and 60; and are all assumed to retire at the age of 65. All policyholders just entered the contract and will pay an annual net premium of 1000 SEK (after fees) as long as they remain in the premium paying state.

5.3 Portfolio investment

The entire fund value is assumed to be invested in short duration bonds and credited interest based on the short rate.

6 Result

The time value of options and guarantees is calculated as the difference between the best estimate of the liabilities using a central scenario and the mean of the stochastic set of scenarios. It has been modeled in the financial projection software MoSes, according the assumptions presented in Chapter 5.

6.1 Guaranteed rate of 0%

In the central scenario with a guaranteed rate of 0%; the maximum of the fund value and guaranteed value will always be equal to the fund value. This is not the case for the stochastic scenario set due to the volatility underlying the scenarios. However, since LMM and CIR doesn't allow for negative interest rates the option values generated for those are low. The reason they end up with an option value is due to the guarantee charge on the fund value, if the guarantee charge would have been zero; the option value would have been zero. HW on the other hand allows for negative interest rates; and this yields a significantly higher option value compared to both LMM and CIR.

Figure 9 illustrates the differences between the different models graphically. The downward slope of the curve is caused by two main factors; policyholders entering the contract at a high age have paid fewer premiums by the time of retirement; secondly, the volatility between the different outcomes in the scenario set grows larger over time.

Females live longer, by the assumptions made in M90, which in average generates a higher guarantee; this explains that females have a slightly higher TVOG.

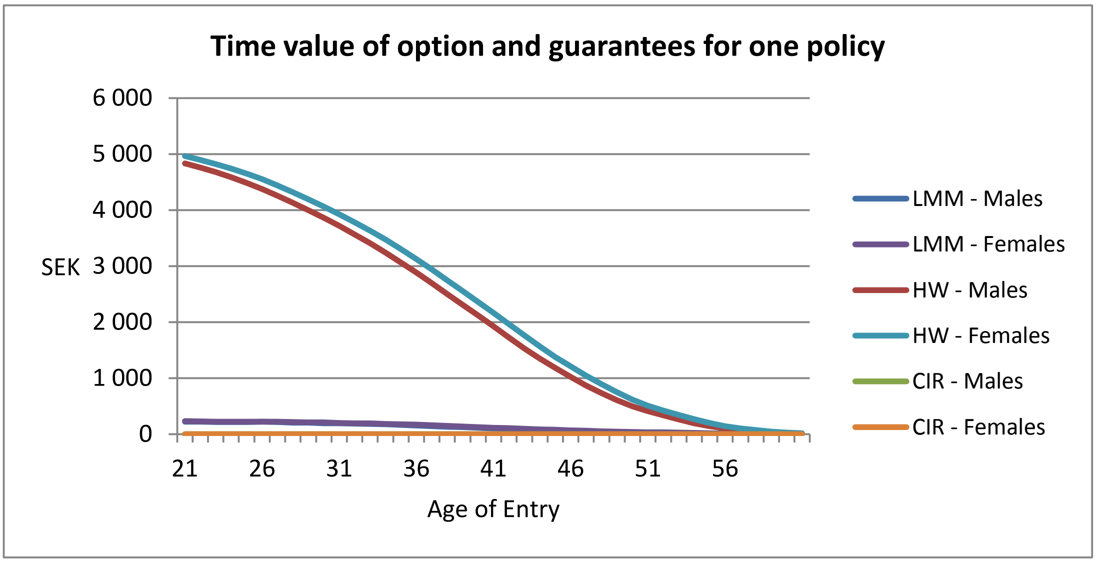


Figure 12: An illustration of how the TVOG varies between different ages and models with a guaranteed rate of 0%. For one female policyholder who enters at the age of 21; the TVOG value for her would almost be 5 000 SEK with the HW-model.

6.2 Guaranteed rate of 2%

With a guaranteed rate of the 2%; the value of the guarantees in central scenario is 0 (similar to 6.1). As illustrated in Figure 10, the HW model gives the highest TVOG. The TVOG calculated using the LMM model generates a lower value than HW for the policyholders entering at a low age, while the value for those entering at a higher age is more similar.

The CIR model generates a value close to 0; it should be noted that the volatility underlying the scenarios is very different from HW and LMM due to the limitation with only one volatility parameter (see section 4.2). This limitation could be causing the low TVOG.

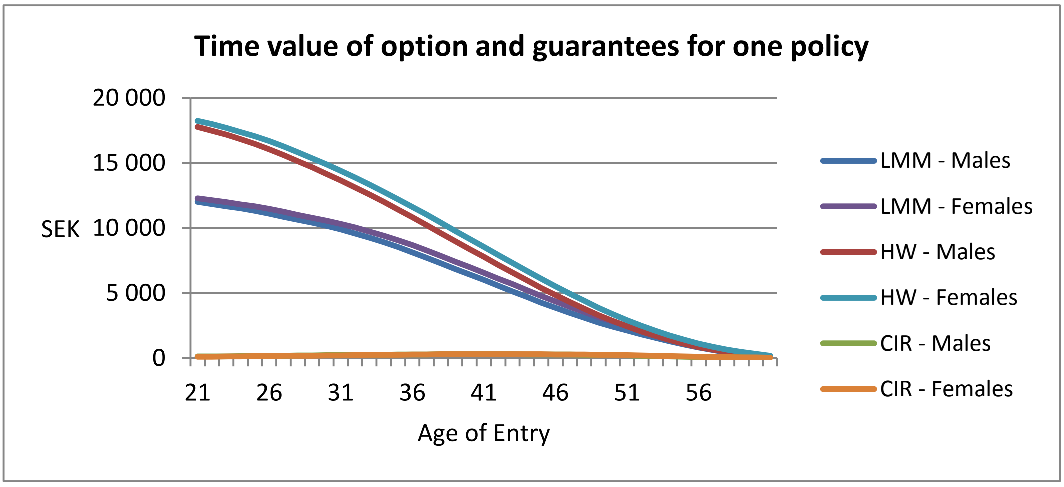


Figure 13: An illustration of how the TVOG varies between different ages and models with a guaranteed rate of 2%. For one female policyholder who enters at the age of 21; the TVOG value for her would be 18 000 SEK with the HW-model.

6.3 Guaranteed rate of 4%

In the central scenario; with a guaranteed rate of 4%; the maximum of the fund value and guaranteed value will not always be equal to the fund value. The HW model generates the largest TVOG; the values are closer to LMM for the policyholders entering the policy at a higher age. For CIR the values increased compared to the previous guaranteed rates of 0% and 2% (see 6.1 and 6.2); however, the limitations of CIR volatility match could potentially still be the main source of the differences.

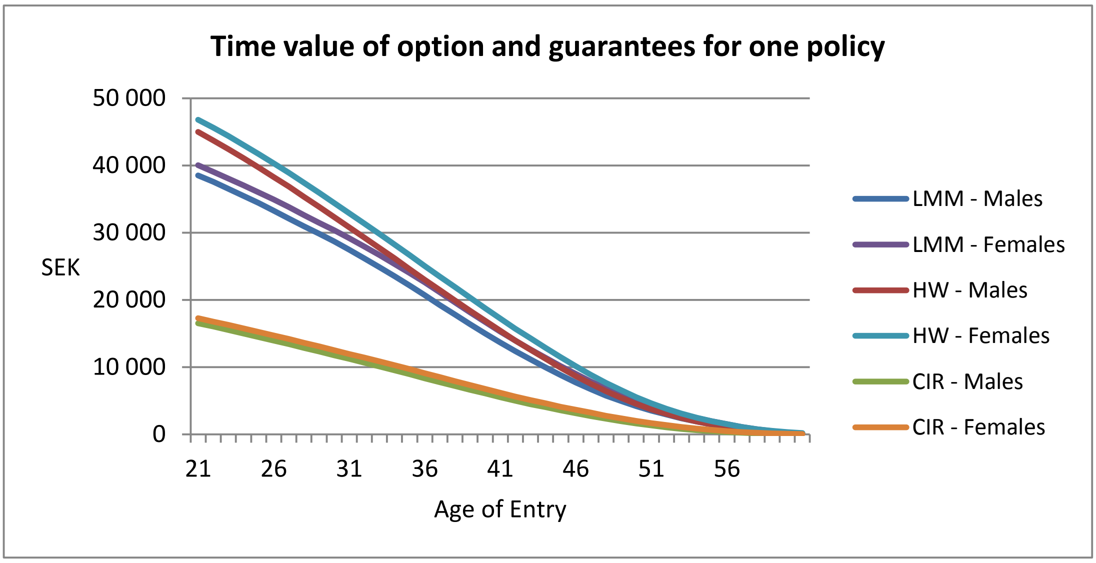


Figure 14: An illustration of how the TVOG varies between different ages and models with a guaranteed rate of 4%. For one female policyholder who enters at the age of 21; the TVOG value for her would almost be 47 000 SEK with the HW-model.

7 Discussion and conclusion

The main findings in this master thesis, commented in more details are the following:

- The choice of interest rate model used for the valuation impacts the TVOG.

- The differences between the interest rate models seem to the most significant for products with a guaranteed rate of 0% due to the allowance of negative rates in HW.

- The interest rate models fail to fully replicate the market prices, especially the CIR.

An initial inspection of the numbers indicates that the choice of interest rate model has a significant impact on the TVOG. The differences could potentially cause management decisions which wouldn't have been taken if another interest model would have been used; giving the management a different view of the liabilities. It's important to understand the differences and be aware of its limitations when applying them; an interest rate model used properly for one purpose doesn't necessarily need to be ideal for another. This work is limited to a very particular guaranteed product, three different interest rate models with certain calibrations modifications and market data from a specific date. The differences illustrated in Chapter 6 can be a result of these conditions; in this case HW always ends up with a higher option value than both LMM and CIR. The conclusion shouldn't be that HW will generate a higher TVOG; it's rather the fact that these different models will result in different option values and this will directly have an impact on the balance sheet.

This work has been limited to a simple guaranteed product with a single payment; it's to be expected that products with other structures could have a very different outcome. If one would consider any ratcheting product where the guaranteed amount increases depending on the market's development; the path of the fund value up until retirement would have a crucial impact of the TVOG. Another example of these path dependencies is by increasing the number of payments from a single payment; if one would pay a discretionary benefit at the age of 65; the fund value might not be able to cover the guarantees at the age of 70 due to the payment at 65.

In general; equities have a higher volatility than bonds; and the volatility is the main source of any financial option value. Therefore an obvious limitation in this work to assume 100% short duration bonds when investigating the sensitivities of the TVOG. Nevertheless, it's still interest rate models that's underlying any index asset model which is typically added to model equity returns and the cash flows will always be discounted by the short rate.

It's noted that the CIR model is using a single volatility parameter and therefore fails to capture the full volatility surface of the market. As a result; it's very difficult to draw any conclusion in regards to the differences in the generated TVOG using CIR compared to HW and LMM; other than the fact that using CIR with a single volatility parameter is too simplistic to capture the TVOG. CIR could potentially be used to capture it by using different sets of stochastic scenarios; depending on the duration of guarantees; no further analysis has been done in this respect. Another option would be to use another modified version of CIR which allows for a fluctuating volatility term.

The figures (Figure 9, 10 and 11) illustrating the short rate index explains the outcome of the modeled guaranteed product very well. It's possible to directly spot the result of the non-negative rates in CIR and LMM; all values are above 1. On the other hand, HW has several index values below 1; this could be linked the Figure 12 where a guaranteed rate of 0% has been applied. In short and slightly simplified, the policyholder is guaranteed to end up with an index value of 1 and any values below 1 will create an option value. This is no longer the case with the guaranteed rate of 2%; instead (still simplified) the policyholder is guaranteed $I_g^2(30) = (1 + 0.02)^{30} = 1.8$. The value of 1.8 explains the low option value for CIR with the guaranteed rate of 2%; one could see in Figure 9 that CIR got almost its entire distribution above 1.8. For the guaranteed rate of 4% we end up with a simplified guaranteed $I_g^4(30) = (1 + 0.04)^{30} = 3.2$ and all distribution are covering that value and as it's calculated all models ends up with a significant option value.

Finally, the focus in this work has been to present and illustrate that the choice of interest model is important and will affect the results. It's left for the reader to select which model would suit ones purpose. Further research studies could potentially be done to conclude the most appropriate model for this specific product. Additionally, it would be interesting to look into the impact for other guaranteed products, e.g. ratcheting products.

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