Convex Risk Measures Beyond Bounded Risks

Gregor Svindland

Dissertation an der Fakultät für Mathematik, Informatik und Statistik der

Ludwig-Maximilians-Universität München

vorgelegt am 12.12.2008

Erstgutachter:	Prof. Dr. D. Filipović
Zweitgutachter:	Prof. Dr. H. Föllmer
Tag des Rigorosums:	19.02.2009

Zusammenfassung

Seit ihrer Einführung durch Artzner et al. [3], Föllmer und Schied [20] sowie Frittelli und Rosazza-Gianin [22] sind kohärente bzw. konvexe Risikomaße ein wichtiges Mittel zur konsistenten Risikobewertung. Beispiele sind das entropische Riskomaß oder der Average Value at Risk, welche sich breiter Anwendung in der Versicherungswirtschaft erfreuen. Die in den Anwendungen vorherrschende Klasse konvexer Risikomaße hat die Eigenschaft verteilungsinvariant zu sein, d.h. Positionen mit derselben Verteilung wird dasselbe Risiko zugesprochen. Die vorliegende Arbeit widmet sich insbesondere dem Studium verteilungsinvarianter konvexer Risikomaße.

Üblicherweise werden konvexe Risikomaße auf dem Raum $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ über einem gegebenen Wahrscheinlichkeitsraum $(\Omega, \mathcal{F}, \mathbb{P})$ definiert. Ein Vorteil dieses Modellraums ist, daß die einem konvexen Risikomaß zugrunde liegende Axiomatik auf L^∞ automatisch Lipschitzstetigkeit impliziert. Ein weiteres häufig angeführtes Argument für die Wahl des Modellraums L^∞ ist die Invarianz dieses Raumes unter äquivalenter Maßtransformation, so daß das grundlegende Modell $(\Omega, \mathcal{F}, \mathbb{P})$ nur bis auf Äquivalenz bestimmt werden muß. Beim Studium von verteilungsinvarianten konvexen Risikomaßen greift dieses Argument aber nicht, denn Verteilungsinvarianz setzt die Festlegung eines verteilungsbestimmenden Referenzmaßes $\mathbb{P}$ voraus. Vielmehr ist der Modellraum L^∞ für etliche praktische Anwendungen, in denen sehr häufig mit unbeschränkten Verteilungen, beispielsweise mit Normalverteilten, modelliert wird, zu klein. Es stellt sich also die Frage, ob es möglich ist, den Modellraum zu erweitern, ob dabei zusätzliche Anforderungen an Risikomaße gestellt werden müssen und ob dadurch die auf L^∞ zur Verfügung stehende Vielfalt an Risikomaßen stark eingeschränkt wird. Auffällig ist, daß sämtliche hinlänglich bekannten Beispiele verteilungsinvarianter konvexer Risikomaße auch auf L^p für $p \geq 1$ wohldefiniert sind, wenn man den Funktionswert ∞ zuläßt. In der vorliegenden Arbeit wird gezeigt, daß dies kein Zufall ist, sondern daß die verteilungsinvarianten unterhalbstetigen konvexen Funktionen auf L^∞ genau denjenigen auf L^p entsprechen. Damit ist gezeigt

Aufbauend auf diesem Resultat werden im weiteren Verlauf optimale Risikotransfers studiert. Die Problemstellung ist wie folgt. Gegeben sind n Agenten mit Risiken $X_i \in L^p$, $i = 1, \ldots, n$. Jeder Agent bewertet sein Risiko mittels eines verteilungsinvarianten unterhalbstetigen konvexen Risikomaßes ρ_i auf L^p. Das aggregierte Risiko ist $X := X_1 + \ldots + X_n$. Eine Allokation von X ist eine Neuverteilung des aggregierten Risikos, d.h. jeder Agent nimmt ein neues Risiko $Y_i \in L^p$ unter der Bedingung, daß $Y_1 + \ldots + Y_n = X$. Der Risikowert einer solchen Allokation ist $\rho_1(Y_1) + \ldots + \rho_n(Y_n)$. Eine optimale Allokation

von X ist eine Allokation von X, welche den Risikowert unter allen Allokationen von X minimiert. Eine solche optimale Allokation ist insbesondere optimal im Sinne von Pareto. Die Frage der Existenz und gegebenenfalls der Charakterisierung von optimalen Allokationen ist Bestandteil des Kapitels 3. Es wird gezeigt, daß unter den genannten Voraussetzungen optimale Allokationen immer existieren und diese sogar komonoton gewählt werden können, also insbesondere als Verträge basierend auf dem aggregierten Risiko X.

Abschließend beschäftigt sich die vorliegende Arbeit mit Subgradienten von verteilungsinvarianten konvexen Risikomaßen auf L^1. Subgradienten und die durch sie gegebenen Preisregeln spielen unter anderem in der Equilibriumtheorie eine bedeutende Rolle. Im Kapitel 4 wird ein verallgemeinerter Subgradientenbegriff eingeführt und der Zusammenhang mit optimalen Allokationen und Equilibria erläutert. Die zentrale Aufgabe dieses Abschnitts ist eine Charakterisierung der Punkte, an denen nicht-leere verallgemeinerte Subgradienten existieren.

Abstract

This work addresses three main issues: Firstly, we study the interplay of risk measures on L^∞ and L^p, for $p \geq 1$. Our main result is a one-to-one correspondence between law-invariant closed convex risk measures on L^∞ and L^1. This proves that the canonical model space for the predominant class of law-invariant convex risk measures is L^1.

Secondly, we provide the solution to the existence and characterisation problem of optimal capital and risk allocations for law-invariant closed convex risk measures on the model space L^p, for any $p \in [1, \infty]$. Our main result says that the capital and risk allocation problem always admits a solution via contracts whose payoffs are defined as increasing Lipschitz continuous functions of the aggregate risk. This result holds without requiring the monotonicity of the risk measures involved.

Finally, we study subgradients of law-invariant convex risk measures on L^1. Here we introduce the notion of a generalised subgradient and point out its connection with optimal risk sharing and equilibria. Our main result is a simple condition guaranteeing the existence of a non-empty generalised subgradient.

Contents

Introduction

This work is based on three papers [17, 18, 19] by the author and D. Filipović. We study law-invariant convex measures of risk which is the predominant class of risk measures in use. Three major problems are addressed:

- the extension of the model space (chapter 2),
- existence and characterisation of optimal risk sharings amongst n agents (chapter 3),
- subgradients of law-invariant convex risk measures (chapter 4).

In the following we give a short outline and describe our contributions to solving these problems.

Artzner et al. [3] introduced the seminal axioms of a coherent risk measure, which then were further generalised to the convex case by Föllmer and Schied [20] and Frittelli and Rosazza-Gianin [22]. Convex risk measures are usually defined on $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ for some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. But there is a growing mathematical finance literature dealing with convex risk measures beyond L^∞, see e.g. [6, 9, 13, 25, 26, 35]. This extended approach is vital since important risk models, such as normal distributed random variables, are not contained in L^∞. However, the interplay between convex risk measures on L^∞ and some extended model space, say L^p for $p \geq 1$, has not been addressed yet. In particular, the question arises wether there is a somehow canonical way to extend risk measures from L^∞ to L^p. Fortunately there is, at least for the predominant class of law-invariant convex risk measures. This is the topic of chapter 2 in which we more generally study extensions of closed convex functions from L^∞ to L^p. The main result is that every law-invariant closed convex function, so in particular every law-invariant convex risk measure, on L^∞ is well-defined on L^1 too. In other words, there is a one-to-one correspondence between law-invariant closed convex functions on L^∞ and L^1.

Another important issue is the existence of optimal allocations. The setting is as follows. A number of n agents determine their preferences by means of closed convex risk measures ρ_i. Each agent has an initial endowment (risk) X_i. The aggregate endowment is $X := X_1 + \ldots + X_n$. The agents may rearrange their portfolios by mutually exchanging risk under the restriction that the aggregate endowment is left unchanged, that is each agent may take a new position Y_i such that $Y_1 + \ldots + Y_n = X$. The (re-)distribution

$(Y_1, \dots, Y_n)$ is called an allocation of X. The optimal allocation problem is to find an allocation $(\widetilde{Y}_1, \dots, \widetilde{Y}_n)$ which minimises

$$\rho_1(Y_1) + \ldots + \rho_n(Y_n) \tag{0.1}$$

over all allocations $(Y_1, \dots, Y_n)$. By cash-invariance an optimal allocation $(\widetilde{Y}_1, \dots, \widetilde{Y}_n)$ does not only minimise the group risk (0.1), but by rebalancing the cash we may assume that $\rho(\widetilde{Y}_i) \leq \rho(X_i)$ too. Thus, we obtain a reduction of group and individual risk. In particular optimal allocations are optimal in Pareto's sense. The optimal allocation problem has been studied by several authors, see e.g. [5, 23, 16, 7, 1, 29]. We devote chapter 3 to it. Here it is proved that if our model space is L^p and if the convex risk measures ρ_i are in addition law-invariant, then there always exists an optimal allocation which is given by contracts whose payoffs are defined as increasing Lipschitz-continuous functions of the aggregate endowment X. Moreover, we do not require monotonicity of the risk measures involved. Similar results have been derived in [23], but only for the model space L^∞ and monotone risk measures.

In chapter 4 we investigate subgradients of law-invariant convex risk measures on L^1. Subgradients of convex risk measures play an important role amongst others in equilibrium theory. In particular, we are interested in subgradients corresponding to probability measures. We introduce a generalised subgradient, give conditions under which this generalised subgradient is non-empty, and point out its relationship to optimal allocations and equilibria. As a byproduct we also illumine the connection of law-invariant convex risk measures with Orlicz spaces and Orlicz hearts. The existence of a link between law-invariant convex risk measures and Orlicz space theory has been observed on the level of examples by several authors, e.g. [9]. However, we prove a systematic connection between law-invariant convex risk measures and certain subspaces of L^1 which are generalised versions of Orlicz spaces.

1. Preliminary Results, Notational Conventions, and Basic Assumptions

In this chapter we collect some basic results and notational conventions which will be used throughtout this text without further explanation. Section 1.1 recalls facts and notation from the field of convex analysis. Here our notation conforms to the usual notation as e.g. applied in [14] or [31]. Every reader being familiar with convex analysis may skip section 1.1. However, in section 1.2 we collect assumptions and notational conventions, some of which are peculiar to this work - a closer look into this section is recommended.

1.1. Some Facts and Notation from Convex Analysis

For the convenience of the reader we collect here some standard definitions and results in convex analysis which will be frequently used throughout this text. For more background we refer to Rockafellar [31] and Ekeland and Témam [14].

Let E denote a Hausdorff locally convex topological vector space with topological dual E^*. A function $f : E \to [-\infty, +\infty]$ is *convex* if

$$f(\lambda X + (1-\lambda)Y) \leq \lambda f(X) + (1-\lambda) f(Y) \quad \forall X, Y \in E, \quad \forall \lambda \in [0,1],$$

whenever the right-hand side is defined. We write dom $f = \{f < \infty\}$ for the *(effective) domain* of f. We call f *proper* if $f > -\infty$ and dom $f \neq \emptyset$.

f is said to be *lower semi-continuous* (l.s.c.) if the level sets $\{X \in E \mid f(X) \leq k\}$ are closed for all $k \in \mathbb{R}$, or equivalently, if for any net $(X_\alpha)_{\alpha \in D} \subset E$ converging to some $X \in E$ we have that $f(X) \leq \liminf_\alpha f(X_\alpha)$. This property is also equivalent to epi $f = \{(X, a) \in E \times \mathbb{R} \mid f(X) \leq a\}$ being a closed set in $E \times \mathbb{R}$ equipped with the product topology (see e.g. [14] proposition 2.3).

A convex set $\mathcal{C} \subset E$ is closed if and only if it is $\sigma(E, E^*)$-closed. As a consequence, a convex function f is l.s.c. if and only if f is l.s.c. with respect to $\sigma(E, E^*)$.

The *closure* of f is denoted by $\mathrm{cl}\,(f)$ and defined as $\mathrm{cl}\,(f) \equiv -\infty$, if $f(X) = -\infty$ for some X, and as greatest convex l.s.c. function majorised by f, else. Hence,

$$\mathrm{cl}(f) = f \tag{1.1}$$

if and only if either $f \equiv \pm\infty$ or f is proper l.s.c. and convex. Any function satisfying (1.1) is *closed*.

The *dual (or conjugate function)* of a function $f : E \to [-\infty, +\infty]$,

$$f^* : E^* \to [-\infty, \infty], \quad f^*(\mu) = \sup_{X \in E} (\langle \mu, X \rangle - f(X)),$$

is a closed convex function on E^*, whereas its *bidual (or biconjugate function)*

$$f^{**} : E \to [-\infty, \infty], \quad f^{**}(X) = \sup_{\mu \in E^*} (\langle \mu, X \rangle - f^*(\mu)),$$

is a closed convex function on E. Moreover, $(\mathrm{cl}\,(f))^* = f^*$, and the following convex duality relation holds (proposition 4.1 in [14])

$$f^{**} = \mathrm{cl}\,(f). \tag{1.2}$$

The set of subgradients of f at X is

$$\partial f(X) := \{\mu \in E^* \mid \forall Y \in E : \ f(Y) \geq f(X) + \langle \mu, Y - X \rangle\}.$$

Clearly, we may have that $\partial f(X) = \emptyset$. If $\partial f(X) \neq \emptyset$, then f is said to be subdifferentiable at X. Moreover, we have (proposition 5.1 in [14])

$$\mu \in \partial f(X) \quad \Leftrightarrow \quad f(X) + f^*(\mu) = \langle \mu, X \rangle. \tag{1.3}$$

The *indicator function* of a set $\mathcal{C} \subset E$ is defined as

$$\delta(X \mid \mathcal{C}) := \begin{cases} 0, & X \in \mathcal{C} \\ +\infty, & X \notin \mathcal{C}. \end{cases}$$

$\delta(\cdot \mid \mathcal{C})$ is closed convex if and only if $\mathcal{C}$ is convex and closed. Its conjugate is the *support function* of $\mathcal{C}$,

$$\delta^*(\mu \mid \mathcal{C}) = \sup_{X \in \mathcal{C}} \langle \mu, X \rangle.$$

Notice that E and E^* can be interchanged in the definition of δ and δ^*.

1.2. Assumptions on the Underlying Probability Space and More Notational Conventions

Throughout this text, we fix a non-atomic standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see section A.3). All equalities and inequalities between random variables are understood in the $\mathbb{P}$-almost sure (a.s.) sense. We write $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$ for any $p \in [0, \infty]$ and $\| \cdot \|_p = \| \cdot \|_{L^p}$ for any $p \in [1, \infty]$. The topological dual space of L^p for $p \in [1, \infty]$ is denoted by L^{p*}. It is well-known that $L^{p*} = L^q$ with $q = \frac{p}{p-1}$ for $1 \leq p < \infty$, and that $L^{\infty *} \supset L^1$ can be identified with *ba*, the space of all bounded finitely additive measures

μ on $(\Omega, \mathcal{F})$ such that $\mathbb{P}(A) = 0$ implies $\mu(A) = 0$. The positive order cone is denoted by L^p_+ and its polar cone by L^{p*}_-. With some facilitating abuse of notation, we shall write $(X, Z) \mapsto E[XZ]$ for the dual pairing on (L^p, L^{p*}) also for the case $p = \infty$. For any $X, Y \in L^0$, by $X \sim Y$ we indicate that X and Y are identically distributed. A function $f : L^0 \supset V \to [-\infty, \infty]$ is said to be *law-invariant* if $X, Y \in V$ and $X \sim Y$ implies $f(X) = f(Y)$.

2. Convex Risk Measures Beyond Bounded Risks

In this chapter we study convex risk measures on L^p-spaces, and in particular the interplay between closed convex risk measures on L^∞ and L^p. We show that every law-invariant convex risk measure on L^∞ can be extended to a law-invariant closed convex risk measure on L^1 (theorem 2.11). Since many of our arguments hold more generally, in section 2.2 we will study the extension of convex functions from L^∞ to L^p, and then apply our results to convex risk measures. However, first of all, in section 2.1 we define a convex risk measure on L^p and state some important properties of these functions.

2.1. Convex Risk Measures on L^p

Let $p \in [1, \infty]$.

Definition 2.1. *A convex function $\rho : L^p \to (-\infty, \infty]$ is called* convex risk measure *if it is*

(i) *cash-invariant: $\rho(0) \in \mathbb{R}$ and $\rho(X + m) = \rho(X) - m$ for all $m \in \mathbb{R}$,*

(ii) *monotone: if $X \geq Y$, then $\rho(X) \leq \rho(Y)$.*

We denote by $\mathcal{A}_\rho := \{X \mid \rho(X) \leq 0\}$ the acceptance set *of ρ.*
A positively homogeneous ($\rho(tX) = t\rho(X)$ for all $t > 0$) convex risk measure ρ is called coherent*.*

Remark 2.2. Due to monotonicity and cash-invariance, every convex risk measure ρ on L^∞ is real-valued (dom $\rho = L^\infty$), and 1-Lipschitz-continuous ([21] lemma 4.3). In general however, for $p \in [1, \infty)$, convex risk measures need not be continuous or even closed. Consider e.g.

$$\rho : L^p \to (-\infty, \infty] \,, \quad \rho(X) = -E[X] + \delta(X^- \mid L^\infty) \,.$$

This law-invariant coherent risk measure assigns to an endowment X the value ∞ in case the possible losses are unbounded, and $E[-X]$ else. Clearly, the acceptance set $\mathcal{A}_\rho$ is not closed, so ρ is not closed. $\Diamond$

The next theorem summarises some fundamental properties of convex risk measures on L^p. To this end, note that we write $\overline{U}^p$ for the closure of a set U in L^p.

Theorem 2.3. *Let ρ be a convex risk measure on L^p.*

(i) $\operatorname{int}\operatorname{dom}\rho \neq \emptyset$ *if and only if ρ is real-valued and continuous on L^p.*

(ii) *$\mathcal{A}_\rho$ is convex with $\mathcal{A}_\rho + L^p_+ \subset \mathcal{A}_\rho$, $\mathbb{R} \cap \mathcal{A}_\rho \neq \emptyset$, and $\overline{\mathcal{A}_\rho}^p \neq L^p$.*

(iii) *For any $\mathcal{A} \subset L^p$ satisfying properties* (ii) *instead of $\mathcal{A}_\rho$,*

$$\rho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}\}$$

defines a convex risk measure on L^p. Moreover, if $\mathcal{A}$ is closed, then $\mathcal{A}_{\rho_{\mathcal{A}}} = \mathcal{A}$ and $\rho_{\mathcal{A}}$ is closed.

(iv) $\operatorname{dom}\rho^* \subset \mathcal{P}^{p*} := \{Z \in L^{p*}_- \mid E[1Z] = -1\}$ *and for all $Z \in \mathcal{P}^{p*}$ we have*

$$\rho^*(Z) = \sup_{X \in \mathcal{A}_\rho} E[ZX]\,.$$

Proof. (i): clearly, if ρ is real-valued and continuous, then e.g. $0 \in \operatorname{int}\operatorname{dom}\rho$. In order to prove the converse implication note that according to [35] proposition 3.1 ρ is continuous on $\operatorname{int}\operatorname{dom}\rho$. Hence, it suffices to prove that $\operatorname{int}\operatorname{dom}\rho = \operatorname{dom}\rho = L^p$. We show this by means of contradiction. Suppose for the moment that there is a $\widetilde{X} \in L^p$ such that $\rho(\widetilde{X}) = \infty$. Since the interior of the convex set $\operatorname{dom}\rho$ is non-empty by assumption and $\widetilde{X} \notin \operatorname{dom}\rho$, an appropriate version of the Hahn-Banach separating hyperplane theorem ensures the existence of a nontrivial $Z \in L^{p*}$ such that

$$\sup_{Y \in \operatorname{dom}\rho} E[ZY] \leq E[Z\widetilde{X}]\,.$$

Since $L^\infty \subset \operatorname{dom}\rho$ we obtain

$$tE[ZX] \leq E[Z\widetilde{X}] \quad \text{for all } X \in L^\infty \text{ and } t \in \mathbb{R},$$

and thus $E[XZ] = 0$ for all $X \in L^\infty$. As L^∞ is dense in L^p, we infer that Z must be trivial. But this is a contradiction. Therefore, $\operatorname{dom}\rho = L^p$.
(ii): convexity, $\forall X \in \mathcal{A}_\rho :\ X + L^p_+ \subset \mathcal{A}_\rho$, and $\mathbb{R} \cap \mathcal{A}_\rho \neq \emptyset$ are obvious by definition of ρ. Suppose we had $\overline{\mathcal{A}_\rho}^p = L^p$. Then for every $n \in \mathbb{N}$ there is a $X_n \in \mathcal{A}_\rho$ such that

$$\|X_n - (-n)\|_p = \|X_n + n\|_p \leq \frac{1}{2^n}\,.$$

By monotonicity we may assume that $X_n + n \geq 0$. Then, the sequence $Y_n := \sum_{k=1}^n X_k + k$, $n \in \mathbb{N}$, converges to

$$Y := \left(\sum_{k=1}^{\infty} X_k + k \right) \in L^p\,, \qquad \|Y\|_p \leq 1,$$

and $Y \geq Y_n \geq X_n + n$ for all $n \in \mathbb{N}$. Thus, by monotonicity, cash-invariance, and $X_n \in \mathcal{A}_\rho$ we infer

$$\rho(Y) \leq \rho(X_n + n) \leq \rho(X_n) - n \leq -n \qquad \text{for all } n \in \mathbb{N}.$$

Consequently $\rho(Y) = -\infty$. But this is a contradiction to the properness of ρ (definition 2.1). Hence, $\overline{\mathcal{A}_\rho}^p \subsetneqq L^p$.

(iii): it is easily verified that $\rho_{\mathcal{A}}$ is a convex, cash-invariant, and monotone function such that $\rho_{\mathcal{A}}(0) < \infty$. In order to prove that $\rho_{\mathcal{A}}$ is proper, it suffices to show that $\rho_{\overline{\mathcal{A}}^p}$ is proper because $\rho_{\overline{\mathcal{A}}^p} \leq \rho_{\mathcal{A}}$. Observe that $\overline{\mathcal{A}}^p$ satisfies properties (ii) because $\mathcal{A}$ does. Hence, $\rho_{\overline{\mathcal{A}}^p}$ is convex, cash-invariant, and monotone too, and $\rho_{\overline{\mathcal{A}}^p}(0) < \infty$. If we had $\rho_{\overline{\mathcal{A}}^p}(0) = -\infty$, then it follows that $\mathbb{R} \subset \overline{\mathcal{A}}^p$, and thus $L^\infty \subset \overline{\mathcal{A}}^p$, so actually $\overline{\mathcal{A}}^p = L^p$ which is a contradiction to the assumption $\overline{\mathcal{A}}^p \neq L^p$. Consequently, $\rho_{\overline{\mathcal{A}}^p}(0) > -\infty$. It is easily verified that $\overline{\mathcal{A}}^p = \mathcal{A}_{\rho_{\overline{\mathcal{A}}^p}}$, so $\rho_{\overline{\mathcal{A}}^p}$ is l.s.c. Since any l.s.c. convex function which assumes the value $-\infty$ cannot take any finite value (see [14] proposition 2.4.), and since $\rho_{\overline{\mathcal{A}}^p}(0) \in \mathbb{R}$, we conclude that $\rho_{\overline{\mathcal{A}}^p} > -\infty$, i.e. $\rho_{\overline{\mathcal{A}}^p}$ is proper and thus closed.

(iv): for a proof of dom $\rho^* \subset \mathcal{P}^{p*}$, please consult [15] lemma 3.2. The stated representation of $\rho^*(Z)$ for any $Z \in \mathcal{P}^{p*}$ follows from $E[1Z] = -1$, thence $E[ZX] - \rho(X) = E[Z(X + \rho(X))]$, and cash-invariance of ρ. □

Part (i) of theorem 2.3 yields a remarkable dichotomy for convex risk measures ρ on L^p: either ρ is continuous on L^p or $\operatorname{int}\operatorname{dom}\rho = \emptyset$.

2.2. L^p-closures

In this section we study under which conditions a proper closed convex function defined on L^∞ may be extended to some L^p for $p \in [1, \infty)$, thereby preserving convexity and achieving closedness on $(L^p, \|\cdot\|_p)$. The derived results are in particular valid for convex risk measures on L^∞.

We fix $p \in [1, \infty]$ and some function $f : L^\infty \to [-\infty, \infty]$. Its conjugate

$$f^*(Z) = \sup_{X \in L^\infty} (E[ZX] - f(X))$$

is a closed convex function on $L^{\infty*}$, and hence on L^{p*}. The following is thus well defined.

Definition 2.4. *The* L^p-closure *of* f *is defined as*

$$\overline{f}^p(X) := \sup_{Z \in L^{p*}} (E[ZX] - f^*(Z)), \quad X \in L^p. \tag{2.1}$$

Note that f is trivially extended to a function $\widetilde{f}$ on L^p by letting $\widetilde{f} = f$ on L^∞ and $\widetilde{f} = \infty$ else. Then, it is easily verified that $(\widetilde{f})^* = f^*$ on L^{p*}. In other words, $\overline{f}^p$ is the well-known *convex closure* (or *l.s.c. convex regularisation*) of $\widetilde{f}$ (see e.g. [14] section 3.2 or [32] section 3). We chose the notation $\overline{f}^p$ in order to put emphasis on the dependence on p, because in general $\overline{f}^p$ will differ with varying p (example 2.28). However, in theorem 2.11 we show that in case of law-invariance the L^p-closure of f is independent of p.

Next we recall some properties of $\overline{f}^p$ (compare to [14] section 3.2 or [32] section 3).

Lemma 2.5. (i) $\overline{f}^p$ *is the greatest closed convex function on* L^p *majorised by* f *on* L^∞.

(ii) $(\overline{f}^p)^* = f^*|_{L^{p*}}$.

(iii) $\overline{f}^p$ *is proper if and only if* f *is proper and* $\operatorname{dom} f^* \cap L^{p*} \neq \emptyset$.

(iv) *If either* f *is real-valued or* $\overline{f}^p$ *is proper then* $\operatorname{epi} \overline{f}^p = \overline{\operatorname{co} \operatorname{epi} f}^p$, *where the right hand side denotes the* $L^p \times \mathbb{R}$*-closure of the convex hull of* $\operatorname{epi} f$.

Proof. By construction, $\overline{f}^p$ is a closed convex function on L^p with

$$\overline{f}^p \leq f \text{ on } L^\infty. \tag{2.2}$$

Now let g be any closed convex function on L^p with $g \leq f$ on L^∞. Then, for all $Z \in L^{p*}$,

$$g^*(Z) = \sup_{X \in L^p} E[XZ] - g(X) \geq \sup_{X \in L^\infty} E[XZ] - f(X) = f^*(Z). \tag{2.3}$$

Hence,

$$g(X) = g^{**}(X) \leq \sup_{Z \in L^{p*}} E[XZ] - f^*(Z) = \overline{f}^p(X),$$

and (i) is proved.

Now let $Z \in L^{p*}$. By definition we obtain

$$(\overline{f}^p)^*(Z) = \sup_{X \in L^p} (E[XZ] - \sup_{Y \in L^{p*}} (E[XY] - f^*(Y))) \leq f^*(Z).$$

On the other hand, from (2.3) we infer $(\overline{f}^p)^*(Z) \geq f^*(Z)$. This proves (ii).

Property (iii) is obvious.

As for (iv), inequality (2.2) implies $\operatorname{epi} f \subset \operatorname{epi} \overline{f}^p$ on $L^\infty \times \mathbb{R}$. By convexity and closedness of $\operatorname{epi} \overline{f}^p$ we thus have $\overline{\operatorname{co} \operatorname{epi} f}^p \subset \operatorname{epi} \overline{f}^p$. To show the converse inclusion, we note that

$$g(X) = \inf\{a \mid (X, a) \in \overline{\operatorname{co} \operatorname{epi} f}^p\}$$

defines a l.s.c. convex function on L^p ($\operatorname{epi} g = \overline{\operatorname{co} \operatorname{epi} f}^p$) with $f \geq g$ on L^∞ and $g \geq \overline{f}^p$. If either f is real-valued or $\overline{f}^p$ is proper, then g is closed. In view of the first part of the lemma, we conclude that $g = \overline{f}^p$ and thus $\overline{\operatorname{co} \operatorname{epi} f}^p = \operatorname{epi} \overline{f}^p$. □

Remark 2.6. Lemma 2.5(iv) does not hold without requiring that f be real-valued or $\overline{f}^p$ be proper. Indeed, example 2.29 below shows that $\overline{\operatorname{co} \operatorname{epi} f}^p \subsetneqq \operatorname{epi} \overline{f}^p$ is possible for a positively homogeneous monotone closed convex function f. ◊

Lemma 2.5(iv) is of conceptual interest, because a natural approach to extending a convex risk measure ρ from L^∞ to L^p is simply to close the acceptance set $\mathcal{A}_\rho$ in L^p, i.e. to study the risk measure given by $\overline{\mathcal{A}_\rho}^p$. Hence, the following corollary:

Corollary 2.7. *Let ρ be a convex risk measure on L^∞. Then $\overline{\rho}^p$ is a closed convex risk measure on L^p if and only if $\overline{\mathcal{A}_\rho}^p \neq L^p$. In either case, $\overline{\rho}^p = \rho_{\overline{\mathcal{A}_\rho}^p}$ and $\mathcal{A}_{\overline{\rho}^p} = \overline{\mathcal{A}_\rho}^p$.*

Proof. Since any convex risk measure ρ on L^∞ is real-valued, lemma 2.5(iv) implies

$$\overline{\operatorname{epi} \rho}^p = \operatorname{epi} \overline{\rho}^p, \tag{2.4}$$

$$\overline{\mathcal{A}_\rho}^p = \{X \in L^p \mid (X, 0) \in \overline{\operatorname{epi} \rho}^p\} = \{X \in L^p \mid (X, 0) \in \operatorname{epi} \overline{\rho}^p\} = \mathcal{A}_{\overline{\rho}^p}. \tag{2.5}$$

Hence, if $\overline{\rho}^p$ is a closed convex risk measure on L^p, then $\overline{\mathcal{A}_\rho}^p$ must have the properties stated in theorem 2.3(ii), in particular $\overline{\mathcal{A}_\rho}^p \neq L^p$.

Conversely, suppose that $\overline{\mathcal{A}_\rho}^p \neq L^p$. We claim that $\overline{\mathcal{A}_\rho}^p$ satisfies the conditions of theorem 2.3(ii). Convexity and $\overline{\mathcal{A}_\rho}^p \cap \mathbb{R} \neq \emptyset$ are obvious, and we have $\overline{\mathcal{A}_\rho}^p \neq L^p$ by assumption. In order to verify the yet missing condition $\overline{\mathcal{A}_\rho}^p + L^p_+ \subset \overline{\mathcal{A}_\rho}^p$, let $X \in \overline{\mathcal{A}_\rho}^p$ and $Y \in L^p_+$. Choose $(X_n)_{n\in\mathbb{N}} \subset \mathcal{A}_\rho$ converging to X. As $Y \wedge n \in L^\infty_+$ for all $n \in \mathbb{N}$, we have $Y_n := Y \wedge n + X_n \in \mathcal{A}_\rho$ for all $n \in \mathbb{N}$. Since Y_n converges to $X + Y$ w.r.t. the $\|\cdot\|_p$-norm, we conclude that $(X+Y) \in \overline{\mathcal{A}_\rho}^p$. Consequently, according to theorem 2.3(ii), $\rho_{\overline{\mathcal{A}_\rho}^p}$ is a closed convex risk measure on L^p. As $\operatorname{epi} \rho_{\overline{\mathcal{A}_\rho}^p} = \overline{\operatorname{epi} \rho}^p$, we infer from (2.4) and (2.5) that $\overline{\rho}^p = \rho_{\overline{\mathcal{A}_\rho}^p}$ and $\mathcal{A}_{\overline{\rho}^p} = \overline{\mathcal{A}_\rho}^p$. □

Definition 2.8. *A function $g : L^p \to (-\infty, \infty]$ is called an* extension *of f to L^p if $g = f$ on L^∞.*

In the following we elaborate on the existence and uniqueness of closed convex extensions. Let us first discuss uniqueness. For illustration, consider a closed convex function g on L^p. Obviously, $\overline{g|_{L^\infty}}^p$ is a closed convex extension of $g|_{L^\infty}$ to L^p. That is, $g = \overline{g|_{L^\infty}}^p$ on L^∞. However, in general we only have $g \leq \overline{g|_{L^\infty}}^p$ on L^p, and this inequality may be strict, as is illustrated by example 2.25. Hence, uniqueness does not hold in general. In fact, an immediate consequence of lemma 2.5(ii) is the following:

Corollary 2.9. *Let g be a closed convex function on L^p. Then $g = \overline{g|_{L^\infty}}^p$ if and only if $g^* = (g|_{L^\infty})^*$ on L^{p*}.*

As for existence of an extension, examples 2.26–2.28 below show convex risk measures on L^∞ which admit no closed convex extension to L^p. We now give necessary and sufficient conditions for the existence of closed convex extensions and illustrate the particular role of the L^p-closure.

Lemma 2.10. *The following properties are equivalent:*

(i) *There exists a closed convex extension of f to L^p.*

(ii) *$\overline{f}^p$ is an extension of f to L^p.*

(iii) *f is convex and $\sigma(L^\infty, L^{p*})$-closed.*

In either case, $\overline{f}^p$ is the greatest closed convex extension of f to L^p.

Proof. (i) $\Leftrightarrow$ (ii): let g be a closed convex extension of f to L^p. Lemma 2.5(i) implies $g \le \overline{f}^p$ and $f = g \le \overline{f}^p \le f$ on L^∞. This proves (ii) and also the last statement of the theorem. The converse implication is trivial.
(ii) $\Rightarrow$ (iii): this is obvious since $\overline{f}^p|_{L^\infty}$ is convex and $\sigma(L^\infty, L^{p*})$-l.s.c.
(iii) $\Rightarrow$ (ii): for all $X \in L^\infty$, the Fenchel–Moreau theorem yields

$$f(X) = \sup_{Z \in L^{p*}} \left(E[XZ] - f^*(Z)\right) = \overline{f}^p(X).$$

□

If we restrict to law-invariant closed convex functions, then existence and uniqueness of closed convex extensions always holds. This is the message of the following theorem, the proof of which requires some closer studies of law-invariant convex functions and is given in section 2.3 below.

Theorem 2.11. *For any law-invariant closed convex function $f : L^\infty \to [-\infty, \infty]$ and $p \in [1, \infty)$, the L^p-closure $\overline{f}^p$ is the unique law-invariant closed convex extension of f to L^p. Moreover, $\overline{f}^p = \overline{f}^1|_{L^p}$, and f is $\sigma(L^\infty, L^\infty)$-closed.*

It is easily verified that the L^p-closure preserves monotonicity and cash-invariance. Hence, according to theorem 2.11, there exists a one-to-one correspondence between law-invariant closed convex risk measures on L^∞ and L^1. In this sense, we may conclude that the canonical model space for law-invariant convex risk measures is L^1.

2.3. Law-Invariant Convex Functions on $\mathrm{L^p}$

In the following we collect and prove results on law-invariant convex functions which will play a fundamental role in the proof of theorem 2.11 which is stated at the end of this section. Throughout this section we let $p \in [1, \infty]$.

One of the main ingredients to proving theorem 2.11 is lemma 2.13 below. In fact, this lemma is an extension of results by Jouini, Schachermayer, and Touzi in [24].

Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. As in [24] we define the conditional expectation on $L^{\infty *}$ as a function $E[\cdot \mid \mathcal{G}] : L^{\infty *} \to L^{\infty *}$ where $E[\mu \mid \mathcal{G}]$ is given by

$$E[E[\mu \mid \mathcal{G}]X] := E[\mu E[X \mid \mathcal{G}]] \quad \forall\, X \in L^{\infty} .$$

Clearly, this definition is consistent with the ordinary conditional expectation in case $\mu \in L^1 \subset L^{\infty *}$.

Remark 2.12. If $\mathcal{G} = \sigma(A_1, \dots, A_n)$ is finite, then $E[\mu \mid \mathcal{G}] \in L^{\infty}$. In order to verify this, note that for all $X \in L^{\infty}$ we have

$$E[E[\mu \mid \mathcal{G}]X] = E[\mu E[X \mid \mathcal{G}]] = \sum_{i=1}^{n} E[X 1_{A_i}] \frac{\mu(A_i)}{\mathbb{P}(A_i)}.$$

Hence, $E[\mu \mid \mathcal{G}] = \sum_{i=1}^{n} \frac{\mu(A_i)}{\mathbb{P}(A_i)} 1_{A_i} \in L^{\infty}$. ◊

Note that (L^{∞}, L^r) is a dual pair for every $r \in [1, \infty]$.

Lemma 2.13. (i) *Let $D \subset L^{\infty}$ be a $\|.\|_{\infty}$-closed convex law-invariant set. Then D is $\sigma(L^{\infty}, L^r)$-closed for every $r \in [1, \infty]$.*

(ii) *A law-invariant convex function $f : L^{\infty} \to [-\infty, \infty]$ is closed if and only if it closed w.r.t. any $\sigma(L^{\infty}, L^r)$-topology for every $r \in [1, \infty]$.*

Proof. (i): if $D = \emptyset$, the assertion is obvious. For the remainder of this proof, we assume thus that $D \neq \emptyset$.

According to lemma 4.2 in [24], for all $Y \in D$ and all sub-σ-algebras $\mathcal{G} \subset \mathcal{F}$ we have

$$E[Y \mid \mathcal{G}] \in D. \tag{2.6}$$

Now let $(X_i)_{i \in I}$ be a net in D converging to some $X \in L^{\infty}$ in the $\sigma(L^{\infty}, L^r)$-topological sense, i.e. $E[ZX_i] \to E[ZX]$ for all $Z \in L^r$. Then, in view of remark 2.12, if $\mathcal{G}$ is finite, we have $E[E[\mu \mid \mathcal{G}]X_i] \to E[E[\mu \mid \mathcal{G}]X]$ for all $\mu \in L^{\infty *}$. But by definition this equals $E[\mu E[X_i \mid \mathcal{G}]] \to E[\mu E[X \mid \mathcal{G}]]$ for all $\mu \in L^{\infty *}$. In other words, the net $(E[X_i \mid \mathcal{G}])_{i \in I}$ converges to $E[X \mid \mathcal{G}]$ in the $\sigma(L^{\infty}, L^{\infty *})$-topology. Since, according to (2.6), $E[X_i \mid \mathcal{G}] \in D$ for all $i \in I$, we conclude that $E[X \mid \mathcal{G}] \in D$, because D is closed and convex and thus $\sigma(L^\infty, L^{\infty *})$-closed. Hence, $E[X \mid \mathcal{G}] \in D$ for all finite sub-σ-algebras $\mathcal{G} \subset \mathcal{F}$. Recalling that we can approximate X in $(L^\infty, \|\cdot\|_\infty)$ by a sequence of conditional expectations $(E[X \mid \mathcal{G}_n])_{n\in\mathbb{N}}$ in which the $\mathcal{G}_n$s are all finite, we conclude by means of the norm-closedness of D that $X \in D$. Thus D is $\sigma(L^\infty, L^r)$-closed, and (i) is proved.

(ii): suppose f is closed. Then, for every $k \in \mathbb{R}$ the level sets $\{X \in L^\infty \mid f(X) \le k\}$ are $\|.\|_\infty$-closed, convex, and law-invariant. Hence, (i) yields the $\sigma(L^\infty, L^r)$-closedness of the level sets, i.e. f is closed with respect to the $\sigma(L^\infty, L^r)$-topology. The converse implication is trivial. □

Recall that the (left continuous) quantile function of a random variable X is

$$q_X : (0,1) \to \mathbb{R}\,, \quad q_X(s) = \inf\{x \in \mathbb{R} \mid \mathbb{P}(X \le x) \ge s\}\,. \tag{2.7}$$

Lemma 2.14. *Let* $q := \frac{p}{p-1}$ *where* $\frac{1}{0} := \infty$ *and* $\frac{\infty}{\infty-1} := 1$. *If* $f : L^p \to [-\infty, \infty]$ *is a closed convex function, then the following conditions are equivalent:*

(i) f *is law-invariant.*

(ii) f *is* $\sigma(L^p, L^q)$*-closed and* f^* *(resp.* $f^*|_{L^1}$ *if* $p = \infty$*) is law-invariant.*

Moreover, if either holds, then:

$$f^*(Z) = \sup_{X\in L^p} \int_0^1 q_X(s) q_Z(s)\, ds - f(X)\,, \quad Z \in L^q\,,$$

and

$$f(X) = \sup_{Z\in L^q} \int_0^1 q_X(s) q_Z(s)\, ds - f^*(Z)\,, \quad X \in L^p\,.$$

Proof. (i) ⇒ (ii): the first statement is trivial if $p \in [1,\infty)$ and proved in lemma 2.13 if $p = \infty$. Moreover, for any $Z \in L^q$ we gather from lemma A.2 that

$$\begin{aligned} f^*(Z) &= \sup_{X\in L^p} E[XZ] - f(X) = \sup_{X\in L^p}\left(\sup_{\hat{X}\sim X} E[\hat{X}Z]\right) - f(X) \\ &= \sup_{X\in L^p} \int_0^1 q_X(s) q_Z(s)\, ds - f(X) \end{aligned}$$

in which the latter expression depends on the law of Z only.
(ii) ⇒ (i): again by lemma A.2

$$\begin{aligned} f(X) = f^{**}(X) &= \sup_{Z\in L^q}\left(\sup_{\hat{Z}\sim Z} E[XZ]\right) - f^*(Z) \\ &= \sup_{Z\in L^q} \int_0^1 q_X(s) q_Z(s)\, ds - f^*(Z) \end{aligned}$$

for all $X \in L^p$. Hence, f is law-invariant. □

Next we introduce two orders on L^1 which are well-known from utility theory. To this end, recall that a utility function is a strictly concave and strictly increasing function $u : \mathbb{R} \to \mathbb{R}$.

Definition 2.15. *For any two $X, Y \in L^1$ we define*

(i) *the concave order:*

$$X \succeq_c Y \;\Leftrightarrow\; E[u(X)] \geq E[u(Y)] \text{ for all concave functions } u : \mathbb{R} \to \mathbb{R},$$

(ii) *the second order stochastic dominance:*

$$X \succeq Y \quad \Leftrightarrow \quad E[u(X)] \geq E[u(Y)] \text{ for all utility functions } u.$$

A function $f : L^p \to [-\infty, \infty]$ is said to be $\succeq_c$($\succeq$)-monotone if $f(X) \leq f(Y)$ whenever $X \succeq_c Y$ ($X \succeq Y$).

We will need the following two facts: for $X, Y \in L^1$ we have

$$X \succeq Y \quad \Leftrightarrow \quad \int_0^t q_X(s)\, ds \geq \int_0^t q_Y(s)\, ds \quad \text{for all } 0 < t \leq 1, \tag{2.8}$$

and

$$X \succeq_c Y \quad \Leftrightarrow \quad X \succeq Y \text{ and } E[X] = E[Y]\,. \tag{2.9}$$

For a proof of (2.8) and (2.9), we refer to theorem 2.58 and corollary 2.62 in [21]. The following lemmas 2.16 and 2.17 are essentially proved in [11]. However, for the sake of completeness, we give proofs here too.

Lemma 2.16. *For $X, Y \in L^1$:*

(i) *$X \succeq Y$ if and only if*

$$\int_0^1 q_X(s) f(s)\, ds \leq \int_0^1 q_Y(s) f(s)\, ds$$

for all increasing $f : (0, 1) \to (-\infty, 0]$ such that both integrals exist.

(ii) *$X \succeq_c Y$ if and only if*

$$\int_0^1 q_X(s) f(s)\, ds \leq \int_0^1 q_Y(s) f(s)\, ds$$

for all increasing $f : (0, 1) \to \mathbb{R}$ such that both integrals exist.

Proof. (i): " $\Leftarrow$ ": note that the functions $-1_{(0,t)}(\cdot)$ are increasing for all $0 < t \leq 1$. Now apply (2.8).
" $\Rightarrow$ ": let $X \succeq Y$.
1. In a first step we assume that f is a simple function, i.e.

$$f(s) = \sum_{i=1}^{n-1} a_i 1_{(t_{i-1},t_i]}(s) + a_n 1_{(t_{n-1},t_n)}(s)$$

where $t_0 = 0 < t_1 < \ldots < t_n = 1$ is a finite partition of $(0,1)$, and $a_i \in \mathbb{R}$ such that $a_1 \leq a_2 \leq \ldots \leq a_n \leq 0$. Applying (2.8) we have

$$a_n \int_0^1 q_X(s)\, ds \leq a_n \int_0^1 q_Y(s)\, ds\,,$$

and for $j = 1, \ldots, n-1$:

$$(a_j - a_{j+1}) \int_0^{t_j} q_X(s)\, ds \leq (a_j - a_{j+1}) \int_0^{t_j} q_Y(s)\, ds\,.$$

Summing up these inequalities we arrive at

$$\sum_{j=1}^{n} a_j \int_{t_{j-1}}^{t_j} q_X(s)\, ds \leq \sum_{j=1}^{n} a_j \int_{t_{j-1}}^{t_j} q_Y(s)\, ds$$

and thus the desired

$$\int_0^1 q_X(s) f(s)\, ds \leq \int_0^1 q_Y(s) f(s)\, ds\,.$$

2. Now for general f, we approximate f by simple functions

$$f_k(s) := \begin{cases} -\frac{i-1}{2^k} & \text{if } f(s) \in (-\frac{i}{2^k}, -\frac{i-1}{2^k}] \text{ for a } i = 1, \ldots, k2^k \\ -k & \text{else} \end{cases}\,, \quad k \in \mathbb{N}\,.$$

The functions f_k converge to f pointwise, and $|f_k| \leq |f|$ for all $k \in \mathbb{N}$. Hence, the dominated convergence theorem in conjunction with 1. yields

$$\begin{aligned} \int_0^1 q_Y(s) f(s)\, ds &= \lim_{k\to\infty} \int_0^1 q_Y(s) f_k(s)\, ds \quad \geq \quad \lim_{k\to\infty} \int_0^1 q_X(s) f_k(s)\, ds \\ &= \int_0^1 q_X(s) f(s)\, ds\,. \end{aligned}$$

(ii): " $\Rightarrow$ ": let $X \succeq_c Y$. By (2.9) we know that $X \succeq Y$ and $\int_0^1 q_X(s)\, ds = \int_0^1 q_Y(s)\, ds$. Hence, by (i), adding up inequalities, we deduce that

$$\int_0^1 q_X(s) f(s)\, ds \leq \int_0^1 q_Y(s) f(s)\, ds$$

for all increasing $f : (0,1) \to \mathbb{R}$ which are bounded from above. Finally, the usual monotone approximation argument from integration theory yields the assertion for any increasing $f : (0,1) \to \mathbb{R}$.
"$\Leftarrow$": again, simply apply (2.9) and (i). $\square$

The subsequent lemma does not only play an important role in the proof of theorem 2.11, but it is also essential to the proof of theorem 3.4, which is the main result in chapter 3. Note that both $X \geq Y$ and $X \succeq_c Y$ imply $X \succeq Y$. However, $\geq$ and $\succeq_c$ are not related in general.

Lemma 2.17. *Let $f : L^p \to [-\infty, \infty]$ be a closed convex function. Equivalent are:*

(i) *f is law-invariant.*

(ii) *f is $\succeq_c$-monotone.*

Moreover, if in addition f is monotone, then (i) and (ii) are equivalent to

(iii) *f is $\succeq$-monotone.*

In particular, if either of the conditions (i), (ii) or (iii) holds, then

(iv) *$f(E[X \mid \mathcal{G}]) \leq f(X)$ for all $X \in L^p$ and all sub-σ-algebras $\mathcal{G} \subset \mathcal{F}$.*

Proof. Let $q := \frac{p}{p-1}$ where $\frac{1}{0} := \infty$ and $\frac{\infty}{\infty - 1} := 1$.
(i) $\Rightarrow$ (ii): let $X \succeq_c Y$. Then by lemmas 2.14 and 2.16:

$$\begin{aligned} f(X) &= \sup_{Z \in L^q} \int_0^\infty q_X(s) q_Z(s)\, ds - f^*(Z) \\ &\leq \sup_{Z \in L^q} \int_0^\infty q_Y(s) q_Z(s)\, ds - f^*(Z) \quad = \quad f(Y)\,. \end{aligned}$$

(ii) $\Rightarrow$ (i): conversely, suppose that f is $\succeq_c$-monotone and let $X \sim Y$. Trivially, $X \succeq_c Y$ and $Y \succeq_c X$, so $f(X) = f(Y)$.
(i) $\Rightarrow$ (iii): let $X \succeq Y$. Since f is monotone, we have that $\operatorname{dom} f^* \cap L^q \subset L^q_-$ (see [15] lemma 3.2). Hence, by lemmas 2.14 and 2.16,

$$\begin{aligned} f(X) &= \sup_{Z \in L^q_-} \int_0^1 q_X(s) q_Z(s)\, ds - f^*(Z) \\ &\leq \sup_{Z \in L^q_-} \int_0^1 q_Y(s) q_Z(s)\, ds - f^*(Z) = f(Y)\,. \end{aligned}$$

(iii) $\Rightarrow$ (i): if $X \sim Y$, then $X \succeq Y$ and $Y \succeq X$. Thus, $f(X) = f(Y)$.
(iv): for any concave function $u : \mathbb{R} \to \mathbb{R}$ Jensen's inequality yields

$$E[u(E[X \mid \mathcal{G}])] \geq E[E[u(X) \mid \mathcal{G}]] = E[u(X)]\,.$$

Hence, $E[X \mid \mathcal{G}] \succeq_c X$. Now apply (ii). $\square$

Corollary 2.18. *Let $\rho : L^p \to (-\infty, \infty]$ be a law-invariant closed convex cash-invariant function. Then, $\rho(X) \geq -E[X] + \rho(0)$ and $\rho^*(-1) = -\rho(0)$.*

Proof. Cash-invariance and lemma 2.17 imply that $\rho(X) \geq -E[X] + \rho(0)$. On the one hand, since $0 \in L^p$, we have that $\rho^*(-1) \geq -\rho(0)$, and on the other hand

$$\rho^*(-1) = \sup_{X \in L^p} (E[-X] - \rho(X)) \leq \sup_{X \in L^p} (E[-X] + E[X] - \rho(0)) = -\rho(0).$$

□

Remark 2.19. Note that the proof of lemma 2.17 relies on lemma 2.13, and thus on lemma 4.2 in [24], only in case $p = \infty$. We recall that lemma 4.2 in [24] states that if $\emptyset \neq D \subset L^p$ is a convex law-invariant and $\|\cdot\|_p$-closed set, then $E[X \mid \mathcal{G}] \in D$ for all $X \in D$ and all sub-σ-algebras $\mathcal{G} \subset \mathcal{F}$. Indeed, for every such set D, the indicator function $\delta(\cdot \mid D)$ is a law-invariant closed convex function. Therefore, according to lemma 2.17(iv), $\delta(E[X \mid \mathcal{G}] \mid D) \leq \delta(X \mid D)$. This implies $E[X \mid \mathcal{G}] \in D$ whenever $X \in D$. Hence, we have derived an alternative proof of lemma 4.2 in [24] for the cases $p \in [1, \infty)$ (but clearly not for $p = \infty$). ♢

On the basis of the previous results, we are now able to prove theorem 2.11:

Proof of Theorem 2.11. According to lemma 2.13 any law-invariant closed convex function $f : L^\infty \to [-\infty, \infty]$ is $\sigma(L^\infty, L^\infty)$-closed. Hence, by lemma 2.10 $\overline{f}^p$ is a closed convex extension of f to L^p. Lemmas 2.5 and 2.14 yield the law-invariance of $\overline{f}^p$. In order to prove that $\overline{f}^p$ is the unique law-invariant closed convex extension of f to L^p, let g be any such extension. For every $X \in L^p$ and all $m \in \mathbb{N}$ there exists a finite partition $A_1^m, \ldots, A_n^m$ of Ω such that the L^p-distance between X and the simple random variable $X_m := E[X \mid \sigma(A_1^m, \ldots, A_n^m)] \in L^\infty$ is less than $1/m$. On the one hand, lemma 2.17(iv) implies that $g(X_m) \leq g(X)$ for all $m \in \mathbb{N}$. On the other hand, by l.s.c. of g, we know that $g(X) \leq \liminf_{m\to\infty} g(X_m)$. Hence, $g(X) = \lim_{m\to\infty} g(X_m)$. Since the latter observation in particular holds for $\overline{f}^p$, we obtain

$$g(X) = \lim_{m\to\infty} g(X_m) = \lim_{m\to\infty} f(X_m) = \lim_{m\to\infty} \overline{f}^p(X_m) = \overline{f}^p(X)$$

and uniqueness is proved. Finally, letting $g = \overline{f}^1|_{L^p}$ yields $\overline{f}^p = \overline{f}^1|_{L^p}$. □

2.4. Examples

In this section, first of all, we present some extensions of well-known law-invariant convex risk measures on L^∞. Then, we go on illustrating some pitfalls and difficulties which have been mentioned in the previous sections. In particular, we show the necessity of many of our assumptions.

2.4.1. Extensions of Law-invariant Convex Risk Measures

Example 2.20. Average Value at Risk: Let $\alpha \in (0,1]$ and

$$\mathcal{Z} := \{Z \in \mathcal{P}^{1*} \mid Z \geq -\frac{1}{\alpha}\}\,.$$

According to theorems 2.3 and 2.11 the corresponding real-valued support function

$$\mathrm{AVaR}_\alpha(X) := \max_{Z \in \mathcal{Z}} E[XZ] = -\frac{1}{\alpha}\int_0^\alpha q_X(s)\,ds, , \quad X \in L^p\,,$$

is a continuous coherent risk measure on L^p which is the unique extension of the well-known Average Value at Risk on L^∞. ◊

The following example is the extension of the entropic risk measure (see example 4.33 in [21]) to L^p. In case $p \in [1,\infty)$, we will see that the entropic risk measure is not continuous.

Example 2.21. Let $M_\infty(\mathbb{P})$ be the set of all probability measures $\mathbb{Q}$ on $(\Omega,\mathcal{F})$ such that $\mathbb{Q} \ll \mathbb{P}$ and $d\mathbb{Q}/d\mathbb{P}$ is bounded. The entropic risk measure of parameter $\beta > 0$ is

$$\mathrm{Entr}_\beta(X) = \frac{1}{\beta}\log E[e^{-\beta X}] = \sup_{\mathbb{Q}\in M_\infty(\mathbb{P})} E_{\mathbb{Q}}[-X] - \frac{1}{\beta}H(\mathbb{Q} \mid \mathbb{P})\,, \quad X \in L^p,$$

where $H(\mathbb{Q} \mid \mathbb{P}) = E_{\mathbb{Q}}[\log \frac{d\mathbb{Q}}{d\mathbb{P}}]$ denotes the relative entropy. Entr_β is a law-invariant closed convex risk measure on L^p (theorem 2.11). If $p \in [1,\infty)$, then $\{X \in L^p \mid \mathrm{Entr}_\beta(X) = \infty\} \neq \emptyset$, whereas, if $p = \infty$, then $\mathrm{dom}\,\mathrm{Entr}_\beta = L^\infty$. Hence, by theorem 2.3, Entr_β is continuous if and only if $p = \infty$. ◊

Another common class of law-invariant convex risk measures are the semi-deviation risk measures which are discussed in the following example.

Example 2.22. For any $r \in [1,\infty]$, the semi-deviation risk measure is

$$\mathrm{Dev}_r(X) := -E[X] + \|(X - E[X])^-\|_r\,, \quad X \in L^p.$$

It is a law-invariant closed coherent risk measure on L^p. If $1 \leq r \leq s$, the Hölder inequality implies $\mathrm{Dev}_r \leq \mathrm{Dev}_s$. Thus Dev_r is real-valued and $\|\,.\,\|_p$-continuous whenever $r \leq p$ (theorem 2.3). If $r > p$, then $\{X \in L^p \mid \mathrm{Dev}_r = \infty\} \neq \emptyset$, so Dev_r is merely closed but not continuous. ◊

Clearly, as on L^∞, the worst case risk measure is the most conservative convex risk measure on any L^p:

Example 2.23. $-\mathrm{essinf}(X)$, $X \in L^p$, is a law-invariant closed coherent risk measure on L^p. By cash-invariance and monotonicity, any convex risk measure ρ on L^p satisfies $\rho(X) \leq \rho(0) - \mathrm{essinf}(X)$. Obviously, if $p \in [1,\infty)$, then $\{X \in L^p \mid \mathrm{essinf}(X) = -\infty\} \neq \emptyset$. Hence, $-\mathrm{essinf}(\cdot)$ is continuous if and only if $p = \infty$. ◊

2.4.2. Non-uniqueness of Closed Convex Extensions

The following lemma will be used to show that closed convex extensions of convex risk measures, if they exist, are not unique in general.

Lemma 2.24. *Let $T \in L^1$ satisfy $\operatorname{essinf} T < 0$ and $\operatorname{esssup} T = \infty$. Denote by $\mathcal{A}$ the closed convex cone generated by T and L^1_+. Then $\rho_{\mathcal{A}}$ is a closed coherent risk measure on L^1 such that $\rho_{\mathcal{A}} \neq -\operatorname{essinf}$ but $\rho_{\mathcal{A}}|_{L^\infty} = -\operatorname{essinf}|_{L^\infty}$.*

Proof. Theorem 2.3 implies that $\rho_{\mathcal{A}}$ is a closed coherent risk measure on L^1 such that $\mathcal{A}_{\rho_{\mathcal{A}}} = \mathcal{A}$. Moreover, $\rho_{\mathcal{A}} \neq -\operatorname{essinf}$ on L^1 since $\rho_{\mathcal{A}}(T) \leq 0 < -\operatorname{essinf} T$ by construction.

We claim that

$$\mathcal{A} \cap L^\infty = L^\infty_+, \tag{2.10}$$

implying that $\rho_{\mathcal{A}}|_{L^\infty} = -\operatorname{essinf}|_{L^\infty}$. As for the proof of (2.10), note that $\mathcal{A} = \overline{A}^1$ where

$$A = \{tX \mid X \in B\,,\, t \geq 0\} \quad \text{and} \quad B = \operatorname{conv}(T, L^1_+) + L^1_+.$$

The inclusion $L^\infty_+ \subset \mathcal{A} \cap L^\infty$ follows from construction. To show the converse, $L^\infty \setminus L^\infty_+ \subset L^\infty \setminus (\mathcal{A} \cap L^\infty)$, we choose any $S \in L^\infty$ such that $\mathbb{P}(S < 0) > 0$. Since $S \notin L^1_+$ is bounded whereas T is unbounded from above, S cannot be an element of the convex hull $\operatorname{conv}(T, L^1_+)$, and neither of its monotone hull B, because any convex combination in $\operatorname{conv}(T, L^1_+)$ is either $\mathbb{P}$-a.s. positive or unbounded from above, so it cannot be dominated by S. But then, clearly $S \notin A$ too. Now suppose $S \in \mathcal{A} \setminus A$. Then there would be a sequence $(S_n)_{n\in\mathbb{N}} \subset A$ converging to S in L^1. By monotonicity of A we may assume that $S_n \geq S$ for all $n \in \mathbb{N}$ (otherwise $\hat{S}_n := S_n \vee S \in A$ will do), and shifting to a subsequence if necessary, we may assume that $S_n \to S$ $\mathbb{P}$-a.s. Clearly, there is some $N_0 \in \mathbb{N}$ such that $\mathbb{P}(S_n < 0) > 0$ for all $n \geq N_0$. By construction of A there are $t_n \geq 0$, $\alpha_n \in (0, 1]$ and $X_n, Z_n \in L^1_+$ such that $S_n = t_n(\alpha_n T + (1 - \alpha_n)X_n + Z_n)$. Thus $\{S_n < 0\} \subset \{T < 0\}$, and consequently we have $\mathbb{P}$-a.s. that

$$\{S < 0\} = \bigcup_{k\in\mathbb{N}} \bigcap_{m\geq k} \{S_n < 0\} \subset \{T < 0\}\,.$$

Let $\epsilon := E[-S 1_{\{S<0\}}] > 0$ and $\delta := E[-T 1_{\{s<0\}}] > 0$. Choose $N_1 \geq N_0$ such that for all $n \geq N_1$: $\|S - S_n\|_1 = E[S_n - S] \leq \frac{\epsilon}{2}$. Then, for $n \geq N_1$ we have:

$$\begin{aligned}
\frac{\epsilon}{2} \geq E[S_n - S] \quad &\geq \quad E[(S_n - S) 1_{\{S<0\}}] \\
&= \quad E[(t_n \alpha_n T + t_n(1 - \alpha_n)X_n + t_n Z_n - S) 1_{\{S<0\}}] \\
&\geq \quad t_n \alpha_n E[T 1_{\{S<0\}}] + E[-S 1_{\{S<0\}}] \\
&= \quad -t_n \alpha_n \delta + \epsilon\,.
\end{aligned}$$

Consequently, $t_n \alpha_n \geq \frac{\epsilon}{2\delta} =: r > 0$, and thus $S_n \geq rT$, for all $n \geq N_1$. But this contradicts the boundedness of S. Hence $S \notin \mathcal{A}$ and (2.10) is proved. □

Example 2.25. Let T and $\rho_{\mathcal{A}}$ be as in lemma 2.24. It is easily verified that $\overline{\rho_{\mathcal{A}}|_{L^\infty}}^1 = -\operatorname{essinf}$. Hence both $-\operatorname{essinf}$ and $\rho_{\mathcal{A}}$ are closed convex risk measures that extend $-\operatorname{essinf}$ to L^1. This shows that such extensions are not unique in general. $\Diamond$

2.4.3. Non-extendable Convex Risk Measures

Example 2.26. Let $p \in [1, \infty)$ and $Z \in \mathcal{P}^{\infty *} \setminus \mathcal{P}^{p*}$. Define $\rho(X) := E[XZ]$. Then ρ is a convex risk measure on L^∞ with L^p-closure $\overline{\rho}^p \equiv -\infty$. Hence, we know by lemma 2.10 that there exists no closed convex extension of ρ to L^p. $\Diamond$

The next example shows a coherent risk measure on L^∞ which cannot be extended to any L^p, $p \in [1, \infty)$.

Example 2.27. Let $(\Omega, \mathcal{F}, \mathbb{P}) = ((0,1], \mathcal{B}(0,1], \lambda)$ where λ denotes the Lebesgue measure restricted to the Borel-σ-algebra $\mathcal{B}(0,1]$, and let $A_n := (0, \frac{1}{2^n}]$, $n \in \mathbb{N}$. Moreover, let $\mathbb{P}_n(\cdot) := \mathbb{P}(\cdot \mid A_n)$, and we denote by $\operatorname{essinf}_{\mathbb{P}_n}(X)$ the essential infimum of a random variable X under the measure $\mathbb{P}_n$. Define

$$\rho(X) := \lim_{n\to\infty} -\operatorname{essinf}_{\mathbb{P}_n}(X)\,, \quad X \in L^0\,.$$

In fact, the function $\rho_\infty := \rho|_{L^\infty}$ is a coherent risk measure on L^∞. However, it is easily verified that for $p \in [1, \infty)$ there are $X \in L^p$ such that $\rho(X) = -\infty$, so ρ fails to be proper on L^p. Moreover, $\overline{\mathcal{A}_{\rho_\infty}}^p = L^p$. The domain of ρ_∞^* is a subset of $L^{\infty *} \setminus L^1$, because for any $Z \in \mathcal{P}^{\infty *} \cap L^1$ we have

$$\rho_\infty^*(Z) = \sup_{X\in\mathcal{A}_\rho} E[XZ] \geq \sup_{k,n\in\mathbb{N}} E[-k1_{A_n^c} Z] = \sup_{k,n\in\mathbb{N}} k(1 + E[Z1_{A_n}]) = \infty\,.$$

That is condition (iii) of lemma 2.5 is not satisfied. Hence, $\overline{\rho_\infty}^p = -\infty$, and, according to lemma 2.10, ρ_∞ is a coherent risk measure on L^∞ which admits no closed convex extension to L^p, $p \in [1, \infty)$. $\Diamond$

Next we illustrate by a simple example that we cannot expect the L^p-closure to be an extension in general, even if it is a closed convex function on L^p.

Example 2.28. Let $\rho(X) := \max\{E[-X], E[ZX], E[\widetilde{Z}X]\}$, $X \in L^\infty$, for some $Z \in \mathcal{P}^{2*} \setminus L^\infty$ and $\widetilde{Z} \in \mathcal{P}^{\infty *} \setminus L^1$. It is easily verified that ρ is a coherent risk measure on L^∞ and that $\operatorname{dom}\rho^*$ is the convex hull $\operatorname{co}\{-1, Z, \widetilde{Z}\}$. We have $\overline{\rho}^p(X) = \max\{E[-X], E[ZX]\}$ for all $p \in [2, \infty)$, but $\overline{\rho}^1 = E[-X]$. Clearly, $\overline{\rho}^1 \neq \overline{\rho}^2$ on L^2 and $\overline{\rho}^p \neq \rho$ on L^∞ for all $p \in [1, \infty)$. Hence, $\overline{\rho}^p$ is no extension of ρ although $\overline{\rho}^p$ is a continuous coherent risk measure on L^p. Moreover, we observe that L^p-closures for different p may differ. $\Diamond$

2.4.4. A Counter-Example Related to Lemma 2.5

The next example shows that the requirements in lemma 2.5(iv) cannot be dropped in general.

Example 2.29. Recall example 2.27 and define

$$f : L^\infty \to (-\infty, \infty], \quad X \mapsto \rho(X) + \delta(X \mid \mathcal{C})$$

where $\mathcal{C} := \{X \in L^1 \mid X \geq 0 \ \mathbb{P}\text{-a.s. on } [1/2, 1]\}$. Clearly, $\mathcal{C} \cap L^p$ is convex and closed for every $p \in [1, \infty]$. Hence, f is a closed convex function on L^∞, and it is easily verified that f is also monotone and positively homogeneous. Next we prove by similar arguments as in example 2.27 that dom $f^* \cap L^1 = \emptyset$ implying that $\overline{f}^p = -\infty$ for all $p \in [1, \infty)$. To this end, note that $\rho(k1_{[0,1/2]}) = -k$ for all $k \in \mathbb{R}$. Consequently, for any $Z \in L^{\infty *}$ we have

$$f^*(Z) \geq \sup_{k \in \mathbb{R}} k(E[Z1_{[0,1/2]}] + 1),$$

so either $E[Z1_{[0,1/2]}] = -1$ or $Z \neq \text{dom } f^*$. Now let $Z \in L^1$ such that $E[Z1_{[0,1/2]}] = -1$. Then $\mathbb{P}(\{Z < 0\} \cap [0, 1/2]) > 0$, and since

$$X_{k,n} := -k1_{A_n^c} 1_{[0,1/2]} 1_{\{Z<0\}} \in \mathcal{C}$$

satisfy $\rho(X_{k,n}) = 0$ for all $k, n \in \mathbb{N}$, we obtain

$$f^*(Z) \geq \sup_{k,n \in \mathbb{N}} E[ZX_{k,n}] \geq \sup_{k \in \mathbb{N}} -kE[Z1_{\{Z<0\}} 1_{[0,1/2]}] = \infty.$$

Hence, $\overline{f}^p = -\infty$ for all $p \in [1, \infty)$ which is equivalent to epi $\overline{f}^p = L^p \times \mathbb{R}$. However, it is easily verified that $\overline{\text{co epi } f^p} = (\mathcal{C} \cap L^p) \times \mathbb{R} \subsetneq L^p \times \mathbb{R}$. ◊

3. Optimal Capital and Risk Allocations

In this chapter we provide a solution to the existence and characterisation problem of optimal capital and risk allocations for law-invariant closed convex risk measures on the model space L^p, for any $p \in [1, \infty]$. That is, we consider n agents with initial endowments $X_1, \dots, X_n \in L^p$, who assess the riskiness of their positions by means of some law-invariant closed convex risk measures ρ_i on L^p. In order to minimise total and individual risk, the agents redistribute the aggregate endowment $X = X_1 + \dots + X_n$ among themselves. An optimal capital and risk allocation $(Y_1, \dots, Y_n)$ satisfies $Y_1 + \dots + Y_n = X$ and

$$\rho_1(Y_1) + \dots + \rho_n(Y_n) = \inf_{\sum_{i=1}^n Z_i = X} \left(\rho_1(Z_1) + \dots + \rho_n(Z_n)\right).$$

Our main result is theorem 3.4 which states the existence and gives a characterisation of optimal allocations. In fact, the assertion of theorem 3.4 is more general insofar as it does not require monotonicity of the ρ_i.

3.1. Existence of Optimal Allocations

We now formalise the above capital and risk allocation problem. Let $n \geq 2$ and $F_1, \dots, F_n : L^p \to (-\infty, \infty]$ be some proper convex functions. Their (infimal) *convolution* at $X \in L^p$ is defined as

$$\square_{i=1}^n F_i(X) = F_1 \square \dots \square F_n(X) := \inf_{\substack{X_1, \dots, X_n \in L^p \\ \sum_{i=1}^n X_i = X}} \sum_{i=1}^n F_i(X_i).$$

The following properties are well known (see e.g. [31]).

Lemma 3.1. (i) $\square_{i=1}^n F_i : L^p \to [-\infty, \infty]$ *is a convex function,*

(ii) $\mathrm{dom}\, \square_{i=1}^n F_i = \sum_{i=1}^n \mathrm{dom}\, F_i$,

(iii) $(\square_{i=1}^n F_i)^* = \sum_{i=1}^n F_i^*$,

(iv) $\mathrm{dom}\, (\square_{i=1}^n F_i)^* = \bigcap_{i=1}^n \mathrm{dom}\, F_i^*$.

Definition 3.2. *Let $X \in L^p$. An n-tuple $(X_1, X_2, \ldots, X_n) \in L^p \times \ldots \times L^p$ such that $\sum_{i=1}^n X_i = X$ is called an* allocation *of X. The convolution $\square_{i=1}^n F_i$ is said to be* exact at X *if there exists an allocation $(X_1, X_2, \ldots, X_n)$ of X such that $\square_{i=1}^n F_i(X) = \sum_{i=1}^n F_i(X_i)$. Such a minimising allocation is called an* optimal *allocation of X. The convolution is said to be* exact *if it is exact at every point $X \in L^p$.*

Hence, the capital and risk allocation problem outlined in the beginning of this chapter is equivalent to finding an optimal allocation for the convolution $\square_{i=1}^n \rho_i$.

Definition 3.3. *An allocation $(X_1, \ldots, X_n)$ of $X \in L^p$ is called* comonotone *if there exist increasing functions $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ such that $\sum_{i=1}^n f_i = \mathrm{Id}_{\mathbb{R}}$ and $X_i = f_i(X)$ for all i. These functions f_i are necessarily 1-Lipschitz-continuous.*

The following existence theorem is the main result of this chapter. Its proof needs some preparation and is given in section 3.5.

Theorem 3.4. *Let $\rho_1, \ldots, \rho_n : L^p \to (-\infty, \infty]$ be law-invariant closed convex cash-invariant functions. Then, $\square_{i=1}^n \rho_i$ is a law-invariant closed convex cash-invariant function on L^p. Furthermore, for every $X \in L^p$ there exists a comonotone allocation $(X_1, \ldots, X_n)$ such that*

$$\square_{i=1}^n \rho_i(X) = \sum_{i=1}^n \rho_i(X_i)\,.$$

In other words, $\square_{i=1}^n \rho_i$ is exact, and amongst the optimal allocations of any $X \in L^p$ there is always a comonotone one.

Remark 3.5. The economic message of theorem 3.4 is that the capital and risk allocation problem always admits a solution via contracts whose payoffs are defined as (increasing Lipschitz-continuous) functions $f_i(X)$ of the aggregate risk X. $\diamondsuit$

We note that the functions ρ_i in theorem 3.4 do not have to be monotone. In case at least one of them is monotone (i.e. a convex risk measure), we may draw the following stronger conclusion:

Corollary 3.6. *Let $\rho_1, \ldots, \rho_n : L^p \to (-\infty, \infty]$ be law-invariant closed convex cash-invariant functions, of which at least one is a convex risk measure. Then, $\square_{i=1}^n \rho_i$ is a law-invariant closed convex risk measure on L^p. Moreover, for every $X \in L^p$ there exists a comonotone optimal allocation.*

Proof. In view of theorem 3.4 it remains to prove that $\square_{i=1}^n \rho_i$ is monotone. But this follows immediately from lemma 3.1 and the fact that a proper closed convex function $F : L^p \to (-\infty, \infty]$ is monotone if and only if $\mathrm{dom}\, F^* \subset L_-^{p*}$ (see e.g. [15] lemma 3.2). $\square$

Next we apply theorem 3.4 to calculate optimal allocations for Average Value at Risks, entropic risk measures, and semi-deviation risk measures, respectively. These convolutions are discussed thoroughly in e.g. [4] or [24] on L^∞. In contrast, theorem 3.4 allows for a simple approach to computing these convolutions, and we provide our results on L^1.

Example 3.7. Recall the Average Value at Risk from example 2.20, and let $\mathrm{AVaR}_0 := -\operatorname{essinf}$. Let $0 \le \beta \le \gamma \le 1$, then

$$\mathrm{AVaR}_\beta \,\Box\, \mathrm{AVaR}_\gamma = \mathrm{AVaR}_\gamma \,.$$

This is easily verified in view of theorem 3.4, (1.2), lemma 3.1, and the fact that $\operatorname{dom} \mathrm{AVaR}_\gamma^* \subset \operatorname{dom} \mathrm{AVaR}_\beta^*$. ◊

Example 3.8. Recall the entropic risk measure from example 2.21. Let $0 < \beta \le \gamma$. Theorem 3.4 and lemma 3.1 justify the following dual approach, for any $X \in L^1$:

$$\begin{aligned} \mathrm{Entr}_\beta \,\Box\, \mathrm{Entr}_\gamma(X) &= \sup_{\mathbb{Q}\in\mathcal{M}(\mathbb{P})} E_\mathbb{Q}[-X] - \frac{1}{\beta} H(\mathbb{Q} \mid \mathbb{P}) - \frac{1}{\gamma} H(\mathbb{Q} \mid \mathbb{P}) \\ &= \sup_{\mathbb{Q}\in\mathcal{M}(\mathbb{P})} E_\mathbb{Q}[-X] - \frac{\beta+\gamma}{\beta\gamma} H(\mathbb{Q} \mid \mathbb{P}) \\ &= \mathrm{Entr}_{\frac{\beta\gamma}{\beta+\gamma}}(X)\,. \end{aligned}$$

Now, in the search for comonotone optimal allocations, the following ansatz seems natural. We guess that for any $X \in L^1$ there must be an (obviously comonotone) optimal allocation amongst the allocations of type (aX, bX) where $a \in [0,1]$ and $b := 1 - a$. If so, then

$$\frac{\beta+\gamma}{\beta\gamma} \log E[e^{-\frac{\beta\gamma}{\beta+\gamma}X}] = \frac{1}{\beta} \log E[e^{-\beta a X}] + \frac{1}{\gamma} \log E[e^{-\gamma b X}]$$

which is equivalent to

$$\log E[e^{-\frac{\beta\gamma}{\beta+\gamma}X}] = \frac{\gamma}{\beta+\gamma} \log E[e^{-\beta a X}] + \frac{\beta}{\beta+\gamma} \log E[e^{-\gamma b X}]\,.$$

Clearly, $a = \frac{\gamma}{\beta+\gamma}$ and $b = \frac{\beta}{\beta+\gamma}$ satisfy this equation. Hence, $(\frac{\gamma}{\beta+\gamma}X, \frac{\beta}{\beta+\gamma}X)$ is a comonotone optimal allocation of X. ◊

Example 3.9. As for the semi-deviation risk measures (example 2.22), recall that $\mathrm{Dev}_p \le \mathrm{Dev}_r$ for $1 \le p \le r < \infty$. Consequently, $\operatorname{dom} \mathrm{Dev}_p^* \subset \operatorname{dom} \mathrm{Dev}_r^*$, and in conjunction with theorem 3.4 and lemma 3.1 we conclude that $\mathrm{Dev}_p \,\Box\, \mathrm{Dev}_r = \mathrm{Dev}_p$. Hence, $(X, 0)$ is a comonotone optimal allocation of $X \in L^1$. ◊

Remark 3.10. The closedness requirement in theorem 3.4 cannot be dropped, as the following example shows. Let $\rho_1(X) = -E[X] + \delta(X^- \mid L^\infty)$, and $\rho_2 = \mathrm{AVaR}_\alpha$ on L^1, for some $\alpha \in (0,1)$. In remark 2.2 we saw that ρ_1 is not closed. We claim that

$$\rho_1 \Box \rho_2 = -E. \tag{3.1}$$

Indeed, on the one hand, we know that $\rho_1^* = \delta(\cdot \mid \{-1\})$ and that $\rho_2^*(-1) = 0$ (corollary 2.18). Hence, $(\rho_1 \Box \rho_2)^* = \rho_1^* + \rho_2^* = \delta(\cdot \mid \{-1\})$ which implies $\rho_1 \Box \rho_2 \geq -E$. On the other hand,

$$\begin{aligned} \rho_1 \Box \rho_2(X) &= \inf_{X_1+X_2=X} E[-X_1] + \delta(X_1^- \mid L^\infty) + \mathrm{AVaR}_\alpha(X_2) \\ &\leq \inf_{K\in\mathbb{N}} E[-X 1_{\{X>-K\}}] + \mathrm{AVaR}_\alpha(X 1_{\{X\leq -K\}}) \\ &\leq E[-X] + \lim_{K\to\infty} \mathrm{AVaR}_\alpha(X 1_{\{X\leq -K\}}) = E[-X] \end{aligned}$$

because AVaR_α is continuous and $X 1_{\{X\leq -K\}} \to 0$ in L^1 for $K\to\infty$. This proves (3.1).

Now, choose any $X \in L^1$ being unbounded from below. Suppose there is an optimal allocation (X_1, X_2) of X. Then X_1 must be bounded and X_2 unbounded from below, respectively. In view of lemma 3.11 below, we thus have $\mathrm{AVaR}_\alpha(X_2) > E[-X_2]$, and hence

$$\begin{aligned} \rho_1 \Box \rho_2(X) &= \rho_1(X_1) + \rho_2(X_2) \\ &= E[-X_1] + \delta(X_1^- \mid L^\infty) + \mathrm{AVaR}_\alpha(X_2) \\ &> E[-X_1] + E[-X_2] = E[-X], \end{aligned}$$

which contradicts (3.1). Hence, there exists no optimal allocation of X. ◊

Lemma 3.11. *Let $0 \leq \beta < \gamma \leq 1$. Then*

$$\mathrm{AVaR}_\beta(X) \geq \mathrm{AVaR}_\gamma(X),$$

and equality holds if and only if $X \geq c$ a.s. and $\mathbb{P}[X = c] \geq \gamma$ for some constant $c \in \mathbb{R}$. In particular, $\mathrm{AVaR}_\beta(X) = E[-X]$ if and only if X is constant.

Proof. The case $\beta = 0$ is obvious. Suppose $\beta > 0$. Since q_X is increasing, we have

$$(\gamma - \beta)\int_0^\beta q_X(s)\,ds \leq \beta \int_\beta^\gamma q_X(s)\,ds,$$

with equality if and only if $q_X(s) = q_X(\gamma)$ for all $s \leq \gamma$. This proves the claim. □

Remark 3.12. The law-invariance requirement in theorem 3.4 cannot be dropped: let $Z \in L^1_+$ be non-constant with $E[Z] = 1$. Then $\rho_1 = -E[\cdot]$ and $\rho_2 = E[-Z\cdot]$ are convex risk measures on L^∞, and ρ_2 is not law-invariant. Thus theorem 3.4 does not apply. Indeed, the convolution $\rho_1 \Box \rho_2 \equiv -\infty$ is not exact. ◊

3.2. Uniqueness of Optimal Allocations

Let $\rho_1, \dots, \rho_n : L^p \to (-\infty, \infty]$ be convex cash-invariant functions. Due to cash-invariance of ρ_i, uniqueness of an optimal allocation can only hold up to *rebalancing the cash*. That is, $(X_1, \dots, X_n)$ is an optimal allocation of X if and only if $(X_1 + c_1, \dots, X_n + c_n)$ is so, for all cash positions $c_i \in \mathbb{R}$ with $\sum_{i=1}^n c_i = 0$.

The following sufficient condition for uniqueness is straightforward.

Proposition 3.13. *Suppose ρ_i, $i = 1, \dots, (n-1)$, are* strictly convex *in the following sense*

$$\rho_i(\lambda X + (1-\lambda)Y) < \lambda \rho_i(X) + (1-\lambda)\rho_i(Y) \quad \textit{for all } \lambda \in (0,1),$$

for all $X, Y \in \operatorname{dom} \rho_i$ with $X - Y \notin \mathbb{R}$. Then any optimal allocation of $X \in L^p$ with $\square_{i=1}^n \rho_i(X) < \infty$ is unique up to rebalancing the cash.

Proof. Let $X \in L^p$ with $\square_{i=1}^n \rho_i(X) < \infty$. We argue by contradiction and suppose $(X_1, \dots, X_n)$ and $(Y_1, \dots, Y_n)$ are optimal allocations of X with $X_j - Y_j \notin \mathbb{R}$ for a $j \in \{1, \dots, n-1\}$. Then, for any $\lambda \in (0,1)$, $Z_i := \lambda X_i + (1-\lambda)Y_i$ defines an allocation of X with

$$\sum_{i=1}^n \rho_i(Z_i) < \lambda \sum_{i=1}^n \rho_i(X_i) + (1-\lambda) \sum_{i=1}^n \rho_i(Y_i) = \square_{i=1}^n \rho_i(X).$$

But this contradicts the optimality of $(X_1, \dots, X_n)$, whence the claim. $\square$

For instance, the optimal allocation for the entropic risk measure in example 3.8 is unique up to rebalancing the cash. More recent examples of strictly convex risk measures can be found in [8].

Without the strict convexity assumption in proposition 3.13, uniqueness does not hold in general. A trivial example is

$$(-E) \square (-E) = -E.$$

In this case, all allocations of any $X \in L^1$ are optimal allocations of X. Another example is the following.

Example 3.14. Let $0 \leq \beta \leq \gamma < 1$. Choose $A \in \mathcal{F}$ and $\mathbb{Q} \ll \mathbb{P}$ such that $\mathbb{P}[A] > 0 = \mathbb{Q}(A)$ and $d\mathbb{Q}/d\mathbb{P} \leq 1/\gamma \leq 1/\beta$. Then,

$$0 = \mathrm{AVaR}_\beta(1_A) = \mathrm{AVaR}_\gamma(1_A) = \mathrm{AVaR}_\beta \square \mathrm{AVaR}_\gamma(1_A).$$

Hence, both $(1_A, 0)$ and $(0, 1_A)$ are comonotone optimal allocations of 1_A. $\lozenge$

On the other hand, the strict convexity assumption in proposition 3.13 is not necessary for uniqueness up to rebalancing the cash. This is shown by the following example.

Example 3.15. Let $\beta \in (0,1)$. We know that $\mathrm{AVaR}_\beta \Box - E = -E$ (example 3.7). Suppose $(Y, X-Y)$ is an optimal allocation of X. This implies $\mathrm{AVaR}_\beta(Y) + E[-(X-Y)] = E[-X]$, that is, $\mathrm{AVaR}_\beta(Y) = E[-Y]$. In view of lemma 3.11 we conclude that Y must be constant, i.e. a cash position. Hence, the optimal allocation $(0, X)$ is unique up to rebalancing the cash. $\Diamond$

3.3. Problem Reduction

First note that it suffices to prove theorem 3.4 for $n = 2$, because

$$\Box_{i=1}^{n}\rho_i = \left(\Box_{i=1}^{n-1}\rho_i\right)\Box\rho_n.$$

For the sake of simplicity, we will further restrict our studies to the case $p = 1$. By nature of the arguments presented in the proof of theorem 3.4 (section 3.5), it will become clear that they all literally carry over to L^p, simply by replacing L^1 with L^p and choosing the appropriate dual. However, in what follows, we give another justification for the retreat to L^1 by proving in corollary 3.16 that the assertions of theorem 3.4 for the L^p-cases can be derived from the L^1-case. The reason for this is theorem 2.11 and the following observation. Let

$$\mathbb{A} := \{(f,g) \mid f,g : \mathbb{R} \to \mathbb{R} \text{ are increasing}\,,\ f + g = \mathrm{Id}_{\mathbb{R}}\}\,.$$

Clearly, if $(f,g) \in \mathbb{A}$, then both f and g are 1-Lipschitz-continuous. Hence, $|f(X)| \leq |X| + |f(0)|$ and $|g(X)| \leq |X| + |g(0)|$, implying that

$$\text{if } X \in L^p \text{ then } (f(X), g(X)) \in L^p \times L^p. \tag{3.2}$$

Thus, if $X \in L^p$, then the set $\{(f(X), g(X)) \mid (f,g) \in \mathbb{A}\}$ of all comonotone 2-dimensional allocations of X is a subset of $L^p \times L^p$.

Corollary 3.16. *Let $\rho, \mu : L^p \to (-\infty, \infty]$ be two law-invariant closed convex cash-invariant functions. Suppose theorem 3.4 is true for the model space L^1. Let $\rho_\infty := \rho|_{L^\infty}$ and $\mu_\infty := \mu|_{L^\infty}$. Then,*

$$\rho\Box\mu = \overline{\rho_\infty}^1\Box\overline{\mu_\infty}^1|_{L^p}.$$

In particular, the assertions of theorem 3.4 are true for L^p too.

Proof. According to theorem 2.11 we have $\rho = \overline{\rho_\infty}^1|_{L^p}$ and $\mu = \overline{\mu_\infty}^1|_{L^p}$. By assumption, and since the L^1-closure preserves cash-invariance, for any $X \in L^p \subset L^1$ there is a comonotone optimal allocation $(f(X), g(X))$, i.e.

$$\overline{\rho_\infty}^1\Box\overline{\mu_\infty}^1(X) = \overline{\rho_\infty}^1(f(X)) + \overline{\mu_\infty}^1(g(X)).$$

Clearly, $\overline{\rho_\infty}^1 \square \overline{\mu_\infty}^1(X) \leq \rho\square\mu(X)$, and since $(f(X), g(X)) \in L^p \times L^p$ by (3.2), we deduce that

$$\overline{\rho_\infty}^1 \square \overline{\mu_\infty}^1(X) = \rho(f(X)) + \mu(g(X)) = \rho\square\mu(X)\,.$$

Hence, $\rho\square\mu$ is simply the restriction of $\overline{\rho_\infty}^1 \square \overline{\mu_\infty}^1$ to L^p and thus a law-invariant closed (w.r.t. $\|\cdot\|_p$) convex cash-invariant function. (The l.s.c. stems from the fact that $\|\cdot\|_p$-convergence implies $\|\cdot\|_1$-convergence.) Moreover, $\rho\square\mu$ is exact, and there is always a comonotone optimal allocation. $\square$

3.4. Comonotone Concave Order Improvement

Recall the concave order $\succeq_c$ from definition 2.15. It is proved in [21] corollary 2.62 that

$$X \succeq_c Y \quad \Leftrightarrow \quad E[X] = E[Y] \text{ and } E[(X-c)^+] \leq E[(Y-c)^+]\ \forall\, c \in \mathbb{R}\,. \tag{3.3}$$

The following proposition will turn out to be the fundament on which the proof of theorem 3.4 is built. It is based upon the results of Landsberger and Meilijson in [27], and states that every allocation is dominated in concave order by a comonotone allocation. The importance of this result is clear by lemma 2.17 where we established that any law-invariant closed convex function is $\succeq_c$-monotone.

Proposition 3.17. *(see proposition 1 in [27]) For any allocation (Y, Z) of $X \in L^1$, there is $(f, g) \in \mathbb{A}$ such that $f(X) \succeq_c Y$ and $g(X) \succeq_c Z$.*

Unfortunately, Landsberger and Meilijson [27] only proved this result for random variables X supported by a finite set. Therefore, we give a full proof here.

Proof. We divide the proof into three steps.

Step 1: We start out as in [27] by noticing that Jensen's inequality implies that $(E[Y \mid X], E[Z \mid X])$ is an allocation of X which is at least as good as (Y, Z), meaning that $E[Y \mid X] \succeq_c Y$ and $E[Z \mid X] \succeq_c Z$. Let $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ be measurable functions such that $h_1(X) = E[Y \mid X]$, $h_2(X) = E[Z \mid X]$. Clearly, we may assume that $h_1 + h_2 = \mathrm{Id}_\mathbb{R}$. If h_1 and h_2 are increasing, we are done, if not, we go on improving this allocation. However, we have now established that during the remainder of this proof we may restrict ourselves to improve allocations (Y, Z) of type $Y = h_1(X)$ and $Z = h_2(X)$ for some measurable functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ such that $h_1 + h_2 = \mathrm{Id}_\mathbb{R}$.

Step 2: Suppose X is a simple random variable, i.e. $X = \sum_{i=1}^n x_i 1_{A_i}$ for a partition $A_1, \dots, A_n$ of Ω and real numbers x_i such that $x_i \neq x_j$ for $i \neq j$. Let $y_i := h_1(x_i)$ and $z_i := h_2(x_i)$. Then $h_1(X) = \sum_{i=1}^n y_i 1_{A_i}$ and $h_2(X) = \sum_{i=1}^n z_i 1_{A_i}$. We set $x := (x_1, \dots, x_n)$, $y := (y_1, \dots, y_n)$, $z := (z_1, \dots, z_n)$ and $p_k := \mathbb{P}(A_k)$, $k = 1, \dots, n$. Let π be a permutation of $\{1, \dots, n\}$ such that $x_\pi := (x_{\pi(1)}, \dots, x_{\pi(n)}) \in \mathcal{D} := \{\widetilde{x} \in \mathbb{R}^n \mid \widetilde{x}_1 \leq \widetilde{x}_2 \leq \ldots \leq \widetilde{x}_n\}$. Observe that $(h_1(X), h_2(X))$ is comonotone if and only if $y_\pi, z_\pi \in \mathcal{D}$.

For sake of brevity we may and will assume w.l.o.g. that $x \in \mathcal{D}$ already. Supposing that (y,z) is not comonotone, i.e. $y \notin \mathcal{D}$ or $z \notin \mathcal{D}$ or both, the following algorithm by M. Landsberger and I. Meilijson transfers (y,z) into a comonotone allocation:

Since (y,z) is not comonotone, there must exist an i such that $y_1 \leq \ldots \leq y_i$, $z_1 \leq \ldots \leq z_i$ but either $y_{i+1} < y_i$ or $z_{i+1} < z_i$. W.l.o.g. let us assume that $z_{i+1} < z_i$. Then there is a smallest j such that $z_{i+1} < z_j$. For $k = j, \ldots, i$ we set

$$y_k^{new} = y_k + \frac{p_{i+1}}{\sum_{l=j}^{i+1} p_l}(z_j - z_{i+1}) \quad \text{and} \quad z_k^{new} = z_k - \frac{p_{i+1}}{\sum_{l=j}^{i+1} p_l}(z_j - z_{i+1})$$

whereas

$$y_{i+1}^{new} = y_{i+1} - \frac{\sum_{l=j}^{i} p_l}{\sum_{l=j}^{i+1} p_l}(z_j - z_{i+1}) \quad \text{and} \quad z_{i+1}^{new} = z_{i+1} + \frac{\sum_{l=j}^{i} p_l}{\sum_{l=j}^{i+1} p_l}(z_j - z_{i+1})\,.$$

The other coordinates of y and z are left unchanged. Finally, set $y := y^{new}$ and $z := z^{new}$ and repeat the procedure in case the output is not comonotone.

Let $(Y^{new}, Z^{new}) := (\sum_{i=1}^n y_i^{new} 1_{A_i}, \sum_{i=1}^n z_i^{new} 1_{A_i})$. Firstly, (Y^{new}, Z^{new}) is obviously an allocation of X, secondly, we claim that $Y^{new} \succeq_c Y$ and $Z^{new} \succeq_c Z$, i.e. each cycle of the algorithm improves the allocation, and finally, it is easily verified that the algorithm returns a comonotone allocation in at most $n(n-1)/2$ cycles (observe that $z_j^{new} = z_{i+1}^{new}$). In order to show that $Y^{new} \succeq_c Y$ and $Z^{new} \succeq_c Z$, let $u : \mathbb{R} \to \mathbb{R}$ be any concave function. Introducing the abbreviations

$$\alpha := \frac{p_{i+1}}{\sum_{l=j}^{i+1} p_l} \in (0,1) \quad \text{and} \quad \lambda_k := \frac{z_j - z_{i+1}}{z_k - z_{i+1}} \in (0,1]$$

and recalling that concavity is equivalent to

$$\forall a < b < c: \ \frac{u(b) - u(a)}{b - a} \geq \frac{u(c) - u(a)}{c - a} \geq \frac{u(c) - u(b)}{c - b}\,, \tag{3.4}$$

we compute:

$$\begin{aligned}
\sum_{k=j}^{i+1} u(z_k^{new})p_k &= \sum_{k=j}^{i} u((1-\alpha\lambda_k)z_k + \alpha\lambda_k z_{i+1})p_k + u((1-\alpha)z_j + \alpha z_{i+1})p_{i+1} \\
&\geq \sum_{k=j}^{i} \left[(1-\alpha\lambda_k)u(z_k) + \alpha\lambda_k u(z_{i+1})\right] p_k \\
&\quad + \left[(1-\alpha)u(z_j) + \alpha u(z_{i+1})\right] p_{i+1} \\
&= \sum_{k=j}^{i+1} u(z_k)p_k + (1-\alpha)(u(z_j) - u(z_{i+1}))p_{i+1} \\
&\quad - \alpha \sum_{k=j}^{i} \lambda_k (u(z_k) - u(z_{i+1}))p_k \\
&\overset{(3.4)}{\geq} \sum_{k=j}^{i+1} u(z_k)p_k \,,
\end{aligned}$$

because $\lambda_k(u(z_k) - u(z_{i+1})) \leq u(z_j) - u(z_{i+1})$ by inequality (3.4). A similar computation for Y^{new} shows that $Y^{new} \succeq_c Y$ and $Z^{new} \succeq_c Z$.

Step 3: Let X be any integrable random variable. Recalling the usual monotone approximation from Lebesgue integration theory, let $(Y_n)_{n\in\mathbb{N}}$ and $(Z_n)_{n\in\mathbb{N}}$ be sequences of simple random variables converging $\mathbb{P}$-a.s. and in L^1 to Y and Z respectively such that $|Y_n| \leq |Y|$ and $|Z_n| \leq |Z|$ for all $n \in \mathbb{N}$. Then $X_n := Y_n + Z_n$ converges to X $\mathbb{P}$-a.s. and in L^1. By step 2, for each $n \in \mathbb{N}$, there exists a comonotone improvement $(f_n(X_n), g_n(X_n))$ of (Y_n, Z_n). Choose $N \in \mathbb{N}$ such that $\|Y_n\|_1 \leq \|Y\|_1 + 1$, $\|Z_n\|_1 \leq \|Z\|_1 + 1$, and $\|X_n\|_1 \leq \|X\|_1 + 1$ for all $n \geq N$. Since all f_n (and g_n) are 1-Lipschitz-continuous, we have that $|f_n(0)| \leq |X_n| + |f_n(X_n)|$. Taking expectations on both sides yields

$$|f_n(0)| \leq E[|X_n|] + E[|f_n(X_n)|] \leq E[|X_n|] + E[|Y_n|] \,,$$

because $f_n(X_n) \succeq_c Y_n$ and $x \mapsto -|x|$ is concave. Hence, if $n \geq N$, we get $|f_n(0)| \leq \|X\|_1 + \|Y\|_1 + 2 =: K_1$ and similarly $|g_n(0)| \leq \|X\|_1 + \|Z\|_1 + 2 =: K_2$, and thus $f_n(0), g_n(0) \in [-K, K]$ for $K := \max\{K_1, K_2\}$. Therefore, by lemma A.3, there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ and a 1-Lipschitz-continuous increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f(a) = \lim_{k\to\infty} f_{n_k}(a)$, $a \in \mathbb{R}$. Now it is easily verified that $(g_{n_k})_{k\in\mathbb{N}}$ converges pointwise to the 1-Lipschitz-continuous increasing function $g := \mathrm{Id}_{\mathbb{R}} - f$. Hence, the sequence $f_{n_k}(X_{n_k})$ converges $\mathbb{P}$-a.s. to $f(X)$, and $g_{n_k}(X_{n_k}) = X_{n_k} - f_{n_k}(X_{n_k})$ converges $\mathbb{P}$-a.s. to $g(X)$. Since $|f_{n_k}(X_{n_k})| \leq |X_{n_k}| + K \leq |Y| + |Z| + K$ and $|g_{n_k}(X_{n_k})| \leq |Y| + |Z| + K$ for large enough $k \in \mathbb{N}$, we can apply the dominated convergence theorem which yields $f(X), g(X) \in L^1$ and $\|f(X) - f_{n_k}(X_{n_k})\|_1 \to 0$, $\|g(X) - g_{n_k}(X_{n_k})\|_1 \to 0$ for $k \to \infty$.

Moreover, in view of (3.3), we have that

$$E[f(X)] = \lim_{k\to\infty} E[f_{n_k}(X_{n_k})] = \lim_{k\to\infty} E[Y_{n_k}] = E[Y]\,,$$

and for all $c \in \mathbb{R}$:

$$\begin{aligned} E[(f(X)-c)^+] &= \lim_{k\to\infty} E[(f_{n_k}(X_{n_k})-c)^+] \\ &\leq \lim_{k\to\infty} E[(Y_{n_k}-c)^+] \;=\; E[(Y-c)^+]\,, \end{aligned}$$

and similarly for g. Hence, $(f(X), g(X))$ is a comonotone allocation of X satisfying $f(X) \succeq_c Y$ and $g(X) \succeq_c Z$ according to (3.3). □

3.5. Proof of Theorem 3.4

In view of section 3.3 we only have to prove theorem 3.4 for $n = 2$ and $p = 1$. To this end, let $\rho_1, \rho_2 : L^1 \to (-\infty, \infty]$ be law-invariant closed convex cash-invariant functions. We divide the proof into four steps.

Step 1: $\rho_1 \Box \rho_2$ is proper, convex, and cash-invariant.

Proof. It is easily verified that the convolution preserves the convexity of ρ_1 and ρ_2. According to corollary 2.18 we have that $\rho_i(X) \geq -E[X] + \rho_i(0)$ for all $X \in L^1$ and $i = 1, 2$. Hence,

$$\begin{aligned} \rho_1 \Box \rho_2(X) &= \inf_{X_1+X_2=X} \rho_1(X_1) + \rho_2(X_2) \\ &\geq \inf_{X_1+X_2=X} -E[X_1] - E[X_2] + \rho_1(0) + \rho_2(0) \\ &= -E[X] + \rho_1(0) + \rho_2(0)\,, \end{aligned}$$

so $\rho_1 \Box \rho_2(0) = \rho_1(0) + \rho_2(0) < \infty$ and $\rho_1 \Box \rho_2$ is proper. Furthermore, for all $r \in \mathbb{R}$ we obtain

$$\rho_1 \Box \rho_2(X + r) = \inf_{Y \in L^1} \rho_1(X + r - Y) + \rho_2(Y) = \rho_1 \Box \rho_2(X) - r$$

due to the cash-invariance of ρ_1. □

Step 2: $\rho_1 \Box \rho_2(X) = \inf_{(f,g)\in\mathbb{A}} \rho_1(f(X)) + \rho_2(g(X))\,, \quad X \in L^1.$

Proof. This is an immediate consequence of proposition 3.17 and lemma 2.17. □

Step 3: $\rho_1 \Box \rho_2$ is exact, and for each $X \in L^1$ there exists a comonotone optimal allocation.

Proof. Suppose $X \in L^1$ is such that $\rho_1 \Box \rho_2(X) = \infty$. Then every comonotone allocation $(f(X), g(X))$ is optimal.

Now let $X \in \operatorname{dom} \rho_1 \Box \rho_2$ and choose a sequence $(f_n, g_n) \in \mathbb{A}$, $n \in \mathbb{N}$, such that $\rho_1 \Box \rho_2(X) = \lim_{n\to\infty} \rho_1(f_n(X)) + \rho_2(g_n(X))$. By cash-invariance we may assume that $f_n(0) = g_n(0) = 0$ for all $n \in \mathbb{N}$. Hence, by lemma A.3, there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ and a 1-Lipschitz-continuous and increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f(a) = \lim_{k\to\infty} f_{n_k}(a)$ for all $a \in \mathbb{R}$. Clearly, the sequence $f_{n_k}(X)$ converges $\mathbb{P}$-a.s. to $f(X)$ and $g_{n_k}(X) = X - f_{n_k}(X)$ converges $\mathbb{P}$-a.s. to $g(X)$ where $g := \mathrm{Id}_{\mathbb{R}} - f$ is a 1-Lipschitz-continuous increasing function. Since $|f_{n_k}(X)| \le |X|$ and $|g_{n_k}(X)| \le |X|$ for all $k \in \mathbb{N}$, we may apply the dominated convergence theorem which yields $f(X), g(X) \in L^1$ and $\|f(X) - f_{n_k}(X)\|_1 \to 0$, $\|g(X) - g_{n_k}(X)\|_1 \to 0$ for $k \to \infty$. On the one hand, by l.s.c., we have

$$\begin{aligned}\rho_1 \Box \rho_2(X) &= \lim_{k\to\infty} \rho_1(f_{n_k}(X)) + \rho_2(g_{n_k}(X)) \\ &\ge \liminf_{k\to\infty} \rho_1(f_{n_k}(X)) + \liminf_{k\to\infty} \rho_2(g_{n_k}(X)) \\ &\ge \rho_1(f(X)) + \rho_2(g(X)) \,.\end{aligned}$$

On the other hand, by definition of the convolution, we have $\rho_1 \Box \rho_2(X) \le \rho_1(f(X)) + \rho_2(g(X))$. Consequently, the comonotone allocation $(f(X), g(X))$ of X is optimal. □

Step 4: $\rho_1 \Box \rho_2$ is closed and law-invariant.

Proof. We claim that $\mathcal{A}_{\rho_1 \Box \rho_2}$ is closed. To this end, let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{A}_{\rho_1 \Box \rho_2}$ be a sequence converging to some X in L^1. According to step 3 there are $(f_n, g_n) \in \mathbb{A}$, $n \in \mathbb{N}$, such that $0 \ge \rho_1 \Box \rho_2(X_n) = \rho_1(f_n(X_n)) + \rho_2(g_n(X_n))$. By cash-invariance we may assume that $f_n(0) = g_n(0) = 0$ for all $n \in \mathbb{N}$. Similar to step 3, employing lemma A.3, we find a subsequence $(f_{n_k}, g_{n_k})_{k\in\mathbb{N}}$ of $(f_n, g_n)_{n\in\mathbb{N}}$ and $(f, g) \in \mathbb{A}$ such that $f_{n_k}(X_{n_k})$ converges to $f(X)$ in L^1 and $g_{n_k}(X_{n_k})$ converges to $g(X)$ in L^1. By l.s.c. of ρ_1 and ρ_2 we obtain

$$\begin{aligned}\rho_1 \Box \rho_2(X) &\le \rho_1(f(X)) + \rho_2(g(X)) \\ &\le \liminf_{k\to\infty} \rho_1(f_{n_k}(X_{n_k})) + \liminf_{k\to\infty} \rho_2(g_{n_k}(X_{n_k})) \\ &\le \liminf_{k\to\infty} \rho_1(f_{n_k}(X_{n_k})) + \rho_2(g_{n_k}(X_{n_k})) \le 0 \,,\end{aligned}$$

and thus $X \in \mathcal{A}_{\rho_1 \Box \rho_2}$. Hence, $\mathcal{A}_{\rho_1 \Box \rho_2}$ is closed, i.e. $\rho_1 \Box \rho_2$ is closed. The law-invariance of $\rho_1 \Box \rho_2$ follows from lemma 3.1 and the fact that a closed convex function on L^1 is law-invariant if and only if its dual is (lemma 2.14). □

3.6. Optimal Risk Sharing under Constraints

For a comprehensive discussion of risk sharing under constraints we refer to [16]. The setting is as follows: two agents with initial endowments X_1 and X_2 in L^p assess their individual risk by means of law-invariant closed convex risk measures ρ_1 and ρ_2 on L^p, respectively. In order to minimise total and individual risk, they reallocate the aggregate endowment $X = X_1 + X_2$ amongst themselves. As is often the case in practice, this reallocation procedure might be subject to some restrictions in the sense that not every risk sharing of X is admissible. We formalise this by defining the set of admissible risk sharings of X as

$$A_X := \{(Y_1, Y_2) \in M_1 \times M_2 \mid Y_1 + Y_2 \leq X\}$$

where $M_i \subset L^p$ are closed convex law-invariant cash-invariant (that is, $Y \in M_i$ implies $Y + a \in M_i$ for all $a \in \mathbb{R}$) sets such that $X_i \in M_i$, $i = 1, 2$. Note that we allow for "free disposal", i.e. $X - Y_1 - Y_2 \geq 0$ for all $(Y_1, Y_2) \in A_X$. The optimal risk sharing under constraints problem becomes

$$\inf_{(Y_1,Y_2)\in A_X} \rho_1(Y_1) + \rho_2(Y_2). \tag{3.5}$$

In order to solve (3.5), denote $\rho_i^{M_i} := \rho_i + \delta(\cdot \mid M_i)$, $i = 1, 2$. Then

$$\begin{aligned}
(3.5) \quad &= \inf_{Y_1,Y_2\in L^p,\, Y_1+Y_2\leq X} \rho_1^{M_1}(Y_1) + \rho_2^{M_2}(Y_2) \\
&= \rho_1^{M_1} \Box \rho_2^{M_2} \Box \delta(\cdot \mid L^p_+)(X) \\
&= \rho_1^{M_1} \Box \rho_2^{M_2} \Box - \operatorname{essinf}(X)\,.
\end{aligned}$$

Note that $\delta(\cdot \mid M_i)$ is proper, closed, law-invariant, and convex. By lemma 2.17 we know that $\delta(E[Y] \mid M_i) \leq \delta(Y \mid M_i)$ for all $Y \in L^p$. Hence $Y \in M_i$ implies $E[Y] \in M_i$, and thus $\mathbb{R} \subset M_i$ by cash-invariance. We conclude that $\rho_1^{M_1}$ and $\rho_2^{M_2}$ are law-invariant closed convex cash-invariant functions. Since $-$essinf is a law-invariant closed coherent risk measure, we know by corollary 3.6 that $\rho_1^{M_1} \Box \rho_2^{M_2} \Box -\operatorname{essinf}$ is a law-invariant closed convex risk measure, and that this convolution admits a comonotone optimal allocation (Y_1, Y_2, Y_3) of X. If $\rho_1^{M_1}(Y_1) + \rho_2^{M_1}(Y_2) - \operatorname{essinf}(Y_3) = \infty$, then any admissible risk sharing of X is optimal. Otherwise, if $\rho_1^{M_1}(Y_1) + \rho_2^{M_1}(Y_2) - \operatorname{essinf}(Y_3) < \infty$, then we have that $-\operatorname{essinf}(Y_3) < \infty$, and thus $(Y_1 + \operatorname{essinf}(Y_3), Y_2)$ is a solution of the optimisation problem (3.5). Also note that $(Y_1 + \operatorname{essinf}(Y_3), Y_2) = (f(X), g(X))$ for some increasing functions $f, g : \mathbb{R} \to \mathbb{R}$.

4. Subgradients of Law-Invariant Convex Risk Measures on L^1

Subgradients play an important role in equilibrium theory (see section 4.4). Given the model space L^∞, we know that every convex risk measure is everywhere subdifferentiable, simply because it is continuous ([14] corollary 2.5 and proposition 5.2). However, throughout this text, we are interested in model spaces larger than L^∞. As we study *law-invariant* convex risk measures, theorem 2.11 suggests the model space L^1. Therefore, throughout this chapter, all law-invariant convex risk measures will a priori be defined on L^1. We will also assume that all risk measures ρ are normalised, i.e. $\rho(0) = 0$. Moreover, we denote by ρ_∞ the restriction of ρ to L^∞, i.e. $\rho_\infty := \rho|_{L^\infty}$.

4.1. Subgradients and Generalised Subgradients

It is well known that a proper closed convex function on a Banach space is continuous and subdifferentiable on the interior of its domain (see e.g. [14] corollary 2.5 and proposition 5.2). Recalling theorem 2.3, we know that for every convex risk measure ρ on L^1 (which is proper by definition) we have $\operatorname{int}\operatorname{dom}\rho \neq \emptyset$ if and only if ρ is real-valued and continuous. We summarise these results on subdifferentiability in the following lemma.

Lemma 4.1. *Let ρ be a convex risk measure on L^1. Equivalent are:*

(i) *ρ is everywhere subdifferentiable.*

(ii) *ρ is real-valued and continuous.*

(iii) $\operatorname{int}\operatorname{dom}\rho \neq \emptyset$.

An example of a continuous convex risk measure on L^1 is AVaR_α, for some $\alpha \in (0, 1]$ (see example 2.20). But we have already seen that that closed convex risk measures are not continuous on L^1 in general. An example is the entropic risk measure Entr_β, $\beta > 0$, (example 2.21) which is closed, but not continuous. According to lemma 4.19 below we have that

$$\mathrm{Entr}_\beta(X) = E[ZX] - \mathrm{Entr}_\beta^*(Z) \quad \Rightarrow \quad Z = \frac{-e^{-\beta X}}{E\left[e^{-\beta X}\right]}.$$

In view of (1.3), we infer that $\partial\,\mathrm{Entr}_\beta(X) = \emptyset$ for every $X \in L^1$ with $e^{-\beta X} \notin L^\infty$, even though $\operatorname{dom}\mathrm{Entr}_\beta$ includes such X.

This motivates the following extension of the notion of a subgradient.

Definition 4.2. *The generalised subgradient of a convex risk measure ρ on L^1 at $X \in L^1$ is defined as*

$$\delta\rho(X) := \{Z \in L^1 \mid (XZ) \in L^1,\ \forall Y \in L^\infty:\ \rho(Y) \geq \rho(X) + E[Z(Y - X)]\}.$$

Lemmas 4.3 and 4.6 below show that $\delta\rho$ is indeed a generalisation of $\partial\rho$.

Lemma 4.3. *Let ρ be a convex risk measure on L^1. The following conditions hold:*

(i) *for all $X \in L^1$: $\partial\rho(X) \subset \delta\rho(X) \subset \mathrm{dom}\ \rho^*_\infty \cap L^1$,*

(ii) *for all $X \in L^\infty$: $\delta\rho(X) = \partial\rho_\infty(X) \cap L^1$,*

(iii) *for all $X \in L^1$: $\delta\rho(X) \neq \emptyset \ \Rightarrow\ X \in \mathrm{dom}\ \rho$.*

Proof. We only prove the inclusion $\delta\rho(X) \subset \mathrm{dom}\ \rho^*_\infty \cap L^1$, because the rest is obvious by definition of $\delta\rho(X)$. However, this inclusion follows from the fact that $Z \in \delta\rho(X)$ implies

$$\infty > E[ZX] - \rho(X) \geq \sup_{Y \in L^\infty} E[ZY] - \rho(Y) = \rho^*_\infty(Z).$$

□

We remark that $\delta\rho(X) = \emptyset$ is possible even for bounded risks $X \in L^\infty$ (see example 4.5.1). In order to have $\delta\rho(X) \neq \emptyset$ on L^∞ at least, we will have to require that ρ is continuous from below. This property is defined and characterised in the following proposition (see also [21] proposition 4.21). It is in fact a property of the restriction ρ_∞ of ρ to L^∞ only. Note that proposition 4.4(iv) shows that continuity from below is satisfied by most law-invariant convex risk measures of interest, e.g. by Avarage Value at Risks, semi-deviation and entropic risk measures.

Proposition 4.4. *Let ρ be a law-invariant closed convex risk measure on L^1. Then, the following conditions are equivalent:*

(i) *ρ is* continuous from below, *i.e. for every $X \in L^\infty$ and every sequence $(X_n)_{n\in\mathbb{N}} \subset L^\infty$ with $X_n \uparrow X$ we have $\rho(X_n) \downarrow \rho(X)$.*

(ii) $\mathrm{dom}\ \rho^*_\infty \subset L^1$.

(iii) *The level sets $Q_k := \{Z \in L^1 | \rho^*_\infty(Z) \leq k\}$, $k \in \mathbb{R}$, are $\sigma(L^1, L^\infty)$-compact.*

(iv) $\{X \in L^1 \mid \mathrm{essinf}\, X = -\infty\} \cap \mathrm{dom}\ \rho \neq \emptyset$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii): In view lemma 2.13, we may apply theorem 5.2 in [24] which establishes the desired equivalences. These equivalences are also partially proved in [21] proposition 4.21 and [16] theorem C.1.

(i) $\Rightarrow$ (iv): Fix a decreasing sequence of sets $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, such that $\mathbb{P}(A_n) > 0$ and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. Since ρ is continuous from below and $-1_{A_n} \uparrow 0$, there is a $n_1 \in \mathbb{N}$ such that $\rho(-1_{A_{n_1}}) \leq \frac{1}{2}$. Then again, as $-1_{A_{n_1}} - 1_{A_l} \uparrow -1_{A_{n_1}}$ for $l \to \infty$, there is a $n_2 > n_1$ such that $\rho(-1_{A_{n_1}} - 1_{A_{n_2}}) \leq \rho(-1_{A_{n_1}}) + \frac{1}{4}$. Continuing this construction inductively, we find for each $k \in \mathbb{N}$ a $n_{k+1} > n_k$ such that $\rho(\sum_{i=1}^{k+1} -1_{A_{n_i}}) \leq \rho(\sum_{i=1}^{k} -1_{A_{n_i}}) + \frac{1}{2^{k+1}}$. The sequence $X_k := \sum_{i=1}^{k} -1_{A_{n_i}}$ converges monotonously to $X := \sum_{i=1}^{\infty} -1_{A_{n_i}}$ which is unbounded from below. By the monotone convergence theorem, and since $\rho^*(-1) = 0$ (corollary 2.18), we deduce that

$$E[|X|] = \lim_{k\to\infty} E[-X_k] \leq \liminf_{k\to\infty} \rho(X_k) \leq \liminf_{k\to\infty} \sum_{i=1}^{k} \frac{1}{2^i} \leq 1.$$

Hence, $X \in L^1$ and by l.s.c. of ρ we have that $\rho(X) \leq \liminf_{k\to\infty} \rho(X_k) \leq 1$, i.e. $X \in \mathrm{dom}\, \rho$.

(iv) $\Rightarrow$ (ii): Let $X \in \{Y \in L^1 \mid \operatorname{essinf} Y = -\infty\} \cap \mathrm{dom}\, \rho$. According to lemma 4.5 below we may assume that $X \leq 0$. Suppose there were $\mu \in \mathrm{dom}\, \rho^*_\infty \setminus L^1$. Then there exists a decreasing sequence of sets $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, such that $\mathbb{P}(A_n) > 0$ and $\bigcap_n A_n = \emptyset$, but $E[-\mu 1_{A_n}] \downarrow \epsilon$ for some $\epsilon > 0$, because otherwise μ would be σ-additive. By considering to a subsequence if necessary we may assume that $\mathbb{P}(A_n) \leq \mathbb{P}(X \leq -n)$. Let $B_n \subset \{X \leq -n\}$ such that $\mathbb{P}(B_n) = \mathbb{P}(A_n)$, and let $\pi_n : \Omega \to \Omega$ be a measure preserving transformation such that $\pi_n(A_n) = B_n$ $\mathbb{P}$-a.s. (see section A.3). Then $X_n := X \circ \pi_n \sim X$, and $A_n \subset \{X_n \leq -n\}$. Hence, by law-invariance

$$\begin{aligned}
\rho(X) &= \rho(X_n) \quad \geq \quad \rho_\infty(X_n \vee (-n-1)) \\
&\geq E[\mu(X_n \vee (-n-1))] - \rho^*_\infty(\mu) \\
&\geq nE[-\mu 1_{A_n}] - \rho^*_\infty(\mu) \\
&\geq n\epsilon - \rho^*_\infty(\mu)
\end{aligned}$$

for all $n \in \mathbb{N}$ which can only hold if $\rho(X) = \infty$. But this is a contradiction to $X \in \mathrm{dom}\, \rho$, so $\mathrm{dom}\, \rho^*_\infty \subset L^1$. □

In the proof of proposition 4.4 (iv)$\Rightarrow$ (ii) we used the following lemma.

Lemma 4.5. *Let ρ be a law-invariant closed convex risk measure on L^1. Then $X \in \mathrm{dom}\, \rho$ if and only if $-X^- \in \mathrm{dom}\, \rho$.*

Proof. "$\Leftarrow$" follows from $X \geq -X^-$ and monotonicity of ρ. As for "$\Rightarrow$", let $X \in \mathrm{dom}\, \rho$ and suppose that $\mathbb{P}(X > 0) > 0$, otherwise the assertion is trivial. By lemma 2.17 we know that $E[X|X1_{\{X<0\}}] \in \text{dom } \rho$. Clearly,

$$E[X|X1_{\{X<0\}}] = X1_{\{X<0\}} + \frac{E[X1_{\{X\geq 0\}}]}{\mathbb{P}(X \geq 0)} 1_{\{X\geq 0\}}.$$

Hence, by cash-invariance and monotonicity we infer that

$$\begin{aligned} \rho(X1_{\{X<0\}}) &= \rho\left(X1_{\{X<0\}} + \frac{E[X1_{\{X\geq 0\}}]}{\mathbb{P}(X \geq 0)}\right) + \frac{E[X1_{\{X\geq 0\}}]}{\mathbb{P}(X \geq 0)} \\ &\leq \rho(E[X|X1_{\{X<0\}}]) + \frac{E[X1_{\{X\geq 0\}}]}{\mathbb{P}(X \geq 0)} < \infty. \end{aligned}$$

□

We now establish a characterisation of the generalised subgradient which is analogous to (1.3).

Lemma 4.6. *Let ρ be a law-invariant closed convex risk measure on L^1 which is continuous from below, and let $X \in L^1$. The following conditions are equivalent:*

(i) $\widetilde{Z} \in \delta\rho(X)$,

(ii) $\widetilde{Z} \in \{Z \in L^1 \mid (XZ) \in L^1,\ \forall Y \in L^1 :\ \rho(Y) \geq \rho(X) + E[Z(Y - X)]\}$ *with the convention that* $\infty - \infty := \infty$,

(iii) $\widetilde{Z} \in L^1$ *such that* $(X\widetilde{Z}) \in L^1$ *and* $\rho(X) = E[\widetilde{Z}X] - \rho^*_\infty(\widetilde{Z})$.

Moreover, if $Z \in L^1$ is such that $(XZ) \in L^1$, then $E[XZ] - \rho^*_\infty(Z) \leq \rho(X)$.

Proof. (i) ⇒ (ii): Suppose that (i) holds. We will prove that

$$\rho(U) \geq \rho(X) + E[\widetilde{Z}(U - X)] \tag{4.1}$$

for all $U \in L^1$ with the convention that $\infty - \infty = \infty$. Note that lemma 4.3(i) and theorem 2.3 (iv) imply $\widetilde{Z} \in L^1_-$. Let $U \in L^1$ such that $E[-\widetilde{Z}U^-] < \infty$ or $E[\widetilde{Z}U^+] > -\infty$ or both, then by (i), monotone convergence and lemma 4.7 below we obtain that

$$\begin{aligned} \rho(U) &= \lim_{m\to\infty}\lim_{n\to\infty} \rho((U^+ \wedge n) - (U^- \wedge m)) \\ &\geq \lim_{m\to\infty}\lim_{n\to\infty} (\rho(X) - E[\widetilde{Z}X] + E[\widetilde{Z}(U^+ \wedge n)] + E[-\widetilde{Z}(U^- \wedge m)]) \\ &= \rho(X) + E[\widetilde{Z}(U - X)], \end{aligned}$$

so (4.1) holds. If $U \in L^1$ is such that $E[-\widetilde{Z}U^-] = \infty$ and $E[\widetilde{Z}U^+] = -\infty$, then according to our convention, the right hand side of (4.1) equals ∞, so we have to show that $\rho(U) = \infty$ too. However, this follows from lemma 4.5 and the first case.

(ii) $\Rightarrow$ (iii): Since, in particular, (4.1) is true for all $U \in L^\infty$, we have $E[X\widetilde{Z}] - \rho(X) \geq \rho^*_\infty(\widetilde{Z})$. Moreover, lemma 4.7 below and monotone convergence imply that

$$\begin{aligned} E[X\widetilde{Z}] - \rho^*_\infty(\widetilde{Z}) &= \lim_{m\to\infty}\lim_{n\to\infty} E[(-m \vee X \wedge n)\widetilde{Z}] - \rho^*_\infty(\widetilde{Z}) \\ &\leq \lim_{m\to\infty}\lim_{n\to\infty} \rho_\infty(-m \vee X \wedge n) \quad = \quad \rho(X). \end{aligned} \tag{4.2}$$

Hence, we obtain $E[X\widetilde{Z}] - \rho(X) = \rho^*_\infty(\widetilde{Z})$.
(iii) $\Rightarrow$ (i): is obvious.
The final statement of the lemma follows from a computation similar to (4.2). □

The proof of lemma 4.6 relied on the following crucial lemma. We remark that a regularity result similar to (4.3) is stated in [26] for real-valued, and thus continuous, convex risk measures.

Lemma 4.7. *Let ρ be a law-invariant closed convex risk measure on L^1 which is continuous from below and let $H \in L^\infty$, then*

$$\rho(H + X) = \sup_{m\in\mathbb{N}}\inf_{n\in\mathbb{N}} \rho(H + (X^+ \wedge n) - (X^- \wedge m)). \tag{4.3}$$

Proof. Let, $H \in L^\infty_+$, and $X \in L^1$ be bounded from below. Then, $H + (X \wedge n) \in \operatorname{dom} \rho$ for all $n \in \mathbb{N} \cup \{\infty\}$ due to monotonicity of ρ. Again by monotonicity, the sequence $\rho(H + (X \wedge n))$, $n \in \mathbb{N}$, is decreasing and bounded from below by $\rho(H + X)$. We claim that

$$\rho(H + X) = \lim_{n\to\infty} \rho(H + (X \wedge n)). \tag{4.4}$$

In order to prove this, suppose for the moment that there is a $K > 0$ such that $\lim_{n\to\infty} \rho(H + (X \wedge n)) > K > \rho(H + X)$. Note that since $H \geq 0$, we have that

$$\lim_{n\to\infty} \rho(H + X \wedge n) = \lim_{n\to\infty} \rho((H + X) \wedge n).$$

Since $(H + X) \wedge n \in L^\infty$, and as ρ_∞ is everywhere subdifferentiable with $\operatorname{dom} \rho^*_\infty \subset L^1$, we have that for each $n \in \mathbb{N}$ there is a $Z_n \in \partial\rho_\infty((H + X) \wedge n) \subset L^1$, i.e.

$$\rho((H + X) \wedge n) = E[Z_n((H + X) \wedge n)] - \rho^*_\infty(Z_n).$$

By lemma 2.17 we have $\rho^*_\infty(E[Z_n \mid (H + X) \wedge n]) \leq \rho^*_\infty(Z_n)$, so we may assume that Z_n is $\sigma((H + X) \wedge n)$-measurable. Moreover, lemmas A.1, A.2 and law-invariance of ρ^*_∞ imply that

$$\begin{aligned} \rho_\infty((H + X) \wedge n) &= E[Z_n((H + X) \wedge n)] - \rho^*_\infty(Z_n) \\ &\leq \int_0^1 q_{(H+X)\wedge n}(s) q_{Z_n}(s) ds - \rho^*_\infty(Z_n) \\ &\leq \rho_\infty((H + X) \wedge n) \end{aligned}$$

which can only hold if

$$E[Z_n((H+X)\wedge n)] = \int_0^1 q_{(H+X)\wedge n}(s)q_{Z_n}(s)ds.$$

According to lemma A.1, we may assume that $Z_n = f_n(X+H)$ for a measurable function $f_n : \mathbb{R} \to \mathbb{R}_-$ which is increasing on $\{F_{H+X} > 0\}$. As $X + H$ is bounded from below we infer that

$$\begin{aligned} \rho^*_\infty(Z_n) &\leq E[((H+X)\wedge n)Z_n] - \rho((H+X)\wedge n) \\ &\leq -\operatorname{essinf}(H+X) - \rho(H+X) =: r, \end{aligned}$$

so $Z_n \in Q_r$ for all $n \in \mathbb{N}$. Since Q_r is weakly sequentially compact (proposition 4.4) and $L^1(\Omega, \sigma(H+X), \mathbb{P})$ is weakly complete, we may assume, by considering a subsequence if necessary, that $(Z_n)_{n\in\mathbb{N}}$ converges weakly to some $Z \in Q_r$ and that $Z = f(H+X)$ for a measurable function $f : \mathbb{R} \to \mathbb{R}_-$. Since the Hahn-Banach separation theorem implies that there is a sequence of convex combinations of the Z_n which converges $\mathbb{P}$-a.s. to Z ([36] corollary III.3.9), we may also assume that f is increasing on $\{F_{X+H} > 0\}$. Let $\mathcal{G}_k$, $k \in \mathbb{N}$, be a sequence of sub-σ-algebras of $\mathcal{F}$ such that $E[X+H|\mathcal{G}_k] \in L^\infty$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} E[X+H|\mathcal{G}_k] = X+H$ in L^1 and $\mathbb{P}$-a.s. The following estimation shows that the sequence $(ZE[X+H|\mathcal{G}_k])_{k\in\mathbb{N}}$ is uniformly integrable. To this end let $a \in \mathbb{R}$ such that $F_{X+H}(a) > 0$. Then,

$$|ZE[X+H|\mathcal{G}_k]| \leq |Z|\|(X+H)\wedge a\|_\infty + |f(a)|E[|X+H|\ |\mathcal{G}_k] =: Y_k,$$

because $|f|$ is decreasing on $\{F_{X+H} > 0\}$. Since $(Y_k)_{k\in\mathbb{N}}$ is uniformly integrable, so is $(ZE[X+H|\mathcal{G}_k])_{k\in\mathbb{N}}$. Consequently, we obtain

$$\begin{aligned} E[Z(H+X)] - \rho^*_\infty(Z) &= \lim_{k\to\infty} E[ZE[H+X|\mathcal{G}_k]] - \rho^*_\infty(Z) \\ &\leq \lim_{k\to\infty} \rho_\infty(E[H+X|\mathcal{G}_k]) \\ &= \rho(H+X) < K \end{aligned} \tag{4.5}$$

in which the last equality is due to lemma 2.17 and l.s.c. of ρ. On the other hand, we observe that for all $k \geq n$ we have $E[Z_k((H+X)\wedge n)] - \rho^*_\infty(Z_k) > K$ (because $(H+X)\wedge k \geq (H+X)\wedge n$). Hence, by monotone convergence and l.s.c. of ρ^*_∞ we obtain

$$\begin{aligned} E[Z(H+X)] - \rho^*_\infty(Z) &= \lim_{n\to\infty} E[Z((H+X)\wedge n)] - \rho^*_\infty(Z) \\ &\geq \lim_{n\to\infty} \limsup_{k\to\infty} E[Z_k((H+X)\wedge n)] - \rho^*_\infty(Z_k) \\ &\geq K. \end{aligned} \tag{4.6}$$

Clearly, (4.6) contradicts (4.5), and thus (4.4) is proved. For general $H \in L^\infty$, and $X \in L^1$ monotonicity and l.s.c. of ρ imply that $\rho(H+X) = \lim_{m\to\infty} \rho(H + (X \vee -m))$. In conjunction with (4.4) and cash-invariance we obtain

$$\begin{aligned}\rho(H+X) &= \lim_{m\to\infty} \rho((H + \|H\|_\infty) + (X \vee -m)) + \|H\|_\infty \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \rho(H + (n \wedge X \vee -m)).\end{aligned}$$

□

The following theorem gives sufficient conditions ensuring the existence of a non-empty generalised subgradient. It is proved throughout sections 4.2 and 4.3.

Theorem 4.8. *Let ρ be a law-invariant closed convex risk measure on L^1 which is continuous from below. If $X \in L^1$ is bounded from below, then $\delta(X) \neq \emptyset$. If, moreover, ρ satisfies the following tail continuity condition*

$$\lim_{n\to\infty} \rho(Y + X1_{\{X \le -n\}}) = \rho(Y) \text{ for all } X \le 0 \text{ s.t. } (Y+X) \in \operatorname{dom} \rho, \tag{4.7}$$

then for every

$$X \in L^1 \text{ for which there is an } \epsilon > 0 \text{ such that } (1+\epsilon)X \in \operatorname{dom} \rho, \tag{4.8}$$

we have $\delta\rho(X) \neq \emptyset$. In both cases we may assume that $Z \in \delta\rho(X)$ is of type $Z = f(X)$ for a measurable function $f : \mathbb{R} \to \mathbb{R}_-$ which is increasing on $\{F_X > 0\}$.

Note that if ρ is coherent, then condition (4.7) is equivalent to

$$\lim_{n\to\infty} \rho(X1_{\{X \le -n\}}) = 0 \quad \text{for all } X \in \operatorname{dom} \rho$$

and (4.8) is equivalent to $X \in \operatorname{dom} \rho$. In examples 4.5.4 and 4.5.6 we illustrate that we cannot expect any better characterisation of the points at which ρ is generalised subdifferentiable. In particular, example 4.5.6 gives a law-invariant closed coherent risk measure which is continuous from below but does not satisfy (4.7), and which admits $X \in \operatorname{dom} \rho$ (which is unbounded from below) such that $\delta\rho(X) = \emptyset$.

4.2. The Space L^ρ

Throughout this section let ρ be a law-invariant closed convex risk measure on L^1.

Definition 4.9. *For $C > 0$ let*

$$\|X\|_{C,\rho} := \inf\{\lambda > 0 \mid \rho(-|X|/\lambda) \le C\}, \quad X \in L^1,$$

with the usual convention that $\inf \emptyset = \infty$, *and define*

$$L^\rho := \{X \in L^1 \mid \|X\|_{C,\rho} < \infty\}.$$

Clearly, we adopted this idea from Orlicz space theory.

Lemma 4.10. (i) *$\|\cdot\|_{C,\rho} : L^1 \to [0,\infty]$ is a law-invariant sub-linear closed function on $(L^1, \|\cdot\|_1)$.*

(ii) *L^ρ is well-defined, i.e. independent of $C > 0$. Moreover, if $C \in (0,1)$, then*

$$C\|\cdot\|_{C,\rho} \leq \|\cdot\|_{1,\rho} \leq \|\cdot\|_{C,\rho}, \tag{4.9}$$

and if $C \geq 1$, then

$$\|\cdot\|_{C,\rho} \leq \|\cdot\|_{1,\rho} \leq C\|\cdot\|_{C,\rho}. \tag{4.10}$$

If ρ is coherent, then for all $C > 0$:

$$C\|\cdot\|_{C,\rho} = \|\cdot\|_{1,\rho} = \rho(-|\cdot|). \tag{4.11}$$

(iii) *$C \cdot \|X\|_{C,\rho} \leq \|X\|_\infty$ for all $X \in L^\infty$ and $C \cdot \|X\|_{C,\rho} \geq \|X\|_1$ for all $X \in L^1$.*

(iv) *$(L^\rho, \|\cdot\|_{C,\rho})$ is a law-invariant Banach space such that $L^\infty \subseteq L^\rho \subseteq L^1$. The inclusion $L^\infty \subset L^\rho$ is strict if and only if ρ is continuous from below. In particular, we have that $\{-X^- \mid X \in \text{dom}\,\rho\} \subset L^\rho$.*

(v) *If $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$ and $X \in L^\rho$, then $E[X|\mathcal{G}] \in L^\rho$.*

Proof. We define $\Lambda_C(X) := \{\lambda > 0 \mid \rho(-|X|/\lambda) \leq C\}$.
(i): The law-invariance of $\|\cdot\|_{C,\rho}$ follows immediately from law-invariance of ρ. Moreover, it is easily verified that $\|tX\|_{C,\rho} = |t| \cdot \|X\|_{C,\rho}$ for all $t \in \mathbb{R}$. In order to show that $\|X+Y\|_{C,\rho} \leq \|X\|_{C,\rho} + \|Y\|_{C,\rho}$ it suffices to consider $X, Y \in L^\rho$ because if either $\|X\|_{C,\rho} = \infty$ or $\|Y\|_{C,\rho} = \infty$ or both, the assertion is trivial. To this end let $\alpha \in \Lambda_C(X)$ and $\beta \in \Lambda_C(Y)$ for some $X, Y \in L^\rho$. Then, by monotonicity and convexity

$$\begin{aligned}\rho\left(-\frac{|X+Y|}{\alpha+\beta}\right) &\leq \rho\left(-\frac{\alpha}{\alpha+\beta}\frac{|X|}{\alpha} - \frac{\beta}{\alpha+\beta}\frac{|Y|}{\beta}\right)\\ &\leq \frac{\alpha}{\alpha+\beta}\cdot\rho\left(-\frac{|X|}{\alpha}\right) + \frac{\beta}{\alpha+\beta}\cdot\rho\left(-\frac{|Y|}{\beta}\right) \leq C,\end{aligned}$$

so $\Lambda_C(X) + \Lambda_C(Y) \subset \Lambda_C(X+Y)$ which proves the triangle inequality. We claim that $\|\cdot\|_{C,\rho}$ is l.s.c. on $(L^1, \|\cdot\|_1)$. In order to verify this, denote the level sets of $\|\cdot\|_{C,\rho}$ by $E_k = \{Y \mid \|Y\|_{C,\rho} \leq k\}$, $k \geq 0$, and let $(X_n)_{n\in\mathbb{N}} \subset E_k$ for some $k \geq 0$ be a sequence converging to $X \in L^1$ w.r.t. $\|\cdot\|_1$. Note that $\|Y\|_{C,\rho} \leq k$ if and only if $\rho(-|Y|/(k+\epsilon)) \leq C$ for all $\epsilon > 0$. Since $X_n \in E_k$ for all $n \in \mathbb{N}$, l.s.c. of ρ yields

$$\rho(-|X|/(k+\epsilon)) \leq \liminf_{n\to\infty} \rho(-|X_n|/(k+\epsilon)) \leq C$$

for any $\epsilon > 0$, and thus $X \in E_k$. Hence, E_k is closed in $(L^1, \|\cdot\|_1)$ for every $k \geq 0$, i.e. $\|\cdot\|_{C,\rho}$ is l.s.c. on $(L^1, \|\cdot\|_1)$. Hence, we have proved that $\|\cdot\|_{C,\rho}$ is a law-invariant closed sublinear function on L^1.

(ii): Clearly, if (4.9) and (4.10) hold, then L^ρ is well-defined. We only prove (4.9) since the proof of (4.10) is similar and (4.11) is obvious by positive homogeneity. To this end, let $C \in (0,1)$, $X \in L^1$ and $\lambda \in \Lambda_1(X)$, i.e. $\rho(-|X|/\lambda) \leq 1$. Then, convexity of ρ yields $\rho(-C|X|/\lambda) \leq C\rho(-|X|/\lambda) \leq C$. Hence, $\frac{1}{C}\Lambda_1(X) \subset \Lambda_C(X)$, so $C\|X\|_{C,\rho} \leq \|X\|_{1,\rho}$. On the other hand, since $C < 1$, we have $\Lambda_C(X) \subset \Lambda_1(X)$ and thus $\|X\|_{1,\rho} \leq \|X\|_{C,\rho}$, and (4.9) is proved.

(iii) and (iv): (i) and lemma 2.17 yield for all $X \in L^1$:

$$\|X\|_{C,\rho} = \||X|\|_{C,\rho} \geq E[|X|] \cdot \|1\|_{C,\rho} = \frac{1}{C}E[|X|] = \frac{1}{C}\|X\|_1. \tag{4.12}$$

Consequently, $\|X\|_{C,\rho} = 0$ if and only if $X = 0$. Apparently, the properties of $\|\cdot\|_{C,\rho}$ ensure that $(L^\rho, \|\cdot\|_{C,\rho})$ is a normed space. In order to prove that this space is complete and thus a Banach space, let $(X_n)_{n\in\mathbb{N}}$ be a Cauchy-sequence in $(L^\rho, \|\cdot\|_{C,\rho})$. Then by (4.12), $(X_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $(L^1, \|\cdot\|_1)$. Let $X \in L^1$ be the unique $\|\cdot\|_1$-limit of $(X_n)_{n\in\mathbb{N}}$. Since $\|\cdot\|_{C,\rho}$ is l.s.c. on $(L^1, \|\cdot\|_1)$, we obtain $\|X\|_{C,\rho} \leq \liminf_{n\to\infty} \|X_n\|_{C,\rho} < \infty$, i.e. $X \in L^\rho$. Let $\epsilon > 0$ and $N(\epsilon) \in \mathbb{N}$ such that $\|X_n - X_k\|_{C,\rho} \leq \epsilon$ for all $k, n \geq N(\epsilon)$. As $(X_n - X_k)$ converges to $X - X_k$ w.r.t. $\|\cdot\|_1$ for $n \to \infty$, we obtain

$$\|X - X_k\|_{C,\rho} \leq \liminf_{n\to\infty} \|X_n - X_k\|_{C,\rho} \leq \epsilon \quad \text{for } k \geq N(\epsilon).$$

Thus we may conclude that X is the $\|\cdot\|_{C,\rho}$-limit of X_n, i.e. $(L^\rho, \|\cdot\|_{C,\rho})$ is complete. For every $0 \neq X \in L^\infty$ we obtain

$$\rho\left(-\frac{C|X|}{\|X\|_\infty}\right) \leq \rho(-C) = C$$

by monotonicity and cash-invariance. Therefore, $\|X\|_{C,\rho} \leq \frac{1}{C}\|X\|_\infty$ and $L^\infty \subset L^\rho$. Now let $X \in \operatorname{dom}\rho$, then $\rho(-X^-) < \infty$ according to lemma 4.5, which implies that $-X^- \in L^\rho$. Hence, if ρ is continuous from below, then, by proposition 4.4, there is a $X \in \operatorname{dom}\rho$ such that $\operatorname{essinf} X = -\infty$, and $-X^- \in L^\rho$, so $L^\rho \setminus L^\infty \neq \emptyset$. Conversely, suppose that $X \in L^\rho \setminus L^\infty$, then, by definition of $\|\cdot\|_{C,\rho}$, there is a $k > 0$ such that $\rho(-k|X|) < \infty$. Since $X \notin L^\infty$, we have either $\operatorname{essinf} X = -\infty$ or $\operatorname{esssup} X = \infty$ or both, which implies that $(-k|X|) \in \{Y \in L^1 \mid \operatorname{essinf} Y = -\infty\} \cap \operatorname{dom}\rho$. But then ρ must be continuous from below (proposition 4.4).

(v): Let $X \in L^\rho$ and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Then, (i) and lemma 2.17 imply that $\|E[X|\mathcal{G}]\|_{C,\rho} \leq \|X\|_{C,\rho}$, so $E[X|\mathcal{G}] \in L^\rho$. □

The reason for introducing the Banach spaces $(L^\rho, \|\cdot\|_{C,\rho})$ is that we will prove that the domain of $\rho|_{L^\rho}$ has a non-empty interior. Hence, we obtain non-empty subgradients at these interior points. The role of the variable $C > 0$ in the norms $\|\cdot\|_{C,\rho}$ will become clear in (the proof of) lemma 4.12 in which we characterise the interior points of dom $\rho|_{L^\rho}$.

Lemma 4.11. *Let ρ be continuous from below. Denote by $L^{\rho*}$ the dual space of L^ρ and by $\|\cdot\|_{C,\rho*}$ the operator norm corresponding to $\|\cdot\|_{C,\rho}$.*

(i) *$L^\infty \subset L^{\rho*}$ and $L^{\rho*}_\infty \subset L^1$ where $L^{\rho*}_\infty := \{l|_{L^\infty} \mid l \in L^{\rho*}\}$. In particular, if $l \in L^{\rho*}$ and $Z \in L^1$ such that $l(X) = E[ZX]$ for all $X \in L^\infty$, then $E[Z\,\cdot] \in L^{\rho*}$.*

(ii) *$L^{\rho*} \cap L^1$ and $\|\cdot\|_{C,\rho*}|_{L^1}$ are law-invariant.*

(iii) *For every $Z \in L^{\rho*} \cap L^1$ and any sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$ we have $E[Z|\mathcal{G}] \in L^{\rho*}$.*

Proof. (i): Since $L^\rho \subset L^1$ and $C\|\cdot\|_{C,\rho} \geq \|\cdot\|_1$, every element $Z \in L^\infty = L^{1*}$ defines a continuous linear functional on L^ρ via $X \mapsto E[XZ]$. Thus, we may view L^∞ as a subset of $L^{\rho*}$. By $L^\infty \subset L^\rho$ and $C\|\cdot\|_{C,\rho} \leq \|\cdot\|_\infty$ on L^∞ we must have $L^{\rho*}_\infty \subset L^{\infty*}$. Recall the general property of normed spaces (see e.g. [2] lemma 6.14)

$$\|X\|_{C,\rho} = \sup_{\|Z\|_{C,\rho*}=1} |E[ZX]|. \tag{4.13}$$

Suppose we had $Z_\mu \in L^{\rho*}$ such that $Z_\mu|_{L^\infty} \in L^{\rho*}_\infty \setminus L^1$. W.l.o.g. $\|Z_\mu\|_{C,\rho*} = 1$. This Z_μ viewed as a continuous linear functional on L^∞ corresponds to a finitely additive but not σ-additive bounded signed measure μ on $(\Omega, \mathcal{F})$ such that $\mathbb{P}(A) = 0$ implies $\mu(A) = 0$ (see [21] theorem A.50). Consider the bounded finitely additive measure $|\mu|$ on $(\Omega, \mathcal{F})$ given by

$$|\mu|(A) = \sup\left\{\sum_{i=1}^k |\mu(A_i)| \mid A_1, \ldots, A_k \in \mathcal{F} \text{ are disjoint subsets of } A,\ k \in \mathbb{N}\right\},$$

$A \in \mathcal{F}$ (for details on $|\mu|$ consult e.g. [12] section III.1). Since $\sum_{i=1}^k \pm 1_{A_i} \in L^\rho$ for any (disjoint) sets $A_1, \ldots, A_k \in \mathcal{F}$, we infer from (4.13) that

$$|\mu|(A) \leq \|1_A\|_{C,\rho} \quad \text{for every } A \in \mathcal{F}. \tag{4.14}$$

As $|\mu|$ is not σ-additive, there exists a decreasing sequence of sets $B_n \downarrow \emptyset$ such that $|\mu|(B_n) \downarrow \epsilon > 0$. We will show that

$$\|1_{B_n}\|_{C,\rho} \to 0 \tag{4.15}$$

which contradicts (4.14) and thus shows that $L^{\rho*}_\infty \subset L^1$. To this end, note that for every $\delta > 0$ there is an $N(\delta) \in \mathbb{N}$ such that $\rho(-1_{B_n}/\delta) \leq C$ for $n \geq N(\delta)$ because $-1_{B_n} \uparrow 0$ and ρ is continuous from below. Hence, if $n \geq N(\delta)$, then $\|1_{B_n}\|_{C,\rho} \leq \delta$ and (4.15) is proved.

Let $l \in L^{\rho*}$ and $Z \in L^1$ such that $l(X) = E[ZX]$ for all $X \in L^\infty$. By monotonicity of $\|\cdot\|_\rho$ it follows that for any $Y \in L^\rho$ with $Y \geq 0$ we have

$$E[Z^+(Y \wedge n)] = l((Y \wedge n)1_{\{Z \geq 0\}}) \leq \|l\|_{\rho*}\|Y\|_\rho \quad \forall n \in \mathbb{N}.$$

Hence, $E[Z^+Y] \leq \|l\|_{\rho*}\|Y\|_\rho$, and noting that $Y \in L^\rho$ if and only if $Y^\pm \in L^\rho$, and that $\|Y\|_\rho = \||Y|\|_\rho$, we infer that $E[Z^+\cdot]$ defines a continuous linear functional on L^ρ. Similarly we find that $E[Z^-\cdot] \in L^{\rho*}$, which yields $E[Z\,\cdot] \in L^{\rho*}$.

(ii): We claim that

$$Z \in L^{\rho*} \cap L^1 \text{ if and only if } Z^+, Z^- \in L^{\rho*} \cap L^1. \tag{4.16}$$

To this end, note that for every $A \in \mathcal{F}$ and $X \in L^\rho$ monotonicity of ρ yields $\| \pm 1_A X\|_{C,\rho} \leq \|X\|_{C,\rho}$ and thus $\pm 1_A X \in L^\rho$. Now let $Z \in L^{\rho*} \cap L^1$. Choosing $A = \{Z \geq 0\}$ shows that $Z^+ \in L^{\rho*}$, because $L^\rho \ni X \mapsto E[Z^+X] = E[Z1_{\{Z\geq 0\}}X]$ is a real-valued linear function, and

$$|E[Z^+X]| = |E[Z1_{\{Z\geq 0\}}X]| \leq \|Z\|_{C,\rho*}\|X1_{\{Z\geq 0\}}\|_{C,\rho} \leq \|Z\|_{C,\rho*}\|X\|_{C,\rho}.$$

Similar arguments yield $Z^- \in L^{\rho*}$. The converse implication of (4.16) is trivial.

By (4.16) it suffices to prove the law-invariance property of $L^{\rho*} \cap L^1$ for the positive cone $L^{\rho*}_+ = \{Z \in L^{\rho*} \cap L^1 \mid Z \geq 0\}$ only. Hence, let $Z \in L^{\rho*}_+$. By law-invariance of $\|\cdot\|_{C,\rho}$, lemma A.2, and $\|X\|_{C,\rho} = \||X|\|_{C,\rho}$ we obtain

$$\begin{aligned} \infty > \|Z\|_{C,\rho*} &= \sup_{\|X\|_{C,\rho}=1} |E[ZX]| \\ &= \sup_{\|X\|_{C,\rho}=1} \sup_{Y \sim |X|} E[ZY] \\ &= \sup_{\|X\|_{C,\rho}=1} \int_0^1 q_Z(s) q_{|X|}(s) ds \end{aligned} \tag{4.17}$$

in which the latter expression depends on the distribution of Z only. Now it is easily verified that every $\widetilde{Z}$ such that $\widetilde{Z} \sim Z$ defines a continuous linear functional on L^ρ too. The law-invariance of $\|Z\|_{C,\rho*}$ for general $Z \in L^{\rho*} \cap L^1$ follows from a calculation similar to (4.17), using the fact that $\|X\|_{C,\rho} = \||X|1_{\{Z\geq 0\}} - |X|1_{\{Z<0\}}\|_{C,\rho}$.

(iii): Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and $Z \in L^{\rho*}_+$. Then, lemmas 4.10 (v) and 2.17 yield $E[E[Z|\mathcal{G}]X] = E[ZE[X|\mathcal{G}]] \leq \|Z\|_{C,\rho*}\|X\|_{C,\rho}$ for every $X \in L^\rho_+$. Since $X \in L^\rho$ if and only if $X^+, X^- \in L^\rho$, we conclude that $E[Z|\mathcal{G}] \in L^{\rho*}_+$ which, in view of (4.16), completes the proof. □

Lemma 4.12. *Let ρ be continuous from below.*

(i) *The function $\rho|_{L^\rho}$ is a law-invariant closed convex risk measure on L^ρ. Moreover, if ρ is coherent, then $\rho|_{L^\rho}$ is a real-valued continuous coherent risk measure on L^ρ.*

(ii) *For all* $X \in \operatorname{int dom} \rho|_{L^\rho}$*, which is for all* $X \in L^\rho$ *in case* ρ *is coherent, we have* $\partial\rho|_{L^\rho}(X) \neq \emptyset$*. In particular,*

$$\text{if for } X \in L^1 \text{ there is } \epsilon > 0 \text{ such that } -(1+\epsilon)|X| \in \operatorname{dom} \rho, \tag{4.18}$$

then $X \in \operatorname{int dom} \rho|_{L^\rho}$*. Moreover, if* ρ *is coherent and* $X \in L^1$*, then* $X \in \operatorname{int dom} \rho|_{L^\rho} = L^\rho$ *if and only if* $-|X| \in \operatorname{dom} \rho$*.*

Proof. (i): Obviously, $\rho|_{L^\rho}$ is law-invariant, convex, cash-invariant and monotone. Moreover, $\rho|_{L^\rho}$ is l.s.c. because, according to lemma 4.10(iii), if a sequence converges in $(L^\rho, \|\cdot\|_{C,\rho})$, then this sequence converges in $(L^1, \|\cdot\|_1)$ and ρ is l.s.c. on $(L^1, \|\cdot\|_1)$. Suppose that ρ is coherent. Then, $\rho|_{L^\rho}$ is coherent too. Moreover, for every $X \in L^\rho$ there is a $k > 0$ such that $\rho(-|X|/k) \leq 1$. Hence, by positive homogeneity and monotonicity we obtain that $\rho(X) \leq \rho(-|X|) \leq k < \infty$. In other words,

$$\operatorname{dom} \rho|_{L^\rho} = \operatorname{int dom} \rho|_{L^\rho} = L^\rho.$$

We recall that any real-valued closed convex function on a Banach space is continuous (see [14] corollary 2.5).

(ii) Recall that any closed convex function on a Banach space is subdifferentiable on the interior of its domain ([14] corollary 2.5 and proposition 5.2). If $\|X\|_{C,\rho} < 1$, then there exists a $\lambda \in (0,1)$ such that $\rho(-|X|/\lambda) \leq C$, and by convexity

$$\frac{1}{\lambda}\rho(-|X|) \leq \rho(-|X|/\lambda) \leq C.$$

Thus $\rho(X) \leq \rho(-|X|) \leq \lambda C < \infty$, that is

$$\mathcal{B} := \bigcup_{C>0} \{X \in L^\rho \mid \|X\|_{C,\rho} < 1\} \subset \operatorname{int dom} \rho|_{L^\rho}.$$

If there is a $\epsilon > 0$ such that $-(1+\epsilon)|X| \in \operatorname{dom} \rho$, then for $\lambda := 1/(1+\epsilon) \in (0,1)$ we have $\rho(-|X|/\lambda) = \rho(-(1+\epsilon)|X|) =: C < \infty$, so $X \in \mathcal{B}$. If ρ is coherent, then by (4.11) $X \in L^\rho$ if and only if $-|X| \in \operatorname{dom} \rho$. □

Remark 4.13. In view of lemma 4.1 the reader might wonder why on the Banach space $(L^\rho, \|\cdot\|_{C,\rho})$ it is possible that $\operatorname{int dom} \rho|_{L^\rho} \neq \emptyset$ without $\rho|_{L^\rho}$ being real-valued and continuous and thus subdifferentiable on all of L^ρ. The reason is that the proof of lemma 4.1 relies on the fact that L^∞ is dense in $(L^1, \|\cdot\|_1)$. This, however, need not be true for $(L^\rho, \|\cdot\|_{C,\rho})$. In example 4.5.4 we show that for the entropic risk measure L^ρ corresponds to an Orlicz space for which it is known that L^∞ is not dense. That is one of the reasons why many authors prefer Orlicz hearts (see e.g. [9]) which are closed sub-spaces of Orlicz spaces such that L^∞ is dense. However, Orlicz hearts are in general much smaller than the corresponding Orlicz space. But we can imitate Orlicz hearts, i.e. shift to the subspace $M^\rho \subset L^\rho$ given by

$$M^\rho := \{X \in L^1 \mid \rho(-c|X|) < \infty \quad \forall c > 0\}.$$

Then $\rho|_{M^\rho}$ is a law-invariant real-valued continuous convex risk measure on M^ρ, and thus everywhere subdifferentiable (on M^ρ). ♢

Remark 4.14. Since $X \in \operatorname{int}\operatorname{dom} \rho|_{L^\rho}$ implies that $(1+\epsilon)X \in \operatorname{dom} \rho|_{L^\rho}$ for small enough $\epsilon > 0$, we have

$$\operatorname{int}\operatorname{dom} \rho|_{L^\rho} \subset \{X \in L^1 \mid X \text{ satisfies condition (4.8)}\}$$

in which the inclusion is strict unless $L^\rho = L^1$. ♢

4.3. Proof of Theorem 4.8

Proof of theorem 4.8. Let $X \in L^1$ and suppose that there is an $\epsilon > 0$ such that $(1+\epsilon)X \in \operatorname{dom} \rho$. Then, in particular, $-(1+\epsilon)X^- \in \operatorname{dom} \rho$ (lemma 4.5), and thus $-X^- \in \operatorname{int}\operatorname{dom} \rho|_{L^\rho}$ (lemma 4.12). By cash-invariance we may w.l.o.g. assume that $\rho(X) = 0$. Let

$$\rho_{X^+}(U) := \rho(X^+ + U), \quad U \in L^\rho.$$

It is easily verified that ρ_{X^+} is a closed convex risk measure on $(L^\rho, \|\cdot\|_{C,\rho})$. Note that monotonicity implies $\operatorname{dom} \rho|_{L^\rho} \subset \operatorname{dom} \rho_{X^+}$. Hence, $-X^- \in \operatorname{int}\operatorname{dom} \rho_{X^+}$ which implies that $\partial\rho_{X^+}(-X^-) \neq \emptyset$ ([14] corollary 2.5 and proposition 5.2). Let $\mu \in \partial\rho_{X^+}(-X^-)$, i.e.

$$\rho_{X^+}(-X^-) = E[\mu(-X^-)] - \rho^*_{X^+}(\mu), \tag{4.19}$$

and let $Z_\mu \in L^1$ such that $E[\mu X] = E[Z_\mu X]$ for all $X \in L^\infty$ (see lemma 4.11). We claim that

$$(Z_\mu X^+) \in L^1 \quad \text{and} \quad \rho^*_{X^+}(\mu) \geq E[-Z_\mu X^+] + \rho^*_\infty(Z_\mu), \tag{4.20}$$

and

$$E[Z_\mu(-X^-)] \geq E[\mu(-X^-)]. \tag{4.21}$$

Suppose we knew (4.20). Then, (4.19) yields

$$\rho(X) = \rho_{X^+}(-X^-) \leq E[Z_\mu X] - \rho^*_\infty(Z_\mu),$$

or in other words $Z_\mu \in \delta\rho(X)$. In order to verify (4.20), in a first step we compute

$$\begin{aligned}\sup_{U \in L^\infty} E[Z_\mu U] - \rho(X^+ + U) &= \sup_{U \in L^\infty} \lim_{n\to\infty} E[Z_\mu U] - \rho((X^+ \wedge n) + U)\\ &\leq \liminf_{n\to\infty} \sup_{U \in L^\infty} E[Z_\mu U] - \rho((X^+ \wedge n) + U)\\ &\leq \sup_{U \in L^\infty} E[Z_\mu U] - \rho(X^+ + U) \end{aligned}\tag{4.22}$$

in which the first equality follows from lemma 4.7. Hence, all inequalities in (4.22) must in fact be equalities. Secondly, we obtain that

$$\begin{aligned}
\rho^*_{X^+}(\mu) &\geq \sup_{U\in L^\infty} E[Z_\mu U] - \rho(X^+ + U) \\
&= \liminf_{n\to\infty} \sup_{U\in L^\infty} E[Z_\mu U] - \rho((X^+\wedge n) + U) \\
&= \liminf_{n\to\infty} \sup_{U\in L^\infty} E[Z_\mu(U - (X^+\wedge n))] - \rho(U) \\
&= E[-Z_\mu X^+] + \rho^*_\infty(Z_\mu),
\end{aligned}$$

in which the first equality is due to our first step, and the last equality follows from monotone convergence. Thus, as $\rho^*_{X^+}(\mu) < \infty$, we infer that $E[-Z_\mu X^+] < \infty$ and $\rho^*_\infty(Z_\mu) < \infty$, and (4.20) is proved. As for (4.21), note that if X^- is bounded, then $E[Z_\mu(-X^-)] = E[\mu(-X^-)]$, and we are done. Now suppose that X^- is unbounded and that ρ satisfies (4.7), but $\delta := E[(\mu - Z_\mu)(-X^-)] > 0$. This implies

$$E[\mu(-X^-)1_{\{X^-\geq n\}}] \geq E[(\mu - Z_\mu)(-X^-)1_{\{X^-\geq n\}}] = E[(\mu - Z_\mu)(-X^-)] = \delta$$

because monotonicity of ρ implies that

$$\mu, Z_\mu \in \{\nu \in L^{\rho*} \mid \forall Y \in L^\rho_+ : E[\nu Y] \leq 0\},$$

and because $(-X^-)1_{\{X^-<n\}} \in L^\infty$. Since $\rho(X) = 0$ by assumption, (4.19) yields

$$E[\mu(-X^-)] = \rho^*_{X^+}(\mu) = \sup_{Y\in\mathcal{A}_{\rho_{X^+}}} E[\mu Y] \tag{4.23}$$

for $\mathcal{A}_{\rho_{X^+}} = \{Y \in L^\rho \mid \rho_{X^+}(Y) \leq 0\}$ where the last step follows from cash-invariance. Let $X_n := -X^- - \epsilon X^- 1_{\{X^-\geq n\}}$, $n \in \mathbb{N}$. Then $X_n \in \operatorname{dom} \rho_{X^+}$ and $\lim_{n\to\infty} \rho_{X^+}(X_n) = \rho_{X^+}(-X^-) = 0$ due to (4.7). Hence, as cash-invariance implies that $E[\mu 1] = -1$, we obtain by (4.23) that

$$\begin{aligned}
E[\mu(-X^-)] &\geq E[\mu(X_n + \rho_{X^+}(X_n))] \\
&= \epsilon E[\mu(-X^-)1_{\{X^-\geq n\}}] + E[\mu(-X^-)] - \rho_{X^+}(X_n) \\
&\geq \epsilon\delta + E[\mu(-X^-)] - \rho_{X^+}(X_n) \quad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Passing to the limit for $n \to \infty$ yields the desired contradiction to $\delta > 0$. Consequently, (4.21) is proved.

It remains to be shown that Z_μ may be chosen as an increasing function of X. To this end, note that according to lemma 4.6 we have $\rho(X) = E[Z_\mu X] - \rho^*_\infty(Z_\mu)$. By lemma 2.17, which implies that $\rho(X) \leq E[E[Z_\mu|X]X] - \rho^*_\infty(E[Z_\mu|X])$ and thus $E[Z_\mu|X] \in \delta\rho(X)$ (lemma 4.6), we may assume that $Z_\mu = f(X)$ for a measurable function $f : \mathbb{R} \to \mathbb{R}_-$, and still $Z_\mu \in L^{\rho*}$ (lemma 4.11). Moreover, since $-X^- \in L^\rho$, and

$L^{\rho*} \cap L^1$ is law-invariant (lemma 4.11) we have that $(-\widetilde{Z}X^-) \in L^1$ for all $\widetilde{Z} \sim Z_\mu$. Consequently, $E[\widetilde{Z}X]$ is well-defined for all $\widetilde{Z} \sim Z_\mu$, so we may apply lemma A.2 in the following. Recalling that $-(q_X)^- = q_{-X^-}$ we obtain

$$-\infty < E[Z_\mu X] \leq \int_0^1 q_{Z_\mu}(s)q_X(s)ds \leq \int_0^1 q_{Z_\mu}(s)q_{-X^-}(s)ds < \infty$$

in which we applied lemmas A.1 and A.2. If $E[ZX] < \int_0^1 q_{Z_\mu}(s)q_X(s)ds$, then according to lemma A.2 there would be a $\widetilde{Z} \sim Z_\mu$ such that $E[\widetilde{Z}X] > E[Z_\mu X]$. Since $\widetilde{Z}X \in L^1$ (lemma A.1), by law-invariance of ρ^*_∞, and by lemma 4.6, we would have that

$$\rho(X) = E[Z_\mu X] - \rho^*_\infty(Z) < E[\widetilde{Z}X] - \rho^*_\infty(\widetilde{Z}) \leq \rho(X)$$

which is a contradiction. Therefore, $E[XZ_\mu] = \int_0^1 q_Z(s)q_X(s)ds$, so f may be chosen as an increasing function on $\{F_X > 0\}$ (lemma A.1). □

4.4. Optimal Risk Sharing

In this section we pick up the optimal risk sharing problem from chapter 3. We consider n agents with initial endowments $X_i \in L^1$, whose preferences, in contrast to the more general setting of chapter 3, are represented by law-invariant closed convex risk measures ρ_i on L^1 which are continuous from below, $i = 1, \ldots, n$. We write

$$X := X_1 + \ldots + X_n$$

for the aggregate endowment. In this setting, theorem 3.4 extends as follows.

Theorem 4.15. *The convolution $\square_{i=1}^n \rho_i$ is a law-invariant closed convex risk measure on L^1 which is continuous from below. Its restriction to L^∞ satisfies*

$$(\square_{i=1}^n \rho_i)_\infty = \square_{i=1}^n ((\rho_i)_\infty). \tag{4.24}$$

Moreover, for every $X \in L^1$, there exists a comonotone optimal allocation, and the first order condition

$$\delta \square_{i=1}^n \rho_i(X) = \bigcap_{i=1}^n \delta\rho_i(Y_i) \tag{4.25}$$

holds for every comonotone optimal allocation $(Y_1, \ldots, Y_n)$ of X. In particular, if X is bounded from below or if $\square_{i=1}^n \rho_i$ satisfies (4.7) and there is $\epsilon > 0$ such that

$$(1+\epsilon)X \in \sum_{i=1}^n \text{dom } \rho_i \; (= \text{dom } \square_{i=1}^n \rho_i), \tag{4.26}$$

then $\delta \square_{i=1}^n \rho_i(X) \neq \emptyset$.

Proof. According to corollary 3.6 $\Box_{i=1}^n \rho_i$ is a law-invariant closed convex risk measure on L^1 admitting a comonotone optimal allocation $(Y_1, \ldots, Y_n)$ for any $X \in L^1$. The continuity from below of $\Box_{i=1}^n \rho_i$ follows from proposition 4.4 and lemma 3.1. The relation (4.24) follows from corollary 3.16.

As for (4.25), let $(Y_1, \ldots, Y_n)$ be any comonotone optimal allocation of X. Suppose $Z \in \delta \Box_{i=1}^n \rho_i(X)$. Then

$$\begin{aligned}\rho_1(Y_1) + \ldots + \rho_n(Y_n) &= \Box_{i=1}^n \rho_i(X) = E[ZX] - (\Box_{i=1}^n \rho_i)_\infty^*(Z) \\ &= \sum_{i=1}^n E[ZY_i] - (\rho_i)_\infty^*(Z)\end{aligned}$$

by (4.24), lemmas 3.1 and 4.6, and the fact that $ZY_i \in L^1$ due to comonotonicity of the allocation. Now lemma 4.6 implies that $Z \in \bigcap_{i=1}^n \delta\rho_i(Y_i)$. Conversely, let $Z \in \bigcap_{i=1}^n \delta\rho_i(Y_i)$, then again by (4.24), and lemmas 3.1 and 4.6

$$\begin{aligned}\Box_{i=1}^n \rho_i(X) &= \sum_{i=1}^n \rho_i(Y_i) = \sum_{i=1}^n E[ZY_i] - (\rho_i)_\infty^*(Z) \\ &= E[ZX] - (\Box_{i=1}^n \rho_i)_\infty^*(Z).\end{aligned}$$

Whence $Z \in \delta \Box_{i=1}^n \rho_i(X)$. The final statement of theorem 4.15 is simply an application of theorem 4.8. $\Box$

Note that the statement (4.25) may be void ($\emptyset = \emptyset$).

The subgradients $\delta \Box_{i=1}^n \rho_i(X)$ induce equilibrium pricing rules as follows. We identify each $Z \in \mathcal{P}_\mathcal{R}$ with the absolutely continuous probability measure $\mathbb{Q} \ll \mathbb{P}$ given by $d\mathbb{Q}/d\mathbb{P} = Z$, and with the corresponding pricing rule

$$L^1(\mathbb{Q}) := L^1(\Omega, \mathcal{F}, \mathbb{Q}) \ni Y \mapsto E_\mathbb{Q}[Y].$$

Definition 4.16. *An allocation $(\widetilde{Y}_1, \ldots, \widetilde{Y}_n)$ of X together with a pricing rule $\mathbb{Q} \in \mathcal{P}_\mathcal{R}$ is called an* equilibrium *if X_i, $\widetilde{Y}_i \in L^1(\mathbb{Q})$, $E_\mathbb{Q}[\widetilde{Y}_i] \leq E_\mathbb{Q}[X_i]$, and*

$$\rho_i(\widetilde{Y}_i) = \inf\{\rho_i(Y_i) \mid Y_i \in L^1(\mathbb{Q}) \cap L^1,\ E_\mathbb{Q}[Y_i] \leq E_\mathbb{Q}[X_i]\}$$

for all $i = 1, \ldots, n$.

For a thorough discussion of equilibria with respect to convex risk measures we refer to [16]. The following theorem establishes the connection between equilibria, optimal allocations and generalised subgradients.

Theorem 4.17. *The following conditions are equivalent:*

(i) *There exists an equilibrium $(\widetilde{Y}_1, \ldots, \widetilde{Y}_n; \mathbb{Q})$.*

(ii) *There exists a comonotone equilibrium* $(\widetilde{Y}_1, \dots, \widetilde{Y}_n; \mathbb{Q})$.

(iii) *There is* $Z \in \delta \square_{i=1}^n \rho_i(X)$ *such that* $ZX_i \in L^1$ *for all* $i = 1, \dots, n$.

Moreover, if $(\widetilde{Y}_1, \dots, \widetilde{Y}_n)$ *is a comonotone optimal allocation of* X *and* (iii) *holds, then* $(\widetilde{Y}_1 + E_{\mathbb{Q}}[X_1 - \widetilde{Y}_1], \dots, \widetilde{Y}_n + E_{\mathbb{Q}}[X_n - \widetilde{Y}_n]; \mathbb{Q})$ *where* $d\mathbb{Q}/d\mathbb{P} = -Z$ *is an equilibrium.*

Proof. Let $\mathbb{Q} \ll \mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ such that $X_i \in L^1(\mathbb{Q})$ for all $i = 1, \dots, n$. We claim that

$$\inf_{\substack{Y \in L^1(\mathbb{Q}) \cap L^1 \\ E_{\mathbb{Q}}[Y] \le E_{\mathbb{Q}}[X_i]}} \rho_i(Y) = E[ZX_i] - (\rho_i)^*_\infty(Z) \tag{4.27}$$

where $Z := -d\mathbb{Q}/d\mathbb{P}$. In order to verify this, note that by cash-invariance of ρ_i it is obvious that the infimum on the left-hand since of (4.27) equals the infimum taken over those $Y \in L^1(\mathbb{Q}) \cap L^1$ satisfying $E_{\mathbb{Q}}[Y] = E_{\mathbb{Q}}[X_i]$. Now for every $Y \in L^1(\mathbb{Q}) \cap L^1$ such that $E_{\mathbb{Q}}[Y] = E_{\mathbb{Q}}[X_i]$ lemma 4.7 and monotone convergence imply that

$$\begin{aligned} \rho_i(Y) &= \lim_{m \to \infty} \lim_{n \to \infty} \rho_i((Y^+ \wedge n) - (Y^- \wedge m)) \\ &\ge \lim_{m \to \infty} \lim_{n \to \infty} E[Z((Y^+ \wedge n) - (Y^- \wedge m))] - (\rho_i)^*_\infty(Z) \\ &= E[ZY] - (\rho_i)^*_\infty(Z) = E[ZX_i] - (\rho_i)^*_\infty(Z). \end{aligned}$$

Hence, we have established $\ge$ in (4.27). Moreover, since $E_{\mathbb{Q}}[Y + E_{\mathbb{Q}}[X_i - Y]] = E_{\mathbb{Q}}[X_i]$ for every $Y \in L^1(\mathbb{Q})$ and by cash-invariance we obtain

$$\begin{aligned} \inf_{\substack{Y \in L^1(\mathbb{Q}) \cap L^1 \\ E_{\mathbb{Q}}[Y] \le E_{\mathbb{Q}}[X_i]}} \rho_i(Y) &= \inf_{Y \in L^1(\mathbb{Q}) \cap L^1} \rho_i(Y + E_{\mathbb{Q}}[X_i - Y]) \\ &= E_{\mathbb{Q}}[-X_i] - \sup_{Y \in L^1(\mathbb{Q}) \cap L^1} \Big(E_{\mathbb{Q}}[-Y] - \rho_i(Y) \Big) \\ &\le E[ZX_i] - \sup_{Y \in L^\infty} \Big(E[ZY] - \rho_i(Y) \Big) \\ &= E[ZX_i] - (\rho_i)^*_\infty(Z), \end{aligned}$$

and (4.27) is proved.

(i) $\Leftrightarrow$ (ii): suppose there exists an equilibrium $(\widetilde{Y}_1, \dots, \widetilde{Y}_n; \mathbb{Q})$. Let $(Y_1, \dots, Y_n)$ be any comonotone optimal allocation of X, which exists according to theorem 4.15. Then $Y_i \in L^1(\mathbb{Q})$ by comonotonicity and the fact that $X \in L^1(\mathbb{Q})$ by definition of an equilibrium. By rebalancing the cash, this is by adding $c_i = E_{\mathbb{Q}}[X_i - Y_i]$ to each Y_i, we achieve that $E_{\mathbb{Q}}[Y_i + c_i] = E_{\mathbb{Q}}[X_i]$ for all $i = 1, \dots, n$, and the modified allocation $(Y_1 + c_1, \dots, Y_n + c_n)$ is still comonotone and optimal due to $\sum_{i=1}^n c_i = 0$ and cash-invariance of the ρ_i. Consequently, we may w.l.o.g. assume that $(Y_1, \dots, Y_n)$ satisfies $E_{\mathbb{Q}}[Y_i] = E_{\mathbb{Q}}[X_i]$. But then, $\rho_i(\widetilde{Y}_i) \leq \rho_i(Y_i)$ for each $i = 1, \ldots, n$ (definition 4.16), which can only hold if $(\widetilde{Y}_1, \ldots, \widetilde{Y}_n)$ is itself optimal and $\rho_i(\widetilde{Y}_i) = \rho_i(Y_i)$. Hence, $(Y_1, \ldots, Y_n, \mathbb{Q})$ is a comonotone equilibrium. The converse implication is trivial.

(ii) ⇒ (iii): in the fist part of the proof, we established that the allocation given by any equilibrium must be optimal. Hence, in view of (4.27), and lemma 3.1 we conclude that

$$\begin{aligned}\Box_{i=1}^n \rho_i(X) &= \sum_{i=1}^n \rho_i(\widetilde{Y}_i) = \sum_{i=1}^n E[ZX_i] - (\rho_i)^*_\infty(Z)\\ &= E[ZX] - (\Box_{i=1}^n \rho_i)^*_\infty(Z)\end{aligned}$$

where $Z := -d\mathbb{Q}/d\mathbb{P}$. Consequently, we have proved that $Z \in \delta\Box_{i=1}^n \rho_i(X)$ (lemma 4.6).

(iii) ⇒ (ii): suppose there is $Z \in \delta\Box_{i=1}^n \rho_i(X)$ and let $\mathbb{Q}$ be given by $d\mathbb{Q}/d\mathbb{P} := -Z$. Moreover, let $(Y_1, \ldots, Y_n)$ be any comonotone optimal allocation of X such that $E_{\mathbb{Q}}[Y_i] = E_{\mathbb{Q}}[X_i]$ for all $i = 1, \ldots, n$ (theorem 4.15 and rebalancing the cash). The equality (4.25) implies that $Z \in \delta\rho_i(Y_i)$ for all $i = 1, \ldots, n$. This in conjunction with (4.27) and lemma 4.6 yields

$$\rho_i(Y_i) = E[ZY_i] - (\rho_i)^*_\infty(Z) = E[ZX_i] - (\rho_i)^*_\infty(Z) = \inf_{\substack{Y \in L^1(\mathbb{Q}) \cap L^1 \\ E_{\mathbb{Q}}[Y] \leq E_{\mathbb{Q}}[X_i]}} \rho_i(Y),$$

so we infer that

$$\rho_i(Y_i) = \inf_{\substack{Y \in L^1(\mathbb{Q}) \cap L^1 \\ E_{\mathbb{Q}}[Y] \leq E_{\mathbb{Q}}[X_i]}} \rho_i(Y).$$

Consequently, $(Y_1, \ldots, Y_n; \mathbb{Q})$ is an equilibrium. This also proves the closing statement of the theorem. □

Finally, we provide two sufficient conditions for the existence of an equilibrium.

Lemma 4.18. *Suppose that X is bounded from below or that $\Box_{i=1}^n \rho_i$ satisfies condition (4.7) and that there is $\epsilon > 0$ such that $(1+\epsilon)X \in \sum_{i=1}^n \operatorname{dom} \rho_i$. If either*

$$\forall\, \widetilde{\mathbb{P}} \in \mathcal{P}_{\mathcal{R}} :\ X \in L^1(\widetilde{\mathbb{P}}) \ \Leftrightarrow\ X_i \in L^1(\widetilde{\mathbb{P}}) \text{ for all } i = 1, \ldots, n \tag{4.28}$$

or

$$X_i \in L^{\Box_{i=1}^n \rho_i} \text{ for all } i = 1, \ldots, n, \tag{4.29}$$

then there exists an equilibrium.

Proof. In case of (4.28) combine theorems 4.15 and 4.17. In case of (4.29) recall the proof of theorem 4.8 too. □

Condition (4.28) is always satisfied if $X_i \in L^\infty$, $i = 1, \ldots, n$, or if the initial risks X_i may be somehow controlled by the aggregate risk X, which should be satisfied in most interesting cases. Condition (4.29) will be applied in example 4.5.2.

4.5. Examples

In example 4.5.1 we show that a law-invariant closed convex risk measure ρ on L^1 which is not continuous from below may have empty generalised subgradients even for bounded risks. Examples 4.5.2, 4.5.3, and 4.5.4 illustrate our main results, in particular theorems 4.8 and 4.17 by means of Average Value at Risks, semi-deviation risk measures and entropic risk measures. In examples 4.5.2 (Average Value at Risk) and 4.5.3 (Semi-Deviation Risk Measure) the spaces L^ρ will coincide with some Lebesgue space L^p which are a subclass of Orlicz hearts. Orlicz hearts are proposed as model spaces for convex risk measures in [9] in an attempt to enlarge the model space from L^∞ to a space containing unbounded risks. Given a continuous convex function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(x) = 0 \Leftrightarrow x = 0$, the set

$$L^\Phi := \{X \in L^0 \mid \exists c > 0 : E[\Phi(|X|/c)] < \infty\} \tag{4.30}$$

is the Orlicz space generated by Φ whereas

$$M^\Phi := \{X \in L^0 \mid \forall c > 0 : E[\Phi(|X|/c)] < \infty\} \tag{4.31}$$

is the corresponding Orlicz heart. L^Φ is a Banach space under the Luxemburg norm

$$\|X\|_\Phi := \inf\{\lambda > 0 \mid E[\Phi(|X|/\lambda)] \leq 1\}, \tag{4.32}$$

and M^Φ is a closed subspace of L^Φ such that L^∞ is dense. For details on Orlicz spaces and hearts please consult [30]. We will elaborate on the connection between ρ respectively L^ρ and some Orlicz space/heart. In example 4.5.4, in which we study the entropic risk measure, we will see that L^ρ corresponds to an Orlicz space which is strictly larger than the corresponding Orlicz heart, and we will find that the set of points satisfying condition (4.18) is also strictly larger than this Orlicz heart. Example 4.5.5 then shows that, although the above mentioned prominent examples of law-invariant convex risk measures are all linked to certain Orlicz spaces, the class of L^ρ-spaces covers a far greater variety of law-invariant Banach spaces.

The section closes with Example 4.5.6 which illustrates that we need condition condition (4.7) in theorem 4.8.

4.5.1. Essential Infimum

Let $\rho = -\operatorname{essinf}$ and let $X \in L^\infty$ be such that $\mathbb{P}(X = \operatorname{essinf} X) = 0$. Then $\delta\rho(X) = \emptyset$, because for every probability measure $\mathbb{Q} \ll \mathbb{P}$ we have that $\mathbb{Q}(X = \operatorname{essinf} X) = 0$. Supposing we had $-d\mathbb{Q}/d\mathbb{P} \in \delta\rho(X)$, then $E_\mathbb{Q}[X - \operatorname{essinf} X] = 0$, which would imply that $X = \operatorname{essinf} X$ $\mathbb{Q}$-a.s., and thus would be a contradiction. Hence, $\delta\rho(X) = \emptyset$, although $\partial\rho_\infty(X) \neq \emptyset$.

4.5.2. Average Value at Risk

Consider the Average Value at Risk (AVaR_α) at level $\alpha \in (0,1]$. We know that AVaR_α is continuous on L^1 (example 2.20) and thus everywhere subdifferentiable by lemma 4.1. Clearly, $L^{\text{AVaR}_\alpha} = L^1$, and in view of lemma 4.10 and continuity w.r.t. $\|\cdot\|_1$ it is easily verified that $\|\cdot\|_{C,\text{AVaR}_\alpha}$ and $\|\cdot\|_1$ are equivalent. Thus $L^{\rho *} = L^\infty$. According to (the proof of) theorem 4.8 for every $X \in L^1$ there is a $f_\alpha : \mathbb{R} \to \mathbb{R}_-$ which is increasing on $\{F_X > 0\}$ such that $f_\alpha(X) \in \partial\,\text{AVaR}_\alpha(X) \subset \delta\,\text{AVaR}_\alpha(X)$. It is proved in [21] theorem 4.47 and remark 4.48 that

$$f_\alpha(X) = -\frac{1}{\alpha}(1_{\{X<q_X(\alpha)\}} - \kappa 1_{\{X=q_X(\alpha)\}})$$

where κ is defined as

$$\kappa := \begin{cases} 0 & \text{if } \mathbb{P}(X = q_X(\alpha)) = 0, \\ \frac{\alpha - \mathbb{P}(X<q_X(\alpha))}{\mathbb{P}(X=q_X(\alpha))} & \text{otherwise} \end{cases}$$

does the job. Note that f_α is indeed increasing, does depend on X, and is not continuous. Let $\beta_i \in (0,1]$ for $i = 1,\dots,n$, and let $\gamma := \max_{i=1,\dots,n} \beta_i$. According to example 3.7

$$\Box_{i=1}^n \text{AVaR}_{\beta_i} = \text{AVaR}_\gamma\,.$$

Hence, as we are in the situation of (4.29), and assuming w.l.o.g. that $\beta_1 = \gamma$, we obtain that for any initial risks $X_i \in L^1$ and $X := \sum_{i=1}^n X_i$ an equilibrium is given by $(X + c_1, c_2, \dots, c_n; \mathbb{Q})$ where $d\mathbb{Q}/d\mathbb{P} = -f_\gamma(X)$ and $c_1 = E_{\mathbb{Q}}[X_1 - X]$, $c_i = E_{\mathbb{Q}}[X_i]$ for $i = 2,\dots,n$.

4.5.3. Semi–Deviation Risk Measure

Let $p \in [1,\infty)$. Dev_p (examples 2.22 and 3.9) is a law-invariant closed coherent risk measure on L^1 which is continuous from below (proposition 4.4 (iv)) and satisfies (4.7). In fact Dev_p is continuous if restricted to $(L^p, \|\cdot\|_p)$. It easily verified that $L^{\text{Dev}_p} = L^p$, and that $\|\cdot\|_{C,\text{Dev}_p}$ and $\|\cdot\|_p$ are equivalent. Thus we have that $L^{\text{Dev}_p *} = L^q$ for $q \in (1,\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. By theorem 4.8, for every $X \in \text{dom}\,\rho$ there is a $f : \mathbb{R} \to \mathbb{R}_-$ which is increasing on $\{F_X > 0\}$ such that $f(X) \in \delta\,\text{Dev}_p(X) \cap L^q$ (the fact that $f(X) \in L^{\text{Dev}_p *} = L^q$ is shown in the proof of theorem 4.8). It is known (see e.g. [1] or [15]) that

$$f(X) = \begin{cases} -1 & \text{if } X = \text{constant}, \\ -1 + \delta \frac{E[((X-E[X])^-)^{p-1}] - ((X-E[X])^-)^{p-1}}{\|(X-E[X])^-\|_p^{p-1}} & \text{otherwise} \end{cases} \tag{4.33}$$

does the job. Suppose agent 1 uses Dev_p and agent 2 Dev_r for $1 \le p \le r < \infty$ and that the initial risks satisfy $X_1, X_2 \in L^p$. Then, in view of example 3.9 and theorem 4.17, we have that $(X + c_1, c_2; \mathbb{Q})$ is an equilibrium, where $X = X_1 + X_2$, $-d\mathbb{Q}/d\mathbb{P}$ is given by (4.33), and $c_1 = E_{\mathbb{Q}}[X_1 - X]$, $c_2 = E_{\mathbb{Q}}[X_2]$. The extension of this two-agent case to the n-agent case is obvious.

4.5.4. Entropic Risk Measure

Recall the entropic risk measure

$$\mathrm{Entr}_\beta(X) = \frac{1}{\beta} \log E[e^{-\beta X}], \quad X \in L^1,$$

with parameter $\beta > 0$ from example 2.21. According to proposition 4.4 (iv) $\mathrm{Entr}_\beta(X)$ is continuous from below, and it is easily verified that $\mathrm{Entr}_\beta(X)$ satisfies (4.7). For simplicity we consider $\rho := \mathrm{Entr}_1$. Then, $\rho^*_\infty(Z) = E[-Z \log(-Z)]$ for every $Z \in \mathcal{P}^{\infty *} \cap L^1$. In the following we illustrate the quality of condition (4.8). To this end, assume that $X \in L^1$ satisfies condition (4.8), i.e. there exists $k > 1$ such that $kX \in \mathrm{dom}\ \rho$. Then we have $Z := -\frac{e^{-X}}{E[e^{-X}]} \in L^1$, and $XZ \in L^1$ too, because $|X|e^{-X} \leq C + e^{-kX}$ for some constant $C > 0$ and $E[e^{-kX}] < \infty$ by assumption. It is proved in [21] lemma 3.29 and example 4.33 that

$$\rho(X) = E[ZX] - \rho^*_\infty(Z),$$

and thus $Z \in \delta\rho(X)$ by lemma 4.6. Obviously, $Z = f(X)$ for an increasing function $f : \mathbb{R} \to \mathbb{R}_-$. Now we show that condition (4.8) is in some sense the best we can expect. For this purpose, consider an $X \in L^1$ being distributed according to

$$F_X(x) = C \cdot \int_{-\infty}^{-1 \wedge x} \frac{e^u}{u^2} du$$

for an appropriate constant $C > 0$. It is easily verified that $X \in \mathrm{dom}\ \rho$ and $X \in L^\rho$, but $(1 + \epsilon)X \notin \mathrm{dom}\ \rho$ for all $\epsilon > 0$. We claim that $\delta\rho(X) = \emptyset$. Suppose we had $\delta\rho(X) \neq \emptyset$. Then, according to lemma 4.19 below, this would imply that $Z := -\frac{e^{-X}}{E[e^{-X}]} \in \delta\rho(X)$. But this cannot hold because

$$E[ZX] = \frac{E[-Xe^{-X}]}{E[e^{-X}]} = \infty,$$

so we must have $\delta\rho(X) = \emptyset$.

Next we elaborate on the connection with Orlicz spaces and Orlicz hearts. To this end, we let $\Phi(x) = \exp(x) - 1$, $x \geq 0$, and define L^Φ, M^Φ, and $\|\cdot\|_\Phi$ as in (4.30), (4.31), and (4.32) respectively. It is well-known that $L^\infty \subset L^\Phi$ is not dense and that the Orlicz heart $M^\Phi \subsetneqq L^\Phi$ is the $\|\cdot\|_\Phi$-closure of L^∞ in L^Φ. Note that $L^\rho = L^\Phi$, and that $\|\cdot\|_\Phi = \|\cdot\|_{\log 2, \rho}$. In search for subgradients, as an alternative to the space L^ρ, one could think of choosing the Orlicz heart M^Φ, because $\rho|_{M^\Phi}$ is closed and real-valued, and thus continuous and everywhere subdifferentiable ([14] corollary 2.5 and proposition 5.2). However, in doing so, we would neglect a lot of points at which ρ is generalised subdifferentiable. In fact, we have that

$$M^\Phi \subsetneqq \{X \in L^1 \mid X \text{ satisfies condition (4.8)}\}. \tag{4.34}$$

The fact that this inclusion must be strict is easily verified by considering any X being distributed according to

$$F_X(x) = e^{\lambda x} \cdot 1_{(-\infty,0]}(x) \quad \text{for } \lambda > 1.$$

On the one hand, $-k|X| \in \operatorname{dom} \rho$ for every $k \in (1, \lambda)$, so $X \in \operatorname{int} \operatorname{dom} \rho|_{L^\rho}$ by (4.18). On the other hand, for $c \geq \lambda$ we have $E[\Phi(c|X|)] = \infty$, so $X \notin M^\Phi$. The last strict inclusion in (4.34) is justified in remark 4.14.

Suppose that the preferences of agent 1 are given by Entr_β and those of agent 2 by Entr_γ for $\beta, \gamma > 0$. Let X_1, X_2 be the initial endowments and $X := X_1 + X_2$ the aggregate endowment such that $X_1 e^{-\alpha X}, X_2 e^{-\alpha X} \in L^1$ and $\delta\,\mathrm{Entr}_\alpha(X) \neq \emptyset$ for $\alpha := \frac{\beta\gamma}{\beta+\gamma}$. Then, in view of theorem 4.17, example 3.8, lemma 4.19 and proposition 3.13, the unique equilibrium is

$$\left(\frac{\gamma}{\beta+\gamma}X + c_1, \frac{\beta}{\beta+\gamma}X + c_2; \mathbb{Q}\right) \quad \text{in which } \mathbb{Q} \text{ is given by} \quad \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{e^{-\alpha X}}{E\left[e^{-\alpha X}\right]},$$

and $c_1 = E_\mathbb{Q}[X_1 - \frac{\gamma}{\beta+\gamma}X]$, $c_2 = E_\mathbb{Q}[X_2 - \frac{\beta}{\beta+\gamma}X]$.

Lemma 4.19. *Let $\beta > 0$. We have*

$$\forall Y \in L^1: \quad Z \in \delta\,\mathrm{Entr}_\beta(Y) \quad \Rightarrow \quad Z = -\frac{e^{-\beta Y}}{E[e^{-\beta Y}]}.$$

Proof. Let $Y \in \operatorname{dom} \mathrm{Entr}_\beta$ and define the probability measure $\widetilde{\mathbb{P}} \approx \mathbb{P}$ by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \frac{e^{-\beta Y}}{E[e^{-\beta Y}]}.$$

Suppose there is a $Z \in \delta\,\mathrm{Entr}_\beta(Y)$. Then, $\frac{d\mathbb{Q}}{d\mathbb{P}} = -Z$ defines a probability measure $\mathbb{Q} \ll \mathbb{P}$, and we have that

$$\mathrm{Entr}_\beta(Y) = E_\mathbb{Q}[-Y] - \frac{1}{\beta}E_\mathbb{Q}\left[\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right] \tag{4.35}$$

in which both $E_\mathbb{Q}[-Y] < \infty$ and $E_\mathbb{Q}[\log\frac{d\mathbb{Q}}{d\mathbb{P}}] < \infty$ (lemma 4.6). Note that

$$\log\frac{d\mathbb{Q}}{d\mathbb{P}} = \log\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}} + \log\frac{e^{-\beta Y}}{E[e^{-\beta Y}]} = \log\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}} - \beta Y - \beta\,\mathrm{Entr}_\beta(Y),$$

and thus

$$\beta Y + \beta \operatorname{Entr}_\beta(Y) + \log \frac{d\mathbb{Q}}{d\mathbb{P}} = \log \frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}}. \tag{4.36}$$

Since the left hand side of (4.36) is $\mathbb{Q}$-integrable, we obtain $\log \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^1(\mathbb{Q})$ and

$$\operatorname{Entr}_\beta(Y) - \left(E_{\mathbb{Q}}[-Y] - \frac{1}{\beta} E_{\mathbb{Q}} \left[\log \frac{d\mathbb{Q}}{d\mathbb{P}} \right] \right) = \frac{1}{\beta} E_{\mathbb{Q}} \left[\log \frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}} \right].$$

By (4.35) we conclude that $E_{\mathbb{Q}}[\log \frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}}] = 0$ which is equivalent to $\mathbb{Q} = \widetilde{\mathbb{P}}$. □

4.5.5. The Variety of the $\mathbf{L}^\rho$-spaces

The spaces L^ρ of the preceding examples all corresponded to Orlicz spaces. This is no suprise since the presented risk measures are all closely connected to some Orlicz space generating function. However, as we should expect, this is not the case in general. In this example we will show that L^ρ might almost be any law-invariant Banach space of random variables. To this end, let $(L, \|\cdot\|_L)$ be a Banach space satisfying the following conditions:

(i) $\|\cdot\|_L : L^1 \to [0, \infty]$ is a law-invariant closed (w.r.t. $\|\cdot\|_1$) sublinear function such that $\|X\|_L = \||X|\|_L$ and $|X| \geq |Y| \Rightarrow \|X\|_L \geq \|Y\|_L$,

(ii) $\mathbb{R} \subset L = \{X \in L^1 \mid \|X\|_L < \infty\}$.

Consider the law-invariant closed coherent risk measure ρ on L^1 given by

$$\rho(X) = E[-X] + \frac{1}{\|1\|_L} \|(X - E[X])^-\|_L, \quad X \in L^1.$$

It is easily verified that $L^\rho = L$. Note that $(L, \|\cdot\|_L)$ is not necessarily an Orlicz space. Conditions (i) and (ii) are for instance satisfied by any Lorentz space (see [30] section 10.3 for a definition) which do not coincide with Orlicz spaces in general ([30] theorem 10.3.3 and [28]). A concrete example is the space given by

$$\|X\|_L = \frac{1}{2} \int_0^1 \frac{q_{|X|}(s)}{\sqrt{1-s}} ds, \quad X \in L^1.$$

4.5.6. An Example of a Law-Invariant Closed Coherent Risk Measure which is Continuous from Below but does not satisfy (4.7)

Let $(\Omega, \mathcal{F}, \mathbb{P}) = ((0,1], \mathcal{B}(0,1], \lambda)$ where $\mathcal{B}(0,1]$ is the Borel-σ-algebra over $(0,1]$ and λ denotes the Lebesgue-measure restricted to $\mathcal{B}(0,1]$. Let the probability measures $\mathbb{Q}_n$ be given by

$$\frac{d\mathbb{Q}_n}{d\mathbb{P}} = n 1_{(0, \frac{1}{n^2}]} + \frac{n}{n+1} 1_{(\frac{1}{n^2}, 1]},$$

and let $\widetilde{Z}_n := -d\mathbb{Q}_n/d\mathbb{P}$, $n \in \mathbb{N}$. Moreover, let

$$\mathcal{Q} := \{Z \in L^\infty \mid \exists\, n \in \mathbb{N} : Z \sim \widetilde{Z}_n\},$$

and define a law-invariant closed coherent risk measure on L^1 by

$$\rho(X) := \sup_{Z\in\mathcal{Q}} E[ZX] = \sup_{Z\in\mathcal{Q}} \int_0^1 q_X(s) q_Z(s)\, ds, \quad X \in L^1,$$

where the last equality, and thus the law-invariance of ρ, follows from law-invariance of $\mathcal{Q}$ and lemma A.2. Note that the following computations also imply that ρ is continuous from below (proposition 4.4). Consider the point $Y(\omega) := -\frac{1}{\sqrt{\omega}}$, $\omega \in (0,1]$, in L^1. Since the function Y is increasing, it is immediate that

$$\sup_{Z\sim\widetilde{Z}_n} E[ZY] = E[\widetilde{Z}_n Y] = \frac{4n}{n+1}$$

and thus

$$\rho(Y) = \lim_{n\to\infty} \frac{4n}{n+1} = 4.$$

We notice that

$$\rho(Y 1_{\{Y\le -n\}}) \ge E[\widetilde{Z}_n Y] = 2 \quad \text{for all } n \in \mathbb{N},$$

so ρ does not satisfy (4.7). Suppose that there is $\widetilde{Z} \in \delta\rho(Y)$. Since $\widetilde{Z}_n$ converges $\sigma(L^1, L^\infty)$-weakly to -1, and since we may assume that $\widetilde{Z} = f(Y)$ for an increasing function f, and by maximality we deduce that $\widetilde{Z} \in \{\widetilde{Z}_n, n \in \mathbb{N}\} \cup \{-1\}$. However, $E[\widetilde{Z}_n Y] < 4$ for all $n \in \mathbb{N}$ and $E[-Y] = 2 < 4$. Hence, $\delta\rho(Y) = \emptyset$. In particular, the subgradient at Y in L^ρ (which exists according to lemma 4.12) must have a singular part.

A. Appendix

A.1. Hardy-Littlewood Inequalities

Lemma A.1 (theorem A.24 in [21]). *For any two random variables X and Z we have*

$$\int_0^1 q_X(1-s)q_Z(s)ds \leq E[XZ] \leq \int_0^1 q_X(s)q_Z(s)ds,$$

provided that the integrals are well-defined. Moreover, if $Z = f(X)$ for a measurable function $f : \mathbb{R} \to \mathbb{R}$ and the upper(lower) bound is finite, then the upper(lower) bound is attained if and only if f can be chosen as an increasing(decreasing) function on either $\{F_X > 0\}$ if Z is bounded from above, or on $\{0 < F_X < 1\}$ else.

The following lemma is an extension of lemma 4.55 in [21]. For the sake of completeness we provide a self-contained proof.

Lemma A.2. *Let $X, Z \in L^1$.*

(i) *If $E[\widetilde{X}Z]$ is well-defined for every $\widetilde{X} \sim X$ and if $\int_0^1 |q_X(s)q_Z(s)|ds < \infty$, then*

$$\sup_{\widetilde{X}\sim X} E[Z\widetilde{X}] = \int_0^1 q_X(s)q_Z(s)ds. \tag{A.1}$$

(ii) *In particular, condition* (i) is satisfied if $(\widetilde{X}Z) \in L^1$ for all $\widetilde{X} \sim X$.

Proof. **step 1.** Suppose the distribution function F_Z of Z is continuous. Then $U := F_Z(Z)$ has a uniform distribution on $(0,1)$ and $Z = q_Z(U)$ $\mathbb{P}$-a.s.. For $\bar{X} := q_X(U) \sim X$ we have that

$$E[|\bar{X}||Z|] = E[|q_X(U)||q_Z(U)|] = \int_0^1 |q_X(s)||q_Z(s)|ds. \tag{A.2}$$

Thus, if $E[\widetilde{X}Z]$ is well-defined for every $\widetilde{X} \sim X$ and if $\int_0^1 |q_X(s)q_Z(s)|ds < \infty$, then (A.1) follows from

$$E[\bar{X}Z] = \int_0^1 q_X(s)q_Z(s)\,ds$$

and lemma A.1. Moreover, if $(\widetilde{X}Z) \in L^1$ for all $\widetilde{X} \sim X$, then $E[\widetilde{X}Z]$ is well-defined for every $\widetilde{X} \sim X$, and $\int_0^1 |q_X(s)q_Z(s)|ds < \infty$ follows from (A.2).

step 2. Now suppose Z has no continuous distribution. Denote by D the countable set of all $z \in \mathbb{R}$ such that $\mathbb{P}[Z = z] > 0$. W.l.o.g. (by adding a constant to Z if necessary) we may assume that $0 \notin D$. Let $A_z := \{Z = z\}$, $z \in D$. Since $(\Omega, \mathcal{F}, \mathbb{P})$ contains no atoms, for each $z \in D$ there is a random variable U_z being uniformly distributed on $(0, \frac{|z|}{2} \wedge 1)$ under the measure $\mathbb{P}(\cdot \mid A_z)$. We claim that the distributions of

$$Z_n := Z - \frac{1}{n} \sum_{z \in D} \operatorname{sgn}(z) U_z 1_{A_z}, \quad n \in \mathbb{N},$$

are continuous. Indeed, for any $y \in \mathbb{R}$

$$\begin{aligned} \mathbb{P}(Z_n = y) &= \mathbb{P}(Z_n = y, Z \notin D) + \sum_{z \in D} \mathbb{P}(Z = z, U_z = \operatorname{sgn}(z) n(z - y)) \\ &= \mathbb{P}(Z = y, Z \notin D) + \sum_{z \in D} \mathbb{P}(A_z) \mathbb{P}(U_z = \operatorname{sgn}(z) n(z - y) \mid A_z) \\ &= 0\,. \end{aligned}$$

Note that $Z^{\pm} - 1 \le Z_n^{\pm} \le Z^{\pm}$. Hence, for all $n \in \mathbb{N}$ and for every $\widetilde{X} \sim X$

- $E[\widetilde{X} Z]$ is well-defined if and only if $E[\widetilde{X} Z_n]$ is well-defined,
- $(\widetilde{X} Z) \in L^1$ if and only if $(\widetilde{X} Z_n) \in L^1$, and
- $\int_0^1 |q_Z(s) q_X(s)| ds < \infty$ if and only if $\int_0^1 |q_{Z_n}(s)||q_X(s)| ds < \infty$.

Furthermore, we observe that Z_n converges to Z $\mathbb{P}$-a.s. and in L^1. So in particular, the respective quantile functions converge almost everywhere. Therefore, the sequence $(q_X q_{Z_n})_{n \in \mathbb{N}}$ converges almost everywhere to the integrable function $q_X q_Z$, and we have $|q_X q_{Z_n}| \le |q_X q_Z|$. Consequently, the dominated convergence theorem in combination with step 1 yields

$$\begin{aligned} \int_0^1 q_X(s) q_Z(s)\, ds &= \lim_{n \to \infty} \int_0^1 q_X(s) q_{Z_n}(s)\, ds \\ &= \lim_{n \to \infty} \sup_{\widetilde{X} \sim X} E[\widetilde{X} Z_n] = \sup_{\widetilde{X} \sim X} E[\widetilde{X} Z] \end{aligned}$$

where the last equality follows from

$$|E[\widetilde{X} Z_n] - E[\widetilde{X} Z]| \le \frac{1}{n} \|X\|_1 \quad \text{for all } \widetilde{X} \sim X \text{ such that } \widetilde{X} Z \in L^1.$$

Hence, (i) is proved. In order to prove (ii) let

$$\overline{Z} := Z + \sum_{z \in D} \operatorname{sgn}(z) U_z 1_{A_z}$$

and note that $\overline{Z}$ has a continuous distribution and $Z^{\pm} \leq \overline{Z}^{\pm} \leq Z^{\pm} + 1$. Hence, for all $\widetilde{X} \sim X$ we have $(\widetilde{X} Z) \in L^1$ if and only if $(\widetilde{X}\overline{Z}) \in L^1$, and

$$\int_0^1 |q_Z(s) q_X(s)| ds \leq \int_0^1 |q_{\overline{Z}}(s) q_X(s)| ds$$

which, in view of step 1, completes the proof. □

A.2. An Arzela-Ascoli Type Argument

The following lemma is needed for the proofs in sections 3.4 and 3.5.

Lemma A.3. *Let $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of increasing 1-Lipschitz-continuous functions such that $f_n(0) \in [-K, K]$ for all $n \in \mathbb{N}$ where $K \geq 0$ is a constant. Then there is a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ and an increasing 1-Lipschitz-continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{k \to \infty} f_{n_k}(x) = f(x)$ for all $x \in \mathbb{R}$.*

Proof. The Lipschitz-continuity guarantees that $f_n(x) \in [-K, K + x]$ if $x \geq 0$ and $f_n(x) \in [-K + x, K]$ if $x \leq 0$. Hence, by a procedure well-known from the standard proof of the Arzela-Ascoli theorem, we are able to extract a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that $\lim_{k \to \infty} f_{n_k}(q)$ exists for all $q \in \mathbb{Q}$. In fact, we can easily show that the sequences $(f_{n_k}(x))_{k \in \mathbb{N}}$ must converge for all $x \in \mathbb{R}$. To this end, let $\epsilon > 0$ be arbitrary and choose $q \in \mathbb{Q}$ and $N_0 \in \mathbb{N}$ such that $|q - x| < \epsilon/3$ and $|f_{n_k}(q) - f_{n_l}(q)| < \epsilon/3$ for all $k, l \geq N_0$. Then for all $k, l \geq N_0$:

$$\begin{aligned} |f_{n_k}(x) - f_{n_l}(x)| &\leq |f_{n_k}(x) - f_{n_k}(q)| + |f_{n_k}(q) - f_{n_l}(q)| + \\ &\qquad + |f_{n_l}(q) - f_{n_l}(x)| \\ &\leq 2|x - q| + |f_{n_k}(q) - f_{n_l}(q)| < \epsilon \, , \end{aligned}$$

in which we did apply the Lipschitz-continuity twice. Now it is easily verified that $f(x) := \lim_{k \to \infty} f_{n_k}(x)$, $x \in \mathbb{R}$, is a 1-Lipschitz-continuous increasing function. □

A.3. Standard Probability Space

Two proability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{B}, \mathbb{Q})$ are *isomorphic mod 0* if there exists null-sets $A \in \mathcal{F}$ and $B \in \mathcal{B}$ and a bijection $f : \Omega \backslash A \to \Omega' \backslash B$ such that both f and f^{-1} are measurable and measure-preserving (i.e. $\mathbb{P}(C \cap A^c) = \mathbb{Q}(f(C \cap A^c))$ for all $C \in \mathcal{F}$) on the restricted probability spaces. The map f is called *isomorphism mod 0*. An atom-less probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is *standard* if it is isomorphic mod 0 to the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ where $\mathcal{B}([0,1])$ denotes the Borel-σ-algebra over $[0,1]$ and λ is the Lebesgue-measure restricted to $\mathcal{B}([0,1])$ (see [33] section 2). A mapping $\tau : \Omega \to \Omega$ is a *measure preserving transformation* if it is an isomorphism mod 0. Given an atom-less standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two sets $A, B \in \mathcal{F}$ such that $\mathbb{P}(A) = \mathbb{P}(B)$, there exists a measure preserving transformation $\tau : \Omega \to \Omega$ such that $\tau(A) = B$ $\mathbb{P}$-a.s. and $\tau(B) = A$ $\mathbb{P}$-a.s. and $\tau = \mathrm{Id}_\Omega$ on $A^c \cap B^c$ $\mathbb{P}$-a.s. This is a direct consequence of the definition of standardness and the fact that for every subset $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ the restricted probability space with conditional measure is again standard (see [33] section 2, in particular 2.3 and 2.4). For instance, if Ω is a complete separable metric space, $\mathcal{F}$ the corresponding σ-algebra of Borel-sets, and $\mathbb{P}$ a probability measure on $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, \mathbb{P})$ is standard (see e.g. [34] theorem 9, p. 327).

Bibliography

[1] Acciaio, B. (2007), Optimal risk sharing with non-monotone monetary functionals, *Finance and Stochastics*, 11, 267–289.

[2] Aliprantis, C.D. and Border K.C. (1999), *Infinite Dimensional Analysis*, Springer.

[3] Artzner, P., Delbaen, F., Eber, J.M. and Heath, D. (1999), Coherent measures of risk, *Mathematical Finance*, 9, 203–228.

[4] Barrieu, P. and El Karoui, N. (2005), Inf-convolution of Risk Measures and Optimal Risk Transfer, *Finance and Stochastics*, Vol. 9, No. 2, 269-298.

[5] Barrieu, P. and El Karoui, N. (2004), Optimal Derivatives Design under Dynamic Risk Measures, *Yin, G. and Zhang, Q. (eds.): Mathematics of Finance. Contemporary Mathematics - A.M.S.*, 13–25.

[6] Biagini, S. and Frittelli, M. (2006), On Continuity Properties and Dual Representation of Convex and Monotone Functionals on Frechet Lattices, Working Paper.

[7] Burgert, C. and Rüschendorf, L. (2005), Allocations of Risk and Equilibrium in Markets with Finitely Many Traders, *Insurance: Mathematics and Economics*, forthcoming.

[8] Cherny, A. and Kupper, M. (2007), Divergence Utilities, VIF Working Paper.

[9] Cheridito, P. and Li, T. (2007) Risk Measures on Orlicz Hearts, *Mathematical Finance*, forthcoming.

[10] Cherny, A.S. and Madan, D.B. (2006), CAPM, Rewards, and Empirical Asset Pricing with Coherent Risk, SSRN Working Paper.

[11] Dana, R-A. (2005), A Representation Result for Concave Schur Concave Functions, *Mathematical Finance*, 15, 613–634.

[12] Dunford, N. and Schwartz, J. (1976), *Linear Operators. Part I: General theory*, Interscience Publishers.

[13] Delbaen, F. (2000), *Coherent Risk Measures*, Cattedra Galileiana. Scuola Normale Superiore di Pisa.

[14] Ekeland, I. and Témam, R. (1999), *Convex Analysis and Variational Problems, Chapter I*, SIAM.

[15] Filipović, D. and Kupper, M. (2007), Monotone and Cash-Invariant Convex Functions and Hulls, *Insurance: Mathematics and Economics*, 41, 1–16.

[16] Filipović, D. and Kupper, M. (2008), Equilibrium Prices for Monetary Utility Functions, *International Journal of Theoretical and Applied Finance*, 11, 325– 343.

[17] Filipović, D. and Svindland, G. (2008), Optimal Capital and Risk Allocations for Law- and Cash-Invariant Convex Functions, *Finance and Stochastics*, 12, 423–439.

[18] Filipović, D. and Svindland, G. (2008), The Canonical Model Space for Law-invariant Convex Risk Measures is L^1, preprint.

[19] Filipović, D. and Svindland, G. (2008), Subgradients of Law-invariant Convex Risk Measures on L^1, preprint.

[20] Föllmer, H. and Schied, A. (2002), Convex Measures of Risk and Trading Constraints, *Finance and Stochastics*, 6, 429–447.

[21] Föllmer, H. and Schied, A. (2002), *Stochastic Finance, An Introduction in Discrete Time*, de Gruyter Studies in Mathematics 27.

[22] Frittelli, M. and Rosazza Gianin, E. (2002), Putting order in risk measures, *Journal of Banking and Finance*, 26, 1473–1486.

[23] Jouini, E., Schachermayer, W. and Touzi, N. (2008), Optimal Risk Sharing for Law-Invariant Monetary Utility Functions, *Mathematical Finance*, Vol. 18, No. 2, 269–292.

[24] Jouini, E., Schachermayer, W. and Touzi, N. (2006), Law invariant risk measures have the Fatou property, *Advances in Mathematical Economics*, Vol. 9, 49–71.

[25] Kaina, M. and Rüschendorf, L. (2007), On Convex Risk Measures on L^p, *Mathematical Methods in Operations Research*, forthcoming.

[26] Krätschmer, V. (2007), On σ-additive robust representation of convex risk measures for unbounded financial positions in the presence of uncertainty about the market model, SFB 649 Discussion Paper 2007-010, Humboldt University Berlin.

[27] Landsberger, M. and Meilijson, I. (1994), Comonotone allocations, Bickel-Lehmann dispersion and the Arrow-Pratt measure of risk aversion, *Annals of Operations Research*, 52.

[28] Lorentz, G.G. (1961), Relations Between Function Spaces, Proc. A.M.S, 12, 127–132.

[29] Ludkovski, M. and Rüschendorf, L. (2007), On comonotonicity of Pareto optimal risk sharing, *Statistics and Probability Letters*, forthcoming.

[30] Rao, M.M. and Ren, Z.D. (1991), *Theory of Orlicz Spaces*, Marcel Dekker, Inc.

[31] Rockafellar, R.T. (1997), *Convex Analysis*, Princeton University Press.

[32] Rockafellar, R.T. (1974), *Conjugate Duality and Optimization*, SIAM.

[33] Rohlin, V.A. (1962), On the Fundamental Ideas of Measure Theory, *American Mathematical Society: Translations, Series 1*, Vol. 10, 1–54.

[34] Royden, H. L. (1968), *Real Analysis*, Macmillan Publishing, second edition.

[35] Ruszczyński, A. and Shapiro, A. (2006), Optimization of Convex Risk Functions, *Math. Opers. Res.*, 31, 433–451.

[36] Werner, D. (2002), *Funktionalanalysis*, Springer, 4. Auflage.