Determining the capital requirement and its optimal allocation in realistic economic scenarios.

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Abstract

The work presented here is part of the doctoral dissertation successfully defended on 24th of September 2012 with professors Soren Asmussen nad Florin Avram as part of the examining committee. Professor Soren Asmussen's merits, besides his vast investigation work, is John von Neumann Theory Prize (2012) and Sobolev Gold Medal (2011).

Risk is inherent to the insurance business and so is the necessity to quantify it. Insurance companies operating in a branch of insurance business need to cover the claims resulting from a portfolio of the contracts. Since the amounts and the timing of these claims is unknown in advance, the company needs to determine some regular patterns in the uncertain quantities to accrue appropriate funds to cover its liabilities. The decision on the volume of these funds, often called reserves, is a trade-off between the solvency and the efficiency of the capital management. Insufficient reserves will lead a company to bankruptcy, while excessive reserves mean a waste of capital resources and loss of competitiveness in the market. The problem of solvency is not only important for the insurance companies themselves but also for the regulator of the insurance market. The requirement of a minimum obligatory reserves that have to be withhold in a company operating in the insurance industry is indeed one of the three pillars sustaining the new set of the regulatory requirements being prepared for the European Union insurance market: Solvency II. The reliable quantitative tools to assess the adequacy of the monetary requirements are not only interesting on their own sake as a theoretical challenge but are also essential for practical purposes, especially in view of recent international financial crisis that, far from being exclusive to banking sector, may affect the insurance industry as well.

In the first part of the work we develop a method to determine the level of minimum capital necessary to guarantee the solvency requirements (i.e. 99,5% capital sufficiency). In the second part of the work, the optimal allocation of the required capital is examined. Depending on the risk appetite of the particular insurance company, this necessary capital may be invested in a varying bundle of conservative assets (bonds) or risky assets (stocks). This way, an answer to several crucial

questions is given: What is the reasonable level of capital necessary to guarantee the solvency of a company? What is the optimal allocation of this capital? What is the risk the company is undertaking at each moment as a function of its investment strategy, capital availability, market situations and nature of its liabilities?

In this work, a solvency problem of an insurance company is treated in a short-term and long-term horizon. The advantage of the long term horizon approach over year to year basis is that ruin problems, being very unusual in short term, are more difficult to grasp in one year horizon since the phenomena are extremely uncommon. Consequently, the extrapolation approach to long term conclusions from short term estimations could deliver misleading results.

It is important to realise that the model we use in order to obtain the desired answers to the questions stated previously is general enough to reflect the complexity of the real situation scenarios and yet tractable enough to obtain numerical answers with high precision. On one hand, the collection of premiums is modelled by a diffusion process without assuming any regularity pattern ("constant rate" or "all at the beginning of the period" assumptions). The evolution of the macroeconomic variables, such as the inflation rate, interest rate, economic cycles, and others, are included through a markovian environment determining the coefficients of the model. The market investment environment is modelled by the most advanced techniques, namely Lévy processes that go beyond usual geometric brownian motion setting, overcoming its limitations. The additional advantage of Markov-modulated models is that, due to their flexibility, they can adapt any alternative model arbitrarily well. This is technically supported by the denseness property of this type of stochastic models.

The risk associated with a particular portfolio is quantified using several criteria. The most common one, is the probability that ruin happens within a given horizon. Though being the principal quantity of interest, other important aspects related to the reserve process have drawn continuous attention, most notably the deficit at ruin and recovery time. One can argue that the bankruptcy does not really occur until the deficit is important enough so that the company cannot recover through a short term loan. The deficit at ruin is the amount of money the company lacks to cover the claims when ruin occurs. It indicates the severity of the financial insufficiency. Closely related to this is the concept of recovery time that represents how long it takes until the company recovers to positive reserves, or even to required minimum reserves levels. Other quantities that appear in ruin theory literature is the distribution of the surplus just before the ruin, the deficit just after the ruin and the time at which the ruin occurred. The joint distribution of these three events gives detailed insight into how dangerous is a particular financial situation at a given moment.

A general framework to analyse these indicators have been developed in a series of papers by Gerber and Shiu (Gerber and Shiu (1997, 1998a,b)). A general utility function dependent on time and severity of ruin or remaining surplus in case of the survival of a given horizon is introduced and denoted as the *penalty-reward* function. Its expected discounted present value is studied and the above mentioned quantities (ruin and survival probabilities, time of ruin, etc.) are shown to be special cases.

In this work we will focus our attention on characterisation and calculation of the expected penalty-reward function in various scenarios.

The model that will be used in this work to describe the evolution of the reserves in time is a **Lévy diffusion** process. It is general enough to accommodate the realistic behaviour of the underlying phenomenon, as was argued by Morales (2007) yet tractable, at least numerically, to obtain results relevant to possible practical application.

Based on the model and the criteria indicators mentioned above, the insurance company or the regulator body can adjust the controllable variables such as initial reserves, premiums collected, and the investment decisions on the funds kept as reserves, to assure sufficient financial resources to cover the liabilities corresponding to a portfolio of its business. This dissertation sets up a theoretical framework to analyse this decision process and provides quantitative tools to evaluate the impact of possible decisions.

The evolution of the reserves is modelled by a stochastic differential equation

$$\mathrm{d}U_t = [c + \mu_t(\sigma_t, Y_t, U_t)U_t]\,\mathrm{d}t + \sqrt{\rho^2 + U_t^2\sigma_t^2(Y_t)}\mathrm{d}W_t - \mathrm{d}X_t, \qquad U_0 = u.$$

where the collection of premiums has two terms, the drift c and the volatility ρ, the investment is made into an asset with drift μ and volatility σ, the claims are paid according to a compound Poisson process X_t. The investment decision is governed by the selection of the desired risk appetite σ, the drift μ is then implied by the market investment possibilities assuming the maximum expected return at a given volatility. All implied coefficients are subject to Markov process Y_t that reflects the state of all relevant macroeconomic variables.

The objective of risk control is expressed as the following quantity (a penalty-reward function)

$$v^\sigma(T, u, \delta_i) = \mathrm{E}\left[P(U_T, Y_T)\cdot \mathbb{I}_{\{\tau\geq T\}} + L(U_\tau, Y_\tau)\cdot \mathbb{I}_{\{\tau<T\}}|U_0 = u, Y_0 = \delta_i\right].$$

This includes the probabilities of ruin (or survival), the deficit at ruin or the time to ruin as special cases.

In order to determine the capital requirement the following equation needs to be solved

For $i = 1, \ldots, n$

$$\begin{aligned}&\frac{1}{2}\left(\sigma^2 + u^2\kappa^2(u)\right)\frac{\mathrm{d}^2}{\mathrm{d}u^2}\Upsilon_\alpha^i(u) + (c+\delta_i u)\frac{\mathrm{d}}{\mathrm{d}u}\Upsilon_\alpha^i(u) + \\ &+\sum_{j=1}^n q_{ij}\Upsilon_\alpha^j(u) - (\alpha+\lambda)\Upsilon_\alpha^i(u) + \lambda\int_0^u \Upsilon_\alpha^i(u-x)\;f(x)\,\mathrm{d}x + \\ &+\alpha P(u) + \lambda\int_u^\infty \pi(u-x)\;f(x)\,\mathrm{d}x = 0.\end{aligned}$$

where Υ is the Laplace-Carson transform of the penalty-reward function v. In our

work we present a numerical method based on Chebyshev polynomial expansion to obtain this solution.

For the optimal allocation problem, the following optimisation problem is solved

$$J(T, u, y) \equiv \max_{\sigma \in \Pi} v^{\sigma}(T, u, y).$$

analyzing all the admissible investment opportunities Π.

In this work, we treat the problems related to risk theory and stochastic control in the context of Lévy diffusion processes. It was argued that the Lévy diffusions provide a fairly general framework covering varied modelisation paradigms from finance and insurance areas. In particular a compound Poisson process with Markov-modulated parameters is analysed, nevertheless, general results include wider spectrum of stochastic models. The methods and the results obtained here are novel and represent a pioneering work as discussed with leading investigators in the risk theory field.

In the first part we present a new approximation procedure for the calculation of the penalty-reward function in a risk theory context. The importance of the contribution is underlined by the fact that previously no solution was available in the general setting that has been proposed in this work.

The second part demonstrates the power of the approaches presented earlier by solving a stochastic control problem of optimal investment of an insurer facing risk management decisions in a context that has not been treated previously. An example shows how maximum survival probability curve can be obtained for different levels of initial conditions.

Besides the theoretical interest of the presented results we believe that these could become relevant analytical tools in practical applications, in both regulatory bodies and internal control processes within insurance companies. Sensible models that are able to explain the behaviour and quantify the answers posed about solvency, profitability or other nature of insurance business are needed in decision making processes.

1 Introduction

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In this work, a solvency problem of an insurance company is treated in a long term horizon. We develop a method to determine the level of minimum capital necessary to guarantee the solvency requirements (i.e. 99,5% capital sufficiency). At the same time, we study the allocation of this capital. Depending on the risk appetite of the particular insurance company, this necessary capital may be invested in a varying bundle of conservative assets (bonds) or risky assets (stocks). This way, an answer to several crucial questions is given: What is the reasonable level of capital necessary to guarantee the solvency of a company? What is the optimal allocation of this capital? What is the risk the company is undertaking at each moment as a function of its investment strategy, capital availability, market situations and nature of its liabilities?

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The model that will be used in this work to describe the evolution of the reserves in time is a **Lévy diffusion** process. It is general enough to accommodate the realistic behaviour of the underlying phenomenon, as was argued by Morales (2007) yet tractable, at least numerically, to obtain results relevant to possible practical application.

Based on the model and the criteria indicators mentioned above, the insurance company or the regulator body can adjust the controllable variables such as initial reserves, premiums collected, and the investment decisions on the funds kept as reserves, to assure sufficient financial resources to cover the liabilities corresponding to a portfolio of its business. This dissertation sets up a theoretical framework to analyse this decision process and provides quantitative tools to evaluate the impact of possible decisions.

2 Establishing the capital requirement

In this section a problem of establishing the capital requirement necessary to guarantee a given level of solvency is treated. We develop a method to determine the capital necessary to cover the liabilities of an insurer as well as answer some related questions mentioned above.

The evolution of the reserves is modelled by a stochastic differential equation

$$\mathrm{d}U_t = [c + \mu_t(\sigma_t, Y_t, U_t)U_t]\,\mathrm{d}t + \sqrt{\rho^2 + U_t^2\sigma_t^2(Y_t)}\mathrm{d}W_t - \mathrm{d}X_t, \qquad U_0 = u.$$

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In order to determine the capital requirement the following equation needs to be solved

For $i = 1, \ldots, n$

$$\begin{aligned}&\frac{1}{2}\left(\sigma^2 + u^2\kappa^2(u)\right)\frac{\mathrm{d}^2}{\mathrm{d}u^2}\Upsilon_\alpha^i(u) + (c + \delta_i u)\frac{\mathrm{d}}{\mathrm{d}u}\Upsilon_\alpha^i(u) + \\ &+\sum_{j=1}^n q_{ij}\Upsilon_\alpha^j(u) - (\alpha+\lambda)\Upsilon_\alpha^i(u) + \lambda\int_0^u \Upsilon_\alpha^i(u-x)\ f(x)\ \mathrm{d}x + \\ &+\alpha P(u) + \lambda\int_u^\infty \pi(u-x)\ f(x)\ \mathrm{d}x = 0.\end{aligned}$$

where Υ is the Laplace-Carson transform of the penalty-reward function v. In our work we present a numerical method based on Chebyshev polynomial expansion to obtain this solution.

The details about the development of the numerical procedure is deferred to the appendix. In the following section we present important examples that illustrate the method.

$\Upsilon_t^i(u)$ is the Laplace-Carson transform in time of the expected penalty-reward function in a jump-diffusion process. This function has a probabilistic interpretation as the penalty-reward function in an exponentially killed time horizon H_α. The ultimate case is also unveiled by a straightforward application of the Tauberian theorem

$$\lim_{\alpha\to 0}\Upsilon_\alpha^i = \lim_{\alpha\to 0}\int_0^\infty \alpha e^{-\alpha t}\phi_t^i(u)\,dt = -\phi_0^i(u) + \lim_{\alpha\to 0}\int_0^\infty e^{-\alpha t}\frac{d}{dt}\phi_t^i(u)\,dt \qquad (1)$$
$$= -\phi_0^i(u) + \int_0^\infty \frac{d}{dt}\phi_t^i(u)\,dt = \phi_\infty^i(u)\,.$$

For the more challenging finite time horizon penalty-reward, a numerical inversion of the Laplace transform recovers the original function $\phi_\alpha^i(u)$, see Usábel (1999). The relationship $C(s) = sL(s)$ between the Laplace transform $L(s)$ and the Laplace-Carson transform $C(s)$ applies.

2.2.1 Ultimate Survival Probability

The survival probability is a special case of the function $\Upsilon_\alpha^i(u)$. For $\pi(x) \equiv 0$ and $P(x) \equiv 1$

$$\phi_\infty^i(u) = \mathrm{E}\left[\mathbb{I}(\tau = \infty) \mid U_0 = u, \Delta_0 = \delta_i\right].$$

The premium collection rate is $c = 11$, the volatility of premium accruals $\sigma^2 = 0.04$, the intensity of claim arrivals $\lambda = 4$, and claims follow a Gamma distribution *Gamma* $(5;2)$. The interest rate is assumed to be fixed at 3% with no volatility ($\sigma_r^2 = 0$). The ultimate survival probability $\phi_\infty^i(u)$ is considered in this context and thus $\alpha = 0$ as motivated by (1). For the change of variables, the function $h(v) = -\ln(1-v)$ was used. The following table shows the approximations for various starting reserves and precision levels (order of Chebyshev polynomials).

N – precision level

		200	250	300	350	400	450
u	1	0.318081594	0.318079845	0.318079373	0.318079219	0.318079161	0.318079137
	2	0.435631392	0.43562899	0.435628343	0.435628132	0.435628053	0.43562802
	5	0.753759689	0.753755453	0.753754322	0.753753953	0.753753813	0.753753756
	10	0.987580029	0.987573342	0.987571486	0.98757086	0.987570616	0.987570511
	15	0.999982762	0.999973643	0.99997087	0.999969864	0.999969447	0.999969256

2.2.2 Markov-modulated Interest Rate Structure

The second example presents an interest rate structure driven by a Markov process and a reserve dependent volatility. Let us assume two regimes (high interest rate and low interest rate) comprising several interest rate levels. The intensity matrix Q, characterising the Markov process, governs the evolution of the interest rate:

$$
\begin{array}{cc} \delta_i & \begin{array}{ccccc} 1\% & 2\% & 7\% & 8\% & 9\% \end{array} \end{array}
$$

$$
Q = \begin{array}{c} 1\% \\ 2\% \\ 7\% \\ 8\% \\ 9\% \end{array} \begin{pmatrix} -2 & 2 & 0 & 0 & 0 \\ 1.9 & -2 & 0.1 & 0 & 0 \\ 0 & 0.1 & -3 & 2.9 & 0 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 3 & -3 \end{pmatrix}.
$$

The low interest rate regime embeds two levels 1% and 2% while the high interest rate regime considers three levels 7%, 8%, and 9%. Let the premium collection rate be 1 with the volatility of premium accruals 0.25, the intensity of claims arrival $\frac{1}{3}$ (one claim every three time periods on average), the distribution of claim size lognormal $\ln \mathcal{N}(0.5; 1)$. The volatility of the return on investment, dependent on the reserves level, is $\kappa^2(u) = \frac{\sigma_r^2}{u}$, as motivated in the introduction, with $\sigma_r^2 = 0.81$. The probability of survival of a random horizon of 20 years on average is approximated ($\alpha = 0.05$, $\pi(x) \equiv 0$ and $P(x) \equiv 1$). Regarding the change of variables, the function $h(v) = -\ln(1 - v)$ was used again. In the following table the survival probabilities conditional on various initial interest rates and starting reserve levels are presented.

u	δ_i	250	300	350	400	450
		$N-$ number of polynomials used for the approximation				
1	1%	0.144815222	0.144815829	0.144815469	0.144814893	0.144814330
	2%	0.146306443	0.146307016	0.146306644	0.146306063	0.146305499
	7%	0.188906830	0.188906028	0.188905174	0.188904470	0.188903928
	8%	0.190954404	0.190953560	0.190952690	0.190951981	0.190951439
	9%	0.191794388	0.191793534	0.191792659	0.191791949	0.191791406
10	1%	0.676382452	0.676390197	0.676389970	0.676387433	0.676384493
	2%	0.689328522	0.689335954	0.689335662	0.689333147	0.689330254
	7%	0.855051985	0.855051380	0.855048719	0.855045981	0.855043652
	8%	0.865060563	0.865059778	0.865057124	0.865054446	0.865052186
	9%	0.870653629	0.870652847	0.870650244	0.870647620	0.870645408
15	1%	0.845057051	0.845074744	0.845078953	0.845078338	0.845076134
	2%	0.864897819	0.864914116	0.864918001	0.864917442	0.864915420
	7%	0.977203365	0.977208141	0.977208683	0.977207860	0.977206693
	8%	0.981995614	0.981999965	0.982000462	0.981999717	0.981998658
	9%	0.984633935	0.984638092	0.984638607	0.984637937	0.984636962
20	1%	0.949938439	0.949967119	0.949978339	0.949982157	0.949982771
	2%	0.967826609	0.967849612	0.967858765	0.967861995	0.967862632
	7%	0.999402042	0.999408967	0.999411851	0.999412960	0.999413267
	8%	0.999715443	0.999721564	0.999724130	0.999725128	0.999725416
	9%	0.999823318	0.999828873	0.999831216	0.999832138	0, 999832412
25	1%	0.993079369	0.993110103	0.993125249	0.993132811	0.993136514
	2%	0.998186486	0.998205547	0.998215150	0.998220073	0.998222575
	7%	0.999986074	0.999991170	0.999993799	0.999995183	0.999995911
	8%	0.999988832	0.999993290	0.999995592	0.999996805	0.999997445
	9%	0.999990114	0.999994091	0.999996147	0.999997232	0.999997804

Figure 1 unveils the impact of the initial conditions on the survival probability. Each curve represents different initial interest rate, the lowest curve corresponds to $\Delta_0 = 1\%$ and the uppermost to $\Delta_0 = 9\%$. The horizontal axis shows the initial reserves level U_0, the vertical axis the survival probability $\Upsilon^i_\alpha(u)$.

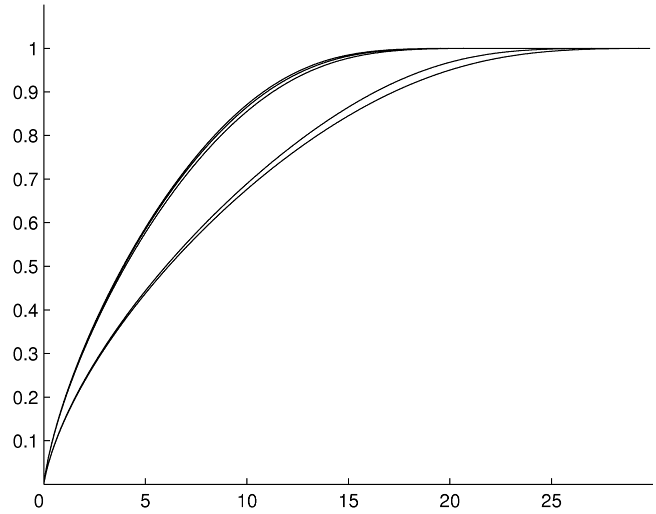

Fig. 1. Survival probability curves as a function of initial reserves. Each curve represents different initial interest level, the lowest curve corresponding to 1%, the uppermost to 9%.

3 Capital allocation problem

Once the capital requirement has been established, the insurance company needs to decide on its allocation. This itself depends on the risk apetite of the insurer and affects the solvency level. Conservative decisions lead to more stable situations, aggressive decissions (investments into risky assets) usually reward higher investment returns but at higher risk.

3.1 The Model

The risk process driving the reserves of an insurance company is assumed to evolve according to the following stochastic differential equation

$$\mathrm{d}R_t = c\mathrm{d}t + \rho \mathrm{d}W_t^{(1)} - X_t, \qquad R_0 = u.$$

This represents premiums collected at constant rate c perturbed by a diffusion with volatility ρ and claims payment that follows a compound Poisson X_t process with intensity λ and jump density function f. $W_t^{(1)}$ is a standard Wiener process independent of X_t.

It is assumed that reserves are invested into a portfolio of assets with expected return μ and volatility σ. The insurance company selects a combination of return and volatility amongst its investment possibilities. The investment possibilities for an insurer are modelled subject to two factors: the general state of economic environment, and the level of funds available for investment. The economic environment is represented by a homogeneous Markov process Y_t with finite state space $\{\delta_1, \ldots, \delta_n\}$ and intensity matrix $Q = \{q_{ij}\}$ and summarises macroeconomic factors that determine investment options such as risk-free rate, inflation rate or economic cycle. The level of funds available for investment U_t conditions the investment options due to transaction costs, divisibility constraints or as a consequence of rational behaviour of market agents as argued by Berk and Green (2004).

Assuming rational behaviour of the investor only Pareto optimal pairs of (μ, σ) will be considered. That is, for a given level of expected return the company would choose the smallest possible level of volatility and, similarly, for a given value of volatility the highest possible level of expected return. Therefore, we can assume the existence of a function reflecting the efficient frontier of the investment possibilities that relates the parameters μ and σ on one-to-one basis. The natural point of view of an insurer is to control the level of risk or the selection of its apetite for risk. For that reason, the volatility is assumed to be chosen based on the parameters of the model and the economic environment Y_t. The value of the corresponding expected return μ is a function of the chosen volatility and the investment opportunities of the insurer (Y_t, U_t).

The stochastic differential equation representing the total reserves process U_t including investment is expressed as

$$\mathrm{d}U_t = [c + \mu_t(\sigma_t, Y_t, U_t)U_t]\,\mathrm{d}t + \sqrt{\rho^2 + U_t^2\sigma_t^2(Y_t)}\mathrm{d}W_t - \mathrm{d}X_t, \qquad U_0 = u. \quad (2)$$

In this work it is assumed that μ_t is a continuous function of time.

3.2 Stochastic Control

Once the insurer selects the desired level of volatility $\sigma_t(Y_t)$ the corresponding expected return $\mu_t(\sigma_t, Y_t, U_t)$ is implied. This way the control variable of the optimisation problem has been reduced to the selection of σ_t. The objective function v to be maximised is expressed as an expected penalty-reward function

$$v^\sigma(T, u, \delta_i) = \mathrm{E}\left[P(U_T, Y_T)\cdot \mathbb{I}_{\{\tau \geq T\}} + L(U_\tau, Y_\tau)\cdot \mathbb{I}_{\{\tau < T\}} | U_0 = u, Y_0 = \delta_i\right] \quad (3)$$

where $\tau = \inf\{t : U_t \notin S\}$ is the exit time of the process U_t form the solvency region S (typically $S = [0, \infty)$)

The concept of the expected penalty-reward function presented in Gerber and Shiu (1997) and Gerber and Shiu (1998a) is a general framework comprising several quantities of interest as special cases, such as the time to ruin, the amount at and immediately prior to ruin or survival probabilities. Let us denote $J(T,u,y)$ the optimal value of the maximisation problem

$$J(T,u,y) \equiv \max_{\sigma\in\Pi} v^{\sigma}(T,u,y). \tag{4}$$

The set Π contains all admissible strategies σ_t, that is the strategies for which a solution of (2) exists. We will focus only on Markov strategies, that is, σ_t depends on the process $\{U_s\}_{0\le s\le\infty}$ only through U_t. Suppose that σ^* is the maximising value for $v^{\sigma}(T,u,y)$ then $J(T,u,y) = v^{\sigma^*}(T,u,y)$.

The details about the development of the numerical procedure is deferred to the appendix. In the following section we present important examples that illustrate the method

Solving the problem (4) directly is not feasible since an explicit expression for $v^{\sigma}(T,u,y)$ is not available in the most general case. However, in order to follow the dynamic programming approach, one can introduce the starting time t an write the value function as

$$J(t,T,u,y) \equiv \max_{\sigma\in\Pi} v^{\sigma}(t,T,u,y).$$

where

$$v^{\sigma}\left(t,T,u,\delta_i\right) = \mathrm{E}\left[P(U_T,Y_T)\cdot \mathbb{I}_{\{\tau\ge T\}} + L(U_\tau,Y_\tau)\cdot \mathbb{I}_{\{\tau<T\}} | U_t = u, Y_t = \delta_i\right].$$

Then the Hamilton-Jacobi-Bellman equation that the value function $J(t,T,u,y)$ satisfies is

$$\sup_{\sigma\in\Pi}\Bigg\{-\frac{\partial J}{\partial t} + \frac{1}{2}(\rho^2 + \sigma^2(\delta_i)u^2)\frac{\partial^2}{\partial u^2}J + (c + \mu(\sigma(\delta_i),\delta_i,u)u)\frac{\partial}{\partial u}J$$
$$+\sum_{j=1}^{n} q_{ij} J\left(t,T,u,\delta_i\right) + \lambda\int_0^{\infty}\left(J\left(t,T,u-x,\delta_i\right) - J\left(t,T,u,\delta_i\right)\right) f\left(x\right)\mathrm{d}x\Bigg\} = 0. \tag{5}$$

This equation is too complex to be solved analytically in the most general case. The optimal strategy $\sigma^*(\delta_i)$ is found for each δ_i differentiating (5) with respect to $\sigma(\delta_i)$ as a solution to

$$\sigma(\delta_i)u^2 J_1^{(uu)} + \frac{\mathrm{d}}{\mathrm{d}\sigma}\mu\left(\sigma,\delta_i,u\right)uJ_1^{(u)} = 0,$$

where $J_1^{(u)}$ and $J_1^{(uu)}$ is the first and the second derivative with respect to u. In case of a linear relationship between volatility σ and expected return μ, as for example in the case of CAPM efficient frontier, this reduces to

$$\sigma(\delta_i)u^2 J_1^{(uu)} + \mu\left(\delta_i,u\right)uJ_1^{(u)} = 0 \tag{6}$$

whence

$$\sigma^*(\delta_i) = -\mu(\delta_i, u)\frac{J_1^{(u)}}{uJ_1^{(uu)}}.$$

In particular, assuming that the investment possibilities do not depend on available capital one gets

$$\sigma^*(\delta_i) = -\mu(\delta_i)\frac{J_1^{(u)}}{uJ_1^{(uu)}}$$

a solution found in Hipp and Plum (2000) and Bauerle and Rieder (2004). The latter authors realise that the optimal strategy is constant for CRRA utility functions and linear specification of the underlying risk process. In general, however, the solution is hard to find. Irgens and Paulsen (2004) study the optimal investment (and other solvency variables) under exponential utility, Yang and Zhang (2005) give an explicit solution for exponential utility under simplified market model.

In this work we present a numerical method to approximate the value function $J(T, x, y) \equiv J(0, T, x, y)$. The idea of Carr (1998) to approximate a fixed horizon T by a series of consecutive exponential intervals (random horizon with Erlang-n distribution) will be applied assuming that the strategy σ_n^* is constant on each interval. It will be shown that this solution converges to the optimal solution as the number of intervals n approaches infinity.

The details about the development of the numerical procedure is deferred to the appendix. In the following section we present the example that illustrates the method.

3.3 Example

In this section we will illustrate the application of the theorems proved above. The risk process considered follows

$$\mathrm{d}R_t = c\mathrm{d}t + \rho\mathrm{d}W_t - \mathrm{d}X_t, \qquad R_0 = u \tag{7}$$

where X_t is a compound Poisson process with intensity $\lambda = \frac{1}{3}$ and lognormal claim size distribution $\mathcal{LN}(1, 2)$. This process represents claims collected at a constant rate $c = 3$ perturbed by a diffusion with volatility $\rho^2 = 0.25$ that can be interpreted as aggregate small claims and claims collection accruals. The Poisson process then represents catastrophic claims (with average occurrence once every 3 periods) with lognormal (heavy-tail) severity distribution. The investment opportunities will be represented by a riskless asset $\mathrm{d}S_t^{(1)} = r\mathrm{d}t$ and a risky asset $\mathrm{d}S_t^{(2)} = \nu\mathrm{d}t + \xi\mathrm{d}W_t$. The proportion invested into a risky asset will be denoted as π. No short-selling is allowed, therefore $\pi \in [0, 1]$.

The parameters of available assets are taken as follows $r = 1, \nu = 2, \xi = 1$. Altogether, the reserve process, including investment, can be written as

$$\mathrm{d}U_t = [c + (r + \pi(\nu - r))U_t]\,\mathrm{d}t + \sqrt{\rho^2 + \pi^2\xi^2U_t^2}\mathrm{d}W_t - \mathrm{d}X_t \qquad U_0 = u \quad (8)$$

In the view of the equation (2) this implies the following linear relationship between the volatility $\sigma = \pi\xi$ and expected return on investment $\mu\sigma$

$$\mu(\sigma) = r + \pi(\nu - r) = r + \sigma\frac{\nu - r}{\xi}. \quad (9)$$

Notice that no Markov modulation is considered in this example (what is equivalent to taking Y_t = constant) as no further insight would be added besides more complex notation. The penalty-reward function considered is $P(u) = 1$, $L(u) = 0$

$$v^\sigma(T, u) = \mathrm{E}\left[\mathbb{I}_{\{\tau \geq T\}}|U_0 = u\right] = \mathrm{P}\left[\tau \geq T|U_0 = u\right] \equiv \varphi(u, T) \quad (10)$$

what represents the survival probability. The optimisation problem (4) then turns to maximisation of survival probability in a fixed horizon T. Similar problems have been treated in Hipp and Plum (2003) and others but no closed form solution exist. Following the development presented above, in order to be able to apply the iterative scheme from Theorem 6.3, the fixed horizon T will be approximated by a series of n exponential horizons with parameter $\frac{n}{T}$

$$\varphi^*(u, \alpha) = \int_0^\infty \varphi(u, T)\alpha\mathrm{e}^{-\alpha T}\mathrm{d}t. \quad (11)$$

The fixed horizon T will be approximated by a series of n exponential horizons with parameter $\frac{n}{T}$, whereas in each of the horizons the problem to be solved is

$$J_i(\frac{n}{T}, u) \equiv \max_{\pi \in \Pi} \Upsilon^\sigma(\alpha, u, J_{i-1}). \quad (12)$$

with $J_0 = P = 1$. As proved in the theorem 6.4, to achieve convergence, it is sufficient to consider strategies π constant on each exponential interval. The function Υ for a constant π satisfies satisfies the following integro-differential equation

$$\frac{1}{2}(\rho^2 + \pi^2\xi^2u^2)\frac{\partial^2}{\partial u^2}\Upsilon + (c + (r + \pi(\nu - r))u)\frac{\partial}{\partial u}\Upsilon \\ - (\lambda + \alpha)\Upsilon + \lambda\int_0^u \Upsilon(\alpha, u - x)f(x)\mathrm{d}x + \alpha J_{i-1}(u) = 0. \quad (13)$$

as derived in section 2 where we also propose an approximation method by chebyshev polynomials to calculate the solution to this problem. Since feasible strategies are bounded it is possible to evaluate Υ for a grid of possible values of $\pi \in [0, 1]$ and take the maximum value as an approximation to the solution of (12). In this example we took equidistant grid of granularity 0.1. The table 1 shows the results of approximated value function $J(u, T)$ for various values of initial reserves u and number of exponential intervals n that approximate the fixed horizon $T = 10$. The convergence is achieved to up to 3 decimal places for as few as 100 intervals.

n – number of intervals

		1	2	5	10	20	50	100
u	0.1	0.354370	0.306438	0.287529	0.288241	0.291537	0.295415	0.297638
	0.5	0.413865	0.361477	0.341409	0.341409	0.346224	0.349630	0.350752
	1	0.427487	0.377567	0.359446	0.361308	0.365365	0.369168	0.370406
	2	0.456898	0.412100	0.397678	0.400614	0.405446	0.410055	0.411589
	5	0.570571	0.542250	0.536840	0.541189	0.547234	0.554050	0.556663
	10	0.886121	0.882775	0.883860	0.885951	0.888517	0.891156	0.891655
	15	0.999024	0.998997	0.999008	0.999027	0.999049	0.999072	0.999074

Figure 2 depicts $J(u, T)$ (maximal survival probability in horizon $T = 10$) as a function of u for $n = 1, 2, 5, 10, 20, 50, 100$. The optimal strategy that leads to the value function can be recovered using the relationship between the value function and the optimal strategy given by (6).

4 Conclusions

In this work, we have treated the problems related to risk theory and stochastic control in the context of Lévy diffusion processes. It was argued that the Lévy diffusions provide a fairly general framework covering varied modelisation paradigms from finance and insurance areas. In particular a compound Poisson process with Markov-modulated parameters has been analysed, nevertheless, general results include wider spectrum of stochastic models.

In the first part we have presented a new approximation procedure for the calculation of the penalty-reward function in a risk theory context. The importance of the contribution is underlined by the fact that previously no solution was available in the general setting that has been proposed in this work.

The second part demonstrates the power of the approaches presented earlier by solving a stochastic control problem of optimal investment of an insurer facing

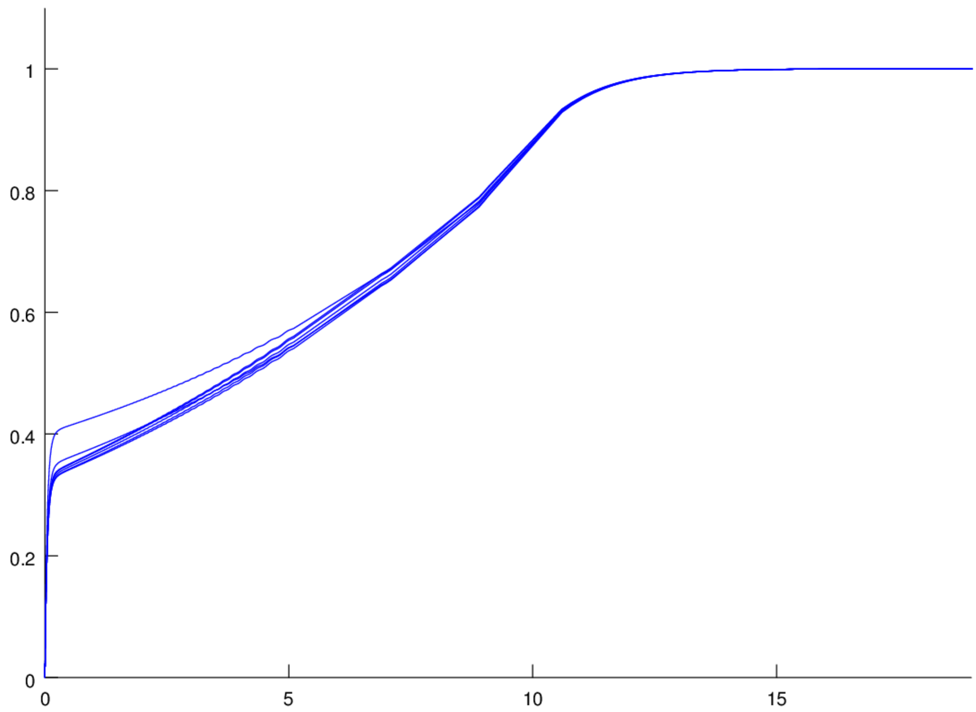

Fig. 2. Convergence of the maximal survival probability in the horizon $T = 10$ as a function of the initial reserve.

risk management decisions in a context that has not been treated previously. An example shows how maximum survival probability curve can be obtained for different levels of initial conditions.

Besides the theoretical interest of the presented results we believe that these could become relevant analytical tools in practical applications, in both regulatory bodies and internal control processes within insurance companies. Sensible models that are able to explain the behaviour and quantify the answers posed about solvency, profitability or other nature of insurance business are needed in decision making processes.

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6 Appendix

6.1 Capital Requirement Calculation

The risk process presented by Gerber (1970) extends the classical model of risk theory introducing a Brownian diffusion. The total claims follow a compound Poisson process $\{X_t, t \geq 0\}$ with Lévy measure $\lambda f(x)\, \mathrm{d}x$, λ being the intensity of arrivals and f the density of jumps. The collection of premiums is driven by a Wiener process W_t^c independent of X_t with drift c and volatility σ, thus the perturbed risk process with initial surplus u is given by

$$\mathrm{d}R_t = c\mathrm{d}t + \sigma \mathrm{d}W_t^c - \mathrm{d}X_t, \qquad R_0 = u. \tag{14}$$

This process has been considered by Dufresne and Gerber (1991) where a defective renewal equation was derived for the probability of ruin $\psi(u) = \Pr(\tau < \infty)$ where $\tau = \inf\{t \geq 0 : R_t < 0\}$. A review of the research on this type of processes can be found in Asmussen and Albrecher (2010), Chapter 11. Generalisation of the model are treated in Li and Garrido (2005), Sarkar and Sen (2005), and Morales (2007), whereas Ren (2005) gives explicit formulae to calculate the ruin probability and related quantities for phase-type distributed claims.

Let us now allow the insurer to invest the reserves U_t into an asset with time-dependent Markov modulated return rate (drift) Δ_t and volatility $\kappa(U_t)$, that possibly depends on the amount invested U_t, driven by a Wiener process W_t^I independent of the risk process R_t

$$\mathrm{d}U_t = \left(\Delta_t \mathrm{d}t + \kappa(U_t)\, \mathrm{d}W_t^I\right) U_t + dR_t, \qquad U_0 = R_0 = u \tag{15}$$

The drift parameter Δ_t is governed by a finite state homogeneous Markov process with state space $\{\delta_1, \ldots, \delta_n\}$, intensity matrix $Q = (q_{ij})_{n\times n}$ and initial state δ_i. For example, Δ_t can be used to model the risk free rate announced by a central bank that evolves according to the Markov process by, for instance, 25 basis points jumps. The state space would be in this case e.g.,

$$1.00\%, 1.25\%, 1.50\%, 1.75\%, 2.00\%, \ldots, 9.00\%.$$

This environment offers considerable versatility in capturing the evolution of interest rates since any diffusion model to forecast the yield curve can be approximated arbitrarily well by continuous time Markov chains, see Kushner and Dupuis (1992). Variation of the volatility according to the size of the funds invested is justified, for example, by Berk and Green (2004) as an implication of their study of the performance of mutual funds and resulting rational capital flows. A particular shape of κ suggested in the cited paper, $\kappa(u) = \dfrac{\sigma_r}{\sqrt{u}}$, yields a surplus process in the form of an affine diffusion that was studied by Avram and Usábel (2008) in this context. Many practical ideas support a fund-dependent volatility, for instance the possibility to obtain more efficient portfolios, due to transaction costs, when more money is available. Model (15) is a generalisation of the process considered most frequently in the literature where the return rate and the volatility are constant in time, $\Delta_t = \delta$, $\kappa(\cdot) = \sigma_r$, like in Paulsen (1993), Paulsen and Gjessing (1997), Wang (2001), Ma and Sun (2003), Gaier and Grandits (2004), Grandits (2005), Cai and Yang (2005), Wang and Wu (2008).

The stochastic differential equation (15) can be arranged into

$$\mathrm{d}U_t = (c + \Delta_t U_t)\,\mathrm{d}t + \sqrt{\sigma^2 + \kappa^2 (U_t)\, U_t^2}\mathrm{d}W_t - \mathrm{d}X_t \tag{16}$$

with initial condition $(U_0, \Delta_0) = (u, \delta_i)$. The expected penalty-reward function, see Gerber and Landry (1998), is introduced

$$\phi_t^i (u) = \mathrm{E}\left[\pi (U_\tau)\, \mathbb{I} (\tau \le t) + P (U_t)\, \mathbb{I} (\tau > t) \mid U_0 = u, \Delta_0 = \delta_i\right] \tag{17}$$

where $\tau = \inf\{s \ge 0 : U_s < 0\}$. If ruin occurs before the time horizon t, the penalty $\pi (U_\tau)$ applies to the overshoot U_τ at the ruin. Otherwise, the reward function $P (U_t)$ applies to the reserves at time t. The concept of the expected penalty-reward function presented in Gerber and Shiu (1997) and Gerber and Shiu (1998a) is a quite general framework comprising several quantities of interest as a special case, such as the time to ruin, the amount at and immediately prior to ruin or survival probabilities.

For further analysis the smoothed version of the function $\phi_t^i(u)$ will be considered, namely its Laplace-Carson transform in time defined as

$$\Upsilon_\alpha^i (u) = \int_0^\infty \alpha \mathrm{e}^{-\alpha t} \phi_t^i (u)\, \mathrm{d}t.$$

Further, letting H_α be an exponentially distributed random variable with parameter α, the former expression may be viewed as a penalty-reward function with an exponentially killed time horizon, see expression (6) in Avram and Usábel (2008),

$$\begin{aligned} \Upsilon_\alpha^i (u) &= \int_0^\infty \alpha \mathrm{e}^{-\alpha t} \phi_t^i (u)\, \mathrm{d}t = \mathrm{E}\left(\phi_{H_\alpha}^i (u)\right) \\ &= \mathrm{E}\left(\pi (U_\tau)\, \mathbb{I} (\tau \le H_\alpha) + P (U_{H_\alpha})\, \mathbb{I} (\tau > H_\alpha) \mid U_0 = u, \Delta_0 = \delta_i\right) \end{aligned} \tag{18}$$

where the last equality comes from substituting the definition of $\phi_t^i (u)$, in (17).

The function $\Upsilon_\alpha^i(u)$ is analytically more tractable than the original function while, at the same time, retains a probabilistic interpretation as a penalty-reward function considering an exponential random time horizon H_α.

6.1.1 Integro-differential System

This section presents further treatment of the transformed expected penalty-reward function defined by (18). The function $\Upsilon_\alpha^i(u)$ is dependent on the initial reserves $U_0 = u$ and the starting return rate $\Delta_0 = \delta_i$. Since the process driving the return rate Δ_t has a finite state space, the number of initial conditions is also finite. Therefore, one can consider the set of functions $\Upsilon_\alpha(u) = (\Upsilon_\alpha^1(u), \Upsilon_\alpha^2(u), \dots \Upsilon_\alpha^n(u))$, each corresponding to different starting return rate from the state space $\{\delta_1, \dots, \delta_n\}$. Below, a Volterra integro-differential system of equations for the functions $\Upsilon_\alpha^1(u), \Upsilon_\alpha^2(u), \dots \Upsilon_\alpha^n(u)$ is derived and, applying the result of Le and Pascali (2009), sufficient conditions for the existence of the solution are established.

Theorem 6.1 *For all $\alpha \geq 0$, functions $\Upsilon_\alpha^i : [0,\infty) \to \mathbb{R}$ defined in (18) satisfy the following system of integro-differential equations*

$$\begin{aligned}
&\textit{For } i = 1, \dots, n \\
&\qquad \frac{1}{2}\left(\sigma^2 + u^2\kappa^2(u)\right)\frac{\mathrm{d}^2}{\mathrm{d}u^2}\Upsilon_\alpha^i(u) + (c + \delta_i u)\frac{\mathrm{d}}{\mathrm{d}u}\Upsilon_\alpha^i(u) + \\
&\qquad + \sum_{j=1}^{n} q_{ij}\Upsilon_\alpha^j(u) - (\alpha + \lambda)\Upsilon_\alpha^i(u) + \lambda\int_0^u \Upsilon_\alpha^i(u-x)\; f(x)\;\mathrm{d}x + \\
&\qquad + \alpha P(u) + \lambda\int_u^\infty \pi(u-x)\; f(x)\;\mathrm{d}x = 0. \qquad (19)
\end{aligned}$$

Given that $\lim_{u\to\infty} P(u)$ exists, $\sigma > 0$ and assuming positive security loading for the reserve process (15), the boundary conditions of the system are

$$\begin{aligned}
\Upsilon_\alpha^i(0) &= \pi(0-) \qquad (20)\\
\lim_{u\to\infty}\Upsilon_\alpha^i(u) &= \lim_{u\to\infty} P(u) \equiv P(\infty)
\end{aligned}$$

Moreover, if $f \in C^2[0,\infty)$, $P(u)$ and $\kappa(u)$ are continuous for $u \geq 0$ and $\pi(u)$ integrable, then the system of equations (19) has a solution $\Upsilon_\alpha^i \in C^2[0,\infty)$, $i = 1, \dots, n$.

Proof First, a straightforward application of Ito's lemma yields the infinitesimal generator of the process U_t, which applied to the functions $\phi_t^i(u)$, $i = 1, ..., n$ defined by (17), yields

$$\mathcal{A}\phi_t^i(u) = \frac{1}{2}\left(\sigma^2 + u^2\kappa^2(u)\right)\frac{\mathrm{d}^2}{\mathrm{d}u^2}\phi_t^i(u) + (c+\delta_i u)\frac{\mathrm{d}}{\mathrm{d}u}\phi_t^i(u) + \sum_{j=1}^{n} q_{ij}\phi_t^j(u) +$$
$$+\lambda\int_0^\infty \left(\phi_t^i(u-x) \quad - \phi_t^i(u)\right) f(x)\,\mathrm{d}x.$$

Functions $\phi_t^i(u)$ satisfy the Fokker-Planck equation, see e.g. Risken (1996),

$$\mathcal{A}\phi_t^i(u) - \frac{\partial\phi_t^i(u)}{\partial t} = 0 \tag{21}$$

with boundary conditions

$$\phi_0^i(u) = P(u) \qquad u > 0 \tag{22a}$$
$$\phi_t^i(u) = \pi(u) \qquad u < 0 \text{ and } t \geq 0 \tag{22b}$$

for each $i = 1, 2\ldots, n$. Using (22b) the following holds

$$\int_0^\infty \phi_t^i(u-x) f(x)\,\mathrm{d}x = \int_0^u \phi_t^i(u-x) f(x)\,\mathrm{d}x + \int_u^\infty \pi(u-x)\; f(x)\,\mathrm{d}x. \tag{23}$$

Substituting the infinitesimal generator and (23) into the Fokker-Planck equation yields

$$\frac{1}{2}\left(\sigma^2 + u^2\kappa^2(u)\right)\frac{\mathrm{d}^2}{\mathrm{d}u^2}\phi_t^i(u) + (c+\delta_i u)\frac{\mathrm{d}}{\mathrm{d}u}\phi_t^i(u) + \sum_{j=1}^{n} q_{ij}\phi_t^j(u) - \lambda\phi_t^i(u) +$$
$$+\lambda\int_0^u \phi_t^i(u-x) f(x)\,\mathrm{d}x + \lambda\int_u^\infty \pi(u-x)\; f(x)\,\mathrm{d}x - \frac{\partial\phi_t^i(u)}{\partial t} = 0.$$

The system (19) is obtained taking the Laplace-Carson transform with respect to t on both sides and expanding the last term integrating by parts

$$\int_0^\infty \alpha\mathrm{e}^{-\alpha t}\frac{\partial\phi_t^i(u)}{\partial t}\mathrm{d}t = -\alpha P(u) + \alpha\int_0^\infty \alpha\mathrm{e}^{-\alpha t}\phi_t^i(u)\,\mathrm{d}t = -\alpha P(u) + \alpha\Upsilon_\alpha^i(u)$$

where the first boundary condition (22a) of the Fokker-Planck equation was used.

Concerning the boundary conditions of the integro-differential system, when the initial reserves are 0 and $\sigma > 0$, the presence of the Wiener fluctuation in premiums causes immediate crossing of 0 level – see for example the proof of Theorem 2.1 in Paulsen and Gjessing (1997). The second condition is the asymptotic case $u \to \infty$ when under the assumption of positive security loading $\lim_{u\to\infty}\Upsilon_\alpha^i(u) = \lim_{u\to\infty} P(u) < \infty$.

To prove the existence of the solution, an equivalent system will be considered. A change of variable is now introduced in the System (19), $h(v) = u$, where $h : [0, 1] \to [0, \infty)$ is an arbitrary strictly monotone, twice continuously differentiable function. The system can now be written in terms of the functions $\Gamma^i_\alpha(v) = \Upsilon^i_\alpha(h(v))$.

For $i = 1, \dots, n$

$$\begin{aligned}
&A(v)\frac{\mathrm{d}^2}{\mathrm{d}v^2}\Gamma^i_\alpha(v) + B_i(v)\frac{\mathrm{d}}{\mathrm{d}v}\Gamma^i_\alpha(v) + \sum_{j=1}^{n} q_{ij}\Gamma^j_\alpha(v) - \\
&-(\alpha+\lambda)\Gamma^i_\alpha(v) + \lambda\int_0^v \Gamma^i_\alpha(y) f(h(v)-h(y)) h'(y)\,\mathrm{d}y + \\
&+\lambda S(v) + \alpha P(h(v)) = 0 \qquad (24)
\end{aligned}$$

where

$$\begin{aligned}
A(v) &= \frac{\sigma^2 + h^2(v)\kappa^2(h(v))}{2[h'(v)]^2} \\
B_i(v) &= \frac{c + \delta_i h(v)}{h'(v)} - \frac{[\sigma^2 + h^2(v)\kappa^2(h(v))]h''(v)}{2[h'(v)]^3} \\
S(v) &= \int_v^1 \pi(h(v)-h(y)) f(h(y)) h'(y)\,\mathrm{d}y
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
\Gamma^i_\alpha(0) &= \pi(0-) \qquad (25) \\
\Gamma^i_\alpha(1) &= \lim_{u\to\infty} P(u).
\end{aligned}$$

Here h' and h'' denote the first and the second derivative of function h. Finally, by integration

$$\begin{aligned}
\Gamma^i_\alpha(s) &= \Gamma^i_\alpha(0) + \int_0^s \frac{h'(v)}{B_i(v)}\left[H(v) - \lambda\int_0^v f(h(v)-h(y))\frac{h'(y)}{h'(v)}\Gamma^i_\alpha(y)\,\mathrm{d}y\right]\mathrm{d}v \\
H(v) &= \frac{-1}{h'(v)}[A(v)\frac{\mathrm{d}^2}{\mathrm{d}v^2}\Gamma^i_\alpha(v) + \sum_{j=1}^{n} q_{ij}\Gamma^i_\alpha(v) - \\
&\quad -(\alpha+\lambda)\Gamma^i_\alpha(v) + \alpha P(h(v)) + \lambda S(v)].
\end{aligned}$$

The existence of the solution $\Gamma^i_\alpha \in C^2[0, 1]$ is guaranteed by Theorem 2 in Le and Pascali (2009), as $H(v)$ is a continuous function and $f(h(v) - h(y))\frac{h'(y)}{h'(v)}$ is integrable. The integrability is immediate as f is a density function and

$\frac{h'(y)}{h'(v)}$ is a bounded function of y on $[0, v]$ for all v. This implies that $\Upsilon_\alpha^i(u) = \Gamma_\alpha^i(h^{-1}(u))$, a solution to (19), exists and $\Upsilon_\alpha^i \in C^2[0, \infty)$. $\square$

6.1.2 Numerical Solution

The second order system of integro-differential equations (19) that characterises the Laplace-Carson transform of the expected penalty-reward function (18) does not have an explicit solution. In Akyuz-Dascioglu and Sezer (2005) and Akyuz-Dascioglu (2007) a numerical method was proposed for fairly general families of Fredholm-Volterra integro-differential systems of higher order which include the system treated in this chapter as a special case. The authors approximate the solution to the system by shifted Chebyshev polynomials on the interval $[0, 1]$. A collocation method is used to fit the Chebyshev expansion of the solution. In order to adapt the procedure to system (19), we need to transform the domain of the unknown functions Υ_α^i, as was done in the proof of Theorem 6.1, from the interval $[0, \infty)$ to $[0, 1]$. First, the solution Γ_α^i of the transformed system is found and then, applying the inverse transform, the functions of interest Υ_α^i are recovered. The convergence of the method is treated in the original article along with the illustrative examples that compare the approximation and the exact solutions showing outstanding performance. The following section describes the method adapted to the setting of this chapter to keep it self-contained. The presentation follows the development in Akyuz-Dascioglu and Sezer (2005).

6.1.3 Approximation by Chebyshev Polynomials

In matrix notation the transformed system is given by

$$\mathbf{P}_2(v)\frac{\mathrm{d}^2}{\mathrm{d}v^2}\mathbf{\Gamma}_\alpha(v) + \mathbf{P}_1(v)\frac{\mathrm{d}}{\mathrm{d}v}\mathbf{\Gamma}_\alpha(v) + \mathbf{P}_0(v)\mathbf{\Gamma}_\alpha(v) = \\ = \mathbf{g}(v) + \int_0^v \mathbf{K}(v, y)\mathbf{\Gamma}_\alpha(y)\,\mathrm{d}y \tag{26}$$

where $\mathbf{\Gamma}_\alpha(v)$ is the column vector of unknown functions $\mathbf{\Gamma}_\alpha(v) = \left(\Gamma_\alpha^1(v), \Gamma_\alpha^2(v), \ldots, \Gamma_\alpha^n(v)\right)^\top$. Coefficient matrices are as follows

$$
\begin{aligned}
\mathbf{P}_2(v) &= \frac{A(v)}{h'(v)} \cdot \mathrm{I}_n \\
\mathbf{P}_1(v) &= h'(v)^{-1}\, diag\,(B_i(v)) \\
\mathbf{P}_0(v) &= h'(v)^{-1}\left[Q - (\alpha + \lambda) \cdot \mathrm{I}_n\right] \\
\mathbf{K}(v, y) &= -\lambda f\,(h(v) - h(y))\, \frac{h'(y)}{h'(v)} \cdot \mathrm{I}_n \\
\mathbf{g}(v) &= -h'(v)^{-1}\left[\alpha P\,(h(v)) + \lambda S(v)\right] \cdot 1_n \\
S(v) &= \int_v^1 \pi\,(h(v) - h(y))\, f\,(h(y))\, h'(y)\, \mathrm{d}y,
\end{aligned}
$$

where I_n is the identity matrix of order $n \times n$ and 1_n is the column vector of ones of order $n \times 1$. The transform is performed with an arbitrary strictly monotone, twice continuously differentiable function $h : [0, 1] \to [0, \infty)$.

The aim of the method is to approximate the solution by a truncated Chebyshev expansion

$$
\Gamma_\alpha^i(v) = \sum_{r=0}^{N} a_{ir}^* T_r^*(v) \qquad i = 1, \ldots, n
$$

on the interval $[0, 1]$, where $T_r^*(v)$ are shifted Chebyshev polynomials of the first kind (see, for example, Boyd (2001)) and a_{ir}^* are the unknown coefficients to be determined. In matrix notation

$$
\Gamma_\alpha^i(v) = T^*(v)\, A_i^*,
$$

where $T^*(v) = (T_0^*(v), T_1^*(v), \ldots, T_N^*(v))$ is a row vector of shifted Chebyshev polynomials up to degree N and $A_i^* = (a_{i0}^*, a_{i1}^*, \ldots a_{iN}^*)^\top$ is a column vector of the corresponding coefficients. Similarly, the $n - th$ derivative of $\Gamma_\alpha^i(v)$ can be expanded into

$$
\frac{\mathrm{d}^n}{\mathrm{d}v^n} \Gamma_\alpha^i(v) = T^*(v)\, A_i^{*(n)}. \tag{27}
$$

The link between coefficients $A_i^{*(n)}$ and A_i^* from Sezer and Kaynak (1996) is

$$
A_i^{*(n)} = 4^n M^n A_i^*, \tag{28}
$$

where

$$M = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \cdots & \frac{N}{2} \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(N+1)\times(N+1)} \quad \text{for odd } N$$

$$M = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \cdots & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & N \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \end{pmatrix}_{(N+1)\times(N+1)} \quad \text{for even } N$$

yields the expansion of the $n-th$ derivative $\frac{\mathrm{d}^n}{\mathrm{d}v^n}\Gamma_\alpha^i(v)$ in terms of Chebyshev coefficients A_i^*.

On the other hand, functions $K_{ij}(v,y)$ can be expanded in variable y into a Chebyshev series

$$K_{ij}(v,y) = \sum_{r=0}^{N} k_r^{*ij}(v)\, T_r^*(y)$$

where the Chebyshev coefficients k_r^{*ij} are functions of v. Using matrix notation for convenience

$$K_{ij}(v,y) = k^{*ij}(v)\, T^*(y)^\top, \tag{29}$$

where k^{*ij} is the row vector of coefficients determined by Clenshaw-Curtis quadrature, see Clenshaw and Curtis (1960).

Substituting (27), (28) and (29), the $i-th$ equation $(i = 1,\ldots,n)$ of the system (26) is finally obtained:

$$\begin{aligned} & h'(v)^{-1} A(v)\, 16M^2 T^*(v)\, A_i^* + h'(v)^{-1} B_i(v)\, 4MT^*(v)\, A_i^* + \\ & + h'(v)^{-1}\left[\sum_{j=1}^{n} q_{ij} - (\alpha+\lambda)\right] T^*(v)\, A_i^* = g_i(v) - \int_0^v k^{*ij}(v)\, T^*(y)^\top T^*(y)\, A_i^* \mathrm{d}y. \end{aligned}$$

The matrix of the inner product of Chebyshev polynomials

$$Z^*(v) = \left(z_{ij}^*(v)\right) \equiv \int_0^v T^*(y)^\top T^*(y)\,\mathrm{d}y =$$
$$= \frac{1}{2}\int_{-1}^{2v-1} T(x)^\top T(x)\,\mathrm{d}x = \frac{1}{2}\left(z_{ij}(2v-1)\right) = \frac{1}{2}Z(2v-1)$$

can be computed as shown in Akyuz-Dascioglu (2007), where

$$z_{ij}(v) = \frac{1}{4}\begin{cases} 2v^2 - 2 & \text{for } i+j=1 \\ \frac{T_{i+j+1}(v)}{i+j+1} - \frac{T_{i+j-1}(v)}{i+j-1} - \frac{1}{i+j+1} + \frac{1}{i+j-1} + v^2 - 1 & \text{for } |i-j|=1 \\ \frac{T_{i+j+1}(v)}{i+j+1} + \frac{T_{1-i-j}(v)}{1-i-j} + \frac{T_{1+i-j}(v)}{1+i-j} + \frac{T_{1-i+j}(v)}{1-i+j} + 2\left(\frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2}\right) & \text{for even } i+j \\ \frac{T_{i+j+1}(v)}{i+j+1} + \frac{T_{1-i-j}(v)}{1-i-j} + \frac{T_{1+i-j}(v)}{1+i-j} + \frac{T_{1-i+j}(v)}{1-i+j} - 2\left(\frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2}\right) & \text{for odd } i+j \end{cases},$$

which yields the system

$$h'(v)^{-1} A(v)\, 8M^2 T^*(v)\, A_i^* + h'(v)^{-1} B_i(v)\, 4M T^*(v)\, A_i^* + \tag{30}$$
$$+h'(v)^{-1}\left[\sum_{j=1}^{n} q_{ij} - (\alpha+\lambda)\right] T^*(v)\, A_i^* = g_i(v) - k^{*ij}(v)\, Z^*(v)\, A_i^*,$$

for all $i = 1, \dots, n$. The only unknown values are Chebyshev expansion coefficients A_i^*. The collocation method proposed by the authors fits the solution through the collocation points

$$x_s = \frac{1}{2}\left(1 + \cos\left(\frac{s}{N}\pi\right)\right), \qquad s = 1, 2, \dots, (N-1).$$

Each of the $N-1$ collocation points x_s is substituted into the system (30) and yields n linear equations of unknown variable A_i^*, whence $n(N-1)$ equations are obtained. The boundary conditions (25) for $i = 1, \dots, n$,

$$T^*(0)\, A_i^* = \pi(0-)$$
$$T^*(1)\, A_i^* = P(\infty),$$

yield another $2n$ equations. A linear system of $n(N+1)$ equations is constructed and solved for the Chebyshev coefficients A_i^*. Once the approximation $\widetilde{\Gamma}_\alpha^i(v) = \sum_{r=0}^{N} a_{ir}^* T_r^*(v)$ is obtained, the relationship between the solution of the transformed and the original system from the Theorem 6.1 yields the approximation of the expected penalty-reward function $\widetilde{\Upsilon}_\alpha^i(u) = \widetilde{\Gamma}_\alpha^i(h^{-1}(u))$.

6.2 Capital allocation problem

6.2.1 Exponential horizon

As a first step, the maximisation problem (4) will be considered in a hypothetical exponential random horizon H_α instead of a fixed horizon T. Let H_α be an exponentially distributed random variable with parameter α. If one lets $\alpha = \frac{1}{T}$ then $\mathrm{E}(H_\alpha) = T$, that is, in expected terms, the random horizon H_α and the fixed horizon T coincide.

Let us denote Υ the objective expected penalty-reward function that is to be maximised in a random horizon H_α, that is

$$\begin{aligned}\Upsilon^\sigma(\alpha, u, y) \equiv & \, \mathrm{E}\left[v^\sigma(H_\alpha, u, y)\right] = \\ = & \, \mathrm{E}\left[P(U_{H_\alpha}, Y_{H_\alpha}) \cdot \mathbb{I}_{\{\tau \geq H_\alpha\}} + L(U_\tau, Y_\tau) \cdot \mathbb{I}_{\{\tau < H_\alpha\}} | U_0 = u, Y_0 = y\right].\end{aligned}$$

Notice that the second expectation is taken with respect to random horizon H_α and the stochastic process U_t. The optimisation problem is similar to (4), J_1 will represent the optimal value

$$J_1(\alpha, u, y) \equiv \max_{\sigma \in \Pi} \Upsilon^\sigma(\alpha, u, y). \tag{31}$$

The objective function Υ can be seen as a Laplace-Carson transform in time of v defined in (3), indeed

$$\Upsilon^\sigma(\alpha, u, y) = \mathrm{E}\left[v^\sigma(H_\alpha, u, y)\right] = \int_0^\infty v^\sigma(t, u, y)\, \alpha \mathrm{e}^{-\alpha t} \mathrm{d}t. \tag{32}$$

Laplace-Carson transform $C(s)$ of an integrable function is closely related to its Laplace transform $L(s)$ by the relationship $C(s) = sL(s)$. This fact can be exploited to obtain the solution of the problem (31) since the Laplace transform of the function v is more easily obtained in many scenarios.

6.2.2 Erlangian horizon

The next step is to approximate the fixed horizon T in the problem (4) by a series of consecutive exponential horizons. The distribution of a sum of k independent variables with identical exponential distribution of parameter α is the Erlang (α, k) distribution. Its density function is given by

$$p(x) = \frac{\alpha^k}{(k-1)!} x^{k-1} \mathrm{e}^{-\alpha x}.$$

Its mean is $\frac{k}{\lambda}$ and variance $\frac{k}{\lambda^2}$.

Let H^n_α be a random variable with Erlang(α, n) distribution. Let us consider a series of random variables

$$H^n_{\frac{n}{T}} \sim \mathrm{Er}(\frac{n}{T}, n). \tag{33}$$

One can observe that $\mathrm{E}(H^n_{\frac{n}{T}}) = T$ and

$$\mathrm{E}(H^n_{\frac{n}{T}} - T)^2 = \frac{T^2}{n} \to 0 \text{ as } n \to \infty$$

that is $H^n_{\frac{n}{T}}$ indeed converges to T in L_2 and therefore in probability.

Let us state the optimisation problem similar to (31) with a horizon that has Erlang distribution H^n_α. It is assumed that the termination of the exponential horizons composing H^n_α is observable. For that purpose a Poisson process H_t, independent of (U_t, Y_t), with jump intensity α is introduced. Then $H^n_\alpha = \inf\{t : H_t \geq n\}$ is a stopping time. The expected penalty-reward function to be maximised in the horizon H^n_α is

$$\begin{aligned}\Upsilon^\sigma_n(\alpha, u, y) &\equiv \mathrm{E}\left[v^\sigma\left(H^n_\alpha, u, y\right)\right] = \\ &= \mathrm{E}\left[P(U_{H^n_\alpha}, Y_{H^n_\alpha}) \cdot \mathbb{I}_{\{\tau \geq H^n_\alpha\}} + L(U_\tau, Y_\tau) \cdot \mathbb{I}_{\{\tau < H^n_\alpha\}} | U_0 = u, Y_0 = y\right].\end{aligned}$$

Taking into account the density function of Erlang distribution, one can write

$$\begin{aligned}\Upsilon^\sigma_n(\alpha, u, y) &= \mathrm{E}\left[v^\sigma\left(H^n_\alpha, u, y\right)\right] = \\ &= \int_0^\infty v^\sigma(t, u, y) \frac{\alpha^n}{(n-1)!} t^{n-1} \mathrm{e}^{-\alpha t} \mathrm{d}t\end{aligned}$$

The optimisation problem is

$$J_n(\alpha, u, y) \equiv \max_{\sigma \in \Pi} \Upsilon^\sigma_n(\alpha, u, y) \tag{34}$$

where J_n represents the optimal value. Only markovian strategies with respect to (U_t, Y_t, H_t) are considered. The next theorem establishes the relationship between the value function $J_n(\alpha, u, y)$ and the value function $J(T, u, y)$ defined by (4).

Theorem 6.2 *Let $J(T, u, y)$ be the value function of the problem (4) and $J_n(\alpha, u, y)$ the value function of the problem (34) then*

$$\lim_{n \to \infty} J_n\left(\frac{n}{T}, u, y\right) = J(T, u, y).$$

Proof (Following the proof of Theorem 2 in Liu and Loewenstein (2002).) Let σ be any feasible strategy in Π such that $v^{\sigma}(t,u,y)$ is continuous in t on $[0,\infty)$. Then

$$\mathrm{E}\left[v^{\sigma}\left(H^{n}_{\alpha},u,y\right)\right]=\Upsilon^{\sigma}_{n}\left(\alpha,u,y\right)\leq J_{n}\left(\alpha,u,y\right),$$

in particular for $\alpha=\frac{n}{T}$, taking limit $n\to\infty$

$$v^{\sigma}\left(T,u,y\right)=\lim_{n\to\infty}\mathrm{E}\left[v^{\sigma}\left(H^{n}_{\frac{n}{T}},u,y\right)\right]\leq\lim_{n\to\infty}J_{n}\left(\frac{n}{T},u,y\right)\tag{35}$$

where the first equality comes from a variant of Helly-Bray Theorem (see Chow and Teicher (2003, Corollary 8.1.6)). Taking maximum over all admissible strategies on the left side of (35) yields

$$J\left(T,u,y\right)\leq\lim_{n\to\infty}J_{n}\left(\frac{n}{T},u,y\right).\tag{36}$$

On the other hand, notice that

$$\begin{aligned}J_{n}\left(\alpha,u,y\right)&\equiv\max_{\sigma\in\Pi}\Upsilon^{\sigma}_{n}\left(\alpha,u,y\right)=\max_{\sigma\in\Pi}\int_{0}^{\infty}v^{\sigma}\left(t,u,y\right)\frac{\alpha^{n}}{(n-1)!}t^{n-1}\mathrm{e}^{-\alpha t}\mathrm{d}t\\&\leq\int_{0}^{\infty}\max_{\sigma\in\Pi}v^{\sigma}\left(t,u,y\right)\frac{\alpha^{n}}{(n-1)!}t^{n-1}\mathrm{e}^{-\alpha t}\mathrm{d}t\\&=\int_{0}^{\infty}J\left(t,u,y\right)\frac{\alpha^{n}}{(n-1)!}t^{n-1}\mathrm{e}^{-\alpha t}\mathrm{d}t.\end{aligned}$$

The inequality comes from the fact that the optimal σ^{*} that maximises the whole integral in the first line is a feasible strategy in the maximisation problem under the integral in the second line for each t. Letting $\alpha=\frac{n}{T}$, taking the limit on both sides of the inequality, and applying the Helly-Bray Theorem again the complementary inequality to (36) follows

$$\lim_{n\to\infty}J_{n}\left(\frac{n}{T},u,y\right)\leq J\left(T,u,y\right)$$

thus completing the proof.

Theorem 6.2 provides a tool to approximate the fixed horizon by a series of consecutive exponential horizons. As will be shown in the next Theorem, this translates the original problem of maximisation in a fixed horizon T into a series of optimisation problems in exponential horizon. If $H^{n}_{\frac{n}{T}}$ is an Erlangian random horizon as defined in (33) it can be expressed as

$$H^{n}_{\frac{n}{T}}=\sum_{i=1}^{n}T^{n}_{i}\tag{37}$$

where $T_1^n, T_2^n, \ldots, T_n^n$ are independent random variables with common exponential distribution with parameter $\frac{n}{T}$. Variables T_i^n can be interpreted as consecutive random exponential horizons that compose the Erlangian horizon $H^n_{\frac{n}{T}}$. This in limit converges to the fixed horizon T. In order to state the next theorem formally we need to introduce the following notation

$$\begin{aligned} J_n(\alpha, u, y, P) &\equiv J_n(\alpha, u, y) \\ \Upsilon_n^\sigma(\alpha, u, y, P) &\equiv \Upsilon_n^\sigma(\alpha, u, y) \end{aligned}$$

when the reward function P needs to be specified explicitly.

Theorem 6.3 *Let P be a reward function, u and y the initial conditions, $\alpha > 0$ a real parameter. For every natural $k \geq 2$ we have*

$$J_k(\alpha, u, y, P) = J_1(\alpha, u, y, P_{k-1}) \tag{38}$$

where $P_{k-1}(w, z) \equiv J_{k-1}(\alpha, w, z, P)$.

Proof Let σ_t^* be the optimal strategy for $J_k(\alpha, u, y, P)$. Conditioning $v^\sigma(H_\alpha^n, u, y)$ on the instant of the first jump of the process H_t, which will be denoted T_1, the value of (U_t, Y_t) at T_1 and occurrence of the ruin one can write

$$J_k(\alpha, u, y, P) = \Upsilon_n^{\sigma^*}(\alpha, u, y) = \mathrm{E}\left[v^{\sigma^*}(H_\alpha^n, u, y, P)\right] = \tag{39}$$

$$= \mathrm{E}\left[\mathrm{E}\left[v^{\sigma^*}(H_\alpha^n, u, y) \mid T_1, (U_{T_1}, Y_{T_1}), \tau\right]\right] \tag{40}$$

$$\begin{aligned} &= \mathrm{E}\left[\mathrm{E}\left[v^{\sigma^*}(H_\alpha^{n-1}, U_{T_1}, Y_{T_1})\right] \mathbb{I}_{\{\tau \geq T_1\}}\right] \\ &\quad + \mathrm{E}\left[L(U_\tau, Y_\tau)\mathbb{I}_{\{\tau < T_1\}} \mid U_0 = u, Y_0 = y\right] \end{aligned} \tag{41}$$

where the first term of (41) comes from the Markovian nature of the process (U_t, Y_t). Given that the ruin did not occur before T_1 the future is independent of the past of the process conditional on the current state U_{T_1}, Y_{T_1}. Moreover, the horizon $H_\alpha^n \sim \mathrm{Er}(\alpha, n)$ is reduced by $T_1 \sim \mathrm{Exp}(\alpha)$ what yields a new horizon $H_\alpha^{n-1} \sim \mathrm{Er}(\alpha, n-1)$. In the second term, given that the ruin occurred before T_1, the expected loss $L(U_\tau, Y_\tau)$ is incurred given the initial state of the process. Developing the first term yields

$$\mathrm{E}\left[v^{\sigma^*}(H_\alpha^{n-1}, U_{T_1}, Y_{T_1})\right] = \Upsilon_{n-1}^{\sigma^*}(\alpha, U_{T_1}, Y_{T_1}) = J_{k-1}(\alpha, U_{T_1}, Y_{T_1}, P) \tag{42}$$

where for the last equality remember that the optimisation is made on Markovian strategies, that is σ^* depends on process (U_s, Y_s, H_s) only through (U_t, Y_t, H_t). Therefore σ^* is the optimal the strategy for Υ^{σ}_{n-1}. Substituting into (41) one gets

$$\begin{aligned} J_k(\alpha, u, y, P) &= \mathrm{E}\Big[J_{k-1}(\alpha, U_{T_1}, Y_{T_1}, P)\mathbb{I}_{\{\tau \geq T_1\}} \\ &\qquad + L(U_\tau, Y_\tau)\mathbb{I}_{\{\tau < T_1\}} \mid U_0 = u, Y_0 = y\Big] \\ &= \Upsilon_1^{\sigma^*}(\alpha, u, y, P_{k-1}) \leq J_1(\alpha, u, y, P_{k-1}). \end{aligned} \tag{43}$$

Notice that in the first line term $J_{k-1}(\alpha, U_{T_1}, Y_{T_1}, P)$ can be included in the conditioning since it is independent of U_0, Y_0.

On the other hand, assume that σ_1 is the optimal strategy for J_1. Let us consider a strategy

$$\sigma_1^* = \begin{cases} \sigma_1 & \text{if } H_t = 0 \\ \sigma^* & \text{if } H_t > 0 \end{cases}. \tag{44}$$

Since σ_1^* is admissible for Υ_n, applying (43) one can write

$$\begin{aligned} J_1(\alpha, u, y, P_{k-1}) &= \Upsilon_1^{\sigma_1^*}(\alpha, u, y, P_{k-1}) \\ &= \Upsilon_n^{\sigma_1^*}(\alpha, u, y, P) \leq J_k(\alpha, u, y, P) \end{aligned}$$

what completes the proof.

Previous theorems provide an approximation method for cases when the solution to the stochastic control problem in exponential time is available. Theorem 6.3 presents a recursive procedure to approximate the value function in Erlangian time by iterating through n exponential horizons. The value function J_k is updated in each step until the final J_n is calculated. Theorem 6.2 guarantees the convergence of value function J_n to its fixed horizon counterpart J as n goes to infinity.

6.2.3 Value function approximation

Since the exponential horizon can be seen as a Laplace transform of time, as illustrated by (32), the solution of the stochastic control in that case tends to be more tractable, since the dependence on time is eliminated (Avram et al. (2002)). Nevertheless, for complex models, in particular for the model defined by (2) that is treated in this chapter, the explicit solution to the stochastic control problem is not available even in exponential horizon. In the next section we present a tool to treat this cases by approximating not only the fixed horizon by a convergent series of Erlangian distributions but also approximating admissible optimal controls σ by a class of controls that are piecewise constant.

For an Erlangian horizon H_α^n determined as hitting time of $\{n\}$ by a Poisson process H_t and a control σ we define piecewise constant control σ_n that changes only with jumps of H_t. In theorem 6.4 it will be shown that the value function of the stochastic control problem in exponential horizon constrained to the class of controls σ_n converges to the value function of the unconstrained problem. Let us denote $\overline{J_n}(\alpha, u, y)$ the solution to the problem (34) constrained to the piecewise constant strategies defined as admissible strategies in Π that remain constant unless a jump of the process H_t^n occurs.

Theorem 6.4 *Let $J(T, u, y)$ be the solution to the problem* (3) *then*

$$\lim_{n\to\infty} \overline{J_n}(\alpha, u, y) = J(T, u, y) \tag{45}$$

Before we prove Theorem 6.4 some notation is introduced. Let us consider the Erlangian horizon H_α^n as a sum of n independent exponential distributions with parameter $\alpha = \frac{n}{T}$. Let V_t^n be a Poisson process independent of (W_t, X_t) with intensity $\frac{n}{T}$, let $T_1^n, \dots, T_n^n$ be the first n jump times of V_t^n and define process $G_t^n = Y_{T_k^n}$ where $k = \max\{i : T_i^n < t\}$. That is G_t^n is a process that remains constant on exponential horizon intervals $\left[T_i^n, T_{i-1}^n\right)$. If we consider the stochastic differential equation

$$\mathrm{d}U_t^n = (\mu_t U_t^n + c)\,\mathrm{d}t + \sqrt{(U_t^n)^2\sigma_t^2(G_t^n, Y_t) + \rho^2}\mathrm{d}W_t - \mathrm{d}X_t, \qquad U_0^n = u \tag{46}$$

then the optimisation problem (4) of the expected penalty reward function (3) under the process U_t^n in an Erlangian horizon yields the value function $\lim_{n\to\infty} \overline{J_n}(\alpha, u, y)$.

Let us state the following Lemma

Lemma 6.1 *Let U_t^n be the solution to* (46) *and U_t the solution to* (2) *then U_t^n converges in Law to U_t.*

Proof of Lemma 6.1 Process G_t^n can be written as $G_t^n = U_0 + (U_{t-}^n - G_{t-}^n)\cdot V_t^n$. Since $\frac{V_t^n}{n}$ converges in Law to $\frac{t}{T}$ and the equation

$$G_t^n = U_0 + \frac{n}{T}\int_0^t (U_{t-}^n - G_{t-}^n)\mathrm{d}t \tag{47}$$

has the solution

$$G_t^n = \mathrm{e}^{\frac{nt}{T}} U_t^n - \mathrm{e}^{-\frac{nt}{T}} \int_0^t \mathrm{e}^{\frac{nt}{T}} U_t^n \mathrm{d}t \to U_t^n \text{ as } n \to \infty \tag{48}$$

applying the Theorem 6.9 from Jacod and Shiryaev (2002, pg. 578) the result follows.

Proof of Theorem 6.4 Since P and L are continuous functions of U_t^n or U_t, by Proposition 3.8 Jacod and Shiryaev (2002, pg. 348) yields the convergence of the expectation of continuous functionals $\overline{J_n}$ to J.

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