Diplomarbeit

zur Erlangung des akademischen Grades
Diplom-Wirtschaftsmathematikerin an der Fakultät für Mathematik
und Wirtschaftswissenschaften der Universität Ulm

Risk-Neutral Valuation of Participating Life Insurance Contracts in a Stochastic Interest Rate Environment

vorgelegt von

Katharina Zaglauer

Gutachter:

Prof. Dr. Hans-Joachim Zwiesler

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Ulm, May 26th 2006 Katharina Zaglauer

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List of Abbreviations

CIR	Cox-Ingersoll-Ross
cp.	compare
e.g.	for example
EMM	equivalent martingale measure
etc.	et cetera
i.e.	that is
IFRS	International Financial Reporting Standards
ODE	ordinary differential equation
OU	Ornstein-Uhlenbeck
p.	page
PDE	partial differential equation
SDE	stochastic differential equation
VAG	Gesetz über die Beaufsichtigung der Versicherungsunternehmen (German insurance supervisory law)
wlog	without loss of generality
ZRQuotenV	Verordnung über die Mindestbeitragsrückerstattung in der Lebensversicherung (Regulation about the minimum premium refund in German life insurance)

Chapter 1

Introduction

1.1 Motivation

'Participating' or 'with-profits' life insurance contracts are policies which include an annual minimum rate of return guarantee as well as a surplus distribution mechanism determined by management decisions of the insurance company. Regarding the market share, these participating contracts are dominant in many insurance markets such as, for example (e.g.), the German one. They contain several options and guarantees such as bonus options, surrender options, interest rate guarantees, et cetera (etc.), which present liabilities to the insurer and hence constitute possible risks to the company's solvency (compare (cp.) [21]). Thus, it is very important to use adequate and accurate valuation and risk management methods.

Until today, policies have generally not been priced using stochastic models. The insurance industry has rather relied on deterministic pricing models, in particular on the principle of equivalence[1], presumably due to the fact that in the past, the minimum interest rate used to be much lower than market rates. This led to great profits and to the embedded options being so far out of the money[2] that life insurance companies did not recognize the necessity of taking embedded options and guarantees into account for pricing purposes. However, market interest rates have fallen in the

[1]The principle of equivalence is an expected value principle based on the assumption that the expected value of future premiums to receive should equal the expected value of future benefits plus expenses to be paid out under prudent assumptions.

[2]An out-of-the-money option currently has no or almost no intrinsic value. A call option, for example, is out of the money if the strike price is higher than the current underlying price. An in-the-money option conversely does have a considerable intrinsic value. The strike price of an in-the-money call option is lower than the current underlying price (see [31]).

1990s, and today the guaranteed rates from old contracts even exceed the market rates. Thus, the issued interest rate guarantee option has greatly increased in value and finding investment opportunities the returns of which enable the insurers to meet the guaranteed minimum rate of return has lately become challenging. Many companies had to cut their bonuses and some even went bankrupt, as for instance the Mannheimer Lebensversicherungs AG, which closed to new business in 2003 and had to transfer its policy portfolio to the Protektor Lebensversicherungs AG, a safety institution for German insurers (see [33]).

As a consequence, insurance companies recently started to focus more and more on their internal financial risk management. The deficiency of the traditional deterministic actuarial pricing techniques for the valuation of the insurer's liabilities led to an increasing interest for "fair" pricing methods employing methods from modern financial mathematics. Moreover, new accounting standards such as the International Financial Reporting Standards (IFRS) 4 call for a stronger integration of financial methods into actuarial valuation mechanisms.

Regarding the above, the stochastic treatment of interest rates plays a major role, since controlling the risk induced by interest rate fluctuations can reduce pricing errors and hence improve the profitability and solvency of insurance companies.

1.2 Objectives and literature

The analysis of participating life insurance contracts with a minimum interest rate guarantee requires a realistic model of bonus payments. In [20], Grosen and Jørgensen model a bonus account establishing some general principles. They propose that the returns on the assets in the present year have to influence the policy interest rate in the following years, that is (i.e.), if returns are high, the policyholder is credited a rather large excessive bonus, if returns are low, the credited excess bonus is low. Moreover, they argue that life insurance policies should provide a low-risk, stable and yet competitive investment opportunity. In particular, the surplus distribution rules should reflect the so-called average interest principle which states that insurers are to build up reserves in years of good returns and use the accumulated reserves to keep the surplus stable in years of low returns without jeopardizing the company's solvency. Grosen and Jørgensen's model further includes the option to surrender the contract and "walk away". Since the value of their bonus account is path-dependent and thus solutions in closed form cannot be presented, they calculate the risk-neutral value of an insurer's liabilities using Monte Carlo methods.

Kling et al. (see [27]) introduce a distribution scheme which is also based on the average interest principle and includes compulsory payments as well as payments due to management decisions of the company. They investigate how the shortfall probability, i.e. the physical probability that the company's liabilities exceed the assets, depends on the characteristics of the contract, on the insurer's reserve situation, and the company's asset allocation. Bauer extends this model in [6]. Besides employing the distribution scheme from [27], he includes a bonus policy based only on compulsory payments. In contrast to [27], he uses risk-neutral valuation to compute the value of the insurance contract, implementing a comprehensive program which uses Monte Carlo methods and a discretization method based on the solution of a Black-Scholes partial differential equation (PDE).

This thesis also addresses the pricing of participating life insurance contracts with the risk-neutral valuation approach. We apply the model of Bauer (cp. [6]), but since interest rates fluctuate over time and insurance contracts are long term investments, we do not use a constant interest rate assumption for the calculations, but embed more consistent models for the behavior of interest rates into his model.

The idea of embedding stochastic short rate models into the pricing of insurance contracts and guarantees is not new. In [29], Miltersen and Persson introduce stochastic interest rates into a model dealing with minimum rate of return guarantees using the general Heath-Jarrow-Morton approach. They employ two well-known short rate models (Vasicek and Cox-Ingersoll-Ross (CIR)) to derive pricing formulas for point-to-point and cliquet style rate of return guarantees on both stock market return processes and short-term interest rate return processes. Briys and de Varenne (see [9]) develop a continuous-time valuation model for life insurance liabilities with point-to-point guarantees, which accounts for both interest rate risk and default risk. They choose to model the instantaneous short rate by an Ornstein-Uhlenbeck (OU) process and derive a closed form solution for the price of certain life insurance liabilities. In [5], Barbarin and Devolder choose the same short rate model and integrate both a risk-neutral and a risk-management approach in the pricing of life insurance products. They introduce a so-called cash-bond-stock model, i.e. an asset model consisting of a portfolio of three different asset forms: a money market process, a fixed interest bond process, and a stock process, and furthermore use fair valuation principles to compute the market value of an insurance contract.

Due to the complexity of considering adequate accounting rules in a multi-asset model, we employ a simpler two asset market model and consider the composition of the insurer's asset portfolio implicitly by choosing adequate volatilities and correlations between the asset process and the interest rate process. We use an Ornstein-

Uhlenbeck as well as a Cox-Ingersoll-Ross model for the instantaneous risk-free interest rate and embed these models in the framework developed in [6] in order to determine the risk-neutral value of a participating life insurance contract.

1.3 Structure of the thesis

The general model for the participating contract is based on Bauer (see [6]) who develops a bonus policy for obligatory payments to the policyholder and further employs the bonus policy proposed in [27] in which payments depend on both regulatory requirements and management decisions of the company. The model and the two bonus policies are introduced in Chapter 2.

In Chapter 3, the concept of risk-neutral valuation is outlined. Following [1] and [6], we construct a market which consists of both financial and biometric events and thus presents a model for an insurance market. As in [6], we give a risk-neutral valuation formula for the price of an insurance contract, decompose the contract into implicit options and furthermore explain how a so-called "walk-away" option, where the policyholder has the opportunity to surrender the contract at certain time points within the lifetime of the contract, can be included.

Chapter 4 is detached from the previous chapters and introduces the theory of stochastic interest rates. We choose two different models for the instantaneous short rate: an Ornstein-Uhlenbeck process and a Cox-Ingersoll-Ross process. The stochastic differential equations which describe the evolution of the processes are introduced and the distributions of the short rates are determined for both stochastic interest rate models.

Subsequently, we introduce financial market models for the numerical valuation. We derive Monte Carlo algorithms for the calculation of the fair value of the contracts for both interest rate models. Additionally, we introduce another approach. A type of Black-Scholes partial differential equation is derived from a stochastic differential equation and the numerical solution of the PDE is employed to calculate the fair value of the contract. Chapter 5 closes with a brief discussion on the imperfections of the asset model.

In Chapter 6, we describe how the model is implemented and discuss the parameter choices. We carry out calculations and discuss the numerical results. Furthermore we investigate the sensitivity of the fair value of the insurance contract to changes of various parameters. In addition, we compare our results with those in [6] to study how stochastic interest rates influence the value of the contract.

Chapter 7 gives a summary of the thesis and presents the most important conclusions and results. Furthermore, problems are discussed and an outlook for future research is provided.

Chapter 2

Model of a participating life insurance contract

Participating life insurance contracts are often very complex as they include several financial options. Interest rate guarantees, bonus distribution schemes, and surrender possibilities are examples of implicit option elements in these policies. In many countries as, e.g., Germany, life insurance contracts are subject to numerous legal and regulatory requirements. Hence, it is very complex to include all details in a model framework. We present a simplified model for German life insurance contracts, which nevertheless possesses the most important features.

At first, we introduce the structure of the contract. Two different surplus distribution schemes are included. The first one, introduced by Bauer (cp. [6]), reflects compulsory payments which the insurance company has to make due to the legal and regulatory framework in Germany, whereas the second one, introduced by Kling et al. (cp. [27]), models the actual behavior of typical German insurers.

We merely give a short summary of the surplus distribution schemes as they are discussed in detail in [6] and [27].

2.1 The contract

We use the simplified balance sheet given in Table 2.1 to model the insurance company's financial situation. Here, S_t denotes the market value of the insurer's asset portfolio at time t, L_t the policyholder's account balance at time t, and $R_t = S_t - L_t$ the time t bonus reserve. We assume that the asset portfolio consists of a variety of different entries, such as fixed-interest bonds, stock etc., and that this portfolio meets

the regulations according to § 54 of the German insurance supervisory law (VAG, cp. [40]). The bonus reserve R is assumed to contain valuation reserves, equity, and other elements.
We further assume that dividend payments d_t to the shareholders occur at the policy anniversaries $t \in \{1, \ldots, T\}$.

Assets	**Liabilities**
S_t	L_t
	R_t
S_t	S_t

Table 2.1: Simplified balance sheet

We merely consider one very simple life insurance contract, namely a single premium term-fix insurance contract issued at time 0 and maturing after T years ignoring any charges. Within this contract, the insurer is obliged to pay a specified amount of money to the policyholder at maturity T in any case, that is, if the insured person dies within the term of the contract or survives until the maturity date T. Hence, the benefit does not depend on biometric circumstances, but merely on the development of the insurer's liabilities. We assume that at the conclusion of the contract, the policyholder pays a single premium P and thereby acquires a contract of nominal value P. Thus, at maturity T the policyholder receives $P\frac{L_T}{L_0}$.

The introduced term-fix insurance is a simple version of a German participating life insurance contract which enables us to focus on essential mechanisms. However, other more complex types of contracts can be easily embedded in the model. Additionally, the choice of this simple contract permits the nice interpretation of an insurer in a steady state: under the assumption that death benefit payments and risk premiums as well as maturity payments and premiums of new contracts neutralize each other in every period, the development of the insurer's liabilities for one contract can be interpreted as a model for the development of the insurer's liabilities as a whole.

2.2 Surplus distribution schemes

The question of how the surplus is distributed to the policyholders in practice is highly delicate and demands political, legal, and strategic considerations within the insurance company. Even though our general model allows for various models, we focus on the German market. We present two different bonus schemes that represent

two different cases: the so-called MUST-case models the situation, where the life insurance company only makes compulsory payments, whereas the IS-case models the actual behavior of typical German insurers in recent years.
In either case the compulsory payments are included, since they are required by German legislation.

The MUST-case

According to § 65 VAG (cp. [40]), insurance companies are required to grant a minimum rate of interest g, which is currently fixed at 2.75% [1]. However, since the minimum guaranteed rate has changed over the years and since it is locked in for each contract at the inception of the policy, the companies' portfolios of policies also contain contracts with higher minimum guaranteed interest rates.

The interest rate guarantee implies that

$$L_t \geq L_{t-1}(1+g), \ t = 1, \ldots, T. \tag{2.1}$$

Furthermore, the German legislation requires that at least a minimum participation rate $\delta = 90\%$ of the earnings on book values has to be passed on to the policyholders (cp. Regulation about the minimum premium refund in German life insurance (ZRQuotenV), § 1, section 2, ([41])).

The insurers' earnings are subject to accounting rules and thus the earnings on book value are in general not equal to the earnings on market value $S_t^- - S_{t-1}^+$, where S_t^- and S_t^+ denote the value of the portfolio at time t shortly prior to and shortly after the payment of the dividends d_t, respectively. We assume that at least a portion y of the earnings on market value has to be displayed as earnings on book value in the balance sheet. This results in

$$L_t \geq L_{t-1} + \left[\delta y \left(S_t^- - S_{t-1}^+\right)\right]^+, \ t = 1, \ldots, T. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain the relation

$$L_t = (1+g) L_{t-1} + \left[\delta y \left(S_t^- - S_{t-1}^+\right) - g L_{t-1}\right]^+, \ t = 1, \ldots, T. \tag{2.3}$$

[1] To be more precise, g is the maximum interest rate at which the policy reserves are allowed to be discounted. However, since this rate is used by almost all insurance companies and since the policy reserves are almost equal to the credit on the policyholders' accounts, this implies a year-by-year interest rate guarantee.

Assuming that the remaining portion of the earnings on book values is paid out as dividends, we obtain that

$$\begin{aligned} d_t \;=\; & (1-\delta)\, y\left(S_t^- - S_{t-1}^+\right) 1_{\left\{\delta y\left(S_t^- - S_{t-1}^+\right) > gL_{t-1}\right\}} \\ & + \left[y\left(S_t^- - S_{t-1}^+\right) - gL_{t-1}\right] 1_{\left\{\delta y\left(S_t^- - S_{t-1}^+\right) \le gL_{t-1} \le y\left(S_t^- - S_{t-1}^+\right)\right\}}. \end{aligned} \tag{2.4}$$

In addition, it is assumed that the policyholders are allowed to cancel the contract at the policy anniversaries and "walk away" at no cost, where the insurance companies are forced to pay out the current account value L_t [2].

The IS-case

The IS-case models the typical behavior of German insurance companies in recent years. Obviously, in the IS-case bonus payments are at least as high as in the MUST-case. Thus, from (2.3) we obtain, that

$$L_t \;\ge\; (1+g)\, L_{t-1} + \left[\delta y\left(S_t^- - S_{t-1}^+\right) - gL_{t-1}\right]^+,\; t = 1, \ldots, T. \tag{2.5}$$

In the past, insurance companies have tried to grant the policyholders a stable interest rate. In years with high capital market yields, asset reserves have been accumulated and utilized in years with poor returns to keep the granted rate at a stable level. Only if the reserves dropped beneath or rose above a certain level would the companies reduce or increase the surplus, respectively.

In what follows we give an outline of the distribution rule introduced by Kling et al. (cp. [27]).
We define the reserve quota as the ratio of reserves and the policyholder's credit account

$$x_t := \frac{R_t}{L_t} = \frac{S_t^+ - L_t}{L_t} = \frac{S_t^- - d_t - L_t}{L_t},$$

and let $z > g$ denote the target interest rate. Furthermore, we assume that at times $t \in \{1, \ldots, T\}$ a portion α of the surplus above the guaranteed level g is distributed to the shareholders. The bonus policy in [27] states that if the target interest rate z leads to a reserve quota between a and b, i.e. if

$$a \le x_t \le b,$$

[2] Here, contracts without surrender or "walk away" option are called European contacts, whereas contracts including the possibility to surrender are referred to as non-European contracts.

for

$$\begin{aligned} L_t &= (1+z)\, L_{t-1}, \\ d_t &= \alpha\,(z-g)\, L_{t-1}, \\ S_t^+ &= S_t^- - d_t, \\ R_t &= S_t^+ - L_t, \end{aligned}$$

then exactly the target interest rate z is credited to the policyholder.

If the reserve quota falls below a when crediting the interest rate z, i.e. $x_t < a$, then we distinguish between two cases:

(i) If crediting the guaranteed interest rate g leads to a reserve rate above a, i.e. if

$$a \le x_t = \frac{S_t^- - (1+g)\, L_{t-1}}{(1+g)\, L_{t-1}},$$

then the company credits exactly the rate of interest that leads to $x_t = a$. Hence, we have

$$\begin{aligned} L_t &= (1+g)\, L_{t-1} + \frac{1}{1+a+\alpha}\left[S_t^- - (1+g)\,(1+a)\, L_{t-1}\right], \\ d_t &= \frac{\alpha}{1+a+\alpha}\left[S_t^- - (1+g)\,(1+a)\, L_{t-1}\right]. \end{aligned}$$

(ii) If even the interest rate g leads to a reserve rate below a, i.e. if

$$\begin{aligned} \frac{S_t^- - (1+g)\, L_{t-1}}{(1+g)\, L_{t-1}} = x_t &< a \\ \Leftrightarrow\ S_t^- &< (1+a)\,(1+g)\, L_{t-1}, \end{aligned}$$

then the policyholder is credited exactly the guaranteed interest rate g and no dividends are paid. Therefore, we obtain

$$L_t = (1+g)\, L_{t-1}, \quad d_t = 0.$$

If crediting the target interest rate z leads to a reserve quota above b, then we grant exactly the rate of interest such that $x_t = b$:

$$\begin{aligned} L_t &= (1+g)\, L_{t-1} + \frac{1}{1+b+\alpha}\left[S_t^- - (1+g)\,(1+b)\, L_{t-1}\right], \\ d_t &= \frac{\alpha}{1+b+\alpha}\left[S_t^- - (1+g)\,(1+b)\, L_{t-1}\right]. \end{aligned} \tag{2.6}$$

Finally, we have to verify that the granted bonus meets the regulative requirements, i.e.

$$L_t \overset{!}{>} L_{t-1} + \delta y \left(S_t^- - S_{t-1}^+\right).$$

If this condition is not fulfilled, the company increases the surplus so that

$$\begin{aligned} L_t &= (1+g)\, L_{t-1} + \left[\delta y \left(S_t^- - S_{t-1}^+\right) - g L_{t-1}\right]^+, \\ d_t &= \alpha \left[\delta y \left(S_t^- - S_{t-1}^+\right) - g L_{t-1}\right]^+. \end{aligned}$$

By summarizing all cases and conditions we obtain (see [6]) that

$$\begin{aligned} L_t = (1+g)\, L_{t-1} + \max\{\ & \left[\delta y \left(S_t^- - S_{t-1}^+\right) - g L_{t-1}\right]^+, \\ & (z-g)\, L_{t-1} 1_{\left\{((1+a)(1+z)+\alpha(z-g))L_{t-1} \le S_t^- \le ((1+b)(1+z)+\alpha(z-g))L_{t-1}\right\}} \\ & + \frac{1}{1+a+\alpha}\left[S_t^- - (1+g)\,(1+a)\, L_{t-1}\right] \\ & \qquad 1_{\left\{(1+a)(1+g)L_{t-1} < S_t^- < ((1+a)(1+z)+\alpha(z-g))L_{t-1}\right\}} \\ & + \frac{1}{1+b+\alpha}\left[S_t^- - (1+g)\,(1+b)\, L_{t-1}\right] 1_{\left\{((1+b)(1+z)+\alpha(z-g))L_{t-1} < S_t^-\right\}}\ \}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} d_t = \max\{\ & \alpha \left[\delta y \left(S_t^- - S_{t-1}^+\right) - g L_{t-1}\right]^+, \\ & \alpha\,(z-g)\, L_{t-1} 1_{\left\{((1+a)(1+z)+\alpha(z-g))L_{t-1} \le S_t^- \le ((1+b)(1+z)+\alpha(z-g))L_{t-1}\right\}} \\ & + \frac{\alpha}{1+a+\alpha}\left[S_t^- - (1+g)\,(1+a)\, L_{t-1}\right] \\ & \qquad 1_{\left\{(1+a)(1+g)L_{t-1} < S_t^- < ((1+a)(1+z)+\alpha(z-g))L_{t-1}\right\}} \\ & + \frac{\alpha}{1+b+\alpha}\left[S_t^- - (1+g)\,(1+b)\, L_{t-1}\right] 1_{\left\{((1+b)(1+z)+\alpha(z-g))L_{t-1} < S_t^-\right\}}\ \}. \end{aligned} \tag{2.8}$$

This reserve corridor $[a, b]$ can be linked to the actual situation in practice: a lower bound for the bonus reserve is required for solvency reasons, an upper bound is justified by the need to stay competitive when markets take off.

Chapter 3

Risk-neutral valuation

In order to price life insurance contracts, we apply methods from modern financial mathematics, in particular the theory of risk-neutral valuation. Below, we give an introduction into the preliminaries of risk-neutral valuation and discuss the difference between financial securities and insurance contracts[1]. Subsequently, we summarize the valuation approach introduced in [6].

3.1 General aspects and principles of risk-neutral valuation

We assume that there exists a probability space $(\Xi, \mathcal{G}, \mathcal{P})$ equipped with the filtration $\mathbf{G} = (\mathcal{G}_t)_{t\in[0,T]}$, so that $\mathbf{G}$ satisfies the usual conditions, i.e. it is right-continuous and the initial information $\mathcal{G}_0$ contains all $\mathcal{P}$-null sets of $\mathbf{G}$ [2]. Let the state space Ξ describe all possible events in the financial market. Then the filtration $\mathbf{G} = (\mathcal{G}_t)_{t\in[0,T]}$ contains all information about the financial market at time t. We further assume that securities described by real stochastic processes exist, which are adapted to the filtration $\mathbf{G}$. In particular, we assume the existence of a numéraire process $B = (B_t)_{t\in[0,T]}$ [3] as well as the asset process $(S_t)_{t\in[0,T]}$ [4].

[1] For a comprehensive introduction into the theory of financial mathematics and especially risk-neutral pricing, see for example [8],[18],[32].

[2] We call $N \subset \Xi$ a $\mathcal{P}$-null set, if there is $B \in \mathbf{G}$ such that $N \subseteq B$ and $\mathcal{P}(B) = 0$.

[3] A numéraire process is a price process $(B_t)_{t\in[0,T]}$ which is strictly positive $\forall t \in [0,T]$. We assume that $B_0 = 1$.

[4] We assume that S is continuous.

Additionally, we assume that a measure $\tilde{\mathcal{Q}}$ equivalent to $\mathcal{P}$ exists, so that the discounted securities and particularly $\left(\frac{S_t}{B_t}\right)_{t\in[0,T]}$ are $\tilde{\mathcal{Q}}$-local martingales. The existence of the equivalent martingale measure implies that the financial market contains no arbitrage opportunities (see [8], page (p.) 112)[5].

If we can find a security portfolio as well as a self-financing trading strategy[6] so that the value of the portfolio equals the payoff of an arbitrary derivative at maturity, then the prices of the derivative and the portfolio must coincide in an arbitrage-free market. Thus, the "fair" or "risk-neutral" price of the derivative security equals the initial costs of the portfolio strategy when existing and, by the risk-neutral pricing formula, is given by the discounted expectation of the derivative's payoff under the equivalent martingale measure $\tilde{\mathcal{Q}}$ (see [8], p. 115).

We assume completeness of the financial market, i.e. we assume that all contingent claims are attainable, that is, for each contingent claim there exists an admissible trading strategy so that the cash flow of the claim can be duplicated with existing assets. According to the Fundamental Theorem of Asset Pricing, completeness of the market is equivalent to the uniqueness of the equivalent martingale measure (see [8], p. 116).

Unlike a financial derivative, an insurance contract does not merely depend on changes in the financial market but also on biometric events such as the survival of individuals. Let $(\Theta, \mathcal{H}, \mathcal{L}, \mathbf{H})$ be a stochastic basis, where Θ describes the state space of biometric events, $\mathbf{H} = (\mathcal{H})_{t\in[0,T]}$ the filtration, and $\mathcal{L}$ the associated probability measure. In order to model a market in which insurance contracts are traded, we have to construct an environment or, more precisely, a filtered probability space, in which the financial market and the market of biometric events are modelled simultaneously. We assume that there are large classes of similar individuals, i.e. individuals with same age, gender and health status. Furthermore, any two persons with similar biometric conditions must pay the same price for the same kind of contract and the biometric states of individuals are assumed to be independent (see [1]). Additionally, we assume that financial and mortality risks are uncorrelated, and that survival probabilities are known.

By taking these assumptions into account, the insurance company is asymptotically

[5]Note, that $\mathcal{P}$ denotes the so-called real-world measure, whereas $\tilde{\mathcal{Q}}$ denotes the risk-neutral probability measure, which is a martingale measure equivalent to $\mathcal{P}$ (EMM) .

[6]A trading strategy without withdrawals or deposits is called self-financing (see [8], p. 230).

risk-neutral with regard to biometric events (see [1])[7], which enables us to derive prices for insurance contracts.

Let us define the common filtered probability space

$$(\Omega, \mathcal{F}, \mathcal{Q}, \mathbf{F}) = (\Xi, \mathcal{G}, \tilde{\mathcal{Q}}, \mathbf{G}) \otimes (\Theta, \mathcal{H}, \mathcal{L}, \mathbf{H}) \tag{3.1}$$

as the product space of financial and biometric events. $\Omega = \Xi \times \Theta$ denotes the state space of financial and biometric events, $\mathcal{F}_t = \mathcal{G}_t \otimes \mathcal{H}_t$ the information about the insurance market up to time t, and $\mathcal{Q} = \tilde{\mathcal{Q}} \otimes \mathcal{L}$ the product measure.
According to [1], the measure $\mathcal{Q}$ is again a martingale measure on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{Q}, \mathbf{F})$ under the assumptions above.

Let X be an insurance benefit, i.e. an $\mathcal{F}$-measurable payoff, depending on both the financial market and biometric events. The risk-neutral valuation formula (see [8], p. 250) then states that the arbitrage price process $\left(P_t^X\right)_{t\in[0,T]}$ of X is given by

$$P_t^X \;=\; B_t \mathbb{E}_{\mathcal{Q}}\left[X B_T^{-1} \mid \mathcal{F}_t\right]. \tag{3.2}$$

Even if risk-neutrality of the insurer with respect to biometric events is not assumed, there are still reasons for employing this measure for valuation purposes (see [15], [30]).

3.2 Special aspects and the model of Bauer

In Chapter 2 we introduced the considered insurance contract. With this particular contract, the policyholder receives the benefit in any case, that is, if he dies within the lifetime of the contract or survives until the maturity date; the evolution of the contract is independent of biometric events. Hence, we do not have to distinguish between the equivalent martingale measure for the financial market $\tilde{\mathcal{Q}}$ and the equivalent martingale measure $\mathcal{Q}$ for the product space. Assuming that the policyholder receives a payoff of $P\,\frac{L_T}{L_0}$ at maturity T of the contract, where P denotes the nominal value of the single premium and L_t the policyholder's account balance at time t, the risk-neutral price of the insurance benefit (without a surrender option) is given by

$$P^* \;=\; \mathbb{E}_{\mathcal{Q}}\left[B_T^{-1} P \frac{L_T}{L_0}\right] \overset{L_0=P}{=} \mathbb{E}_{\mathcal{Q}}\left[B_T^{-1} L_T\right], \tag{3.3}$$

[7] We are disregarding systematic mortality risk; including stochastic mortalities could be one of the next steps.

where B denotes the price process of the numéraire.

However, this price is only adequate if the insurance company takes the results of risk-neutral valuation into account and applies hedging strategies, which presents a difficulty in this model: in contrast to unit-linked products, the underlying security in our case is not traded on the financial market since it depends on the whole asset side of the insurance company. It is an asset portfolio composed due to management decisions within the insurance company. Nevertheless, it is possible to approximate the insurer's asset portfolio by a benchmark portfolio, i.e. a synthetic portfolio consisting of various instruments which are actually traded on the market. This portfolio could be employed for the risk-neutral valuation approach[8]. However, since the underlying is the company's asset side which is changed when changing the asset allocation, to hedge inside the company's balance sheet (cp. [6]) is not possible for the insurer.

To overcome this "feedback effect", we choose a different approach and assume that the insurer invests his money in the reference portfolio S and leaves it there. The reference portfolio is not modified, except for the occurrence of one of the following two cases:

(i) If dividends are paid out to the shareholders, these payments leave the insurance company and thereby reduce the value of the reference portfolio. However, they do not change the composition of the portfolio.

(ii) If the return of the reference portfolio is so poor that the interest rate guarantee cannot be fulfilled with the company's funds even if the reserve is dissolved completely, then the insurance company requires capital to fulfil its obligations. We assume that the company receives a capital shot c_t at time t, which thus increases the value of the reference portfolio but again does not change its composition. We assume that these capital shots come from an investor who is not restricted in its asset management decisions.

Equation (3.3) can be used to calculate the value of the contract as a whole, but it is not possible to valuate implicit options, such as for example the interest rate guarantee. Isolating the embedded options is particularly important for securitization purposes: a financial intermediator would hedge the options rather than the contract as a whole.

Below, we give an outline of how the cash-flows are priced. For a detailed description see [6].

[8]In the following we call the the synthetic portfolio the reference portfolio S.

Let c_t be the capital shots the insurance company receives at time t, $1 \leq t \leq T$. Then the risk-neutral value at time 0 of the capital shots is given by

$$\mathrm{C}_0 \;=\; \mathbb{E}_{\mathcal{Q}} \left[\sum_{t=1}^{T} B_t^{-1} c_t \right], \tag{3.4}$$

and C_0 can be interpreted as the value of the interest rate guarantee. Similarly, the risk-neutral value at $t = 0$ of the dividend payments is

$$\mathrm{D}_0 \;=\; \mathbb{E}_{\mathcal{Q}} \left[\sum_{t=1}^{T} B_t^{-1} d_t \right], \tag{3.5}$$

where d_t is the dividend paid to the shareholders at time t.

Furthermore, the risk-neutral value of the change of reserve is given as

$$\Delta R_0 \;=\; \mathbb{E}_{\mathcal{Q}} \left[B_T^{-1} R_T \right] - R_0, \tag{3.6}$$

where R_t is the reserve account at time t.

For a "fair" contract, the value of the interest rate guarantee C_0 should equal the sum of the value of the dividend payments D_0 and the change of reserve ΔR_0, i.e.

$$\mathrm{C}_0 \stackrel{!}{=} \mathrm{D}_0 + \Delta R_0, \tag{3.7}$$

since if, on the one hand,

$$\mathrm{C}_0 \;>\; \mathrm{D}_0 + \Delta R_0,$$

then the contract will be of greater value for the policyholder, because the interest rate guarantee will be more valuable than his losses. If, on the other hand,

$$\mathrm{C}_0 \;<\; \mathrm{D}_0 + \Delta R_0,$$

the contract would bring a financial disadvantage for the policyholder, since the interest rate guarantee would be of less value than the dividend payments plus undistributed final reserves exceeding the initial reserves.

Hence, (3.7) represents an equilibrium condition for a fair contract. Taking into account that for the value of the contract we have

$$P^* \;=\; \mathbb{E}_{\mathcal{Q}} \left[B_T^{-1} P \frac{L_T}{L_0} \right] \;=\; P + \mathrm{C}_0 - \mathrm{D}_0 - \Delta R_0, \tag{3.8}$$

the equilibrium condition (3.7) is equivalent to

$$P^* \stackrel{!}{=} P. \tag{3.9}$$

In the following we can thus use (3.3) to calculate the fair value of the insurance contract and (3.4) - (3.6) to calculate implicit options for both, MUST-case and IS-case.

As mentioned earlier, the policyholder has the possibility to surrender his contract at any discrete time point $t_0 \in \{0, 1, \ldots, T\}$ and "walk away" with the account value L_{t_0} [9]. Although policyholders often surrender the contracts because of personal rather than financial reasons, we have to make an assumption about the surrender behavior of the customer. To stay on the safe side, we assume that he surrenders the contract when it is profitable for him financially.

If a policyholder surrenders his contract at time t_0, he receives his time t_0 account value L_{t_0}. The remaining reserves and the remaining value of the capital shots C_0 that were required to ensure the minimum guaranteed rate of interest remain with the insurance company.

Using the same notation as in [7], the policyholder's gain from surrendering at time t_0 is

$$w_{t_0} = \max\left\{\mathrm{D}_{t_0} + B_{t_0}\mathbb{E}_{\mathcal{Q}}\left[B_T^{-1}R_T \mid \mathcal{F}_{t_0}\right] - R_{t_0} - \mathrm{C}_{t_0}, 0\right\},$$

where D_{t_0} denotes the value of dividend payments in $[t_0, T]$ at t_0 and C_{t_0} the value of future capital shots at t_0. Hence, the value of the surrender or walk-away option at $t = 0$ is given by

$$W_0 = \sup_{\tau \in \Upsilon_{[0,T]}} \mathbb{E}_{\mathcal{Q}}\left[B_\tau^{-1} w_\tau\right], \tag{3.10}$$

where $\Upsilon_{[0,T]}$ denotes all stopping times with values in $\{0, 1, \ldots, T\}$.

Thus, the equilibrium condition (3.7) becomes

$$\mathrm{C}_0 + W_0 \stackrel{!}{=} \mathrm{D}_0 + \Delta R_0, \tag{3.11}$$

which again has the equivalent representation

$$\begin{aligned} P^* = \mathbb{E}_{\mathcal{Q}}\left[B_T^{-1} P \frac{L_T}{L_0}\right] + W_0 &= P + \mathrm{C}_0 - \mathrm{D}_0 - \Delta R_0 + W_0 \\ &\stackrel{!}{=} P. \end{aligned} \tag{3.12}$$

Hence, the value of the contract can be calculated in two different ways: directly as a discounted expectation or by summing up the implicit options. Note, that if a walk-away option is included in the insurance contract, the insurance company not only has to finance the interest rate guarantee at the beginning of the contract, but also the surrender option.

[9] Surrendering at $t = 0$ is equivalent to not concluding the contract.

Chapter 4

Stochastic interest rates

Many valuation and pricing models for life insurance contracts assume deterministic or even constant interest rates. However, due to long contract periods in life insurance business this assumption is not adequate. It is more realistic to assume that interest rates can change over the lifetime of the contract. Therefore, it is of great interest to obtain suitable models describing the dynamics of interest rates.

Here, we focus on the class of single-factor time-homogeneous models for the short rate process r of the form

$$\mathrm{d}r_t = \kappa\,(\xi - r_t)\,\mathrm{d}t + \sigma_r r_t^{\gamma}\mathrm{d}W_t, \tag{4.1}$$

where κ, ξ, σ_r, and γ are constants, and $(W_t)_{t\in[0,T]}$ is a one-dimensional standard Brownian motion under the equivalent risk-neutral probability measure $\tilde{\mathcal{Q}}$ on the complete filtered probability space $(\,\Xi\,,\mathcal{G}\,,\tilde{\mathcal{Q}}\,,\mathbf{G})$.

From this general setting we focus on two different specific models: the Ornstein-Uhlenbeck model for $\gamma = 0$ and the Cox-Ingersoll-Ross model for $\gamma = \frac{1}{2}$.

4.1 The Ornstein-Uhlenbeck model

Setting $\gamma = 0$ in (4.1), we obtain the Ornstein-Uhlenbeck model (cp. [8], p. 340). Here, the dynamics of the short term interest rate r are given by the stochastic differential equation (SDE)

$$\mathrm{d}r_t = \kappa\,(\xi - r_t)\,\mathrm{d}t + \sigma_r\mathrm{d}W_t, \tag{4.2}$$

where $\sigma_r > 0$ denotes the volatility of the process r_t, ξ can be interpreted as a long term equilibrium value for the interest rate, and κ determines the speed at

which r_t is pulled towards ξ (reversion rate). Therefore, the considered Ornstein-Uhlenbeck model describes the time evolution of the risk-free interest rate r via a mean reverting process, where the mean reversion is affected by the size of κ and the interest rate is reverted toward ξ.
It is usually assumed that κ and ξ are strictly positive constants.

In order to solve the SDE in (4.2), we let $f(t,x) := e^{\kappa t}x$.
Then, we have

$$f_t = \kappa f, \quad f_x = e^{\kappa t}, \text{ and } f_{xx} = 0.$$

Applying Itô's formula (see [8], p. 194) with $x = r_t$ gives

$$\begin{aligned} \mathrm{d}f\left(t, r_t\right) &= f_t\left(t, r_t\right)\,\mathrm{d}t + f_x\left(t, r_t\right)\,\mathrm{d}r_t \\ &= \kappa e^{\kappa t} r_t\,\mathrm{d}t + e^{\kappa t}\,\mathrm{d}r_t \\ &= e^{\kappa t}\left(\kappa\xi\,\mathrm{d}t + \sigma_r\,\mathrm{d}W_t\right). \end{aligned}$$

Hence,

$$e^{\kappa t} r_t = r_0 + \kappa\xi \int_0^t e^{\kappa s}\,\mathrm{d}s + \sigma_r \int_0^t e^{\kappa s}\,\mathrm{d}W_s,$$

and we obtain that

$$r_t = e^{-\kappa t} r_0 + \xi\left(1 - e^{-\kappa t}\right) + \int_0^t \sigma_r e^{-\kappa(t-s)}\,\mathrm{d}W_s \tag{4.3}$$

is the solution of the SDE in (4.2).

It is easy to see that

$$r_t | r_u \sim N\left(e^{-\kappa(t-u)}\left(r_u - \xi\right) + \xi\ ,\ \frac{\sigma_r^2}{2\kappa}\left(1 - e^{-2\kappa(t-u)}\right)\right), \quad u < t. \tag{4.4}$$

Since the interest rates are normally distributed, they can become negative with positive probability, which may limit the applicability of the model.

4.2 The Cox-Ingersoll-Ross model

Setting $\gamma = \frac{1}{2}$ in (4.1), we obtain the Cox-Ingersoll-Ross model. It was proposed by Cox, Ingersoll, and Ross (cp. [10]) and leads to the following modification of the mean reverting process of Ornstein-Uhlenbeck, known as the *square root process*:

$$\mathrm{d}r_t = \kappa\left(\xi - r_t\right)\,\mathrm{d}t + \sigma_r\sqrt{r_t}\,\mathrm{d}W_t, \tag{4.5}$$

where the parameters κ, ξ, and σ_r are again assumed to be strictly positive. They can be similarly interpreted as in (4.2). $(W_t)_{t\in[0,T]}$ denotes again a one-dimensional standard Brownian motion under $\tilde{\mathcal{Q}}$.
Due to the presence of the square root in the term $\sigma_r\sqrt{r_t}$, the process only takes positive values. It can reach zero if $2\kappa\xi < \sigma_r^2$, yet never becomes negative. If $2\kappa\xi \geq \sigma_r^2$, then r_t remains strictly positive for all t (see [14]).

As in the Ornstein-Uhlenbeck model, the form of the drift in (4.5) suggests that r_t is pulled towards a long-term value ξ at a rate controlled by κ. In contrast to the Ornstein-Uhlenbeck model, where negative interest rates are possible, the term $\sigma_r\sqrt{r_t}$ decreases to zero as r_t approaches the origin and thus negative interest rates are excluded. Furthermore, the absolute variance of the interest rate increases as the interest rate itself increases. Due to these more realistic properties, the CIR model generally presents better features than the OU model when modelling interest rates.

The solution of (4.5) cannot be written in an explicit form like (4.3), but we can write the SDE as

$$r_t = e^{-\kappa t}r_0 + \xi\left(1 - e^{-\kappa t}\right) + \int_0^t \sigma_r e^{-\kappa(t-s)}\sqrt{r_s}\,\mathrm{d}W_s \tag{4.6}$$

by applying Itô's formula.

4.2.1 Distribution of r in the Cox-Ingersoll-Ross model

Although we cannot represent the solution of the SDE (4.5) in an explicit form, it can be shown (see [18]) that r_t given r_u for some $u < t$ is up to a scale factor noncentral chi-square distributed:

By letting

$$\begin{cases} c(u,t) &= \dfrac{2\kappa}{\sigma_r^2\left(1 - e^{-\kappa(t-u)}\right)} \\ d &= \dfrac{4\kappa\xi}{\sigma_r^2} \\ q(u,t) &= 2c(u,t)\, r_u\, e^{-\kappa(t-u)},\ u < t, \end{cases}$$

(4.5) can be expressed as

$$r_t = \frac{1}{2c(u,t)}\chi^2_{d,\,q(u,t)}, \quad u < t, \tag{4.7}$$

where $\chi^2_{d,\,q(u,t)}$ is a chi-square random variable with d degrees of freedom and non-centrality parameter $q(u,t)$.

Thus, r_t given r_u is distributed as $\frac{1}{2c(u,t)}$ times a noncentral chi-square random variable with d degrees of freedom and non-centrality parameter $q(u,t)$.

Chapter 5

Evaluation approach for a simple asset model

In what follows we apply the general model from Chapter 2 to obtain numerical results. At first, we introduce a specific stochastic model for the asset process. Then the two stochastic short rate models introduced in Chapter 4 are embedded in the financial market model.

The considered contracts and options are very complex, path-dependent derivatives. Since considering stochastic interest rates further complicates the existing valuation, it is not possible to present analytical solutions for the risk-neutral value of the contract. Hence, we have to rely on numerical methods for the evaluation: we introduce an "exact"[1] Monte-Carlo approach as well as a "discretized" Monte Carlo approach, which enable us to numerically calculate the value of European contracts and the implicit options. Subsequently, we introduce a discrete lattice approach which allows us to evaluate both European and non-European contracts.

Finally, we discuss the imperfections of the chosen asset model.

5.1 Financial market

In the following we always assume that investors can trade continuously in a complete, frictionless, arbitrage-free financial market with finite time horizon T. We suppose that the single premium, which is paid at the conclusion of the contract, is invested in a diversified reference portfolio S consisting of various asset classes.

[1] "Exact" in the sense that we can exactly determine and thus simulate the distribution of the involved random variables.

To determine the fair value of the insurance contract, the valuation has to be carried out under the risk-neutral probability measure $\tilde{\mathcal{Q}}$. Therefore, we let $(W_t)_{t\in[0,T]}$ and $(Z_t)_{t\in[0,T]}$ be two independent standard Brownian motions under $\tilde{\mathcal{Q}}$ on the complete, filtered probability space $(\Xi, \mathcal{G}, \tilde{\mathcal{Q}}, \mathbf{G})$.

We consider a stochastic process for the risk-free interest rate r and assume that disregarding the dividend payments the process S follows a geometric Brownian motion, i.e. it evolves according to the following stochastic differential equation under $\tilde{\mathcal{Q}}$[2]:

$$\frac{\mathrm{d}S_t}{S_t} = r_t\,\mathrm{d}t + \sigma_S\left[\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}Z_t\right], \qquad S_0 > 0, \tag{5.1}$$

where $\sigma_S > 0$ denotes the volatility of the asset process S and $\rho \in [0,1]$ characterizes the correlation between S and the risk-free rate r.

Later on we will see that the volatility of the interest rate process, the volatility of the asset process, and the correlation between interest rate process and asset process have a tremendous influence on the risk-neutral value of the insurance contract.

In order to solve the SDE in (5.1) we let

$$X_t := \begin{pmatrix} W_t \\ Z_t \end{pmatrix},$$

and

$$f(t,x) = f(t,x_1,x_2) := \exp\left(\int_0^t r_s\,\mathrm{d}s - \frac{\sigma_S^2 t}{2} + \rho\sigma_S x_1 + \sqrt{1-\rho^2}\sigma_S x_2\right).$$

Then we obtain with $x_1 = W_t$, $x_2 = Z_t$, and by applying Itô's formula (see [8], p. 194) that

[2]Note however, that we do not have to distinguish between the measures $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ (see Section 3.2).

$$\begin{aligned}
\mathrm{d}f\left(t, X_t\right) &= \frac{\partial f\left(t, X_t\right)}{\partial t}\,\mathrm{d}t + \frac{\partial f\left(t, X_t\right)}{\partial x_1}\,\mathrm{d}W_t + \frac{\partial f\left(t, X_t\right)}{\partial x_2}\,\mathrm{d}Z_t \\
&\quad + \frac{1}{2}\left(\frac{\partial^2 f\left(t, X_t\right)}{\partial x_1^2} + \frac{\partial^2 f\left(t, X_t\right)}{\partial x_2^2}\right)\mathrm{d}t \\
&= f\left(t, X_t\right)\left[r_t\,\mathrm{d}t - \frac{\sigma_S^2}{2}\,\mathrm{d}t + \sigma_S\rho\,\mathrm{d}W_t + \sigma_S\sqrt{1-\rho^2}\,\mathrm{d}W_t\right] \\
&\quad + \frac{1}{2}f\left(t, X_t\right)\left[\sigma_S^2\rho^2 + \sigma_S^2\left(1-\rho^2\right)\right]\,\mathrm{d}t \\
&= f\left(t, X_t\right)\left[r_t\,\mathrm{d}t + \sigma_S\rho\,\mathrm{d}W_t + \sigma_S\sqrt{1-\rho^2}\,\mathrm{d}Z_t\right].
\end{aligned}$$

Hence, the asset price dynamics as given in (5.1) can be integrated as

$$S_t = S_0 \exp\left(\int_0^t r_s\,\mathrm{d}s - \frac{\sigma_S^2 t}{2} + \rho\sigma_S W_t + \sqrt{1-\rho^2}\sigma_S Z_t\right). \tag{5.2}$$

If at time $t \in \{1, 2, \ldots, T\}$ an amount d_t is paid out to the shareholders, then S_t^- and S_t^+ denote the value of the asset immediately prior to the payment and immediately after the payment of the dividends d_t respectively, where at all times at least the liability towards the insured person has to be fulfilled.

Thus, equation (5.2) implies that

$$S_t^- = S_{t-1}^+ \exp\left(\int_{t-1}^t r_s\,\mathrm{d}s - \frac{\sigma_S^2}{2} + \int_{t-1}^t \rho\sigma_S\,\mathrm{d}W_s + \int_{t-1}^t \sqrt{1-\rho^2}\sigma_S\,\mathrm{d}Z_s\right), \tag{5.3}$$

and

$$S_t^+ = \max\left\{S_t^- - d_t, L_t\right\}. \tag{5.4}$$

To be able to apply the methods from modern financial mathematics, we furthermore assume the existence of a money market account in the economy, i.e. a financial asset with no instantaneous risk. The value of this asset, B_t, grows at the instantaneous risk-free rate r_t according to the following stochastic differential equation:

$$\mathrm{d}B_t = r_t B_t\,\mathrm{d}t \quad \text{with initial value} \quad B_0 = 1. \tag{5.5}$$

Hence, the value at time t of an initial investment B_0 that is continuously reinvested is given by

$$B_t = B_0 \exp\left\{\int_0^t r_s\,\mathrm{d}s\right\}.$$

5.2 Valuation using Monte Carlo simulation

We can use the results of the previous section together with the results of Chapter 2 to determine the fair prices of the contract and of the implicit options.

Let $r_t^S := \dfrac{S_t^- - S_{t-1}^+}{S_{t-1}^+}$ denote the rate of return of the asset portfolio S in the time period $[t-1,t)$.[3]

From (5.3) we obtain that

$$r_t^S = \exp\left(\int_{t-1}^t r_s\,\mathrm{d}s - \frac{\sigma_S^2}{2} + \int_{t-1}^t \rho\sigma_S\,\mathrm{d}W_s + \int_{t-1}^t \sqrt{1-\rho^2}\sigma_S\,\mathrm{d}Z_s\right) - 1. \quad (5.6)$$

Let furthermore $x_t := \dfrac{R_t}{L_t}$ denote the reserve quota as it was introduced in Chapter 2. Then the following relations hold in the MUST - case, $1 \le t \le T$ (see [6]):

$$\begin{aligned}
L_t &= \left(1 + \max\left\{\delta y r_t^S\left(1+x_{t-1}\right), g\right\}\right) L_{t-1},\\[2ex]
d_t &= (1-\delta) y S_{t-1}^+ r_t^S 1_{\{\delta y r_t^S (1+x_{t-1}) > g\}}\\
&\quad + \left(y r_t^S\left(1+x_{t-1}\right) - g\right) L_{t-1} 1_{\{\delta y r_t^S (1+x_{t-1}) \le g \le y r_t^S (1+x_{t-1})\}},\\[2ex]
S_t^+ &= S_{t-1}^+\left(1 + r_t^S\left(1-y+\delta y\right)\right) 1_{\{\delta y r_t^S (1+x_{t-1}) > g\}}\\
&\quad + \left(S_{t-1}^+ + L_{t-1}\left(r_t^S\left(1+x_{t-1}\right)(1-y) + g\right)\right) 1_{\{\delta y r_t^S (1+x_{t-1}) \le g \le y r_t^S (1+x_{t-1})\}}\\
&\quad + S_{t-1}^+\left(1+r_t^S\right) 1_{\{y r_t^S (1+x_{t-1}) \le g \le r_t^S + x_{t-1}\left(1+r_t^S\right)\}}\\
&\quad + (1+g) L_{t-1} 1_{\{g > r_t^S + x_{t-1}\left(1+r_t^S\right)\}},\\[2ex]
R_t &= S_t^+ - L_t\\
&= \left[R_{t-1} + S_{t-1}^+ (1-y) r_t^S\right] 1_{\{g \le y r_t^S (1+x_{t-1})\}}\\
&\quad + \left[R_{t-1} + \left(r_t^S\left(1+x_{t-1}\right) - g\right) L_{t-1}\right] 1_{\{y r_t^S (1+x_{t-1}) \le g \le r_t^S + x_{t-1}\left(1+r_t^S\right)\}},\\[2ex]
x_t &= \frac{R_t}{L_t}\\
&= \frac{x_{t-1} + (1+x_{t-1})(1-y) r_t^S}{1 + \delta y r_t^S (1+x_{t-1})} 1_{\{\delta y r_t^S (1+x_{t-1}) > g\}}\\
&\quad + \frac{x_{t-1} + (1+x_{t-1})(1-y) r_t^S}{1+g} 1_{\{\delta y r_t^S (1+x_{t-1}) \le g \le y r_t^S (1+x_{t-1})\}}\\
&\quad + \frac{x_{t-1} + (1+x_{t-1}) r_t^S - g}{1+g} 1_{\{y r_t^S (1+x_{t-1}) \le g \le r_t^S + x_{t-1}\left(1+r_t^S\right)\}}. \quad (5.7)
\end{aligned}$$

[3] Note that r_t denotes the risk-free interest rate at time t, whereas r_t^S denotes the rate of return of the asset portfolio in the period $[t-1,t)$.

In the IS - case we obtain the following for $1 \leq t \leq T$ (see [6]):

$$\begin{aligned}
S_t^- &= S_{t-1}^+ \left(1 + r_t^S\right), \\
L_t &= \Bigg(1 + \max\Big\{g, \delta yr_t^S \left(1 + x_{t-1}\right), \\
&\qquad z 1_{\left\{(1+a)(1+z)+\alpha(z-g) \leq (1+x_{t-1})\left(1+r_t^S\right) \leq (1+b)(1+z)+\alpha(z-g)\right\}} \\
&\qquad + \frac{\left(1 + r_t^S\right)\left(1 + x_{t-1}\right) - 1 - a + g\alpha}{1 + a + \alpha} \\
&\qquad\quad \cdot 1_{\left\{(1+a)(1+g) < (1+x_{t-1})\left(1+r_t^S\right) < (1+a)(1+z)+\alpha(z-g)\right\}} \\
&\qquad + \frac{\left(1 + r_t^S\right)\left(1 + x_{t-1}\right) - 1 - b + g\alpha}{1 + b + \alpha} \\
&\qquad\quad \cdot 1_{\left\{(1+b)(1+z)+\alpha(z-g) < (1+x_{t-1})\left(1+r_t^S\right)\right\}}\Big\}\Bigg) L_{t-1}
\end{aligned}$$

$$\begin{aligned}
d_t &= \alpha\left(L_t - (1+g)\, L_{t-1}\right), \\
S_t^+ &= \max\left\{L_t, S_t^- - d_t\right\}, \\
R_t &= S_t^+ - L_t, \\
x_t &= \frac{R_t}{L_t}.
\end{aligned} \tag{5.8}$$

These relations are used to determine the value of the contract. For example, we obtain in the MUST - case:

$$\begin{aligned}
L_T &= \left(1 + \max\left\{\delta yr_T^S \left(1 + x_{T-1}\right), g\right\}\right) L_{T-1} \\
&= \left(1 + \max\left\{\delta yr_T^S \left(1 + x_{T-1}\right), g\right\}\right) \\
&\quad \left(1 + \max\left\{\delta yr_{T-1}^S \left(1 + x_{T-2}\right), g\right\}\right) L_{T-2} \\
&= \ldots \\
&= \left(1 + \max\left\{\delta yr_T^S \left(1 + x_{T-1}\right), g\right\}\right) \\
&\quad \left(1 + \max\left\{\delta yr_{T-1}^S \left(1 + x_{T-2}\right), g\right\}\right) \\
&\quad \left(1 + \max\left\{\delta yr_{T-2}^S \left(1 + x_{T-3}\right), g\right\}\right) \\
&\qquad \vdots \\
&\quad \left(1 + \max\left\{\delta yr_2^S \left(1 + x_1\right), g\right\}\right) \\
&\quad \left(1 + \max\left\{\delta yr_1^S \left(1 + x_0\right), g\right\}\right) L_0 \\
&= L_0 \prod_{k=0}^{T-1} \left(1 + \max\left\{\delta yr_{k+1}^S \left(1 + x_k\right), g\right\}\right),
\end{aligned} \tag{5.9}$$

and the risk-neutral price of the insurance contract is given by

$$P^* \;=\; \mathbb{E}_{\tilde{\mathcal{Q}}}\left[B_T^{-1} L_T\right].$$

In order to calculate the discounted expected value of L_T under $\tilde{\mathcal{Q}}$, we carry out Monte Carlo simulations, i.e. suitable values for an uncertain variable are randomly generated over and over again to simulate a model[4]. The strong law of large numbers ensures that the unbiased estimator of the uncertain variable converges to the correct value as the number of generated values increases (cp. [18]). The method is useful for obtaining numerical solutions to adequate problems which are too complex to solve analytically.

5.2.1 "Exact" Monte Carlo approach

In this subsection we use the knowledge of the distributions of the involved processes, in particular the distribution of the short rate process r, to calculate the fair value of the insurance contract via an "exact" Monte Carlo algorithm. Here, we assume that r follows an Ornstein-Uhlenbeck process.

For the Monte Carlo simulation we require the rate of return r_t^S of the asset portfolio in each time period $[t-1, t)$.
From (5.6) we have that

$$\begin{aligned} r_t^S \;&=\; \frac{S_t^- - S_{t-1}^+}{S_{t-1}^+} \\ &=\; \exp\left\{\int_{t-1}^t r_s \,\mathrm{d}s - \frac{\sigma_S^2}{2} + \rho\sigma_S \int_{t-1}^t \mathrm{d}W_s + \sqrt{1-\rho^2}\sigma_S \int_{t-1}^t \mathrm{d}Z_s\right\} - 1. \end{aligned}$$

We recall from (4.3) that in the Ornstein-Uhlenbeck model the risk-free rate r is given as

$$r_t \;=\; e^{-\kappa t} r_0 + \xi\left(1 - e^{-\kappa t}\right) + \int_0^t \sigma_r e^{-\kappa(t-s)} \,\mathrm{d}W_s,$$

and from (4.4) that

$$r_t | r_u \sim N\left(e^{-\kappa(t-u)}\left(r_u - \xi\right) + \xi\,,\; \frac{\sigma_r^2}{2\kappa}\left(1 - e^{-2\kappa(t-u)}\right)\right), \quad u < t.$$

[4]The risk-neutral values of the other cash flows are determined analogously.

Furthermore,

$$\begin{aligned}\int_{t-1}^{t} r_u \,\mathrm{d}u &= \int_{t-1}^{t} \left[r_{t-1}\, e^{-\kappa(u-t+1)} + \xi \left(1 - e^{-\kappa(u-t+1)}\right) + \int_{t-1}^{u} \sigma_r\, e^{-\kappa(u-s)} \,\mathrm{d}W_s \right] \mathrm{d}u \\ &= r_{t-1} \int_{t-1}^{t} e^{-\kappa(u-t+1)} \,\mathrm{d}u + \xi \int_{t-1}^{t} \left(1 - e^{-\kappa(u-t+1)}\right) \mathrm{d}u \\ &\quad + \int_{t-1}^{t} \int_{t-1}^{u} \sigma_r e^{-\kappa(u-s)} \,\mathrm{d}W_s \,\mathrm{d}u,\end{aligned}$$

and hence, by applying a version of Fubini's theorem for stochastic integrals, we have

$$\int_{t-1}^{t} r_u \,\mathrm{d}u = \frac{(r_{t-1} - \xi)}{\kappa} \left(1 - e^{-\kappa}\right) + \xi + \frac{\sigma_r}{\kappa} \int_{t-1}^{t} \left(1 - e^{-\kappa(t-s)}\right) \mathrm{d}W_s.$$

Thus, the distribution of both r_t and $\int_{t-1}^{t} r_u \,\mathrm{d}u$ are explicitly known under r_{t-1}.

To simulate r_t^S we have to generate the following[5]:

- $r_t = e^{-\kappa}\, r_{t-1} + \xi \left(1 - e^{-\kappa}\right) + \underbrace{\int_{t-1}^{t} \sigma_r\, e^{-\kappa(t-s)} \,\mathrm{d}W_s}_{:=X_1},$
- $\int_{t-1}^{t} r_s \,\mathrm{d}s = \frac{(r_{t-1} - \xi)}{\kappa} \left(1 - e^{-\kappa}\right) + \xi + \underbrace{\frac{\sigma_r}{\kappa} \int_{t-1}^{t} \left(1 - e^{-\kappa(t-s)}\right) \mathrm{d}W_s}_{:=X_2},$
- $\underbrace{\rho\, \sigma_S \left(W_t - W_{t-1}\right)}_{:=X_3}$, and
- $\underbrace{\sqrt{1-\rho^2}\, \sigma_S \left(Z_t - Z_{t-1}\right)}_{:=X_4}.$ (5.10)

The simulation involves the four Gaussian processes X_1, X_2, X_3, and X_4, where X_4 is independent of $X_1, X_2,$ and X_3. The covariance matrix of X_1, X_2, and X_3 is given by

$$K = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

[5]Note that $(W_t)_{t\in[0,T]}$ and $(Z_t)_{t\in[0,T]}$ are two independent standard Brownian motions under the risk-neutral probability measure $\tilde{\mathcal{Q}}$.

with

$$\begin{aligned} A_{ii} &= \mathrm{Var}(X_i),\ 1 \le i \le 3, \\ A_{ij} &= \mathrm{Cov}(X_i, X_j),\ 1 \le i, j \le 3,\ i \ne j. \end{aligned}$$

Using the properties of Gaussian processes we have

$$\begin{aligned} A_{11} &= \mathrm{Var}(X_1) = \sigma_r^2 \int_{t-1}^{t} e^{-2\kappa(t-s)}\,\mathrm{d}s = \frac{\sigma_r^2}{2\kappa}\left(1 - e^{-2\kappa}\right), \\ A_{22} &= \mathrm{Var}(X_2) = \frac{\sigma_r^2}{\kappa^2} \int_{t-1}^{t} \left(1 - e^{-\kappa(t-s)}\right)^2 \mathrm{d}s = \frac{\sigma_r^2\left(2\kappa - 3 + 4e^{-\kappa} - e^{-2\kappa}\right)}{2\kappa^3}, \\ A_{33} &= \mathrm{Var}(X_3) = \rho^2\sigma_S^2, \\ A_{12} &= \mathrm{Cov}(X_1, X_2) = \frac{\sigma_r^2}{\kappa} \int_{t-1}^{t} \left(e^{-\kappa(t-s)} - e^{-2\kappa(t-s)}\right) \mathrm{d}s = \frac{\sigma_r^2}{2\kappa^2}\left(1 - e^{-\kappa}\right)^2, \\ &= \mathrm{Cov}(X_2, X_1) = A_{21}, \\ A_{13} &= \mathrm{Cov}(X_1, X_3) = \rho\sigma_r\sigma_S \int_{t-1}^{t} e^{-\kappa(t-s)}\,\mathrm{d}s = \frac{\rho\sigma_r\sigma_S}{\kappa}\left(1 - e^{-\kappa}\right), \\ &= \mathrm{Cov}(X_3, X_1) = A_{31}, \text{ and} \\ A_{23} &= \mathrm{Cov}(X_2, X_3) = \frac{\rho\sigma_r\sigma_S}{\kappa} \int_{t-1}^{t} \left(1 - e^{-\kappa(t-s)}\right) \mathrm{d}s = \frac{\rho\sigma_r\sigma_S}{\kappa}\left(1 - \frac{1}{\kappa}\left(1 - e^{-\kappa}\right)\right), \\ &= \mathrm{Cov}(X_3, X_2) = A_{32}. \end{aligned}$$

Therefore, using the covariances between X_1, X_2, and X_3 we obtain that

$$\begin{aligned} &\bullet\ X_1 = \sqrt{A_{11}}\left(\tilde{\mu}N_1 + \tilde{\lambda}N_2 + \tilde{\rho}N_3\right), \\ &\bullet\ X_2 = \sqrt{A_{22}}\left(\mu N_1 + \lambda N_2\right), \text{ and} \\ &\bullet\ X_3 = \sqrt{A_{33}}\, N_1, \end{aligned} \tag{5.11}$$

with $N_1, N_2, N_3 \sim N(0,1)$ iid and

$$\begin{aligned} \tilde{\mu} &= \frac{A_{31}}{\sqrt{A_{11}A_{33}}}, \\ \tilde{\lambda} &= \frac{A_{12} - \frac{A_{31}A_{32}}{A_{33}}}{\sqrt{A_{11}A_{22}\left(1 - \frac{A_{32}^2}{A_{22}A_{33}}\right)}}, \\ \tilde{\rho} &= \sqrt{1 - \tilde{\mu}^2 - \tilde{\lambda}^2}, \\ \mu &= \frac{A_{32}}{\sqrt{A_{22}A_{33}}}, \text{ and} \\ \lambda &= \sqrt{1 - \mu^2}. \end{aligned}$$

Then the risk-neutral value of the contract is calculated using the algorithm presented in Table 5.1.

"Exact" Monte Carlo simulation

For $k = 1, 2, \ldots, N$ (N =number of simulations):

For $t = 1, 2, \ldots, T$:

1. Generate $N_{t1}^{(k)}, N_{t2}^{(k)}, N_{t3}^{(k)}, N_{t4}^{(k)} \sim N(0,1)$ iid.

2. Calculate
$$\begin{aligned} X_{t1}^{(k)} &= \sqrt{A_{11}} \left(\tilde{\mu} N_{t1}^{(k)} + \tilde{\lambda} N_{t2}^{(k)} + \tilde{\rho} N_{t3}^{(k)} \right), \\ X_{t2}^{(k)} &= \sqrt{A_{22}} \left(\mu N_{t1}^{(k)} + \lambda N_{t2}^{(k)} \right), \\ X_{t3}^{(k)} &= \sqrt{A_{33}}\, N_{t1}^{(k)}, \\ X_{t4}^{(k)} &= \sqrt{1-\rho^2}\, \sigma_S N_{t4}^{(k)}. \end{aligned}$$

3. Calculate $r_t^{S^{(k)}}, r_t^{(k)}, L_t^{(k)}, x_t^{(k)}, \int_0^t r_u^{(k)}\, \mathrm{d}u$.

4. Calculate $L_T^{(k)} \exp\left(- \int_0^T r_u^{(k)}\, \mathrm{d}u \right)$.

5. Calculate $L(N) = \frac{\sum_{k=1}^N L_T^{(k)} \exp\left(-\int_0^T r_u^{(k)}\, \mathrm{d}u\right)}{N}$.

Since $L(N) \xrightarrow{\text{a.s.}} \mathbb{E}_{\mathcal{Q}} \left[\exp\left(- \int_0^T r_u\, \mathrm{d}u \right) L_T \right]$ $(N \to \infty)$,

let $\mathbb{E}_{\mathcal{Q}} \left[\exp\left(- \int_0^T r_u\, \mathrm{d}u \right) L_T \right] \approx L(N)$ for N sufficiently large.

Table 5.1: "Exact" Monte Carlo simulation

5.2.2 "Discretized" Monte Carlo approach

If the short rate r is modelled by a CIR process, then the distribution of r_t given r_u is

$$r_t | r_u \sim \frac{1}{2c(u,t)}\, \chi^2_{d,\, q(u,t)}, \qquad u < t$$

with $c(u,t)$, d, and $q(u,t)$ as in (4.7).

To determine the fair value of the contract, we again have to calculate the rate of return r_t^S of the asset portfolio in each time period $[t-1, t)$. As in the previous section, r_t^S is given by

$$r_t^S = \exp\left\{ \int_{t-1}^t r_s\, \mathrm{d}s - \frac{\sigma_S^2}{2} + \rho\sigma_S \int_{t-1}^t \mathrm{d}W_s + \sqrt{1-\rho^2}\sigma_S \int_{t-1}^t \mathrm{d}Z_s \right\} - 1.$$

However, since the distribution of $\int_{t-1}^{t} r_s \, \mathrm{d}s$ is not explicitly known (see [18]), we cannot implement an "exact" Monte Carlo algorithm as in Subsection 5.2.1. Thus, we carry out the simulation using a discretization of the model instead.

Thus, we partition each interval $[t-1, t)$, $t = 1, \ldots, T$ in n equidistant time intervals, namely the intervals $[t_{i-1}, t_i)$, $1 \leq i \leq n$, with $t_0 = t-1$, $t_n = t$, $t_i = t_0 + \frac{i}{n}$, and n sufficiently large.

We set

$$r_t^S = \prod_{i=1}^{n} r_{t_i}^S - 1, \tag{5.12}$$

with

$$r_{t_i}^S := \exp\left\{ \int_{t_{i-1}}^{t_i} r_s \, \mathrm{d}s - \frac{\sigma_S^2 (t_i - t_{i-1})}{2} + \rho\sigma_S \int_{t_{i-1}}^{t_i} \mathrm{d}W_s + \sqrt{1-\rho^2}\sigma_S \int_{t_{i-1}}^{t_i} \mathrm{d}Z_s \right\}$$

for $1 \leq i \leq n$.

The integrals $\int_{t_{i-1}}^{t_i} r_s \, \mathrm{d}s$, $1 \leq i \leq n$ are approximated by

$$\int_{t_{i-1}}^{t_i} r_s \, \mathrm{d}s \approx \frac{r_{t_i} + r_{t_{i-1}}}{2n},$$

where the r_{t_i} are generated from the $r_{t_{i-1}}$ using the relation

$$r_{t_i} = e^{-\kappa(t_i - t_{i-1})} \left(r_{t_{i-1}} - \xi \right) + \xi + \sigma_r \int_{t_{i-1}}^{t_i} e^{-\kappa(t_i - s)} \sqrt{r_s} \, \mathrm{d}W_s, \tag{5.13}$$

which is obtained by an application of Itô's formula. The integrals in (5.13) are approximated by their left sums.

Then

$$\int_{t-1}^{t} r_s \, \mathrm{d}s \approx \frac{1}{2n} \sum_{i=1}^{n} \left(r_{t_i} + r_{t_{i-1}} \right). \tag{5.14}$$

The approximation becomes the better, the smaller the increments within the time intervals are. Hence, the number of calculation steps increases with the desired accuracy. The risk-neutral value of the contract is calculated using the algorithm given in Table 5.2.

"Discretized" Monte Carlo simulation

For $k = 1, 2, \ldots, N$ (N =number of simulations):

For $t = 1, 2, \ldots, T$:

For $i = 1, 2, \ldots, n$ (n = number of interval steps):

1. Set $t_{i-1} := t - 1 + \frac{i-1}{n}$ and $t_i := t_{i-1} + \frac{1}{n}$.

2. Generate $N_{t_{i1}}^{(k)}, N_{t_{i2}}^{(k)} \sim N(0,1)$ iid.

3. Generate $r_{t_i}^{(k)}$ from $r_{t_{i-1}}^{(k)}$ and calculate $r_{t_i}^{S^{(k)}}$.

4. Calculate $r_t^{S^{(k)}}$, $\int_0^t r_u^{(k)}\,\mathrm{d}u$, $L_t^{(k)}$, $x_t^{(k)}$.

5. Calculate $L_T^{(k)} \exp\left(-\int_0^T r_u^{(k)}\,\mathrm{d}u\right)$.

6. Calculate $L(N) = \frac{\sum_{k=1}^N L_T^{(k)} \exp\left(-\int_0^T r_u^{(k)}\,\mathrm{d}u\right)}{N}$.

Since $L(N) \xrightarrow{\text{a.s.}} \mathbb{E}_{\mathcal{Q}}\left[\exp\left(-\int_0^T r_u\,\mathrm{d}u\right) L_T\right]$ $(N \to \infty)$,

let $\mathbb{E}_{\mathcal{Q}}\left[\exp\left(-\int_0^T r_u\,\mathrm{d}u\right) L_T\right] \approx L(N)$ for N sufficiently large.

Table 5.2: "Discretized" Monte Carlo simulation

The implicit options C_0, D_0, and ΔR_0 can also be determined via the calculation of certain discounted expected values, using similar algorithms as in Tables 5.1 and 5.2. Therefore, the value of the contract can be calculated in two different ways: directly as in Tables 5.1 and 5.2 or by using the equilibrium condition (3.7). In the latter case the risk-neutral value of the contract is determined by

$$\mathbb{E}_{\mathcal{Q}}\left[\exp\left(-\int_0^T r_u\,\mathrm{d}u\right) L_T\right] = P + \mathrm{C}_0 - \mathrm{D}_0 - \Delta R_0. \tag{5.15}$$

Since the results of Monte Carlo methods vary slightly in general, and since the relation in the equilibrium condition just holds for the expected value and not for the paths, the two ways of calculating the contract value may lead to slightly different results.

In this approach we did not take the walk-away option into account, since the valuation of such options using Monte Carlo methods is both complex and computationally

expensive and therefore not advisable. However, the walk-away option can be valuated using our second numerical approach.

5.3 Valuation using a discrete lattice approach

After having evaluated the insurance contract via Monte Carlo simulations in the previous section, we derive a second numerical method to calculate the risk-neutral value of the contract. In particular, non-European contracts can be valuated with this approach.

The approach is based on the relationship between stochastic differential equations and partial differential equations. We first derive a type of Black-Scholes PDE, the solution of which can be employed to derive a valuation algorithm. We extend the approach of Bauer (see [6]), which itself is an extension of the one presented in Tanskanen and Lukkarinen (see [37]), adjust it to our model and approximate the risk-neutral value of the insurance contract on a three dimensional lattice.

5.3.1 Derivation of a Black-Scholes type PDE

In the following we make some considerations similar to those of Feynman and Kac (see [8], p. 201), but do not focus on mathematical and technical details. For a more detailed description we refer to [8].

Let $(S_t)_{t\in[0,T]}$ be a (continuous) asset process modelled by a geometric Brownian motion, and let $f(S_T)$ be a payoff at maturity T, where f fulfills certain regularity conditions.[6] Let V_t be the risk-neutral price of this payoff at the valuation date t. Since at any valuation date t the dynamics of V only depend on the present states S_t and r_t, V_t is a Markov process[7].

The financial market is complete. Hence, a trading strategy exists, so that $V_T = f(S_T)$, i.e. $f(S_T)$ is attainable. By the risk-neutral valuation formula (see [8], p. 250) the price process $(V_t)_{t\in[0,T]}$ of the payment $f(S_T)$ is given by

[6]Note, that $(S_t)_{t\in[0,T]}$ does not coincide with our insurance portfolio, but rather denotes some traded underlying. Thus, in general $f(S_T)$ does not equal the insurance payoff.

[7]A process X is called *Markov* if for each t, each $A \in \sigma(X(s) : s > t)$ (the 'future') and each $B \in \sigma(X(s) : s < t)$ (the 'past'), $\mathbb{P}(A \mid X(t), B) = \mathbb{P}(A \mid X(t))$ (see [8]).

$$\begin{aligned} V_t &= B_t \mathbb{E}_{\mathcal{Q}}\left[f(S_T) B_T^{-1} \middle| \mathcal{F}_t \right] \\ &= \exp\left(\int_0^t r_u \, \mathrm{d}u \right) \mathbb{E}_{\mathcal{Q}}\left[f(S_T) \exp\left(- \int_0^T r_u \, \mathrm{d}u \right) \middle| \mathcal{F}_t \right] \end{aligned} \tag{5.16}$$

and thus the discounted price process $\exp\left(- \int_0^t r_u \, \mathrm{d}u \right) V\left(t, S_t, r_t\right)$ is a martingale.

Consider the stochastic differential equation

$$\mathrm{d}X_t = \mu\left(t, X_t\right) \, \mathrm{d}t + \sigma\left(t, X_t\right) \, \mathrm{d}\tilde{W}_t \quad (0 \leq t \leq T), \tag{5.17}$$

where $X_t := \begin{pmatrix} S_t \\ r_t \end{pmatrix}$ denotes a 2-dimensional stochastic process and

$$\begin{aligned} \mathrm{d}S_t &= S_t r_t \, \mathrm{d}t + S_t \sigma_S \left[\rho \, \mathrm{d}W_t + \sqrt{1-\rho^2} \, \mathrm{d}Z_t \right], \\ \mathrm{d}r_t &= \kappa\left(\xi - r_t\right) \, \mathrm{d}t + \sigma_r r_t^{\gamma} \, \mathrm{d}W_t. \end{aligned}$$

Then,

$$\mathrm{d}X_t = \mathrm{d}\begin{pmatrix} S_t \\ r_t \end{pmatrix} = \underbrace{\begin{pmatrix} S_t r_t \\ \kappa\left(\xi - r_t\right) \end{pmatrix}}_{\mu(t, X_t)} \mathrm{d}t + \underbrace{\begin{pmatrix} S_t \sigma_S \rho & S_t \sigma_S \sqrt{1-\rho^2} \\ \sigma_r r_t^{\gamma} & 0 \end{pmatrix}}_{\sigma(t, X_t)} \underbrace{\begin{pmatrix} \mathrm{d}W_t \\ \mathrm{d}Z_t \end{pmatrix}}_{\mathrm{d}\tilde{W}_t}, \tag{5.18}$$

and this stochastic differential equation has a unique solution $X = (X_t)_{[0,T]}$ for a given X_0, if μ and σ are suitable functions.

Assuming that V has continuous derivatives of order one in the first and order two in the second component, we obtain by applying Itô's formula (see [8], p. 194) to

$$\exp\left(- \int_0^t r_u \, \mathrm{d}u \right) V(t, X_t),$$

that

$$\begin{aligned} \mathrm{d} \exp\left(- \int_0^t r_u \, \mathrm{d}u \right) V\left(t, X_t\right) = \exp\left(- \int_0^t r_u \, \mathrm{d}u \right) \Bigg(& V_t\left(t, X_t\right) \, \mathrm{d}t - r_t V\left(t, X_t\right) \, \mathrm{d}t \\ & + \left[S_t r_t \, \mathrm{d}t + S_t \sigma_S \rho \, \mathrm{d}W_t + S_t \sigma_S \sqrt{1-\rho^2} \, \mathrm{d}Z_t \right] V_{x_1}\left(t, X_t\right) \\ & + \left[\kappa\left(\xi - r_t\right) \, \mathrm{d}t + \sigma_r r_t^{\gamma} \, \mathrm{d}W_t \right] V_{x_2}\left(t, X_t\right) \\ & + \frac{1}{2} \Big[\sigma_S^2 S_t^2 V_{x_1 x_1}\left(t, X_t\right) + S_t \sigma_S \sigma_r \rho r_t^{\gamma} V_{x_1 x_2}\left(t, X_t\right) \\ & \quad + S_t \sigma_S \sigma_r \rho r_t^{\gamma} V_{x_2 x_1}\left(t, X_t\right) + \sigma_r^2 r_t^{2\gamma} V_{x_2 x_2}\left(t, X_t\right) \Big] \, \mathrm{d}t \Bigg). \end{aligned} \tag{5.19}$$

Hence, by the martingale property and since $\exp\left(-\int_0^t r_u \,\mathrm{d}u\right) > 0$, it follows that

$$\begin{aligned} 0 \;=\; & \frac{\partial V}{\partial t} + \frac{1}{2}\left[\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2S\sigma_S\sigma_r\rho r^\gamma \frac{\partial^2 V}{\partial S \partial r} + \sigma_r^2 r^{2\gamma}\frac{\partial^2 V}{\partial r^2}\right] \\ & + rS\frac{\partial V}{\partial S} + \kappa\left(\xi - r\right)\frac{\partial V}{\partial r} - rV. \end{aligned} \tag{5.20}$$

Therefore, we have shown that for a solution $(X_t)_{[0,T]}$ of the SDE (5.17) the price process $(V_t)_{[0,T]}$ satisfies the PDE (5.20).

Now suppose that a sufficiently smooth function $V(t,x)$ satisfies the PDE (5.20) with boundary condition

$$V\left(T, x_1, x_2\right) = f(x_1, x_2).$$

Then, (5.19) implies that

$$\begin{aligned} \mathrm{d}\exp\left(-\int_0^t r_u \,\mathrm{d}u\right) V(t, X_t) \;=\; & \exp\left(-\int_0^t r_u \,\mathrm{d}u\right)\Big[\sigma_r r_t^\gamma V_{x_2}\left(t, X_t\right)\,\mathrm{d}W_t \\ & + S_t\sigma_S\rho V_{x_1}\left(t, X_t\right)\,\mathrm{d}W_t + S_t\sigma_S\sqrt{1-\rho^2}V_{x_1}\left(t, X_t\right)\,\mathrm{d}Z_t\Big], \end{aligned}$$

which can be written in stochastic-integral form as

$$\begin{aligned} \exp\left(-\int_0^T r_u \,\mathrm{d}u\right) V\left(T, X_T\right) \;=\; & \exp\left(-\int_0^t r_u \,\mathrm{d}u\right) V\left(t, X_t\right) \\ & + \int_t^T \exp\left(-\int_0^s r_u \,\mathrm{d}u\right)\sigma_r r_s^\gamma V_{x_2}\left(s, X_s\right)\,\mathrm{d}W_s \\ & + \int_t^T \exp\left(-\int_0^s r_u \,\mathrm{d}u\right) S_s\sigma_S\rho V_{x_1}\left(s, X_s\right)\,\mathrm{d}W_s \\ & + \int_t^T \exp\left(-\int_0^s r_u \,\mathrm{d}u\right) S_s\sigma_S\sqrt{1-\rho^2}V_{x_1}\left(s, X_s\right)\,\mathrm{d}Z_s. \end{aligned} \tag{5.21}$$

Under suitable conditions[8] the stochastic integrals on the right side of (5.21) are martingales with constant expectation 0. Hence,

[8] $\mu(t,x)$ and $\sigma(t,x)$ have to fulfil certain growth conditions, see for example [8]. It is shown in Subsection 5.3.2 that such conditions are fulfilled for $\gamma = 0$.

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{Q}}\left[\exp\left(-\int_0^T r_u\,\mathrm{d}u\right)V(T,X_T)\Big|\,\mathcal{F}_t\right]\\
&=\mathbb{E}_{\mathcal{Q}}\left[\exp\left(-\int_0^T r_u\,\mathrm{d}u\right)f(X_T)\Big|\,\mathcal{F}_t\right]\\
&=\exp\left(-\int_0^t r_u\,\mathrm{d}u\right)V(t,X_t)\\
&\Leftrightarrow V(t,X_t)=\exp\left(\int_0^t r_u\,\mathrm{d}u\right)\mathbb{E}_{\mathcal{Q}}\left[\exp\left(-\int_0^T r_u\,\mathrm{d}u\right)f(X_T)\Big|\,\mathcal{F}_t\right], \qquad (5.22)
\end{aligned}
$$

and V is the desired price process.

5.3.2 Existence and uniqueness

In this subsection we introduce a theorem which presents conditions for the existence and uniqueness of solutions to certain stochastic differential equations. The existence and uniqueness conditions are necessary for the derivation of the PDE in Subsection 5.3.1. For simplicity, we restrict our considerations to the case, where the instantaneous short rate r follows an Ornstein-Uhlenbeck process ($\gamma = 0$).

Since we have already derived solutions to the stochastic differential equations that describe r and S (see Chapter 4 and Section 5.1), we know that the SDE (5.17) has a solution. It remains to show that this solution is unique which is accomplished with the help of the following theorem (cp. [36]):

Theorem 5.3.1 *Suppose that the functions $\mu_i(t,x)$ and $\sigma_{ij}(t,x), i,j=1,2$ satisfy the following local Lipschitz and growth conditions: For some $C_n < \infty$, and for $t \in \mathbb{R}$, x, $y \in \mathbb{R}^n$, and t, x, and y no larger than n in the Euklidean norm*[9]*,*

$$
\begin{aligned}
||\mu_i(t,x)-\mu_i(t,y)|| &\le C_n||x-y||,\\
||\sigma_{ij}(t,x)-\sigma_{ij}(t,y)|| &\le C_n||x-y||,\\
||\mu_i(t,x)|| &\le C_n(1+||x||), \text{ and}\\
||\sigma_{ij}(t,x)|| &\le C_n(1+||x||). \qquad (5.23)
\end{aligned}
$$

Then for each initial condition $X_0 = x_0$, there is at most one solution to the stochastic differential equation

$$\mathrm{d}X_t = \mu(t,X_t)\,\mathrm{d}t + \sigma(t,X_t)\,\mathrm{d}\tilde{W}_t \quad (0 \le t \le T),$$

[9] If $x=(x_1,\ldots,x_n)$ is a vector, then its Euklidean norm is defined as $||x|| = \sqrt{\sum_i x_i^2}$.

where

$$\mu\left(t, X_t\right) = \begin{pmatrix} \mu_1\left(t, X_t\right) \\ \mu_2\left(t, X_t\right) \end{pmatrix}, \ \sigma\left(t, X_t\right) = \begin{pmatrix} \sigma_{11}\left(t, X_t\right) & \sigma_{12}\left(t, X_t\right) \\ \sigma_{21}\left(t, X_t\right) & \sigma_{22}\left(t, X_t\right) \end{pmatrix}.$$

If $\gamma = 0$, i.e. if r follows an Ornstein-Uhlenbeck process, the following local Lipschitz and growth conditions hold for the SDE (5.17):

$$\begin{aligned}
&||\sigma_{11}\left(t,x\right) - \sigma_{11}\left(t,y\right)|| = ||\sigma_{11}\left(t,x_1,x_2\right) - \sigma_{11}\left(t,y_1,y_2\right)|| = ||x_1\sigma_S\rho - y_1\sigma_S\rho|| \\
&\qquad\qquad \leq \underbrace{||\sigma_S\rho||}_{=:C_{n1}} \, ||x-y||, \\
&||\sigma_{12}\left(t,x\right) - \sigma_{12}\left(t,y\right)|| = ||x_1\sigma_S\sqrt{1-\rho^2} - y_1\sigma_S\sqrt{1-\rho^2}|| \\
&\qquad\qquad \leq \underbrace{||\sigma_S\sqrt{1-\rho^2}||}_{=:C_{n2}} \, ||x-y||, \\
&||\sigma_{21}\left(t,x\right) - \sigma_{21}\left(t,y\right)|| = ||\sigma_r - \sigma_r|| = 0, \\
&||\sigma_{22}\left(t,x\right) - \sigma_{22}\left(t,y\right)|| = 0,
\end{aligned}$$

$$\begin{aligned}
&||\mu_1\left(t,x\right) - \mu_1\left(t,y\right)|| = ||x_1x_2 - y_1y_2|| = ||x_1\left(x_2-y_2\right) + y_2\left(x_1-y_1\right)|| \\
&\qquad\qquad \leq ||x_1||\,||x_2-y_2|| + ||y_2||\,||x_1-y_1|| \leq \underbrace{2n}_{:=C_{n3}} \, ||x-y||, \\
&||\mu_2\left(t,x\right) - \mu_2\left(t,y\right)|| = ||\kappa\left(\xi - x_2\right) - \kappa\left(\xi - y_2\right)|| = ||\kappa\xi - \kappa x_2 - \kappa\xi + \kappa y_2|| \\
&\qquad\qquad \leq \underbrace{||\kappa||}_{=:C_{n4}} \, ||x-y||,
\end{aligned}$$

$$\begin{aligned}
&||\sigma_{11}\left(t,x\right)|| = ||x_1\sigma_S\rho|| \leq \underbrace{||\sigma_S\rho||}_{=:C_{n5}} \, ||x|| \leq C_{n5}(1+||x||), \\
&||\sigma_{12}\left(t,x\right)|| = ||x_1\sigma_S\sqrt{1-\rho^2}|| \leq \underbrace{||\sigma_s\sqrt{1-\rho^2}||}_{=:C_{n6}} \, ||x|| \leq C_{n6}(1+||x||), \\
&||\sigma_{21}\left(t,x\right)|| = ||\underbrace{\sigma_r}_{=:C_{n7}}|| \leq C_{n7}(1+||x||), \\
&||\sigma_{22}\left(t,x\right)|| = 0, \\
&||\mu_1\left(t,x\right)|| = ||x_1x_2|| = ||x_1||\,||x_2|| \leq \underbrace{n}_{=:C_{n8}} \, ||x|| \leq C_{n8}\left(1+||x||\right), \text{ and} \\
&||\mu_2\left(t,x\right)|| = ||\kappa\left(\xi - x_2\right)|| \leq ||\kappa\xi|| + ||\kappa||\,||x_2|| \leq ||\kappa|| + ||\kappa||\,||x_2|| \\
&\qquad\qquad \leq ||\kappa||(1+||x_2||) \leq \underbrace{||\kappa||}_{=:C_{n9}} \left(1+||x||\right). \qquad (5.24)
\end{aligned}$$

Hence, for $C_n := \max\{C_{n1} \ldots, C_{n9}\}$ there exists according to Theorem 5.3.1 at most one solution to the SDE (5.17) on $[0,T] \times \mathbb{R}^2$ for $\gamma = 0$ and the solution derived in Chapter 4 and Section 5.1 is unique.

5.3.3 Numerical evaluation

We have shown that the risk-neutral value of a financial claim is given by the solution of the PDE (5.20). However, so far we have merely considered a one-period approach and have not taken the bonus distribution into consideration.

In the following we present a valuation approach based on Tanskanen and Lukkarinen ([37]), extended by Bauer ([6]), and adjust it to our model; in particular, we show how the PDE (5.20) can be solved numerically and how the solutions can be employed to derive the value of an insurance contract.

The bonus mechanism

According to [6], the value of the policyholder's account L_ν at time $\nu \in \{1, 2, \ldots, T\}$ is given by a known function $\text{Bon}_\nu = (\cdot, \cdot, \cdot, \cdot)$ with

$$L_\nu = \text{Bon}_\nu \left(S_\nu^-, S_{\nu-1}^+, L_{\nu-1}, x_{\nu-1} \right), \tag{5.25}$$

i.e. the policyholder's account at time ν depends on the change in the asset portfolio, the value of the policyholder's account in the previous year, and the reserve quota. However, since only the return on the assets

$$r_\nu^S = \frac{S_\nu^- - S_{\nu-1}^+}{S_{\nu-1}^+}$$

in the time period $[\nu - 1, \nu)$ influences the evolution, and because $x_{\nu-1} = \frac{S_{\nu-1}^+}{L_{\nu-1}} - 1$, Bon_ν simplifies to

$$L_\nu = \text{Bon}_\nu \left(S_\nu^-, S_{\nu-1}^+, L_{\nu-1} \right). \tag{5.26}$$

Analogously, the dividend payments D_ν at time ν are determined by the market value of the assets S_ν^- and the policyholder's time ν account value L_ν. Thus, S_ν^+ is given by a function $\text{Div}_\nu = (\cdot, \cdot, \cdot)$, with

$$S_\nu^+ = \text{Div}_\nu \left(S_\nu^-, S_{\nu-1}^+, L_{\nu-1} \right). \tag{5.27}$$

This leads to the following relations:

$$L_t = \begin{cases} L_{\nu-1} & , t \in [\nu - 1, \nu) \\ \text{Bon}_\nu \left(S_\nu^-, S_{\nu-1}^+, L_{\nu-1} \right) & , t = \nu, \end{cases}$$

$$S_t^+ = \begin{cases} S_t^- = S_{\nu-1}^+ \left(1 + \frac{S_t^- - S_{\nu-1}^+}{S_{\nu-1}^+} \right) & , t \in [\nu - 1, \nu) \\ \text{Div}_\nu \left(S_\nu^-, S_{\nu-1}^+, L_{\nu-1} \right) & , t = \nu, \end{cases} \tag{5.28}$$

where it is always assumed that Div_ν and Bon_ν are continuous in their parameters and positive for $\nu \in \{1, 2, \ldots, T\}$.

In Section 5.2 we merely evaluated European contracts and did not consider a surrender option. Now, we want to evaluate both European and non-European (or Bermuda style) contracts. The fair value of the contract at maturity is equal to its payoff value, that is, $V_T = L_T$ [10]. Then, the risk-neutral values of the European and non-European contracts are given (see (5.16)) by

$$V_t = \begin{cases} V_t^{\mathrm{EUR}} & = \mathbb{E}_{\mathcal{Q}}\left[\exp\left(-\int_t^T r_u \,\mathrm{d}u\right) L_T \middle| \mathcal{F}_t\right], \\ V_t^{\mathrm{NON-EUR}} & = \sup_{\tau \in \Upsilon_{[t,T]}} \mathbb{E}_{\mathcal{Q}}\left[\exp\left(-\int_t^\tau r_u \,\mathrm{d}u\right) L_\tau \middle| \mathcal{F}_t\right] \end{cases}, \tag{5.29}$$

where $\Upsilon_{[t,T]}$ denotes all stopping times with values in $\{\lceil t \rceil, 2, \ldots, T\}$ [11]. Then, the value of the walk-away option is

$$W_t := V_t^{\mathrm{NON-EUR}} - V_t^{\mathrm{EUR}}. \tag{5.30}$$

The value V_t of the contract depends on the value of the states S_t, r_t, and L_t at time t. Since the policyholder's account L remains constant between two valuation dates, we obtain that V_t is a function

$$V_t = V\left(t, S_t^-, S_{\lfloor t \rfloor}^+, L_{\lfloor t \rfloor}, r_t\right),$$

where r_t denotes the instantaneous short rate at time t [12].

In each time interval $[\nu - 1, \nu)$, $\nu = 1, \ldots, T$ we define the function F_ν for all s, l, r by

$$F_\nu(s, l, r) := V(\nu, s, s, l, r). \tag{5.31}$$

It can be shown that the value function has to be almost surely left continuous at ν, i.e.

$$V_t \to V_\nu, (t \to \nu), \text{ a.s.},$$

otherwise there would be an arbitrage opportunity (see [37]).

Thus,

$$\begin{aligned} \lim_{t \to \nu^-} V(t, s', s, l, r) &= F_\nu\left(\mathrm{Div}_\nu(s', s, l), \mathrm{Bon}_\nu(s', s, l), r\right) \\ &= V\left(\nu, \mathrm{Div}_\nu(s', s, l), \mathrm{Div}_\nu(s', s, l), \mathrm{Bon}_\nu(s', s, l), r\right). \end{aligned} \tag{5.32}$$

[10] given $P = L_0$.

[11] $\lceil x \rceil = \min\{n \in \mathbb{N} \mid n \geq x\}$.

[12] $\lfloor x \rfloor = \max\{n \in \mathbb{N} \mid n \leq x\}$

Between two valuation dates the evolution of the value function V only depends on changes in the asset portfolio S and the interest rate r. Hence, according to the results from Subsection 5.3.1, given the values of $S^+_{\nu-1}, L_{\nu-1}$, and given the value function at time $t_0 \in [\nu-1, \nu)$, we know that the value function satisfies the PDE

$$\begin{aligned} 0 \;=\; & \frac{\partial g}{\partial t} + \frac{1}{2}\left[\sigma_S^2 S^2 \frac{\partial^2 g}{\partial S^2} + 2S\sigma_S\sigma_r\rho\frac{\partial^2 g}{\partial S \partial r} + \sigma_r^2\frac{\partial^2 g}{\partial r^2}\right] \\ & + rS\frac{\partial g}{\partial S} + \kappa\left(\xi - r\right)\frac{\partial g}{\partial r} - rg \end{aligned} \tag{5.33}$$

with final condition

$$g(t_0, S, r) \;=\; V\left(t_0, S, S^+_{\nu-1}, L_{\nu-1}, r\right).$$

If we are given a solution g, then we have according to Subsection 5.3.1 that

$$V\left(t, S, S^+_{\nu-1}, L_{\nu-1}, r\right) \;=\; g(t, S, r),\; t \in [\nu-1, t_0].$$

For $t_0 \to \nu$ we obtain with (5.32) a method for the evaluation of the value function $\forall \nu - 1 \le t < \nu$ if V is known at $t = \nu$.

Let $s = S^+_{\nu-1}$, $l = L_{\nu-1}$ and let a solution of the PDE (5.33) be given with the final condition

$$\begin{aligned} g(\nu, S, r) \;&=\; F_\nu\left(\mathrm{Div}_\nu(S, s, l), \mathrm{Bon}_\nu(S, s, l), r\right) \\ &=\; V\left(\nu, \mathrm{Div}_\nu(S, s, l), \mathrm{Div}_\nu(S, s, l), \mathrm{Bon}_\nu(S, s, l), r\right); \end{aligned} \tag{5.34}$$

then we have

$$V(t, S, s, l, r) = g(t, S, r).$$

Thus, we can compute the value function $\forall t \in [0, T]$ using the following algorithm:

Determination of the value function V

- For $t = T : F_T(s, l, r) = V(T, s, s, l, r) = l \ \forall s, l, r$

- For $t = T - k, k \in \{1, 2, \ldots, T-1\}$: $\forall s', l', r' > 0$ evaluate the PDE (5.33) with final condition

$$g(T+1-k, S, r) = F_{T+1-k}\left(\mathrm{Div}_{T+1-k}(S, s', l'), \mathrm{Bon}_{T+1-k}(S, s', l'), r\right)$$

and set

$F_{T-k}(s', l', r') = V(T-k, s', s', l', r') = g(T-k, s', r')$
for a European contract, and

$F_{T-k}(s', l', r') = V(T-k, s', s', l', r') = \max\{g(T-k, s', r'), l'\}$
for a non-European contract.

- For $t = 0 : \forall s', l', r' > 0$ evaluate the PDE (5.33) with final condition

$$g(1, S, r) = F_1\left(\mathrm{Div}_1(S, s', l'), \mathrm{Bon}_1(S, s', l'), r\right)$$

and set

$F_0(s', l', r') = V(0, s', s', l', r') = g(0, s', r')$
for a European contract, and

$F_0(s', l', r') = V(0, s', s', l', r') = \max\{g(0, s', r'), l'\}$
for non-European contract.

Table 5.3: Determination of the value function

Practical implementation of the algorithm

In [37], Tanskanen and Lukkarinen show how a similar algorithm to the one in Table 5.3 can be implemented via a discretization of the value function on a lattice. We present a similar approach, approximating the value function on a three dimensional lattice.

Let $Y_\nu \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ be the set of all possible values of the state vector $y_\nu :=$

(S_ν, L_ν, r_ν) and $\mathrm{Lat}_\nu(S,L,r) \subset Y_\nu$ be a finite subset of Y_ν. $\mathrm{Lat}_\nu(S,L,r)$ can be varied for each valuation date $\nu \in \{0,1,\ldots,T\}$. In order to simplify notation, we choose equidistant lattice points and use the same number of lattice points for each lattice Lat_ν, $\nu \in \{0,1,\ldots,T\}$. The margins of the lattice are determined by the maximal and minimal chosen values of S, L, and r, respectively.

Let $L_\nu^{\min}$, $L_\nu^{\max}$, $S_\nu^{\min}$, $S_\nu^{\max}$, $r_\nu^{\min}$, and $r_\nu^{\max}$ denote the minimal and maximal values for L_ν, S_ν, and r_ν, $\nu \in \{0,1,\ldots T\}$, and Git_S, Git_L, and Git_r the number of lattice points of S, L, and r, respectively.
$L_\nu^{\min}$, $L_\nu^{\max}$, $S_\nu^{\min}$ etc. are chosen such that the respective random variables L_ν, S_ν, r_ν remain below the maximal value and above the minimal value with a probability of more than 0.99. For example, we choose $S_\nu^{\max}$, such that

$$\mathbb{P}\left(S_\nu \leq S_\nu^{\max}\right) \geq 0.99$$

for a specific choice of S_0.

At maturity T, we have

$$F(s,l,r) \;=\; F_T(s,l,r) \;=\; V_T(T,s,s,l,r) \;=\; l \;\; \forall (s,l,r) \in Y_\nu. \tag{5.35}$$

Thus, by numerically solving the PDE (5.33) with this terminal condition, we can determine $F_{T-1}(s,l,r) \;\forall\; (s,l,r) \in \mathrm{Lat}_{T-1}(S,L,r)$. Consequently, we are given the terminal condition for the next iteration step at least on the lattice. However, for given states s', l', r' at $T-2$, the required values of the terminal condition

$$g(T-1,S,r) = F_{T-1}\left(\mathrm{Div}_{T-1}(S,s',l'), \mathrm{Bon}_{T-1}(S,s',l'), r\right),$$

may not be on this lattice, since $\mathrm{Div}_{T-1}(S,s',l')$, and $\mathrm{Bon}_{T-1}(S,s',l')$ are not necessarily located on the lattice. Therefore, we have to interpolate between the given values of F on the lattice to obtain F between the lattice points. Since r is not changed by the distribution scheme, it is sufficient to interpolate in S and L when only values on the lattice are required to solve the PDE. For the other policy anniversaries t the computation can be carried out analogously.

We choose an interpolation scheme which is quite accurate and therefore allows us to use a lattice with a limited number of points. To simplify the presentation, we assume without loss of generality (wlog) that the function values are only known for pairs (x,y) of natural numbers[13].

[13] In the implementation the function values are known for pairs of natural numbers, but in the sense of vector indices instead of function arguments.

We consider the area between four lattice points, and let $(\lfloor x \rfloor, \lfloor y \rfloor)$ denote the lower left point, $(\lfloor x \rfloor, \lfloor y \rfloor + 1)$ the upper left point, $(\lfloor x \rfloor + 1, \lfloor y \rfloor)$ the lower right point, and $(\lfloor x \rfloor + 1, \lfloor y \rfloor + 1)$ the upper right point of the area. In the interior of the area, we determine F by linearly interpolating first in the x- and later in the y-coordinate of F [14].

Hence,

$$F(x,y) \;=\; \varphi(x, \lfloor y \rfloor) + (y - \lfloor y \rfloor)\left(\varphi\left(x, \lfloor y \rfloor + 1\right) - \varphi\left(x, \lfloor y \rfloor\right)\right),$$

where

$$\varphi(x,y) \;=\; F\left(\lfloor x \rfloor, y\right) + (x - \lfloor x \rfloor)\left(F\left(\lfloor x \rfloor + 1, y\right) - F\left(\lfloor x \rfloor, y\right)\right).$$

At the margins we extrapolate linearly.

The solution of the PDE

In order to evaluate the algorithm presented in Table 5.3, it remains to solve the PDE (5.33).

In [28], Mallier and Deakin derive a closed form solution when evaluating convertible bonds. In their publication Laplace and Mellin transformations are used to transform the PDE (5.33) into an ordinary differential equation (ODE). However, we found that their approach is invalid: the Mellin transformation is not properly applied when transforming the PDE. Thus, we are not able to apply their ideas, but have to rely on numerical methods. We use a finite difference scheme for solving the PDE numerically. This scheme is included into the lattice approach for determining the risk-neutral value of the insurance contract.

The solution of the PDE is approximated on a $S \times r$ lattice. For simplification, we choose the same lattice points for S and r as above and denote the lattice points of S and r for the lattice Lat_ν by S_i^ν, $0 \le i \le \mathrm{Git}_S$, and r_j^ν, $0 \le j \le \mathrm{Git}_r$ [15]. We further let $h_S^\nu = S_i^\nu - S_{i-1}^\nu, 1 \le i \le \mathrm{Git}_S$ and $h_r^\nu = r_j^\nu - r_{j-1}^\nu, 1 \le j \le \mathrm{Git}_r$.

For the approximation of the first and second order space derivatives of g, we use symmetric differential coefficients and obtain for the space derivatives of first order

[14] Since the interpolation is commutative, it does not matter whether we interpolate first in x or in y.

[15] In particular, this ensures that is is sufficient for the terminal condition to be known on the lattice.

$$\frac{\partial g}{\partial S}(S_i^\nu, r) \approx \frac{g(S_{i+1}^\nu, r) - g(S_{i-1}^\nu, r)}{S_{i+1}^\nu - S_{i-1}^\nu} = \frac{g(S_{i+1}^\nu, r) - g(S_{i-1}^\nu, r)}{2h_S^\nu}$$

and

$$\frac{\partial g}{\partial r}(S, r_j^\nu) \approx \frac{g(S, r_{j+1}^\nu) - g(S, r_{j-1}^\nu)}{r_{j+1}^\nu - r_{j-1}^\nu} = \frac{g(S, r_{j+1}^\nu) - g(S, r_{j-1}^\nu)}{2h_r^\nu},$$

for $0 \le i \le \mathrm{Git}_S$ and $0 \le j \le \mathrm{Git}_r$, using linear extrapolation at the margins.

For the space derivatives of second order, we obtain analogously

$$\frac{\partial^2 g}{\partial S^2}(S_i^\nu, r) \approx \frac{\frac{g(S_{i+1}^\nu, r) - g(S_i^\nu, r)}{h_S^\nu} - \frac{g(S_i^\nu, r) - g(S_{i-1}^\nu, r)}{h_S^\nu}}{h_S^\nu}$$

$$= \frac{g(S_{i+1}^\nu, r) - 2g(S_i^\nu, r) + g(S_{i-1}^\nu, r)}{h_S^{\nu 2}},$$

$$\frac{\partial^2 g}{\partial r^2}(S, r_j^\nu) \approx \frac{\frac{g(S, r_{j+1}^\nu) - g(S, r_j^\nu)}{h_r^\nu} - \frac{g(S, r_j^\nu) - g(S, r_{j-1}^\nu)}{h_r^\nu}}{h_r^\nu}$$

$$= \frac{g(S, r_{j+1}^\nu) - 2g(S, r_j^\nu) + g(S, r_{j-1}^\nu)}{h_r^{\nu 2}},$$

and

$$\frac{\partial^2 g}{\partial S \partial r}(S_i^\nu, r_j^\nu) \approx \frac{\frac{g(S_{i+1}^\nu, r_{j+1}^\nu) - g(S_{i-1}^\nu, r_{j+1}^\nu)}{2h_S^\nu} - \frac{g(S_{i+1}^\nu, r_{j-1}^\nu) - g(S_{i-1}^\nu, r_{j-1}^\nu)}{2h_S^\nu}}{2h_r^\nu}$$

$$= \frac{g(S_{i+1}^\nu, r_{j+1}^\nu) - g(S_{i-1}^\nu, r_{j+1}^\nu) - g(S_{i+1}^\nu, r_{j-1}^\nu) + g(S_{i-1}^\nu, r_{j-1}^\nu)}{4h_S^\nu h_r^\nu}, \tag{5.36}$$

for $0 \le i \le \mathrm{Git}_S$ and $0 \le j \le \mathrm{Git}_r$.

By setting $\tau = T - t$, we transform (5.33) into the PDE

$$\begin{aligned}\frac{\partial g}{\partial \tau} &= \frac{1}{2}\left[\sigma_S^2 S^2 \frac{\partial^2 g}{\partial S^2} + 2S\sigma_S\sigma_r\rho\frac{\partial^2 g}{\partial S \partial r} + \sigma_r^2\frac{\partial^2 g}{\partial r^2}\right] \\ &\quad + rS\frac{\partial g}{\partial S} + \kappa(\xi - r)\frac{\partial g}{\partial r} - rg\end{aligned}$$

and approximate the time derivative $\dfrac{\partial g}{\partial \tau}$ via a discrete Euler scheme.

For notational reasons, we set

$$\begin{aligned}f(\tau, S, r) := \frac{1}{2}&\left[\sigma_S^2 S^2 \frac{\partial^2 g(\tau,S,r)}{\partial S^2} + 2S\sigma_S\sigma_r\rho\frac{\partial^2 g(\tau,S,r)}{\partial S\partial r} + \sigma_r^2\frac{\partial^2 g(\tau,S,r)}{\partial r^2}\right] \\ &+ rS\frac{\partial g(\tau,S,r)}{\partial S} + \kappa(\xi - r)\frac{\partial g(\tau,S,r)}{\partial r} - rg(\tau,S,r).\end{aligned}$$

Then,

$$\mathrm{d}g(\tau, S, r) = f(\tau, S, r)\,\mathrm{d}\tau$$

and using the Euler scheme, we obtain the approximation

$$g(\tau + \Delta\tau, S, r) \approx g(\tau, S, r) + f(\tau, S, r)\Delta\tau, \tag{5.37}$$

with $\Delta\tau$ sufficiently small.

By making these approximations, we obtain a numerical solution as needed.

5.4 Imperfections of the asset model

So far, we followed the theory of Black and Scholes and described the dynamics of the asset process S by a geometric Brownian motion. In this framework the log returns of the asset, i.e. the logarithms of the asset returns

$$\log\left(\frac{S_t^-}{S_{t-1}^+}\right) = \mu - \frac{\sigma_S^2}{2} + \int_{t-1}^{t}\rho\sigma_S\,\mathrm{d}W_s + \int_{t-1}^{t}\sqrt{1-\rho^2}\sigma_S\,\mathrm{d}Z_s,\ t \in [0,T]$$

follow a normal distribution under the physical probability measure[16].

[16]For the evolution of S we assume $\mathrm{d}S_t = S_t\left(\mu\,\mathrm{d}t + \rho\sigma_S\,\mathrm{d}W_t + \sqrt{1-\rho^2}\sigma_S\,\mathrm{d}Z_t\right)$ under the physical probability measure $\mathcal{P}$.

The geometric Brownian motion is a classical model to describe stock price evolutions. Combined with the concepts of no-arbitrage pricing, the model led to many famous results such as the Black-Scholes formula for the price of a European Call Option and explicit formulas for hedging strategies (cp. [8]).

However, empirical studies show deviations from the properties of the geometric Brownian motion when analyzing market data. This subsection gives a rough empirical analysis of how distinct these deviations are in our framework.

5.4.1 Properties of normally distributed log returns

If the log returns of S follow a normal distribution with empirical mean $\tilde{\mu}$ and standard deviation $\tilde{\sigma}$, their moments have certain features (see [35]):

(i) **Zero Skewness**: The skewness is defined to be the third central moment, divided by the third power of the standard deviation. It measures the degree of asymmetry. The skewness of normally distributed log returns is zero, i.e.

$$\frac{\mathbb{E}\left[\left(\log\left(\frac{S_t^-}{S_{t-1}^+}\right)-\tilde{\mu}\right)^3\right]}{\tilde{\sigma}^3}=0.$$

In general, if the left tail of the distribution is more pronounced than the right, the function is said to have negative skewness, if the reverse is true, it has positive skewness; if the two are equal, the skewness is zero.

(ii) **Kurtosis is 3**: The kurtosis is the degree of peakedness of a distribution, defined as a normalized form of the fourth central moment of a distribution. For the normally distributed log returns, the kurtosis is three, i.e.

$$\frac{\mathbb{E}\left[\left(\log\left(\frac{S_t^-}{S_{t-1}^+}\right)-\tilde{\mu}\right)^4\right]}{\tilde{\sigma}^4}=3.$$

In general, if a distribution has a high kurtosis, it shows a high peak near the mean, declines rapidly and has heavy tails. Distributions with low kurtosis have a flat top near the mean and thinner tails.

As an example, Figure 5.1 shows the density functions of the standard normal distribution with kurtosis 3 and of a Laplace distribution with kurtosis 6.

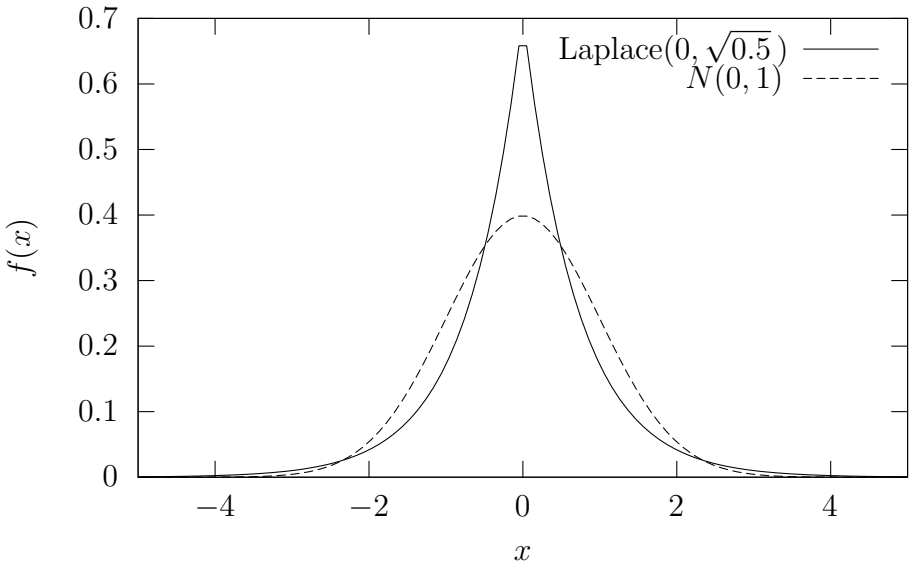


Figure 5.1: Densities of $N(0,1)$ and Laplace $(0, \sqrt{0.5})$ distribution

5.4.2 Empirical assessment

In order to analyze the actual distribution of the log returns of the asset process S, we first estimate skewness and kurtosis of annual log returns of DAX[17] and money market from 1980 to 2005[18]:

Index	**Skewness**	**Kurtosis**
DAX	-0.7334	3.0392
money market	0.7363	2.4768
10% DAX+90% money market	-0.5822	3.0028
$X \sim N(\mu, \sigma^2)$	0	3

Table 5.4: Skewness and kurtosis

The results given in Table 5.4 may lack accuracy, since on the one hand we use a rather small sample, and on the other hand we only consider annual returns. However, the tendency accords with former studies (see [35]).

We observe that for the DAX log returns the skewness is negative, implying that the returns have fatter tails to the left than to the right. For the money market log

[17]DAX (Deutscher Aktienindex) : German stock index.

[18]For the used data see Appendix A.

returns the skewness is positive, implying that the right tail of the distribution is more pronounced than the left. The kurtosis of the DAX log returns lies just slightly above three, whereas the kurtosis of the money market log returns is smaller than three.

We are mainly interested in an asset process consisting of a small portion of stock and the rest money market, since this composition roughly approximates the asset side of a life insurer[19]. According to the current average for life insurance companies, we choose a portfolio consisting of 90% money market and 10% stock.

Table 5.4 shows that for such a portfolio the skewness is still significantly negative. Therefore, even though the kurtosis almost equals three and therefore resembles that of a normal distribution, it is not very accurate to describe the log returns of the asset process S by a normal distribution. But even though there are some discrepancies, a geometric Brownian motion seems to be a more adequate model for the asset side of an insurer than it is for modelling stocks or equity. However, due to the limited amount of data, the rather shallow results of this ad-hoc analysis have to be further scrutinized.

Several authors have recently proposed to replace the Brownian motion by Lévy processes, leading to more sophisticated and realistic models (see e.g. [3], [26]).

[19]For simplicity we neglect other asset classes as, e.g., real estate.

Chapter 6

Implementation and results

In this chapter, we describe how the algorithms of the previous chapter are implemented and present some results. An outline of the program structure containing the most important features is given, and sensitivity analyses with respect to some parameters are allegorized. In particular, we focus on the parameters that come into play due to the stochasticity of the instantaneous short rate.

6.1 Implementation and program structure

The Monte Carlo methods and the discrete lattice approach are implemented in the object-oriented programming language `C++` which supports inheritance and thus gives us the opportunity to implement the basis for the different methods only once and to efficiently reuse already programmed code.
The structure of the program is similar to the one described in [6]. We work with two input files: the *data ini file* contains information about the used parameters, such as guaranteed interest rate, participation rate, volatility of the asset process, etc., and the *method ini file* specifies which method is to be used, i.e. either the "exact" Monte Carlo method, the "discretized" Monte Carlo method, or the discrete lattice method. Furthermore, the *method ini file* provides information about which bonus distribution scheme is relevant, either MUST- or IS-case, and whether to evaluate a European contract or a non-European contract. The structure of the *method ini file* along with two examples is given in Table 6.1.

Table 6.2 presents the structure of the *data ini file*. The example shows that it is possible to carry out calculations for more than just one set of parameters at a time, which is important when performing sensitivity analyses. The information of the *data ini file*

		Example 1	Example 2
input	**explanation**	**MUST DisLat**	**IS MoCa**
$\{1,2\}$ case	1: MUST, 2: IS	1	2
$\{1,2\}$ walk-away option	1: Europ., 2: non-Europ. (2 not for MoCa.)	2	1
		*****	*****
$\{1,2,3\}$ method	1: MoCa exact, 2: MoCa discretized, 3: DisLat	1	2
DiscreteLatticeMethod:	(only in DisLat)		
int GitS	number of lattice points for S	101	
int GitL	number of lattice points for L	101	
int GitR	number of lattice points for r	51	
double Smax	maximal value for S	40400	
double Lmax	maximal value for L	40400	
double rmin	minimal value for r	0	
double rmax	maximal value for r	0.102	
double Soutmin	output minimum S	9000	
double Soutmax	output maximum S	12000	
double Loutmin	output minimum L	9000	
double Loutmax	output maximum L	12000	
double routmin	output minimum r	0	
double routmax	output maximum r	0.102	
$\{0,1\}$ logf	1: in file, 0: to STDOUT	0	
string logfile	if logf=1 file name		
MonteCarloMethods :	(only in MoCa)		
$\{1,2,3\}$ ran	generator 1:mt19937, 2:ranlux389, 3:taus		1
$\{1,2\}$ gaussmethod	Norm. rv – 1:Box-Müller, 2:Ratio Meth.		1
$\{1,2\}$ rateprocess	1: OU, 2: CIR		1
int steps	N – Iterations		250000
$\{0,1\}$ logf	1: in file with option eval., 0: E_Q to STDOUT		1
string logfile	if logf=1 file for output of paths		log
string mergedlog	if logf=1 file for output of options		merged

Table 6.1: Method ini file

is saved in the class `data`, which besides the parameters, contains the constructor, the destructor, and a method `print()` for the output of the parameters. For the specification of the method, we use the features of object-oriented programming: we work with the abstract class `method` which is extended to the classes `montecarlomethodexact`, `montecarlomethoddiscrete`, and `discretelatticemethod`. For each of the three classes, there exist a constructor which initializes the class with the respective data, a destructor, an output method, and a method for the evaluation. The parameters of the `evaluate` method, which carries out the respective algorithms, are a `data` object, a parameter indicating whether a European or a non-European contract is considered, and the functions which determine the respective bonus distribution scheme.

In the `evaluate` method of the class `montecarlomethodexact`, the contract value is determined by generating the required random variables for the distribution of the interest rate and asset process, and then carrying out the Monte Carlo simulation. In the `evaluate` method of the class `montecarlomethoddiscrete` we introduce a discretization approach for the generation of the respective processes, since not all

input	explanation	example
		DATA
		TEST:28.12.05
double[ ] g	guaranteed interest rate g	0.0275:0.035:0.04
double[ ] delta	participation rate δ	0.9:0.9:0.9
double[ ] z	target distribution z	0.05:0.05:0.05
double[ ] y	ident. market value earnings y	0.5:0.5:0.5
double[ ] alpha	dividend rate α	0.05:0.05:0.05
double[ ] sigma	volatility of interest rate σ_r	0.01:0.01:0.01
double[ ] sigmaS	volatility of asset σ_S	0.075:0.075:0.075
double[ ] rho	correlation of asset and interest rate ρ	0.5:0.5:0.5
double[ ] a	lower limit of reserve corridor a	0.05:0.05:0.05
double[ ] b	upper limit of reserve corridor b	0.3:0.3:0.3
double[ ] a1	reversion rate κ	0.14:0.14:0.14
double[ ] b1	reversion level ξ	0.04:0.04:0.04
int[ ] T	time horizon T	10:10:10
double[ ] Pr	initial investment P	10000:10000:10000
double[ ] xnull	initial reserve quota x_0	0.1:0.1:0.1
double[ ] rnull	initial interest rate r_0	0.04:0.04:0.04

Table 6.2: Data ini file

the distributions of the quantities to simulate are known explicitly. In the `evaluate` method of the class `discretelatticemethod`, the contract value is determined by the algorithm presented in subsection 5.2.2. Here FLENS[1] is used to numerically solve the partial differential equations.

The class `run` joins the class `data` with the respective method class. The data from the first part of the *method ini file* determines the distribution mechanism. The `data` object and the `method` object are initialized in the constructor and the evaluation method is started via a call of the function `doit()`.

Appendix B presents the most important source code fragments.

6.2 Sample outputs

There are three possible types of output:

(i) For the **Monte Carlo evaluation without individual options** the output only consists of the risk-neutral value of the insurance contract.
For example, for the "exact" Monte Carlo algorithm we have

```
Monte Carlo Algorithm exact, OU
```

[1]FLENS is a flexible library for efficient numerical solutions in `C++` which is developed at the Department of Numerical Analysis at Ulm University (see [16]).

```
Go #0 :
*****************************************

E_Q[e^{-int_0^T r(s)ds} L_T] = 10498.6

*****************************************
```

and for the "discretized" Monte Carlo method we have

```
Monte Carlo Algorithm discrete, CIR
Go #0 :

*****************************************

E_Q[e^{-int_0^T r(s)ds} L_T] = 10373.3

*****************************************
```

(ii) For the **Monte Carlo evaluation with individual options** the output consists of a list of the values of the single options as well as the contract value (here for the "exact" Monte Carlo method):

```
Monte Carlo evaluation exact/ 2006-03-06 : DIM=1,
Ornstein-Uhlenbeck
***********************************************
***********************************************
1 steps :
250000, T = 10, g  = 0.035, delta = 0.9, y = 0.5, sigma_r = 0.01,
sigma_S = 0.075, rho = 0.51, L_0 = 10000, x_0 = 0.1, r_0 = 0.04,
z= 0.05, [a,b] = [0.05,0.3], a1 = 0.14, b1 = 0.04
*******************************************
i:L_i:S_i^+:S_i^-:D_i:R_i:x_i:FinSpr_i
*******************************************
0:10000:11000:11000:0:1000:0.1:0
1:10038.9:11453.1:11465.9:23.9649:975.481:0.0965695:36.5189
2:10079:11939.7:12007.2:24.0622:998.391:0.0980873:87.6683
3:10121:12505.8:12598:24.3329:1038.52:0.10146:108.681
4:10166.7:13128.3:13234.8:24.8046:1088.7:0.105894:119.448
5:10215.2:13792:13909.5:25.0429:1140.37:0.110526:126.595
6:10267.6:14501.6:14626.6:25.5376:1194.19:0.115354:130.706
7:10321.4:15247.8:15379.5:25.6884:1245.94:0.120003:133.657
8:10377.7:16040.3:16176.3:26.0192:1298.86:0.124744:134.581
9:10436.5:16873.4:17013.8:26.3294:1349.74:0.129243:135.591
10:10498.6:17754.3:17899.3:26.7637:1400.52:0.133665:136.675

values of the implicit options:
E_Q[dis. capital shot] = 1150.12
E_Q[dis. interest option] = 252.546
E_Q[dis. final reserve] = 1400.52
initial reserve=1000
reserve delta = 400.516
E_Q[dis.L _T] = 10498.6
             = L(0) + capital shots - interest rate option - reserve delta = 10497.1
```

Note, that the contract values do not coincide due to Monte Carlo errors as described in Chapter 5.

(iii) For the **discrete lattice method** the output consists of a list of contract values on the $S \times L \times r$ lattice:

```
Discrete lattice algorithm
Go #0 :

S / L / r : 8800 / 8800 / 0...V_0 = 6801.07
S / L / r : 8800 / 8800 / 0.002...V_0 = 10854.5
S / L / r : 8800 / 8800 / 0.004...V_0 = 10753.4
S / L / r : 8800 / 8800 / 0.006...V_0 = 10652.6
S / L / r : 8800 / 8800 / 0.008...V_0 = 10552.4
S / L / r : 8800 / 8800 / 0.01...V_0 = 10452.9

[...]

S / L / r : 11200 / 10000 / 0.026...V_0 = 11106.9
S / L / r : 11200 / 10000 / 0.028...V_0 = 11008.6
S / L / r : 11200 / 10000 / 0.03...V_0 = 10911.8
S / L / r : 11200 / 10000 / 0.032...V_0 = 10816.4
S / L / r : 11200 / 10000 / 0.034...V_0 = 10722.4
S / L / r : 11200 / 10000 / 0.036...V_0 = 10629.9
S / L / r : 11200 / 10000 / 0.038...V_0 = 10538.7
S / L / r : 11200 / 10000 / 0.04...V_0 = 10449
S / L / r : 11200 / 10000 / 0.042...V_0 = 10360.7
S / L / r : 11200 / 10000 / 0.044...V_0 = 10273.7
S / L / r : 11200 / 10000 / 0.046...V_0 = 10188.1
S / L / r : 11200 / 10000 / 0.048...V_0 = 10103.9
S / L / r : 11200 / 10000 / 0.05...V_0 = 10021
S / L / r : 11200 / 10000 / 0.052...V_0 = 9939.39

[...]

S / L / r : 11600 / 11600 / 0.096...V_0 = 9539.05
S / L / r : 11600 / 11600 / 0.098...V_0 = 9468.43
S / L / r : 11600 / 11600 / 0.1...V_0 = 9398.45
```

The contract value is presented for different initial values of S, L, and r, where for example $S = 11,200$, $L = 10,000$, and $r = 0.04$ correspond to a contract with initial investment of $10,000$ units, an initial interest rate of $r_0 = 0.04$, and an initial reserve quota of $\frac{S_0}{L_0} - 1 = \frac{11{,}200}{10{,}000} - 1 = 0.12$.

6.3 Results

The risk-neutral value of a life insurance policy depends on many factors and regulations. Besides the current regulatory and legal requirements, the corporate policy

and the market situation affect the value of the contract.
In this section, we assign reasonable values to the model parameters and discuss the choice. We carry out the valuation procedures and explore the results. Subsequently, we discuss the influence of the diverse model parameters on the risk-neutral value of the contract by performing sensitivity analyses in the MUST- and IS-case. Furthermore, we investigate the interaction of several parameters and the influence of the different models for the short rate. In particular, we compare the results with those of Bauer (see [6]), who uses a constant short rate for his calculations.

6.3.1 Parameter choice

We distinguish between parameters, that are chosen due to the regulatory and legal requirements in Germany, corporate-political parameters, which are chosen in a way to model the actual behavior of typical German insurers fairly close to reality, and other parameters. We choose the parameters according to [6] in order to obtain comparable results.

(i) **Compulsory parameters:**
Currently, the minimum rate of interest g which the German life insurance companies have to guarantee the policyholders is fixed at 2.75%.
However, since the interest rate guarantee has to be granted for the whole term of the contract and because the guaranteed rate has changed over the years, the insurance companies' portfolios of policies still contain contracts with higher minimum guaranteed interest rates such as 3.25% or even 4%. We assume an average guaranteed interest rate over all policies of $g = 3.5\%$.
Furthermore, according to the German regulation, at least a minimum participation rate $\delta = 90\%$ of the earnings on book values have to be credited to the policyholder's account (see ZRQuotenV ([41])).
Finally, the minimum portion of market value earnings that has to be identified as book value earnings in the balance sheet, y, is assumed to be 50% [2].

(ii) **Corporate-political parameters:**
We choose the corporate-political parameters in a way to represent the situation and behavior of typical German insurers in the last couple of years fairly close to reality.

[2] Due to the complexity of the German accounting system, an estimation of y is hard to perform. However, within the adequate ranges, the results are rather insensitive to changes in y (see [6]).

We let the target rate $z = 5\%$, the reserve corridor $[a, b] = [5\%, 30\%]$, the portion of earnings that is provided to equity holders $\alpha = 5\%$, and the volatility of the asset portfolio $\sigma_S = 7.5\%$ as in [6].

The asset portfolio can be composed of various asset forms such as bonds, stock, realty, etc.. For simplicity however, we consider a portfolio which is composed merely of stock and money market. For now, we assume a market average of $10\% - 15\%$ portion of stock in the asset portfolio, where certainly financially strong companies tend to hold a higher portion of stock than smaller insurance companies[3]. The high money market portion in the portfolio leads to a positive correlation between asset return and money market return. However, we do not have an exact idea of the size of the parameter ρ. Here, we estimate it roughly and let $\rho = 0.5$[4].

(iii) **Other parameters:**
We let the time horizon be $T = 10$, the initial investment $P = 10.000$, the insurer's initial reserve quota $x_0 = 10\%$, and the initial interest rate $r_0 = 4\%$ as in [6]. We let the volatility of the Ornstein-Uhlenbeck process $\sigma_r = 1\%$ as in [9] and choose the reversion rate $\kappa = 0.14$ as in [5] and the reversion level $\xi = r_0 = 4\%$. Then, the short rate r_t is an unbiased estimator for the reversion level b under r_0 with $r_0 = b$ for both the Ornstein-Uhlenbeck and the CIR process, since

$$\mathbb{E}\,(r_t|r_0)^{\mathrm{OU}} \;=\; \mathbb{E}\,(r_t|r_0)^{\mathrm{CIR}} \;=\; e^{-\kappa t}\,(r_0 - \xi) + \xi \;=\; \xi$$

We equate the variances of the Ornstein-Uhlenbeck and the Cox-Ingersoll-Ross process with each other under r_0 in order to obtain comparable results for the volatilities in the evaluation: let σ_r denote the volatility of the Ornstein-Uhlenbeck process and $\tilde{\sigma}_r$ the volatility of the Cox-Ingersoll-Ross process. From (4.4) and (4.7) we obtain that

$$\mathrm{Var}\,(r_t|r_0)^{\mathrm{OU}} \;=\; \mathrm{Var}\,(r_t|r_0)^{\mathrm{CIR}}$$

is equivalent to

$$\frac{\sigma_r^2}{2\kappa}\left(1 - e^{-2\kappa t}\right) \;=\; \frac{\tilde{\sigma}_r^2}{2\kappa}\,e^{-2\kappa t}\,(\xi - 2r_0) + \frac{\tilde{\sigma}_r^2}{\kappa}\,e^{-\kappa t}\,(r_0 - \xi) + \frac{\tilde{\sigma}_r^2 \xi}{2\kappa},$$

and therefore, since we assume $\xi = r_0$, we have that

$$\tilde{\sigma}_r \;=\; \sqrt{\frac{\sigma_r^2}{\xi}}\,. \tag{6.1}$$

[3]The Allianz Lebensversicherungs AG currently holds a stock portion of about 18 % (see [22]).

[4]See Appendix A for a more detailed discussion.

6.3.2 Numerical results

The risk-neutral value of a European insurance contract can be calculated in two different ways using the Monte Carlo algorithms introduced in Chapter 5 – directly, or by summation of the individual contract components:

$$\begin{aligned}\text{contract value} \;=\; &\text{initial investment} + \text{value of the interest rate guarantee} \\ &- \text{change of reserve} - \text{value of the dividends.}\end{aligned}$$

The non-European contract with walk-away option is valuated by the algorithm introduced in Section 5.3.

Table 6.3 shows European and non-European contract values for guaranteed interest rates of 2.75%, 3.5% and 4% in the Ornstein-Uhlenbeck case along with the values of the implicit options. The values in parentheses are those obtained with the discrete lattice method.

OU MUST-case	**g=2.75%**	**g=3.5%**	**g=4%**
RN value EUR_MC	10,058.1 (9,978.4)	10,497.0 (10,426.0)	10,829.6 (10,774.5)
RN value NON-EUR_DL	10,273.6	10,595.1	10,886.6
Interest rate guarantee	874.9	1,150.1	1,370.5
Value of dividends	271.8	252.6	237.6
Discounted final reserve	1,545.0	1,400.5	1,303.3
Walk away option	295.2	169.1	112.1
OU IS-case	**g=2.75%**	**g=3.5%**	**g=4%**
RN value EUR_MC	10,827.7(10,722.8)	11,092.4(10,985.8)	11,292.7(11,195.0)
RN value NON-EUR_DL	10,843.2	11,068.0	11,256.2
Interest rate guarantee	1,052.3	1,283.3	1,460.4
Value of dividends	106.9	82.7	67.3
Discounted final reserve	1,117.7	1,108.2	1,100.3
Walk away option	120.4	82.2	61.2

Table 6.3: Contract values, g

Having decomposed the insurance contract we can observe how the implicit options influence its value. With increasing guaranteed rate of interest, the value of the guarantee option rises and the value of the dividends as well as the discounted final reserves decrease. Furthermore, we note that the value of the walk-away option decreases as g increases, which results from the fact that with rising guaranteed interest rate, the probability that the policyholder finds a more profitable investment after surrendering the contract decreases. However, unlike in [6] where the surrender option was 0 in most cases, the walk-away option is of value for all three choices of g, even in the IS-case.

We observe a deviation between the contract values calculated with the Monte Carlo simulation and the values calculated with the discrete lattice method. Assuming that the values obtained with Monte Carlo are correct, the discrete lattice method produces a numerical error mostly significantly smaller than 1%, which is acceptable regarding the chosen lattice accuracy. Moreover, since the discrete lattice value always is inferior to the Monte Carlo value, the errors occur in the same direction at all times, and the value of the walk-away option as the difference of two contract values calculated with the same method should be more accurate.

We now present the influence of the two stochastic short rate models. Table 6.4 presents the risk-neutral values of the European contracts and the implicit options for a constant short rate of $r = 4\%$, and for an Ornstein-Uhlenbeck short rate, and a Cox-Ingersoll-Ross short rate with $r_0 = 4\%$.

MUST-case	$r_{\text{const.}} = 4\%$	**OU,** $r_0 = 4\%$	**CIR,** $r_0 = 4\%$
RN value EUR_MC	10360.40	10497.10	10504.90
Interest rate guarantee	865.92	1150.12	1136.97
Value of dividends	238.08	252.55	251.73
Discounted final reserve	1267.47	1400.52	1380.33
IS-case	$r_{\text{const.}} = 4\%$	**OU,** $r_0 = 4\%$	**CIR,** $r_0 = 4\%$
RN value EUR_MC	10919.1	11092.50	11102.40
Interest rate guarantee	1004.19	1283.34	1273.03
Value of dividends	75.05	82.70	82.76
Discounted final reserve	1010.05	1108.17	1087.88

Table 6.4: Contract values, r_0

We notice that for both stochastic short rate models, the contract values are higher than for a constant short rate, but the difference between the two stochastic models is negligible. Furthermore, we observe that the interest rate option has a huge influence. Comparing the CIR MUST-case with the constant short rate case, we observe a rise in contract value of approximately 140 units but a rise in guarantee option of 270 units.

Since the contract values differ just slightly for the two short rate models, we focus on one model for the further considerations and assume in most cases that the interest rate follows an Ornstein-Uhlenbeck model. In addition, we mainly consider European contracts in order to make comparisons to [6].

In the following we analyze the influences of the model parameters on the risk-neutral value of the insurance contract. We first analyze the sensitivity of the risk-neutral

value with respect to changes in those parameters that our model and the model of Bauer (see [6]) have in common and compare the results. Then we focus on the parameters that come into play due to the influence of the stochastic interest rates.

We use GNUPLOT ([19]) to generate two dimensional plots presenting the influence of one specific parameter on the contract value and three dimensional plots presenting the joint influence of two different parameters at a time.

The Influence of the Guaranteed Rate of Interest

The insurer's portfolio of policies contains contracts with different interest rate guarantees. Contracts with different guaranteed interest rates lead to different liabilities, since a contract with for example $g = 4\%$ represents a higher risk for the insurer than a contract with a guarantee level of $g = 2.75\%$. Figure 6.1 illustrates, how the guaranteed interest rate g influences the value of the insurance contract in the MUST-case, ceteris paribus[5]. The respective contract values have been calculated

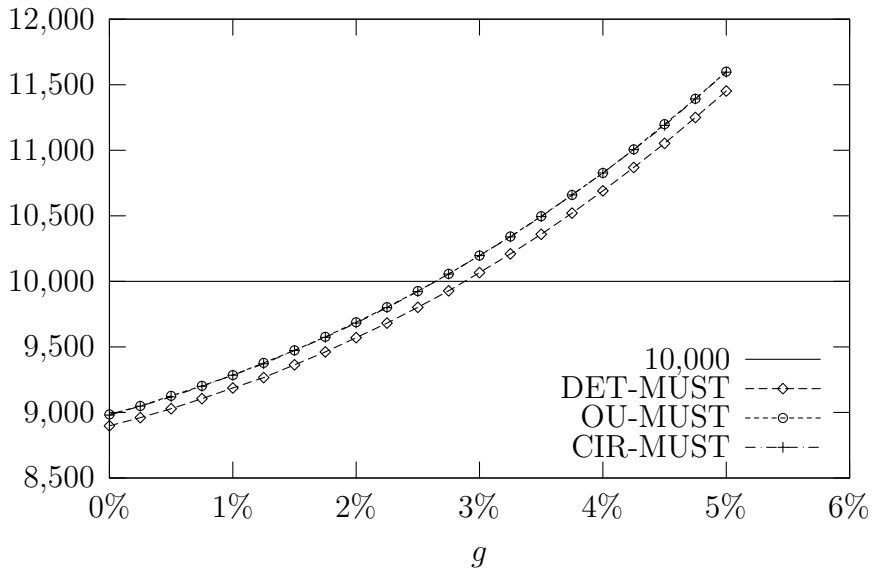


Figure 6.1: Influence of g on the contract value, MUST-case

for a constant short rate of 4% (DET-MUST), an Ornstein-Uhlenbeck short rate model (OU-MUST), and a Cox-Ingersoll-Ross short rate model (CIR-MUST). We observe that for all three short rate processes, the price of the insurance contract increases in an almost equal manner as the guaranteed rate of interest, g, increa-

[5]Unless noted otherwise, we use the parameters as discussed at the beginning of this section.

ses.[6] This is due to the fact that if a higher interest rate guarantee is promised, the probability that the company fails to fulfill the guarantee on its own rises. Thus the value of the interest rate guarantee option increases, since it is more likely that a capital shot is needed. Furthermore, we find that for both stochastic interest rate models the risk-neutral value of the contract exceeds the one for a constant short rate, which is caused by the additional source of uncertainty that comes into play with the stochasticity of the short rate.

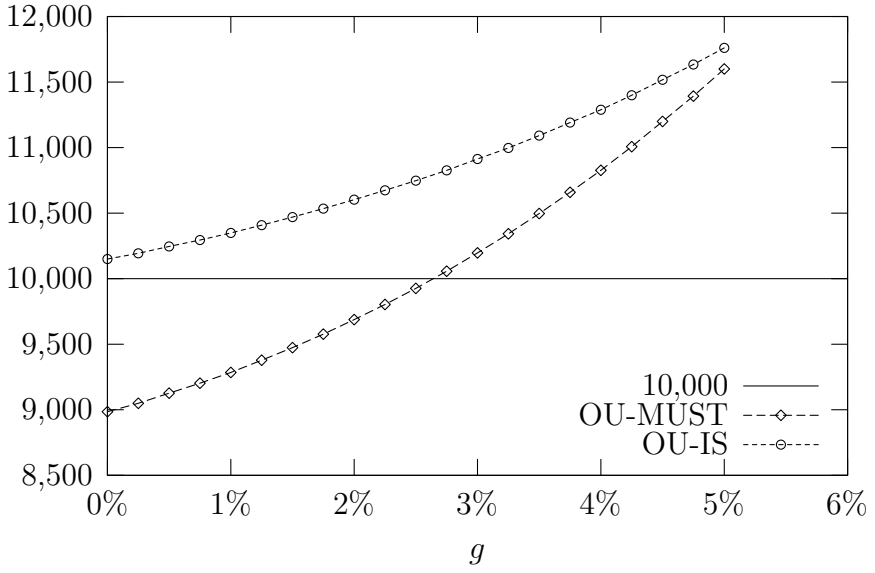


Figure 6.2: Influence of g on the contract value, MUST- and IS-case

Figure 6.2 illustrates that the influence of the guaranteed rate of interest g is stronger in the MUST-case than in the IS-case. This is due to the fact that in the IS-case the target interest rate z rather than the guaranteed rate g is passed on to the policyholders in the majority of cases. We further notice that in the IS-case it is not possible to create a "fair contract", i.e. a contract the value of which equals the initial investment (10,000 units), even if the guaranteed interest rate is zero.

If we alter other parameters such as the minimum participation rate δ, the minimum portion of market value earnings y, the initial reserve quota x_0, the initial short rate r_0, or the volatility of the asset process σ_S, we observe that in general, the value of the insurance contract with stochastic interest rates exceeds the value of a contract using a constant interest rate. However, the tendencies of how the contract

[6] In Figure 6.1 the contract values for Ornstein-Uhlenbeck and CIR short rate differ just very slightly and therefore the curves seem to overlap.

value changes with the individual parameters remain the same as for constant yields (see [6]).

More specifically, if we increase the participation level δ, the value of the contract is increased, too. This results from the fact, that as the insurance company raises the portion of the benefit which is credited to the policyholder, the guarantee option becomes more valuable and the value of the dividends decreases at the same time.

For the minimum portion of market value earnings that has to be identified as book value earnings, y, the value of the contract also increases as we raise y since, if the insurance company has to identify a higher percentage of its market value earnings as book value earnings, the credit on the policyholders account will be higher and thus the value of the contract rises. However, this sensitivity is rather moderate for values of y between 0% and 60% (see [6]).

The value of the contract is increased with rising initial reserve quota x_0 as well. The more reserves the insurance company holds, the higher are the funds on the revenue of which the insured can participate and thus the higher the value of the contract. Hence, potential customers should conclude their contracts primarily at financially strong companies.

The value of an insurance contract decreases with increasing initial short rate r_0, since for the customers the alternative of investing in the money market becomes more attractive with higher yields than investing in an insurance contract with unchanged minimum interest rate g. Therefore, changes of g and the risk-free interest rate r should always go in the same direction in order to keep the contract value stable.

In the following we investigate the influence of the parameters that come into play with the stochasticity of the short rates, namely the volatilities of the interest rate processes, σ_r and $\tilde{\sigma}_r$, and the correlation between interest rate and asset process, ρ. Moreover, we analyze the influence of the asset volatility σ_S. For the sensitivity analyses we only consider European contracts. However, the results are similar for non-European contracts, since the walk-away option plays a minor role, at least in the IS - case (see Table 6.3).

The Influences of the Volatilities of Interest Rate and Asset Process

The volatility of the interest rate influences the risk-neutral value of the insurance contract considerably. Note again, that as a result of (6.1), a volatility of $\sigma_r = 1\%$ for the Ornstein-Uhlenbeck short rate corresponds to a volatility of $\tilde{\sigma}_r = 5\%$ for the

Cox-Ingersoll-Ross short rate. Hence, we choose different ranges for σ_r and $\tilde{\sigma}_r$ in the calculations.

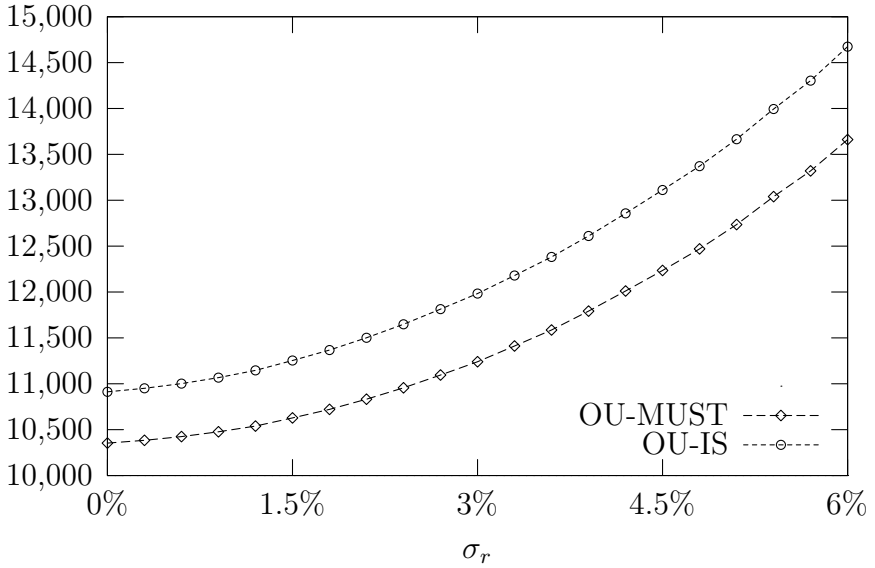


Figure 6.3: Influence of σ_r on the contract value

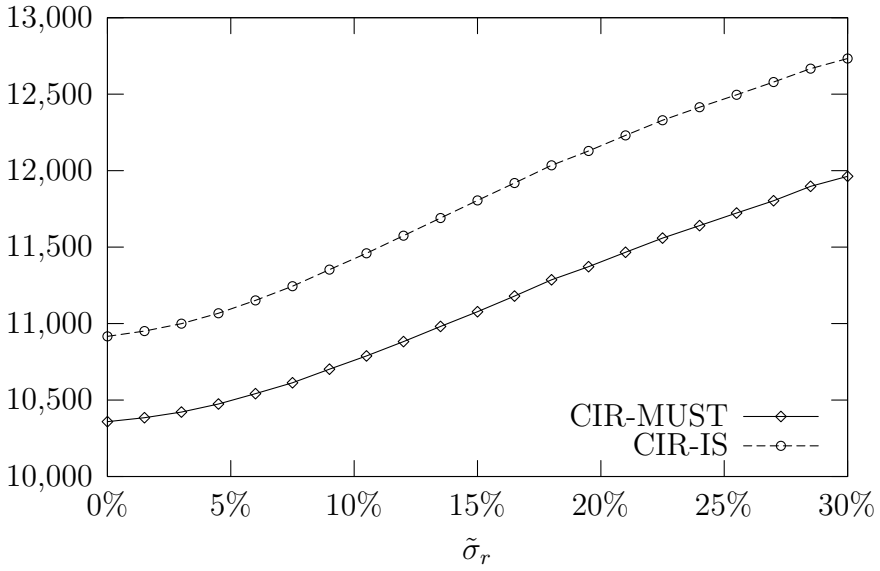


Figure 6.4: Influence of $\tilde{\sigma}_r$ on the contract value

Figure 6.3 shows that a change in the volatility of the OU process influences the value of the contract in the MUST-case and in the IS-case likewise. We observe in both cases a significant increase of the contract value. If the volatility σ_r rises from 1% to 2%, the contract value increases by almost 200 units, if it rises to 5%, then

the value even increases by over another 1,800 units. This strong sensitivity of the contract value is explained by the risk of poor returns on the money market.

For a CIR process, the value of the insurance contract behaves similarly (see Figure 6.4). It rises as the volatility rises. However, the increase is not as strong as in the OU case, which is due to the fact that in the CIR model interest rates are always positive, whereas in the OU model they can become negative.

To make assertions about the influence of model parameters on the value of the insurance contract, it is not only important to examine the influence of a single parameter at a time, but also to determine how certain parameters interact. Hence, we consider pairs of parameters and calculate the contract values for combinations of these parameters ceteris paribus with the help of the two Monte Carlo methods. The resulting data is fitted by a function $f : \mathbb{R}^2 \to \mathbb{R}$ and plotted in a three dimensional grid (see e.g. Figure 6.5).

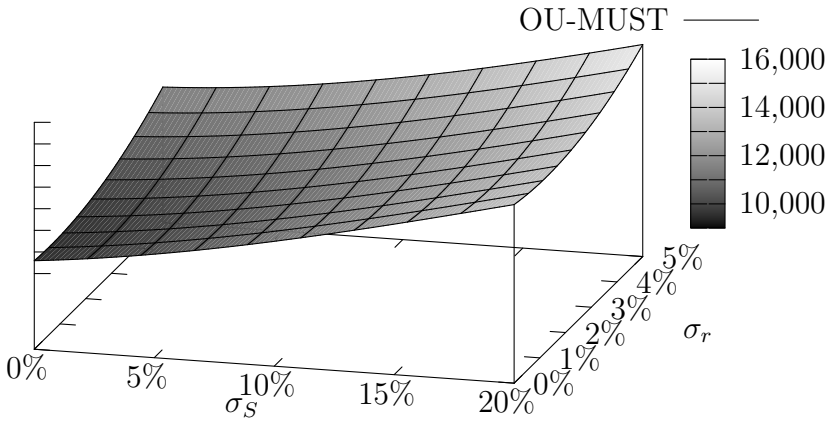


Figure 6.5: Influence of σ_r and σ_S on the contract value, OU-MUST

Figure 6.5 illustrates how the parameters σ_r and σ_S interact in the Ornstein-Uhlenbeck MUST-case. With increasing σ_S and σ_r, the contract value increases which can be explained by the fact that rising volatilities of asset and interest rate processes imply an increasing probability of unfavorable asset returns and of low market interest rates, which increases the value of the guarantee option. The strong influence of the guarantee option on the contract value is displayed in Table 6.5. We clearly observe the disproportionately high increase of the guarantee option compared with the contract value.

OU MUST - case	$\sigma_r = 1\%, \sigma_S = 7\%$	$\sigma_r = 2\%, \sigma_S = 9\%$	$\sigma_r = 3\%, \sigma_S = 11\%$
RN value EUR_MC	10,402.6	11,079.7	11,918.0
guarantee option	1,027.1	1,989.5	3,134.9
OU IS - case	$\sigma_r = 1\%, \sigma_S = 7\%$	$\sigma_r = 2\%, \sigma_S = 9\%$	$\sigma_r = 3\%, \sigma_S = 11\%$
RN value EUR_MC	10,996.3	11,768.5	12,759.0
guarantee option	1,160.7	2,123.2	3,282.9

Table 6.5: Influence of σ_r and σ_S on the guarantee option

We are particularly interested in parameter combinations that lead to the same contract value. Besides the parameter pairs that lead to the value of a "standard contract", i.e. a contract with parameters as discussed at the beginning of this section, the parameter combinations, that lead to a "fair" contract, are of interest. We relate a "fair" contract to an insurance policy for which the initial investment and the contract value correspond (see Chapter 2). Thus, considering an initial investment of 10,000 units leads to a fair contract value of 10,000 and a standard contract value of 10,497.1 units in the Ornstein-Uhlenbeck MUST-case and 11,092.5 units in the Ornstein-Uhlenbeck IS-case. To determine the combinations of σ_r and σ_S that lead to the value of a fair and a standard contract, we cut the plane in Figure 6.5 with the corresponding planes parallel to the $\sigma_S \times \sigma_r$ plane. The resulting plot is given in Figure 6.6.

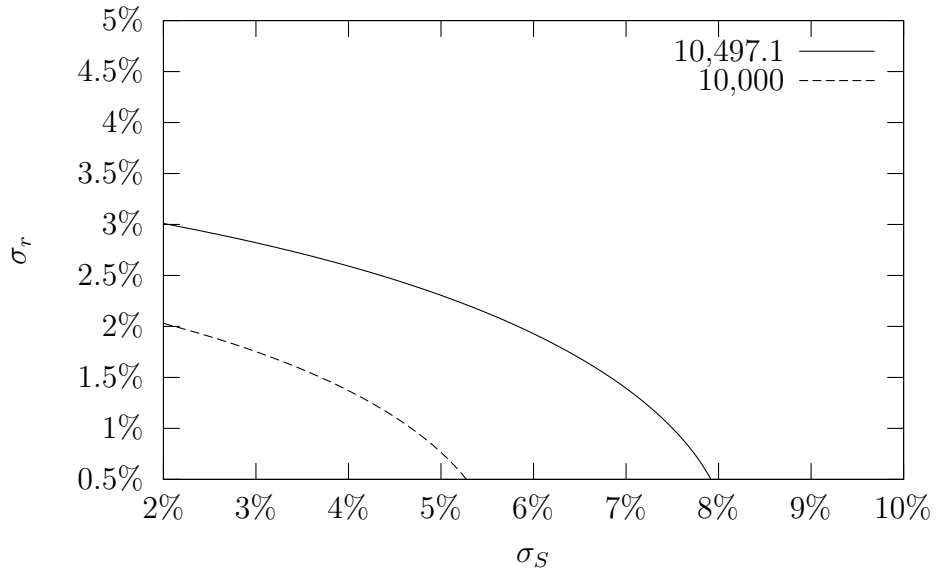


Figure 6.6: Parameter combinations of σ_r and σ_S, OU-MUST

We notice that if we increase the volatility σ_S of the asset process and if we are

interested in keeping the contract value on the same level, then the volatility of the interest rate process has to be lower. However, if σ_r and σ_S exceed 3% and 8% respectively, neither a standard contract nor a fair contract can be created. A fair contract even requires σ_r and σ_S to remain beneath 2% and 5.5%, respectively. This indicates that on the one hand, the insurance company has to make sure that the asset portfolio is not too volatile, which means to invest more in low risk assets such as bonds and money market rather than high-risk assets such as stock. On the other hand, the instantaneous money market short rate must not be too volatile. Hence, if a rising volatility for the money market interest rate is observed, the insurance company should decrease the portion of stock in its asset portfolio in order to decrease the asset volatility σ_S. In times of a low volatility on the money market, the portion of stock and other risky investments with moderate volatilities can be increased. However, even with a very low interest rate volatility or even a constant interest rate, the 8% - bound for the volatility σ_S remains. The results in the IS-case are similar.

At this point, though, it has to be mentioned that in order to properly take changes of the volatility σ_S into consideration, we have to adjust the parameter ρ in equal measure, since a change in volatility implicitly affects the composition of the asset portfolio and thus the correlation between asset portfolio and money market. However, we will postpone this matter to the end of this chapter.

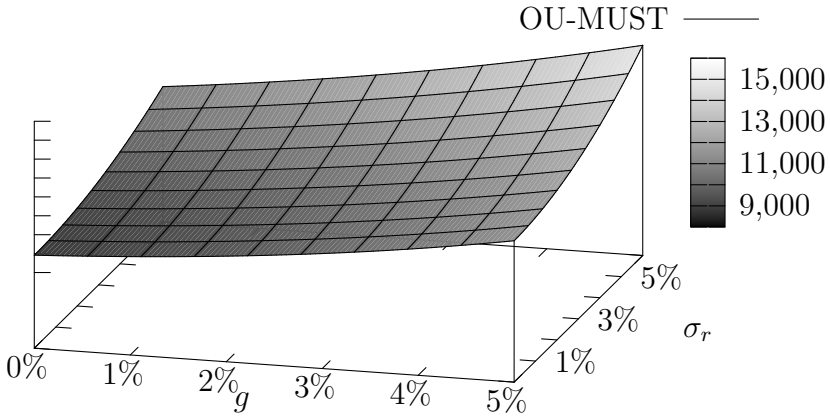


Figure 6.7: Influence of g and σ_r on the contract value, OU-MUST

We have already discussed the influence of the guaranteed interest rate g on the value of the insurance contract. However, we have not yet shown how g interacts

with the interest rate volatility σ_r . These interactions are important since they assess how the volatility in the money market influences the guarantees.

Figures 6.7 and 6.8 show similar interactions in the OU MUST- and the OU IS-case. With rising g and σ_r , the value of the insurance contract is increased as well.

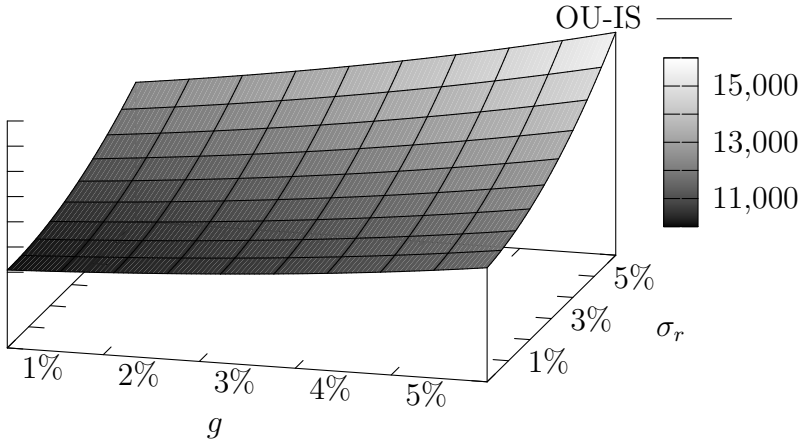


Figure 6.8: Influence of g and σ_r on the contract value, OU-IS

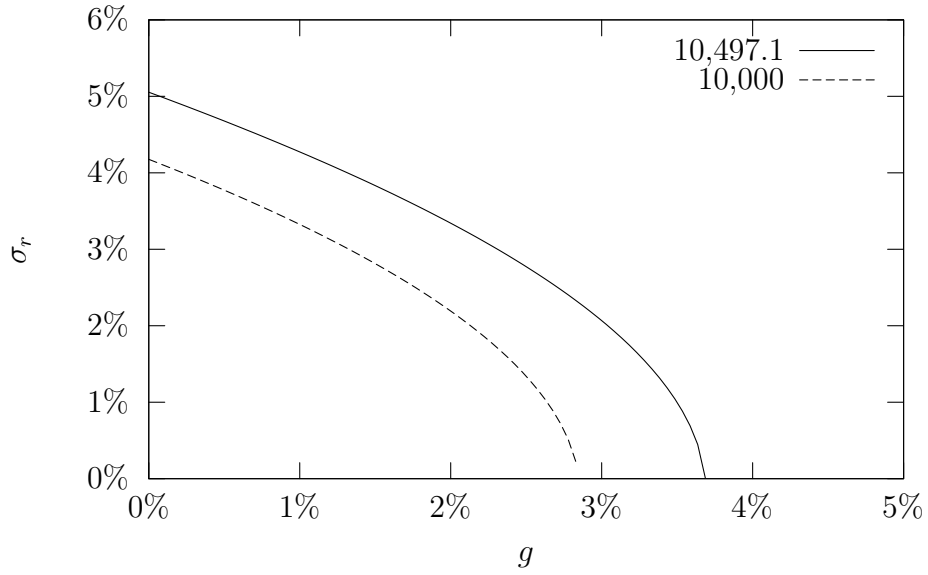


Figure 6.9: Parameter combinations of g and σ_r , OU-MUST

We further observe that in the IS-case, due to the influence of the target interest rate z, the influence of the guaranteed rate of interest is not as strong as in the MUST-case (see also Figure 6.2).

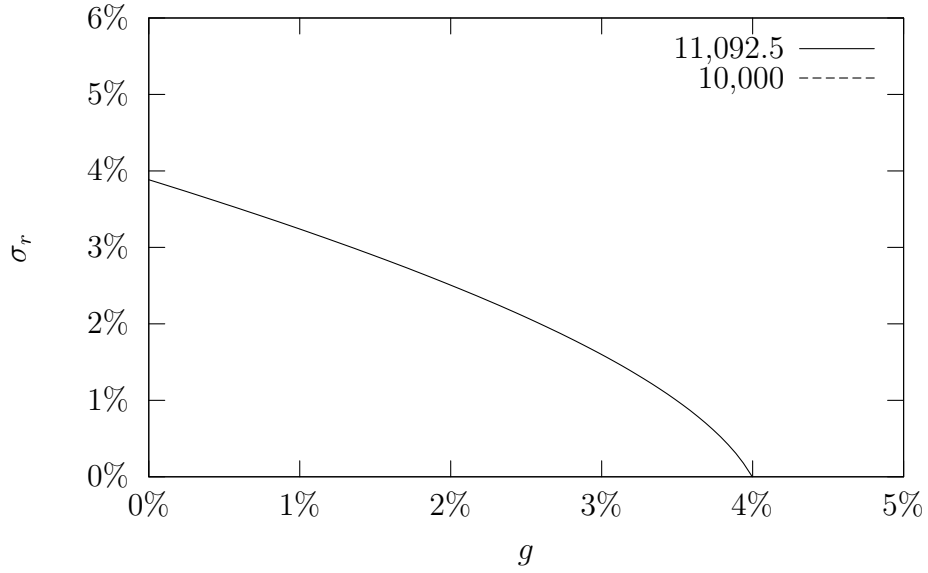


Figure 6.10: Parameter combinations of g and σ_r, OU-IS

Figure 6.9 presents the combinations of g and σ_r that lead to a fair contract with value of $10,000$ units and a standard contract with value of $10,497.1$ units for an Ornstein-Uhlenbeck short rate in the MUST-case. We observe that if the guaranteed interest rate g exceeds the current minimum guaranteed interest rate of 2.75%, then even with the volatility of the short rate approaching zero a fair contract cannot be generated. For a standard contract, g must not exceed 3.7%. Figure 6.10 presents the parameter combinations in the IS-case. We notice that there are no combinations that lead to a fair contract, and in order to generate a standard contract, g must not exceed 4%.

To assess how money market volatility and the target rate z, which can be altered by the management, jointly influence the contract value, we analyze the interaction of z and σ_r (see Figure 6.11). We observe that an increase in both, z and σ_r, goes along with an increase in contract value.

Figure 6.12 illustrates the pairs of z and σ_r that lead to a fair contract and to a contract with standard parameters. We find that if z is considerably higher than 5.5%, then even an interest volatility of zero cannot lead to the value of a standard contract. There are no combinations that lead to a fair contract with a value of

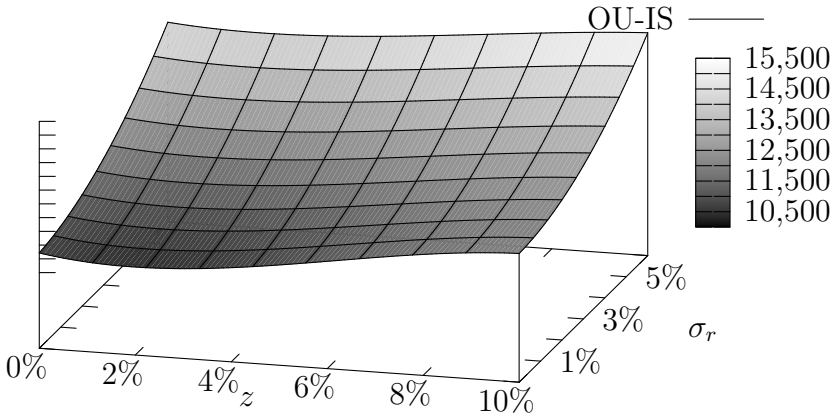


Figure 6.11: Influence of z and σ_r on the contract value, OU-IS

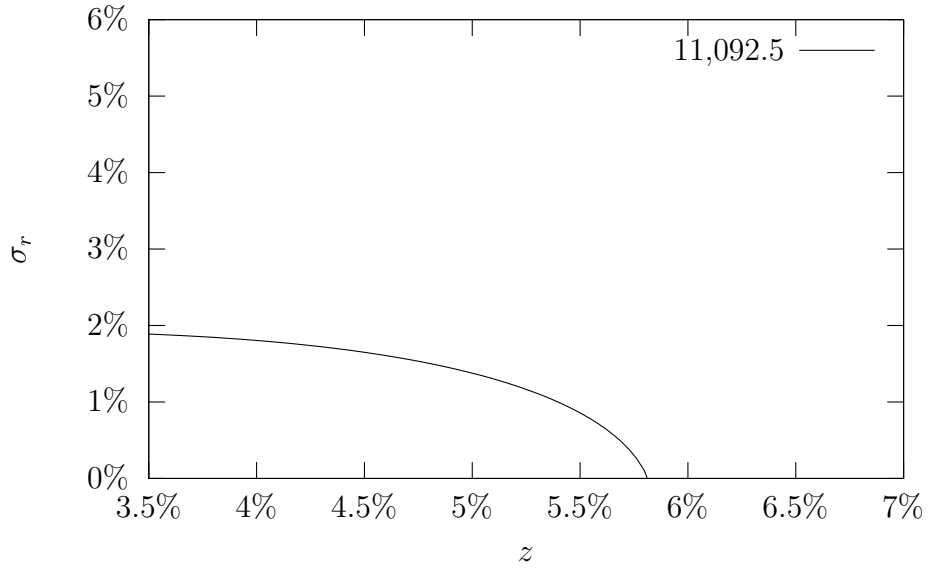


Figure 6.12: Parameter combinations of z and σ_r, OU-IS

10,000 units.

To see how market uncertainty and financial strength relate with respect to their influence on the contract value, Figure 6.13 presents the joint influence of the initial reserve quota x_0 and the volatility σ_r of the instantaneous short rate for the Ornstein-Uhlenbeck MUST-case. Obviously, with increasing σ_r and rising x_0 the contract value is increased as well. We further notice that with increasing initial re-

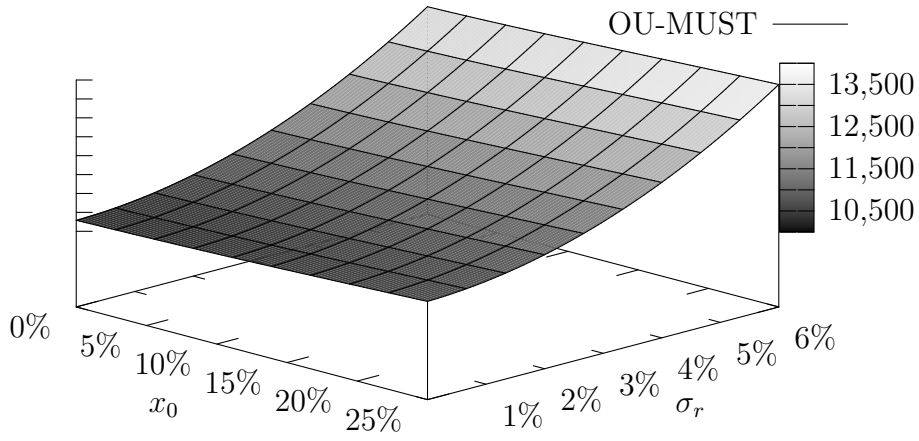


Figure 6.13: Influence of x_0 and σ_r on the contract value, OU-MUST

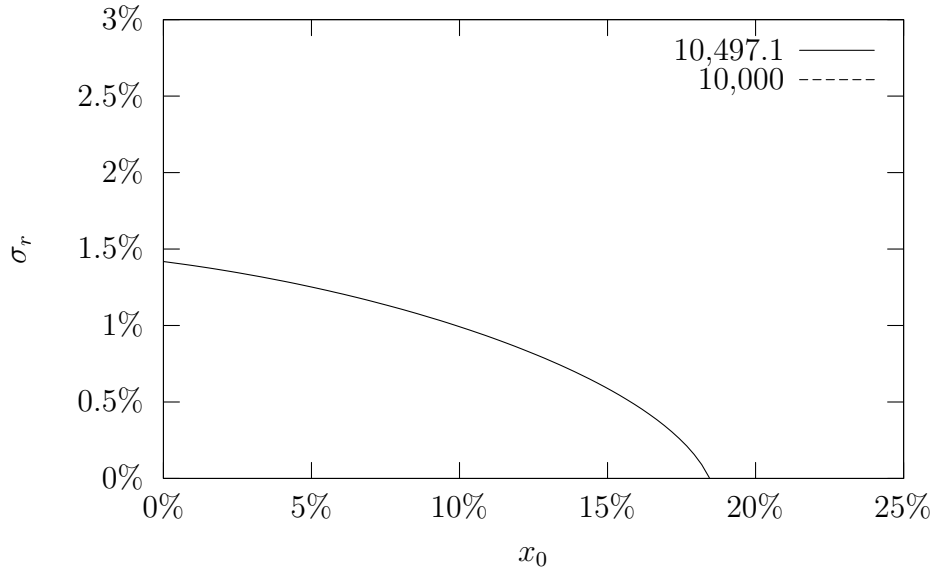


Figure 6.14: Parameter combinations of x_0 and σ_r, OU-MUST

serve quota, the volatility plays a minor role for the value of the contract, i.e. with increasing initial reserve quota, the sensitivity of the value of the contract to changes in the volatility decreases.

This shows that companies with high reserves do not depend as much on changes in the money market interest rate r as companies with low reserves do or, in other words, that financially strong companies are less sensitive to market vacillations.

Figure 6.14 furthermore displays that.

An alarming fact is that with the chosen parameter combination, there are no pairs of x_0 and σ_r which provide the opportunity of generating a fair contract in the MUST-case, i.e. a contract with value of $10,000$ units. Therefore, if the insurance company offers a contract with initial investment of $10,000$ units, then this contract is, ceteris paribus, always underpriced.

By varying the volatility σ_S of the asset process, i.e. by deciding how volatile the asset portfolio is going to be, the contract value can be altered considerably[7]. When examining the interaction of the initial reserve quota x_0 and the volatility of the reference portfolio σ_S in the OU MUST-case, we observe that with rising initial reserve quota and rising volatility, the value of the insurance contract increases as well (see Figure 6.15). This seems evident, since a rising volatility presents a higher

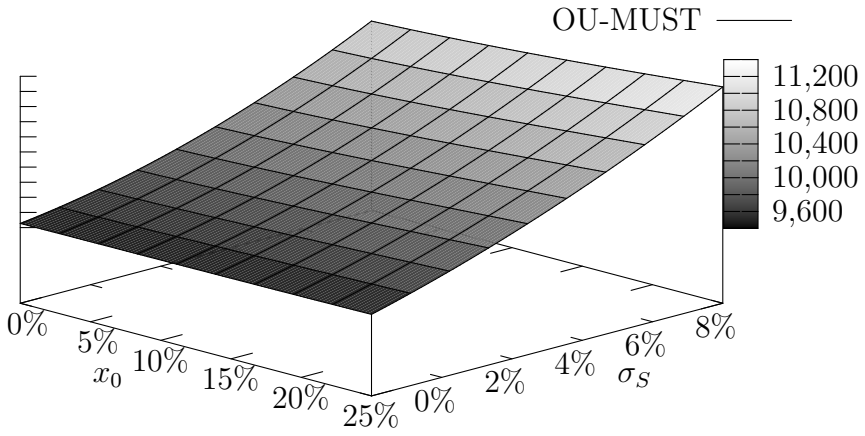


Figure 6.15: Influence of x_0 and σ_S on the contract value, OU-MUST

risk of unfavorable asset returns and thus an increased value of the interest rate guarantee; a rising initial reserve quota leads to a higher amount of money at the interest return of which the policyholders can participate. Both result in a higher contract value.

Figure 6.16 presents parameter combinations that lead to identical contract values in the OU MUST-case, again for insurance policies with a value of 10,000 and 10,497.1 units, respectively. We observe an almost linear relation: in the considered interval,

[7]Note again, that ideally ρ should be altered with σ_S which is postponed to the next section.

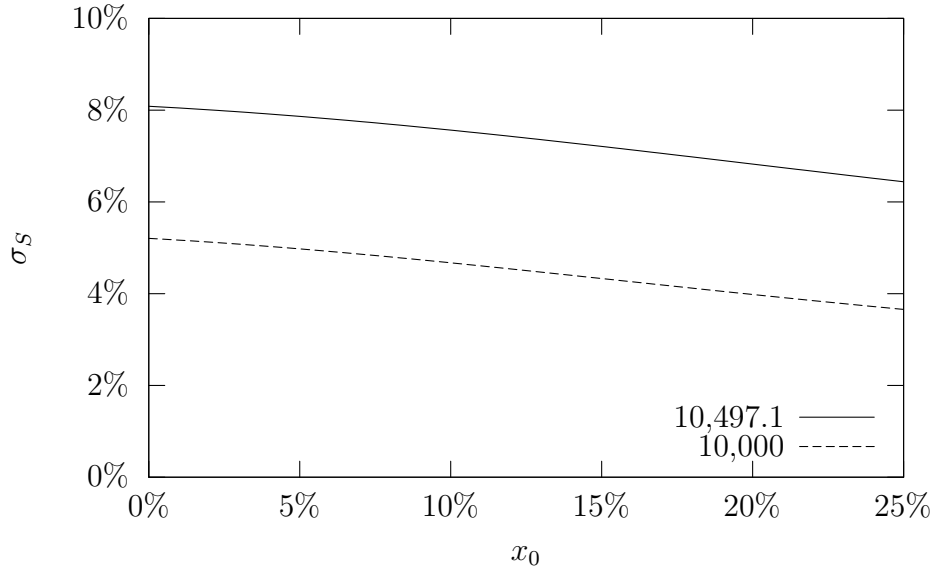


Figure 6.16: Parameter combinations of x_0 and σ_S, OU-MUST

changes in the asset volatility of 0.5% can be compensated by changes in the initial reserve quota of about 10%. All in all, we notice that the contract value is not very sensitive to changes of the initial reserve quota in the MUST-case. However, in the IS-case the contract value increases considerably with x_0 and σ_S (see Figure 6.17). We

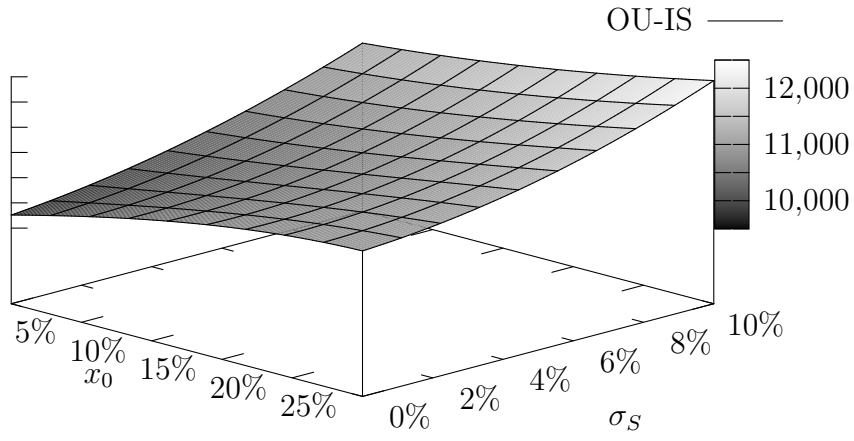


Figure 6.17: Influence of x_0 and σ_S on the contract value, OU-IS

notice that the influence of the volatility seems to be slightly smaller when the initial reserve quota is higher and deduce that an insurance company with high reserves

is able to generate high contract values even with a low-volatility portfolio. This is favorable for the companies, since with such low-risk portfolios it is less likely that capital shots are needed. However, since this high contract value is mainly financed by the difference of initial and final reserves, future customers may not savor the same advantages.

Figure 6.18 illustrates that in the IS-case a fair contract can only be obtained with a very small initial reserve quota and a reference portfolio with very low volatility. For $\sigma_S > 3\%$ or $x_0 > 3\%$, it is not possible to generate a contract with value of 10,000 units at all.

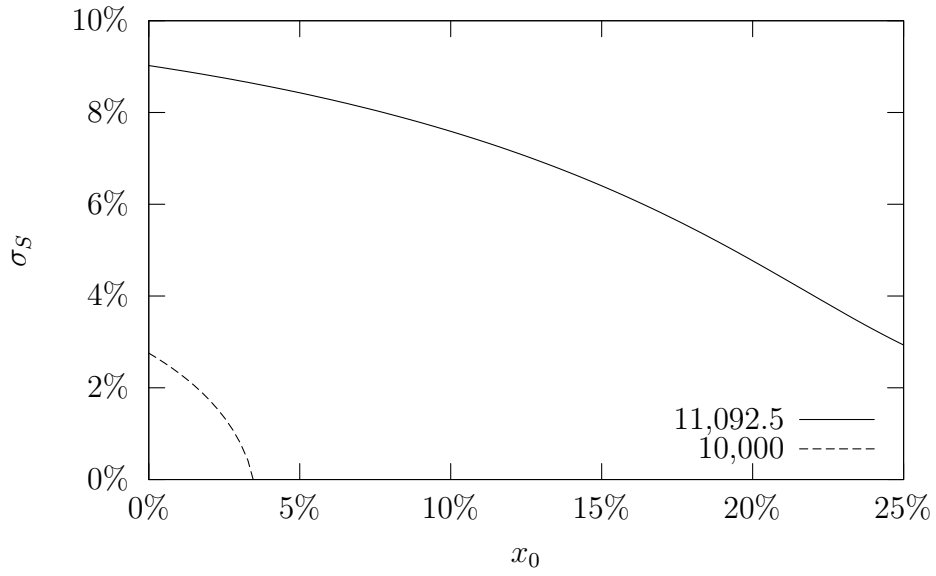


Figure 6.18: Parameter combinations of x_0 and σ_S, OU-IS

The fact that a change in volatility at a higher reserve quota is of less importance than at a low reserve quota is reflected by the concave shape of the curves. Ceteris paribus, a decrease in reserve quota from 20% to 15% has to be compensated by an increase in volatility σ_S of the reference portfolio of about 1.3% to provide a standard contract with value of 11,092.5 units, whereas if the reserve quota is changed from 10% to 5%, the volatility has to be increased by merely 0.8% to keep the price at 11,092.5 units.

To get an idea about the influence of volatilities at different guarantee levels, Figure 6.19 presents the interaction between the guaranteed interest rate g and the volatility of the asset portfolio σ_S in the OU MUST-case. We observe that the contract value rises with g and σ_S.

Again, we are interested in parameter combinations that lead to a fair contract with

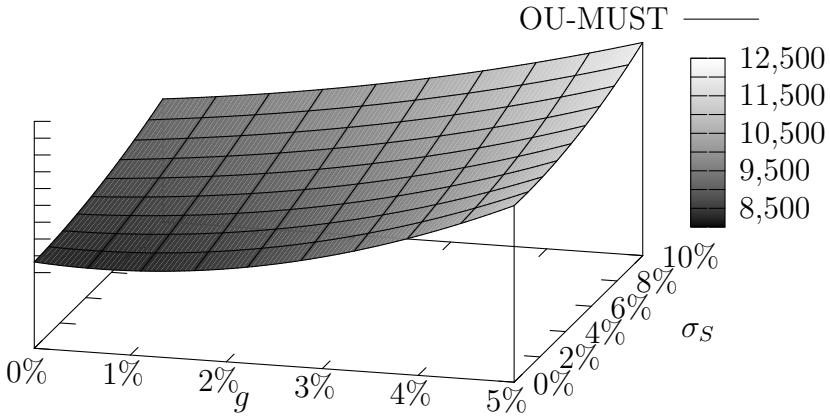


Figure 6.19: Influence of g and σ_S on the contract value, OU-MUST

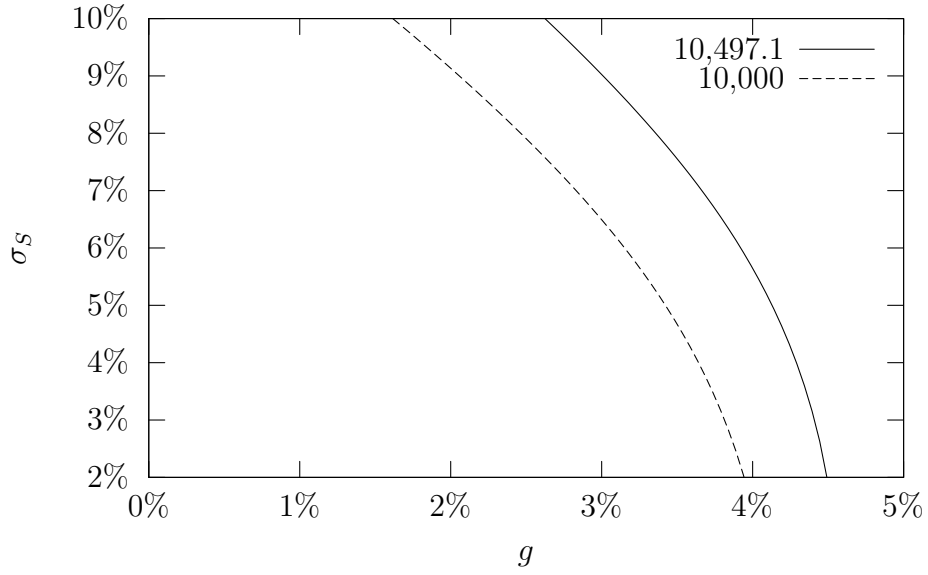


Figure 6.20: Parameter combinations of g and σ_S, OU-MUST

value of $10,000$ units and a standard contract with value of $10,497.1$ units in the MUST-case. Figure 6.20 presents these combinations. An increase of the guaranteed interest rate from 3% to 3.5% has to be compensated by a decrease of the volatility from 6.5% to 5% to maintain a value of $10,000$ units and from 9% to 7.5% to maintain the level of $10,497.1$ units. For the current guaranteed interest rate $g = 2.75\%$, the volatility of the asset portfolio must not exceed 7% in order to provide a fair contract.

Another interesting parameter combination is that of σ_S and the target interest rate z, since these are the parameters that can be adjusted by the company's management. Figure 6.21 presents their influence on the risk-neutral value of the insurance contract.

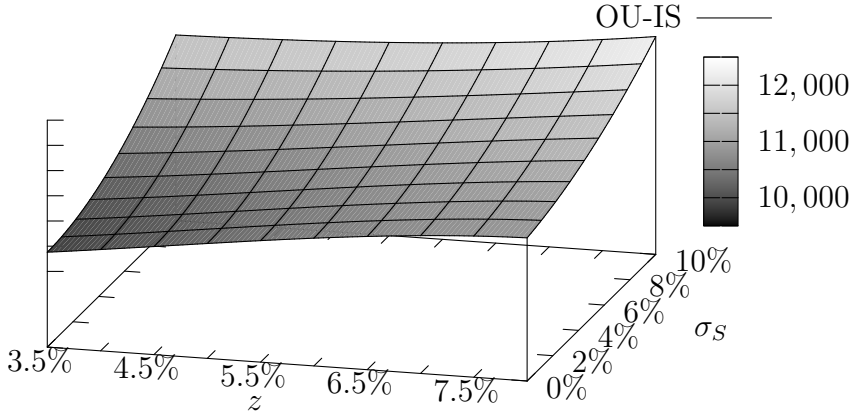


Figure 6.21: Influence of z and σ_S on the contract value, OU-IS

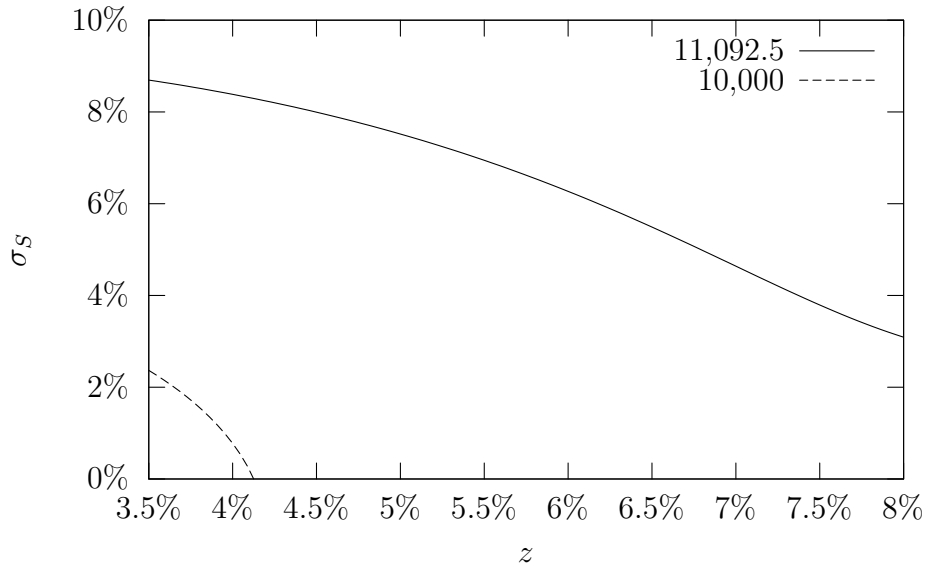


Figure 6.22: Parameter combinations of z and σ_S, OU-IS

At a small target interest rate z, a change in volatility, and hence a change in the structure of the asset portfolio has in general a greater impact on the value of

the contract; this is because the smaller the target rate z, the more relevant the minimum participation rate δ, since it is then less likely that the obligatory 90 % are already passed on to the policyholders, whereas if the target interest rate is high, more than $\delta = 90\%$ are passed on anyway.

Figure 6.22 shows pairs (z, σ_S) that lead to the same contract values, again for contracts with values of 10,000 units and 11,092.5 units, respectively. To be able to generate a fair contract, σ_S must not exceed about 2% and z has to remain beneath about 4%, which is very low compared to the standard values $\sigma_S = 7.5\%$ and $z = 5\%$. For the standard contract, we notice that for σ_S ranging from between 6% and 8%, the relation between σ_S and z is almost linear. A change in the volatility of 3% can be compensated by a change in the target rate of about 2%.

The Influence of the Correlation between Interest Rate and Asset

The monthly report of the German Central Bank from February 2004 (see [12]) states that in the recent past, stock and money market were negatively correlated. The report explains this by the fact that periods with negative correlation often come along with strong fluctuations in stock markets. In such times of uncertainty, investors often show the so-called "flight-to-quality" behavior, which means that they rearrange their investments from riskier to less risky investments. If we therefore constructed a portfolio consisting of a high portion of stock and the rest money market, the correlation between asset return and money market return would be negative. However, as mentioned earlier, we consider an asset portfolio consisting of about $10 - 15\%$ stock and the rest money market. Hence, the correlation between asset return and money market return should be positive.

So far, we have chosen the model parameters so that it has been possible to compare our results with those presented in [6], i.e. the initial interest rate $r_0 = 4\%$ has been chosen correspondingly to the constant short rate r and the asset volatility $\sigma_S = 7.5\%$ correspondingly to the asset volatility in [6]. Moreover, we have chosen $\sigma_r = 1\%$ as in [9]. For ρ we have used the rough estimate $\rho = 0.5$. However, in [6] no empirical studies were made which led to the chosen parameters.

In what follows we use annual DAX and money market returns from 1980 to 2005 to obtain estimates for the parameters r_0, ρ, σ_r, and σ_S. Let $\hat{\rho}$ denote the correlation between returns of the money market and returns of the asset process, $\hat{\sigma}_S$ the standard deviation of the log returns of the asset process, $\hat{\sigma}_r$ the standard deviation of the log returns of the money market and $\hat{r}_0$ the expected log return of the money market.

	10% **stock**	15% **stock**	25% **stock**
$\hat{r}_0$	5.0%	5.0%	5.0%
$\hat{\sigma}_r$	2.4%	2.4%	2.4%
$\hat{\sigma}_S$	3.2%	4.2%	6.5%
$\hat{\rho}$	0.59	0.39	0.17

Table 6.6: Estimations for $\hat{\rho}, \hat{\sigma}_S, \hat{\sigma}_r$, and $\hat{r}_0$.

The estimates of $\hat{\rho}, \hat{\sigma}_S, \hat{\sigma}_r$, and $\hat{r}_0$ are given in Table 6.6 for asset portfolios with different portions of stock. We clearly observe that with rising stock portion, the asset portfolio becomes more volatile and the correlation between asset portfolio and interest rate decreases.

However, the estimated parameters $\hat{\rho}, \hat{\sigma}_S, \hat{\sigma}_r$, and $\hat{r}_0$ do not correspond to the actual parameters ρ, σ_r, σ_S, and r_0, but are given by certain functions. $\hat{\rho}$ for example is described by a function $\hat{\rho} = h(\rho, \sigma_S, \sigma_r, \kappa, \xi)$.[8] The estimated actual parameters for stock portions of 10% and 15%, respectively, are presented in Table 6.7 along with the originally used parameters.

	original parameters	10% **stock**	15% **stock**
r_0	4.0%	5.0%	5.0%
σ_r	1.0%	4.3%	4.3%
σ_S	7.5%	3.2%	4.2%
ρ	0.5	0.67	0.44

Table 6.7: Estimations for ρ, σ_S, σ_r, and r_0.

We observe that according to the used data, the assumed interest rate volatility is much too low. Moreover, the estimated initial interest rate is higher than the assumed 4% and significantly higher than the actual money market rate which currently lies at about 2.63% (April 2006). Both high volatility and deviation in r_0 are explained by the extreme development of the money market since the 1980s. Market rates have fallen considerably and the insurer's portfolio still contains bonds with higher yields. The chosen correlation almost resembles the actual correlation, since $\rho = 0.5$ corresponds to a stock portion somewhere between $10 - 15\%$, which is acceptable regarding the current average of 10% in the German life insurance market (see [2]). The chosen asset volatility $\sigma_S = 7.5\%$ is a little too high.

[8] For the derivation of the functions and the used data, see Appendix A.

However, it must be pointed out here that the used estimation methods are very rough and the used data sample is rather small. Therefore, the estimations may not be very accurate and in order to estimate the parameters more accurately, certainly more sophisticated estimation methods have to be used. However, the target of this thesis is not to estimate parameters, but to study their significance on the value of insurance contracts.

The influence of the correlation between interest rate and asset is connected to the influence of the volatility of the asset process since, if the portfolio structure is altered, both the volatility, as the "riskiness" of the portfolio and the correlation change. Therefore it is not easy to determine how the correlation itself influences the risk-neutral value of the insurance contract. We therefore examine the influences of different portions of stock on the contract value which implicitly implies different correlations and asset volatilities.

According to our estimations, we assume for the following an interest rate volatility of $\sigma_r = 4.3\%$ (OU) and an initial interest rate of $r_0 = \xi = 5\%$. The risk-neutral value of the insurance contract is calculated for different portions of stock and hence for different values of ρ and σ_S. The data is fitted by a function $f : \mathbb{R} \to \mathbb{R}$ and the plot for the OU MUST - case Ornstein-Uhlenbeck is given in Figure 6.23.

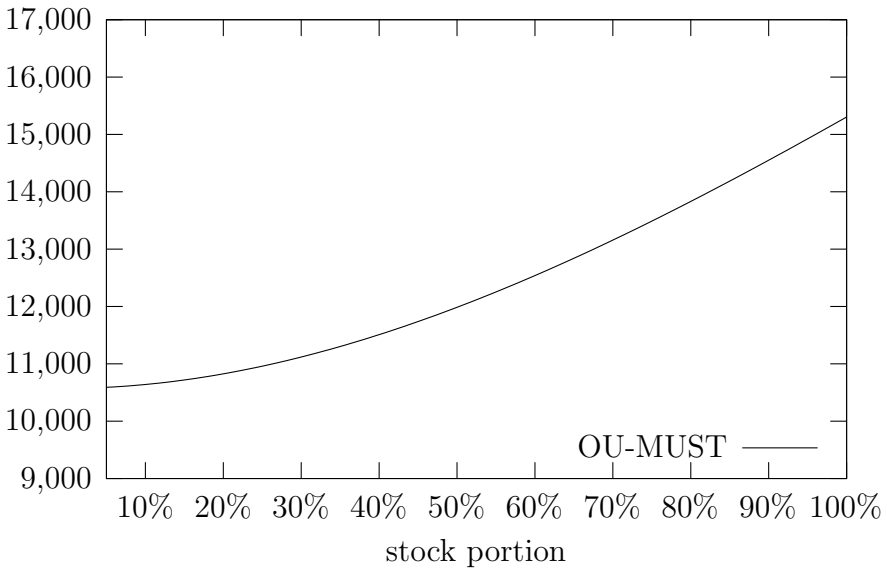


Figure 6.23: Influence of the stock portion on the contract value

Table 6.8 further displays how the correlation changes with the stock portion in the portfolio.

stock portion	0%	5%	10%	15%	20%	25%	30%	40%	50%	60%
correlation $\hat{\rho}$	1	0.86	0.59	0.39	0.25	0.17	0.11	0.03	-0.02	-0.05

Table 6.8: Stock portion/correlation

We observe that with rising stock portion in the asset portfolio, the contract value increases, which is due to the increasing risk of unfavorable asset returns represented by an increasing asset volatility σ_S. If the portfolio consists of 10% stock, which is regarded realistic, the contract value amounts to 10,635.8 units. Even with a stock portion of 0%, it is not possible to create a "fair" contract with value of 10,000 units. This is mainly due to the high estimated volatility of the interest rates.

Table 6.9 shows how the contract value and the implicit options change if we use the estimated parameters compared with the originally used parameters. We clearly see that the rise in contract value for the estimated parameters is mainly due to the rising interest rate guarantee option: For the OU MUST - case, the value of the contract using the estimated parameters is about 140 units higher than the value of the contract with the original parameter choice, whereas the value of the guarantee option rises by over 700 units!

OU MUST-case, $g = 3.5\%$	**original param.**	**estim. param.,** 10% **stock**
RN value EUR_MC	10,497.0	10,635.1
Interest rate guarantee	1,150.1	1,887.3
Value of dividends	252.6	278.7
Discounted final reserve	1,400.5	1,973.5
OU IS-case, $g = 3.5\%$	**original param.**	**estim. param.,** 10% **stock**
RN value EUR_MC	11,092.4	11,576.4
Interest rate guarantee	1,283.3	2,049.9
Value of dividends	82.7	116.4
Discounted final reserve	1,108.2	1,357.1

Table 6.9: Contract values with original parameters/estimated parameters

We have now presented the interaction and influence of several pairs of parameters on the contract value. Since these are shown to be very significant, the companies' managements should take parameter interactions into account in their decisions. Since the implicit options, in particular the interest rate guarantee, have a significant influence on the value of the contract, it is necessary for the companies to hedge those options.

Chapter 7

Summary, problems, and prospect

This thesis presents a valuation model for German life insurance contracts which, in particular, allows for a stochastic evolution of interest rates. In order to focus on the basic effects, only a very simple kind of insurance contract, namely a term fix contract with a single up-front premium is considered. However, more complex contracts could be included in the model. We present two different bonus distribution schemes for the insurance contract, the MUST-case considering only compulsory payments due to legal and regulatory requirements, and the IS-case in which additionally corporate political decisions are taken into account.

The life insurance contract is valuated and analyzed using methods from modern financial mathematics, which require that the prerequisites for risk-neutral valuation are fulfilled. In particular, a specified underlying security and an equivalent martingale measure must exist. Additionally, in order to employ the results from the risk-neutral valuation for risk management purposes, it must be possible for the insurer to hedge its liabilities. Due to the legal situation and the special features of the German insurance industry, these requirements are not automatically fulfilled. This problem is encountered by using a cash-flow model which makes it possible to apply the the concept of risk-neutral valuation and, in particular, to price and hedge the implicit options separately.

For the instantaneous short rate two different stochastic short rate models are considered: an Ornstein-Uhlenbeck model and a Cox-Ingersoll-Ross model. The Ornstein-Uhlenbeck process is easier to handle, since the respective stochastic differential equation has a closed form solution. However, the process can take negative values, which may limit the applicability of the model. The Cox-Ingersoll-Ross short rate does not become negative and therefore describes a "real-world" interest rate in a better way. However, the CIR - model is more delicate to handle.

The insurance contract itself and the implicit options are complex, path-dependent derivatives. Hence, it is not possible to obtain closed form solutions for their risk-neutral values and numerical methods have to be applied. The thesis presents an "exact" Monte Carlo algorithm to price the contract for the Ornstein-Uhlenbeck short rate model; the algorithm is "exact" in the sense, that the distributions of the involved quantities are explicitly known and thus the concerned random variables can be simulated explicitly. In addition, a "discretized" Monte Carlo algorithm is introduced, which provides the opportunity of pricing the contract, even if the distribution of the respective quantities is not known explicitly. In particular, the "discretized" Monte Carlo algorithm is used for the Cox-Ingersoll-Ross process. The Monte Carlo methods admit the valuation of the implicit options, whereas the valuation of non-European contracts, i.e. contracts including a surrender option, proves to be difficult with Monte Carlo methods.

Thus, a second approach is presented which allows for the valuation of Bermuda style walk-away options in non-European contracts: a type of Black Scholes partial differential equation is derived, the solution of which is employed to implement an algorithm determining the risk-neutral value of the insurance contract. The PDE is solved numerically using a finite difference scheme, and the risk-neutral value of the insurance contract is calculated via the approximation of the value function on a discrete lattice. Due to the complexity, the calculations are restricted to Ornstein-Uhlenbeck short rates.

The algorithms for the evaluation are implemented in the object-oriented programming language `C++`, which enabled us to implement the two Monte Carlo methods as well as the discrete lattice method in a single program. The program structure is chosen in a way, that other distribution mechanisms, other interest rate models, and other valuation methods can be easily integrated.

Besides calculating contract values, sensitivity analyses with respect to the most important parameters are performed. We focus on the parameters that come into play due to the stochasticity of the interest rate. Although the influence of the other model parameters is not less interesting, their behavior is only discussed briefly, since this has already been studied in detail by Bauer (see [6]) and most of his insights remain valid in a stochastic interest rate environment.

It turns out that due to the additional source of uncertainty in the model, for a comparable parameter choice the risk-neutral value of an insurance contract with stochastic short rates always exceeds the value of a contract with a constant or deterministic short rate. With rising volatility of the interest rate process, the contract value rises. Even though the values under stochastic and constant interest rates do

not differ tremendously for realistic choices for the interest rate volatility, the decomposition into the various embedded options is altered considerably. In particular, the value of the interest rate guarantee is increased above average. Furthermore, we observe that under the influence of stochastic short rates, the value of the insurance contract as a whole exceeds the initial premium paid by the insured person for realistic parameter choices and thus the insurance products seem to be underpriced. This is alarming and it does not seem surprising that many companies have gotten into financial troubles.

The incorporation of stochastic interest rates makes the model more realistic, since market interest rates do not remain constant over the long lifetimes of insurance contracts. However, it is very difficult to choose an adequate model and, within a given interest rate model, to calibrate the parameters adequately. For instance, the correlation between stock returns and money market returns can change considerably over the years. The monthly report of the German Central Bank from February 2004 (see [12]) states that recent developments at the German capital markets were characterized by an opposite development of stock and bond markets. However, it also shows that this negative correlation is rather an exception in a long-time comparison.

Furthermore, our empirical studies show that the distribution of the log returns of the asset process might differ from the assumed normal distribution. However, a more thorough and detailed study is necessary in order to assess the distributional properties of an insurer's asset portfolio. One of the next steps could be to consider other processes to model the asset portfolio. We could further extend the model by considering an asset portfolio consisting of several different asset processes including the modelling of bonds, real estate etc. instead of a single asset process whose composition is described via correlations. Furthermore, in order to obtain a more applicable model, it would be interesting to determine hedging strategies for the insurance contract and for the implicit options. We could additionally include more complex insurance contracts such as, for example, whole life insurances or pure endowment insurances.

All in all, the thesis models and prices German life insurance contracts under the influence of stochastic short rates. It gives insights into the interaction of the different factors that influence the contract and helps to understand the risks that come along with the insurer's liabilities. Furthermore, the thesis presents ways of managing these risks and offers a solid basis for further extensions.

Appendix A

Determination of $\rho, \sigma_r, \sigma_S,$ and r_0

In Chapter 6, we denoted by $\hat{\rho}$ the correlation between returns of the money market and returns of the asset process, by $\hat{\sigma}_S$ the standard deviation of the log returns of the asset process, by $\hat{\sigma}_r$ the standard deviation of the log returns of the money market and by $\hat{r}_0$ the expected log return of the money market.

In formulas, we have

$$\begin{aligned}
\hat{\sigma}_S &= \sqrt{\mathrm{Var}\left[\log\left(\frac{S_t^-}{S_{t-1}^+}\right)\right]}, \\
\hat{\sigma}_r &= \sqrt{\mathrm{Var}\left[\log\left(\frac{B_t}{B_{t-1}}\right)\right]}, \text{ and} \\
\hat{r}_0 &= \mathbb{E}\left[\log\left(\frac{B_t}{B_{t-1}}\right)\right]. \qquad \text{(A.1)}
\end{aligned}$$

Furthermore,

$$\hat{\rho} = \frac{\mathrm{Cov}\left(r_t^r, r_t^S\right)}{\sqrt{\mathrm{Var}(r_t^r)\mathrm{Var}(r_t^S)}}, \qquad \text{(A.2)}$$

where r_t^r denotes the return on the interest rate r in the time period $[t-1, t)$, i.e.

$$r_t^r = \frac{B_t - B_{t-1}}{B_{t-1}} = \exp\left(\int_{t-1}^{t} r_u \,\mathrm{d}u\right) - 1,$$

and r_t^S denotes the return on the asset process S in $[t-1, t)$, i.e.

$$r_t^S = \frac{S_t^- - S_{t-1}^+}{S_{t-1}^+} = \exp\left(\mu - \frac{\sigma_S^2}{2} + \int_{t-1}^{t} \rho\sigma_S \,\mathrm{d}W_u + \int_{t-1}^{t} \sqrt{1-\rho^2}\sigma_S \,\mathrm{d}Z_u\right) - 1.$$

For the following considerations we have to restrict ourselves to Ornstein-Uhlenbeck short rates, since we need the exact distributions of the involved processes and these cannot be obtained for Cox-Ingersoll-Ross short rates. Thus, using the results also for an asset process with Cox-Ingersoll-Ross short rates is only a rough approximation.

For the subsequent considerations we need the following definition (see [39]):

Definition A.0.1 *The* moment generating function *of a random variable* X *is given by the function*

$$\mathrm{M}_X(t) \;=\; \mathbb{E}\left(e^{tX}\right),\; t \in \mathbb{R}.$$

If $\tilde{X}$ *is normally distributed, that is* $\tilde{X} \sim N(\mu, \sigma)$, *then its moment generating function is given by*

$$\mathrm{M}_{\tilde{X}}(t) \;=\; \exp\left(\mu t + \sigma^2 \frac{t^2}{2}\right).$$

Then, using results from Chapter 4 and Chapter 5, we obtain that at time t, regarding the fact that $r_0 = \xi$, we have

$$\underbrace{\int_{t-1}^{t} r_u \,\mathrm{d}u}_{=:N_1} \;\sim\; N\left(\xi, \underbrace{\frac{\sigma_r^2\left(2\kappa - 3 + 4e^{-\kappa} - e^{-2\kappa}\right)}{2\kappa^3}}_{:=\sigma_1^2}\right),$$

and thus,

$$\begin{aligned} r_0 \;&=\; \xi \;=\; \hat{r}_0, \text{ and} \\ \sigma_r \;&=\; \sqrt{\frac{2\kappa^3\sigma_1^2}{\left(2\kappa - 3 + 4e^{-\kappa} - e^{-2\kappa}\right)}} \;=\; \sqrt{\frac{2\kappa^3\hat{\sigma}_r^2}{\left(2\kappa - 3 + 4e^{-\kappa} - e^{-2\kappa}\right)}}. \end{aligned} \tag{A.3}$$

Furthermore,

$$\begin{aligned} \mathbb{E}\left(r_t^r\right) \;&=\; \mathbb{E}\left(e^{N_1}\right) - 1 \;=\; \exp\left(\xi + \frac{\sigma_1^2}{2}\right) - 1, \\ \mathbb{E}\left[\left(r_t^r\right)^2\right] \;&=\; \mathbb{E}\left[\exp\left(\int_{t-1}^{t} r_u \,\mathrm{d}u\right) - 1\right]^2 \\ &=\; \mathbb{E}\left[\exp\left(2N_1\right) - 2\exp(N_1) + 1\right] \\ &=\; \exp\left(2\xi + 2\sigma_1^2\right) - 2\exp\left(\xi + \frac{\sigma_1^2}{2}\right) + 1, \end{aligned}$$

and hence,

$$\begin{aligned}\mathrm{Var}\left(r_t^r\right) &= \mathbb{E}\left[\left(r_t^r\right)^2\right] - \left[\mathbb{E}\left(r_t^r\right)\right]^2 \\ &= \exp\left(2\xi + 2\sigma_1^2\right) - 2\exp\left(\xi + \frac{\sigma_1^2}{2}\right) + 1 - \left(\exp\left(\xi + \frac{\sigma_1^2}{2}\right) - 1\right)^2 \\ &= \exp\left(2\xi + 2\sigma_1^2\right) - \exp\left(2\xi + \sigma_1^2\right). \end{aligned} \tag{A.4}$$

For

$$\begin{aligned} r_t^S &= \exp\left(\mu - \frac{\sigma_S^2}{2} + \int_{t-1}^{t} \rho\sigma_S \,\mathrm{d}W_u + \int_{t-1}^{t} \sqrt{1-\rho^2}\sigma_S \,\mathrm{d}Z_u\right) - 1 \\ &= \exp\underbrace{\left(\mu - \frac{\sigma_S^2}{2} + \rho\sigma_S N_2 + \sqrt{1-\rho^2}\sigma_S N_3\right)}_{:=N_4} - 1, \end{aligned}$$

with $N_2 \sim N(0,1)$ and $N_3 \sim N(0,1)$ independent,

we obtain that

$$\begin{aligned}\mathrm{Var}\left(N_4\right) &= \mathrm{Var}\left[\mu - \frac{\sigma_S^2}{2} + \rho\sigma_S N_2 + \sqrt{1-\rho^2}\sigma_S N_3\right] \\ &= \rho^2\sigma_S^2 \mathrm{Var}(N_2) + \left(1-\rho^2\right)\sigma_S^2 \mathrm{Var}(N_3) \\ &\quad + 2\mathrm{Cov}\left(\rho\sigma_S N_2, \sqrt{1-\rho^2}\sigma_S N_3\right) \\ &= \sigma_S^2, \end{aligned}$$

and thus,

$$N_4 \sim N\left(\mu - \frac{\sigma_S^2}{2}, \sigma_S^2\right).$$

Hence, it follows that

$$\sigma_S = \hat{\sigma}_S, \tag{A.5}$$

and

$$\begin{aligned}\mathbb{E}\left(r_t^S\right) &= \mathbb{E}\left(e^{N_4}\right) - 1 = \exp\left(\mu\right) - 1, \\ \mathbb{E}\left[\left(r_t^S\right)^2\right] &= \exp\left(2\mu + \sigma_S^2\right) - 2\exp\left(\mu\right) + 1. \end{aligned}$$

Consequently,

$$\begin{aligned}\mathrm{Var}\left(r_t^S\right) &= \mathbb{E}\left[\left(r_t^S\right)^2\right] - \left[\mathbb{E}\left(r_t^S\right)\right]^2 \\ &= \exp\left(2\mu + \sigma_S^2\right) - \exp\left(2\mu\right). \end{aligned} \tag{A.6}$$

To calculate the desired correlation, we further require the covariance between the return on the asset S and the return on the interest rate r. It is given by

$$\begin{aligned}\mathrm{Cov}\left(r_t^r, r_t^S\right) &= \mathrm{Cov}\left(e^{N_1} - 1, e^{N_4} - 1\right) \\ &= \mathbb{E}\left(e^{N_1} e^{N_4}\right) - \mathbb{E}\left(e^{N_1}\right)\mathbb{E}\left(e^{N_4}\right) \\ &= \mathbb{E}\left(e^{N_1+N_4}\right) - \mathbb{E}\left(e^{N_1}\right)\mathbb{E}\left(e^{N_4}\right).\end{aligned}$$

Since we know the distributions of N_1 and N_4, we only need to determine the moments of $N_1 + N_4$ to proceed. From Chapter 5 it follows that

$$\begin{aligned}\mathbb{E}\left(N_1 + N_4\right) &= \mathbb{E}\left(N_1\right) + \mathbb{E}\left(N_4\right) = \underbrace{\xi + \mu - \frac{\sigma_S^2}{2}}_{:=\mu_2}, \\ \mathrm{Var}\left(N_1 + N_4\right) &= \mathrm{Var}\left(N_1\right) + \mathrm{Var}\left(N_4\right) + 2\mathrm{Cov}\left(N_1, N_4\right) \\ &= \sigma_1^2 + \sigma_S^2 + 2\mathrm{Cov}\left(N_1, \mu - \frac{\sigma_S^2}{2} + \rho\sigma_S N_2 + \sqrt{1-\rho^2}\sigma_S N_3\right) \\ &= \sigma_1^2 + \sigma_S^2 + 2\rho\sigma_S\mathrm{Cov}\left(N_1, N_2\right) \\ &= \underbrace{\sigma_1^2 + \sigma_S^2 + \frac{2\rho\sigma_r\sigma_S}{\kappa} - \frac{2\rho\sigma_r\sigma_S}{\kappa^2}\left(1 - e^{-\kappa}\right)}_{:=\sigma_2^2}.\end{aligned}$$

Then,

$$\begin{aligned}\mathrm{Cov}\left(r_t^r, r_t^S\right) &= \mathbb{E}\left(e^{N_1} e^{N_4}\right) - \mathbb{E}\left(e^{N_1}\right)\mathbb{E}\left(e^{N_4}\right) \\ &= \exp\left(\mu_2 + \frac{\sigma_2^2}{2}\right) - \exp\left(\xi + \frac{\sigma_1^2}{2}\right)\exp\left(\mu\right),\end{aligned}$$

and the correlation is given by

$$\hat{\rho} = h(\rho, \sigma_S, \sigma_r, \kappa, \xi) = \frac{\mathrm{Cov}\left(r_t^r, r_t^S\right)}{\sqrt{\mathrm{Var}(r_t^r)\mathrm{Var}(r_t^S)}}.$$

Solving h numerically for ρ, we obtain the desired value.

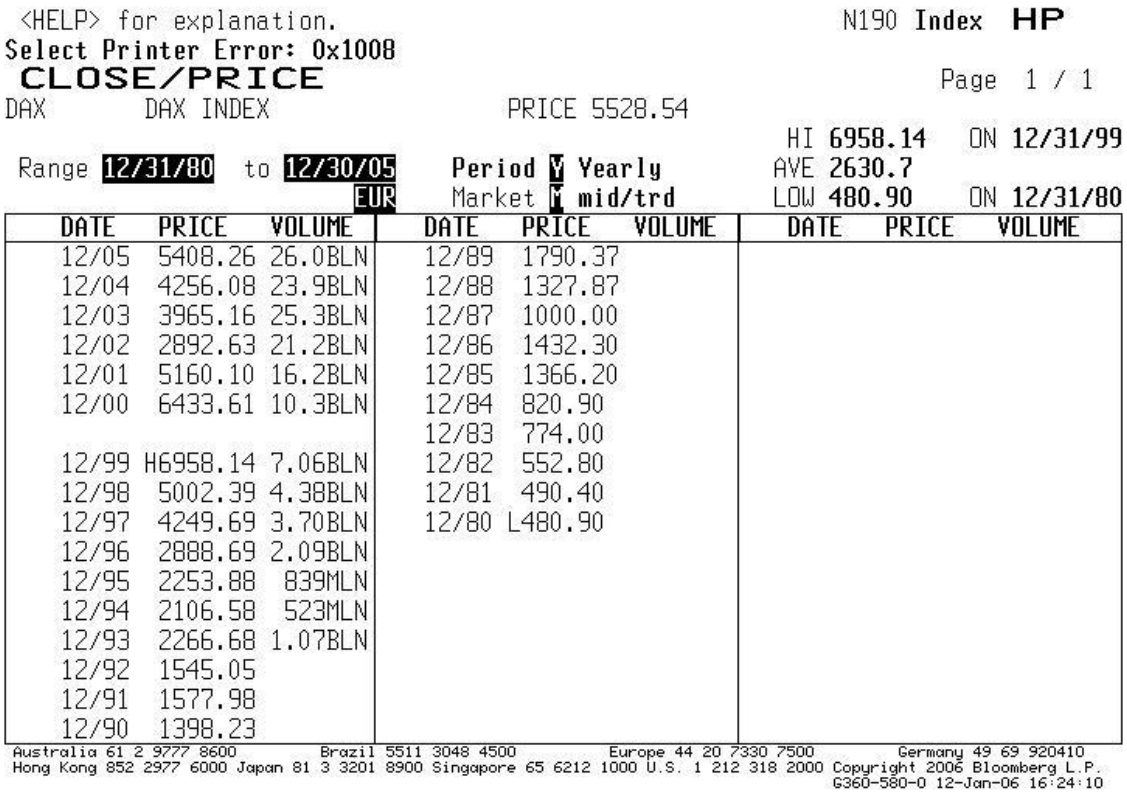
The following tables present extracts from the DAX and money market data that have been used for the parameter estimations in Chapter 6 and for the calculation of skewness and kurtosis in Chapter 5:

Zeitreihe su0101: Geldmarktsätze am Frankfurter Bankplatz / Tagesgeld / Monatsdurchschnitt

Zeitpunkt	Wert	Wertestatus	relative Differenz zur Vorperiode in %	relative Differenz zum Vorjahr in %
2006-04	2,63		4,4	26,4
2006-03	2,52		7,2	22,3
2006-02	2,35		0,9	14,1
2006-01	2,33		2,2	12,6
2005-12	2,28		9,6	11,2
2005-11	2,08		0,5	-0,5
2005-10	2,07		-1,0	-1,9
2005-09	2,09		0,5	2,0
2005-08	2,08		0,5	2,0
2005-07	2,07		0,0	0,5
2005-06	2,07		0,0	2,0
2005-05	2,07		-0,5	2,5
2005-04	2,08		1,0	0,5
2005-03	2,06		0,0	2,5
2005-02	2,06		-0,5	1,5
2005-01	2,07		1,0	2,5
2004-12	2,05		-1,9	2,0
2004-11	2,09		-0,9	5,6
2004-10	2,11		2,9	4,5
2004-09	2,05		0,5	1,0
2004-08	2,04		-1,0	-2,4
2004-07	2,06		1,5	-1,0
2004-06	2,03		0,5	-8,6
2004-05	2,02		-2,4	-21,1
2004-04	2,07		3,0	-19,1
2004-03	2,01		-1,0	-25,8
2004-02	2,03		0,5	-26,7
2004-01	2,02		0,5	-27,6
2003-12	2,01		1,5	-34,5
2003-11	1,98		-2,0	-40,0
2003-10	2,02		-0,5	-38,8

Zeitpunkt	Wert	Wertestatus	relative Differenz zur Vorperiode in %	relative Differenz zum Vorjahr in %
2003-10	2,02		-0,5	-38,8
2003-09	2,03		-2,9	-38,7
2003-08	2,09		0,5	-36,5
2003-07	2,08		-6,3	-37,0
2003-06	2,22		-13,3	-33,7
2003-05	2,56		0,0	-22,7
2003-04	2,56		-5,5	-22,4
2003-03	2,71		-2,2	-16,6
2003-02	2,77		-0,7	-15,3
2003-01	2,79		-9,1	-15,2
2002-12	3,07		-7,0	-7,5
2002-11	3,30		0,0	-6,0
2002-10	3,30		-0,3	-16,7
2002-09	3,31		0,6	-16,6
2002-08	3,29		-0,3	-26,7
2002-07	3,30		-1,5	-26,8
2002-06	3,35		1,2	-26,0
2002-05	3,31		0,3	-28,7
2002-04	3,30		1,5	-34,5
2002-03	3,25		-0,6	-31,9
2002-02	3,27		-0,6	-34,3
2002-01	3,29		-0,9	-30,7
2001-12	3,32		-5,4	-31,1
2001-11	3,51		-11,4	-27,2
2001-10	3,96		-0,3	-16,6
2001-09	3,97		-11,6	-13,3
2001-08	4,49		-0,4	2,0
2001-07	4,51		-0,4	4,9
2001-06	4,53		-2,4	5,8
2001-05	4,64		-7,9	18,4
2001-04	5,04		5,7	37,3

Figure A.1: Money market data



<HELP> for explanation. N190 Index HP
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DAX DAX INDEX PRICE 5528.54
HI 6958.14 ON 12/31/99
Range 12/31/80 to 12/30/05 Period Y Yearly AVE 2630.7
EUR Market M mid/trd LOW 480.90 ON 12/31/80

DATE	PRICE	VOLUME	DATE	PRICE	VOLUME	DATE	PRICE	VOLUME
12/05	5408.26	26.0BLN	12/89	1790.37				
12/04	4256.08	23.9BLN	12/88	1327.87				
12/03	3965.16	25.3BLN	12/87	1000.00				
12/02	2892.63	21.2BLN	12/86	1432.30				
12/01	5160.10	16.2BLN	12/85	1366.20				
12/00	6433.61	10.3BLN	12/84	820.90				
			12/83	774.00				
12/99	H6958.14	7.06BLN	12/82	552.80				
12/98	5002.39	4.38BLN	12/81	490.40				
12/97	4249.69	3.70BLN	12/80	L480.90				
12/96	2888.69	2.09BLN						
12/95	2253.88	839MLN						
12/94	2106.58	523MLN						
12/93	2266.68	1.07BLN						
12/92	1545.05							
12/91	1577.98							
12/90	1398.23							

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G360-580-0 12-Jan-06 16:24:10

Figure A.2: DAX data

Appendix B

Important source code

The program consists of seven files plus the associated header files. The `makefile` gives an overview of the program structure:

```
main:       main.o run.o data.o method.o cases.o hlp.o processes.o
            readlog.o pdesolver.o
            g++ -Wall -o main main.o run.o data.o method.o cases.o hlp.o
            processes.o  readlog.o pdesolver.o -lgsl -lgslcblas -lm -ldonner
            -L/home/katha/flens/lib -latlas -lg2c -lcblas -llapack -lgomp

main.o:     main.cc run.h
            g++ -Wall -c main.cc -lgsl -lgslcblas -lm

run.o:      run.h run.cc data.h method.h cases.h
            g++ -Wall -c run.cc -lgsl -lgslcblas -lm

method.o:   method.h method.cc processes.h data.h readlog.h
            g++ -Wall -c method.cc -I/home/katha/flens/include
            -DNETLIB -DDEBUG -fopenmp readlog.o:  readlog.h readlog.cc data.h
            g++ -Wall -c readlog.cc -lm -lgsl

data.o:     data.cc data.h
            g++ -Wall -c data.cc -lgsl -lgslcblas -lm

cases.o:    cases.h cases.cc hlp.h
            g++ -Wall -c cases.cc

processes.o:    processes.h processes.cc
                g++ -Wall -c processes.cc

hlp.o:          hlp.h hlp.cc
                g++ -Wall -c hlp.cc

pdesolver.o:    pdesolver.h pdesolver.cc
                g++ -Wall -c pdesolver.cc -I/home/katha/flens/include
                -DNETLIB -DDEBUG -fopenmp -lgomp

clean:      rm -f *.o core

realclean:  rm -f *.o main core
```

The files `run.*` and `data.*` implement the classes "run" and "data" introduced in Section 6.1, `hlp.*` contains several auxiliary functions, and `cases.*` the functions that are associated with the bonus mechanisms, such as L_t and x_t.

`processes.*` implements the generation of the interest rate processes used for the "discretized" Monte Carlo method:

```
double rgenerateOrnsteinUhlenbeck(double t, double u, double ru,
                                  double a, double b, double sigmar,
                                  double NW){

       return (exp(-a*(t-u))*(ru-b+sigmar*sqrt(t-u)*NW)+b);

}

double rgenerateCIR(double t, double u, double ru, double a,
                    double b, double sigmar, double NW){

       return (exp(-a*(t-u))*(ru-b+sigmar*sqrt(t-u)*NW*sqrt(ru))+b);


}
```

`method.*` and `pdesolver.*` are the most important files in the program.
`method.*` implements the three introduced evaluation methods, the "exact" Monte Carlo method, the "discretized" Monte Carlo method, and the discrete lattice method. The following code fragments present the most important extracts from each evaluation method.

Monte Carlo method "exact":

```
//MONTE CARLO METHOD ''EXACT''


void montecarlomethodexact::evaluate(data * dat, int euram,
           double (* Lkleint)(double, double, double, double,
                      double, double, double, double,
                      double, double, double, double),
           double (* xkleint)(double, double, double, double,
                      double, double, double, double,
                      double, double, double, double),
           double (* Lgrosst)(double, double, double, double,
                      double, double, double, double,
                      double, double, double, double),
           double (* xgrosst)(double, double, double, double,
                      double, double, double, double,
                      double, double, double, double)){
 cout << "Monte Carlo Algorithm exact" << endl;
 ostream *out;
```

```
  ofstream file;

  if (euram == 2){
    cout << " No evaluation of the Bermuda option with the Monte-Carlo
    method!"<< endl;
    exit(11);
  }

  if (logf){
    file.open(logfile.c_str() , ios::out);
    out = &file;
    *out << logfile << endl;
    *out <<"Monte Carlo evaluation exact/ "<< dat->notes <<  endl;
    *out << "*********************************************" << endl;
    *out << "*********************************************" << endl;

  }
  else{
    out = NULL;
  }
  //Print the contract parameters
  for(int k=0;k < DIM;k++){
    double gloc = dat->g[k];
    double deltaloc = dat->delta[k];
    double zloc = dat->z[k];
    double yloc = dat->y[k];
    double alphaloc = dat->alpha[k];
    double rloc = dat->r[k];
    double sigmaloc = dat->sigma[k];
    double sigmaSloc = dat->sigmaS[k];
    double rholoc = dat->rho[k];
    double aloc = dat->a[k];
    double bloc = dat->b[k];
    double a1loc = dat->a1[k];
    double b1loc = dat->b1[k];
    double Tloc = dat->T[k];
    double Prloc = dat->Pr[k];
    double xnullloc = dat->xnull[k];
    double rnullloc = dat->rnull[k];
    if (logf){
      *out << (k+1) << endl;
      *out << "steps : " << steps <<", T = " << Tloc <<", g  = "
       << gloc << ", delta = " << deltaloc << ", y = "
       << yloc << ", sigma = " << sigmaloc << ", sigmaS = "
       << sigmaSloc << ", rho =  " << rholoc << ", r = "
       << rloc << ", L_0 = " << Prloc << ", x_0 = " << xnullloc
       << ", r_0 = " << rnullloc << ", z= " << zloc << ", [a,b] = ["
       << aloc << "," << bloc
       <<"]" << ", a1 = " << a1loc << ", b1 = " << b1loc <<  endl;
      *out << "*****************************************" << endl;
    }
    else{
      cout << "Go #" << k << " : " << endl;

    }

    double EW = 0;
    double A = 0.0;
```

```
double B = 0.0;
double C = 0.0;
double D = 0.0;
double E = 0.0;
double F = 0.0;
 A = sigmaloc*sigmaloc/(2*a1loc)*(1-exp(-2*a1loc));
 B = sigmaloc*sigmaloc/(2*a1loc*a1loc)*pow(1-exp(-a1loc),2);
 C = rholoc*sigmaloc*sigmaSloc/a1loc*(1-exp(-a1loc));
 D = sigmaloc*sigmaloc/(2*a1loc*a1loc*a1loc)*
     (2*a1loc-3+4*exp(-a1loc)-exp(-2*a1loc));
 E = sigmaloc/a1loc*rholoc*sigmaSloc-sigmaloc/(a1loc*a1loc)*
     rholoc*sigmaSloc*(1-exp(-a1loc));
 F = rholoc*rholoc*sigmaSloc*sigmaSloc;


//number of simulations
for (int i = 0; i < steps;i++){

  double Lsim = Prloc;
  double xsim = xnullloc;
  double Asim = 100.00;
  double rensim = 0.0;
  double rsim = rnullloc;

  double N1 = 0.0;
  double N2 = 0.0;
  double N3 = 0.0;
  double N4 = 0.0;
  double X1 = 0.0;
  double X2 = 0.0;
  double X3 = 0.0;
  double X4 = 0.0;
  double zinshelp = 0.0;

  //if logfile=1, output of all paths
  if (logf){

double Aminsim = Lsim * ( 1+ xsim);
double Aplussim = Aminsim;
double Dsim = 0.0;
double Rsim = xsim * Lsim;
double spritzesim = 0.0;

*out << "s" <<endl;
*out << 0 << ":";
*out << 0 << ":";
*out << Lsim << ":";
*out << Aminsim << ":";
*out << Aplussim << ":";
*out << Dsim << ":";
*out << Rsim << ":";
*out << xsim << ":";
*out << spritzesim << endl;

for (int time =1;time < Tloc;time++){

  //Simulation of the paths
```

```
if (gaussmethod == 1){

  N1 = gsl_ran_gaussian(ran,1);
  N2 = gsl_ran_gaussian(ran,1);
  N3 = gsl_ran_gaussian(ran,1);
  N4 = gsl_ran_gaussian(ran,1);

  X1 = sqrt(A)*(C/sqrt(A*F)*N1+((B-C*E/F)/(sqrt(A*D*(1-E*E/(D*F)))))*N2+
       sqrt(1-C*C/(A*F)-(B-C*E/F)*(B-C*E/F)/(A*D*(1-E*E/(D*F))))*N3);
  X2 = sqrt(D)*(E/sqrt(D*F)*N1+sqrt(1-E*E/(D*F))*N2);
  X3 = sqrt(F)*N1;
  X4 = sqrt(1-rholoc*rholoc)*sigmaSloc*N4;

  rensim =exp((rsim-b1loc)/a1loc*(1-exp(-a1loc))+b1loc+X2-
       sigmaSloc*sigmaSloc/2+X3+X4)-1;

  zinshelp=zinshelp+(rsim-b1loc)/a1loc*(1-exp(-a1loc))+b1loc+X2;

  rsim = exp(-a1loc)*rsim+b1loc*(1-exp(-a1loc))+X1;


}
else{

  N1 = gsl_ran_gaussian_ratio_method(ran,1);
  N2 = gsl_ran_gaussian_ratio_method(ran,1);
  N3 = gsl_ran_gaussian_ratio_method(ran,1);
  N4 = gsl_ran_gaussian_ratio_method(ran,1);


  X1 = sqrt(A)*(C/sqrt(A*F)*N1+((B-C*E/F)/(sqrt(A*D)*sqrt(1-E*E/(D*F))))*N2+
       sqrt(1-C*C/(A*F)-(B-C*E/F)*(B-C*E/F)/(A*D*(1-E*E/(D*F))))*N3);
  X2 = sqrt(D)*(E/sqrt(D*F)*N1+sqrt(1-E*E/(D*F))*N2);
  X3 = sqrt(F)*N1;
  X4 = sqrt(1-rholoc*rholoc)*sigmaSloc*N4;

  rensim =exp((rsim-b1loc)/a1loc*(1-exp(-a1loc))+b1loc+X2-
       sigmaSloc*sigmaSloc/2+X3+X4)-1;

  zinshelp=zinshelp+(rsim-b1loc)/a1loc*(1-exp(-a1loc))+b1loc+X2;

  rsim = exp(-a1loc)*rsim+b1loc*(1-exp(-a1loc))+X1;


}
double hlp = Lsim;
Aminsim = (1 + rensim) * Aplussim;
Lsim = Lkleint(hlp, xsim, Asim, Asim * (1+rensim),
       deltaloc, yloc, gloc, xnullloc,
       alphaloc, zloc, aloc, bloc);
xsim = xkleint(hlp, xsim, Asim, Asim * (1+rensim),
       deltaloc, yloc, gloc, xnullloc,
       alphaloc, zloc, aloc, bloc);
Aplussim = ( 1 + xsim) * Lsim;
Dsim = max(Aminsim - Aplussim,0);
Rsim = xsim * Lsim;
spritzesim = max(Lsim - Aminsim, 0);
Asim = ( 1 + rensim) * Asim;
```

```
    //write in logfile
     *out << time << ":";
     *out << rensim << ":";
     *out << Lsim*exp(-zinshelp)<< ":";
     *out << Aminsim<< ":";
     *out << Aplussim<< ":";
     *out << Dsim*exp(-zinshelp)<< ":";
     *out << Rsim*exp(-zinshelp)<< ":";
     *out << xsim << ":";
     *out << spritzesim*exp(-zinshelp)<< endl;
  }
  //The same for t=T
  if (gaussmethod == 1){

      N1 = gsl_ran_gaussian(ran,1);
      N2 = gsl_ran_gaussian(ran,1);
      N3 = gsl_ran_gaussian(ran,1);
      N4 = gsl_ran_gaussian(ran,1);


      X1 = sqrt(A)*(C/sqrt(A*F)*N1+((B-C*E/F)/(sqrt(A*D)*sqrt(1-E*E/(D*F))))*N2+
              sqrt(1-C*C/(A*F)-(B-C*E/F)*(B-C*E/F)/(A*D*(1-E*E/(D*F))))*N3);
      X2 = sqrt(D)*(E/sqrt(D*F)*N1+sqrt(1-E*E/(D*F))*N2);
      X3 = sqrt(F)*N1;
      X4 = sqrt(1-rholoc*rholoc)*sigmaSloc*N4;

      rensim =exp((rsim-b1loc)/a1loc*(1-exp(-a1loc))+b1loc+X2-
              sigmaSloc*sigmaSloc/2+X3+X4)-1;

      zinshelp=zinshelp+(rsim-b1loc)/a1loc*(1-exp(-a1loc))+b1loc+X2;

      rsim = exp(-a1loc)*rsim+b1loc*(1-exp(-a1loc))+X1;



  }
  else{

      N1 = gsl_ran_gaussian_ratio_method(ran,1);
      N2 = gsl_ran_gaussian_ratio_method(ran,1);
      N3 = gsl_ran_gaussian_ratio_method(ran,1);
      N4 = gsl_ran_gaussian_ratio_method(ran,1);

      X1 = sqrt(A)*(C/sqrt(A*F)*N1+((B-C*E/F)/(sqrt(A*D)*sqrt(1-E*E/(D*F))))*N2+
              sqrt(1-C*C/(A*F)-(B-C*E/F)*(B-C*E/F)/(A*D*(1-E*E/(D*F))))*N3);
      X2 = sqrt(D)*(E/sqrt(D*F)*N1+sqrt(1-E*E/(D*F))*N2);
      X3 = sqrt(F)*N1;
      X4 = sqrt(1-rholoc*rholoc)*sigmaSloc*N4;

      rensim =exp((rsim-b1loc)/a1loc*(1-exp(-a1loc))+b1loc+X2-
              sigmaSloc*sigmaSloc/2+X3+X4)-1;

      zinshelp=zinshelp+(rsim-b1loc)/a1loc*(1-exp(-a1loc))+b1loc+X2;

      rsim = exp(-a1loc)*rsim+b1loc*(1-exp(-a1loc))+X1;

  }
  double hlp = Lsim;
```

```
    Aminsim = (1 + rensim) * Aplussim;
    Lsim = Lgrosst(hlp, xsim, Asim, Asim * (1+rensim),
                deltaloc, yloc, gloc, xnullloc,
                alphaloc, zloc, aloc, bloc);
    xsim = xgrosst(hlp, xsim, Asim, Asim * (1+rensim),
                deltaloc, yloc, gloc, xnullloc,
                alphaloc, zloc, aloc, bloc);
    Aplussim = ( 1 + xsim) * Lsim;
    Dsim = max(Aminsim - Aplussim,0);
    Rsim = xsim * Lsim;
    spritzesim = max(Lsim - Aminsim, 0);
    Asim = ( 1 + rensim) * Asim;
    *out << Tloc << ":";
    *out << rensim << ":";
    *out << Lsim*exp(-zinshelp)<< ":";
    *out << Aminsim<< ":";
    *out << Aplussim<< ":";
    *out << Dsim*exp(-zinshelp)<< ":";
    *out << Rsim*exp(-zinshelp)<< ":";
    *out << xsim << ":";
    //*out << rsim << ":";
    *out << spritzesim*exp(-zinshelp)<< endl;
      }



    [...]


 //if logfile=1, evaluation of the paths, see function mergelog

 if (logf){

   ((ofstream*)out)->close();
   mergelog(steps,dat,logfile,mergedlog);
 }
}
```

Monte Carlo method "discretized":

```
//MONTE CARLO DISCRETE
void montecarlomethoddiscrete::evaluate(data * dat, int euram,
          double (* Lkleint)(double, double, double, double,
                     double, double, double, double,
                     double, double, double, double),
          double (* xkleint)(double, double, double, double,
                     double, double, double, double,
                     double, double, double, double),
          double (* Lgrosst)(double, double, double, double,
                     double, double, double, double,
                     double, double, double, double),
          double (* xgrosst)(double, double, double, double,
                     double, double, double, double,
                     double, double, double, double)){
```

```
[...]



  double rensimhelp = 0.0;
  double Lsimhelp = 0.0;
  double zsimhelp = 0.0;
  double EW = 0;
  int intervalsteps=100;

  //number of simulations
  for (int i = 0; i < steps;i++){

    double Lsim = Prloc;
    double xsim = xnullloc;
    double Asim = 100.00;
    double rensim = 0.0;
    double rensimt = 0.0;
    double rsim = rnullloc;
    double rminsim = rsim;

    double NW = 0.0;
    double NZ = 0.0;
    double zinshelp = 0.0;
    double tnmin = 0.0;
    double tn = 0.0;
    //if logfile=1, output of all paths
    if (logf){

  double Aminsim = Lsim * ( 1+ xsim);
  double Aplussim = Aminsim;
  double Dsim = 0.0;
  double Rsim = xsim * Lsim;
  double spritzesim = 0.0;

  *out << "s" <<endl;
  *out << 0 << ":";
  *out << 0 << ":";
  *out << Lsim << ":";
  *out << Aminsim << ":";
  *out << Aplussim << ":";
  *out << Dsim << ":";
  *out << Rsim << ":";
  *out << xsim << ":";
  *out << spritzesim << endl;


  for (int time =1;time < Tloc;time++){
    double help = 1.0;
    for (int n=1;n<=intervalsteps;n++){
     double doubleintervalsteps=double(intervalsteps);
     tnmin = time+(n-1)/doubleintervalsteps;
     tn = tnmin + 1/doubleintervalsteps;

     if (gaussmethod == 1){
```

```
  NW = gsl_ran_gaussian(ran,1);
  NZ = gsl_ran_gaussian(ran,1);

  if (rateprocess==1){      //Ornstein-Uhlenbeck
   rsim=rgenerateOrnsteinUhlenbeck(tn,tnmin,rminsim,a1loc,b1loc,sigmaloc,NW);
    }
  else{                     //Cox-Ingersoll-Ross
   rsim=rgenerateCIR(tn,tnmin,rminsim,a1loc,b1loc,sigmaloc,NW);
    }

  rensimt = exp(1/(2*doubleintervalsteps)*(rsim+rminsim)-sigmaSloc*sigmaSloc/
          (2*doubleintervalsteps)+rholoc*sigmaSloc*
          sqrt(1/doubleintervalsteps)*NW+sqrt(1-rholoc*rholoc)*
          sigmaSloc*sqrt(1/doubleintervalsteps)*NZ);

}
else{

  NW = gsl_ran_gaussian(ran,1);
  NZ = gsl_ran_gaussian(ran,1);

  if (rateprocess==1){        //Ornstein-Uhlenbeck
   rsim=rgenerateOrnsteinUhlenbeck(tn,tnmin,rminsim,a1loc,b1loc,sigmaloc,NW);
  }

  else{                     //Cox-Ingersoll-Ross
   rsim=rgenerateCIR(tn,tnmin,rminsim,a1loc,b1loc,sigmaloc,NW);
  }
  rensimt = exp(1/(2*doubleintervalsteps)*(rsim+rminsim)-sigmaSloc*sigmaSloc/
          (2*doubleintervalsteps)+rholoc*sigmaSloc*
          sqrt(1/doubleintervalsteps)*NW+sqrt(1-rholoc*rholoc)*
          sigmaSloc*sqrt(1/doubleintervalsteps)*NZ);

}
help = help*rensimt;
zinshelp = zinshelp+1/(2*doubleintervalsteps)*(rsim+rminsim);
rminsim = rsim;

}

rensim = help-1;

double hlp = Lsim;
Aminsim = (1 + rensim) * Aplussim;
Lsim = Lkleint(hlp, xsim, Asim, Asim * (1+rensim),
       deltaloc, yloc, gloc, xnullloc,
       alphaloc, zloc, aloc, bloc);
xsim = xkleint(hlp, xsim, Asim, Asim * (1+rensim),
       deltaloc, yloc, gloc, xnullloc,
       alphaloc, zloc, aloc, bloc);
Aplussim = ( 1 + xsim) * Lsim;
Dsim = max(Aminsim - Aplussim,0);
Rsim = xsim * Lsim;
spritzesim = max(Lsim - Aminsim, 0);
Asim = ( 1 + rensim) * Asim;

//write in logfile
```

```
      *out << time << ":";
      *out << rensim << ":";
      *out << Lsim*exp(-zinshelp)<< ":";
      *out << Aminsim<< ":";
      *out << Aplussim<< ":";
      *out << Dsim*exp(-zinshelp)<< ":";
      *out << Rsim*exp(-zinshelp)<< ":";
      *out << xsim << ":";
     // *out << rsim << ":";
      *out << spritzesim*exp(-zinshelp)<< endl;
  }
  //The same for t=T
    double help = 1.0;
    for (int n=1;n<=intervalsteps;n++){
     double doubleintervalsteps = double(intervalsteps);
     tnmin = Tloc-1+(n-1)/double(intervalsteps);
     tn = tnmin + 1/double(intervalsteps);

     if (gaussmethod == 1){


      NW = gsl_ran_gaussian(ran,1);
      NZ = gsl_ran_gaussian(ran,1);

      if (rateprocess==1){      //Ornstein-Uhlenbeck
       rsim=rgenerateOrnsteinUhlenbeck(tn,tnmin,rminsim,a1loc,b1loc,sigmaloc,NW);
         }
      else{                     //Cox-Ingersoll-Ross
       rsim=rgenerateCIR(tn,tnmin,rminsim,a1loc,b1loc,sigmaloc,NW);
         }


      rensimt = exp(1/(2*doubleintervalsteps)*(rsim+rminsim)-sigmaSloc*sigmaSloc/
                (2*doubleintervalsteps)+rholoc*sigmaSloc*
                sqrt(1/doubleintervalsteps)*NW+sqrt(1-rholoc*rholoc)*
                sigmaSloc*sqrt(1/doubleintervalsteps)*NZ);

  }
  else{

      NW = gsl_ran_gaussian(ran,1);
      NZ = gsl_ran_gaussian(ran,1);

      if (rateprocess==1){      //Ornstein-Uhlenbeck
       rsim=rgenerateOrnsteinUhlenbeck(tn,tnmin,rminsim,a1loc,b1loc,sigmaloc,NW);
         }
      else{                     //Cox-Ingersoll-Ross
       rsim=rgenerateCIR(tn,tnmin,rminsim,a1loc,b1loc,sigmaloc,NW);
         }

      rensimt = exp(1/(2*doubleintervalsteps)*(rsim+rminsim)-sigmaSloc*sigmaSloc/
                (2*doubleintervalsteps)+rholoc*sigmaSloc*
                sqrt(1/doubleintervalsteps)*NW+sqrt(1-rholoc*rholoc)*
                sigmaSloc*sqrt(1/doubleintervalsteps)*NZ);
  }
       help = help*rensimt;
    zinshelp=zinshelp+1/(2*doubleintervalsteps)*(rsim+rminsim);
    rminsim=rsim;
```

```
    }

    rensim = help-1;

    double hlp = Lsim;
    Aminsim = (1 + rensim) * Aplussim;
    Lsim = Lkleint(hlp, xsim, Asim, Asim * (1+rensim),
           deltaloc, yloc, gloc, xnullloc,
           alphaloc, zloc, aloc, bloc);
    xsim = xkleint(hlp, xsim, Asim, Asim * (1+rensim),
           deltaloc, yloc, gloc, xnullloc,
           alphaloc, zloc, aloc, bloc);
    Aplussim = ( 1 + xsim) * Lsim;
    Dsim = max(Aminsim - Aplussim,0);
    Rsim = xsim * Lsim;
    spritzesim = max(Lsim - Aminsim, 0);
    Asim = ( 1 + rensim) * Asim;



  // write in logfile
  *out << Tloc << ":";
  *out << rensim << ":";
  *out << Lsim*exp(-zinshelp)<< ":";
  *out << Aminsim<< ":";
  *out << Aplussim<< ":";
  *out << Dsim*exp(-zinshelp)<< ":";
  *out << Rsim*exp(-zinshelp)<< ":";
  *out << xsim << ":";
  //*out << rsim << ":";
  *out << spritzesim*exp(-zinshelp)<< endl;
    }
  }



  [...]
```

Discrete lattice method:

```
//DISCRETE LATTICE

void discretelatticemethod::evaluate(data * dat, int euram,
                     double (* Lkleint)(double, double,
                            double, double,
                            double, double,
                            double, double,
                            double, double,
                            double, double),
                     double (* xkleint)(double, double,
                            double, double,
                            double, double,
                            double, double,
                            double, double,
```

```
                                double, double),
                        double (* Lgrosst)(double, double,
                                double, double,
                                double, double,
                                double, double,
                                double, double,
                                double, double),
                        double (* xgrosst)(double, double,
                                double, double,
                                double, double,
                                double, double,
                                double, double,
                                double, double)){


cout << "Discrete Lattice Algorithm" << endl;

ostream *out;
ofstream file;
if (logf){
  file.open(logfile.c_str() , ios::out);
  out = &file;
  *out << logfile << endl;
  *out <<"Diskrete lattice evaluation /"<< dat->notes <<  endl;
  *out << "***********************************************" << endl;
  *out << "***********************************************" << endl;

}
else{
  out = NULL;
}


  [...]



  double F[GitS][GitL][GitR];
  double Fbeg[GitS][GitL][GitR];
  double bound[GitS][GitR];

  for(int i = 0;i<GitS;i++){
    for(int j = 0;j<GitL;j++){
  for(int k = 0;k<GitR;k++){
     F[i][j][k] = (Lmax * (double)(j+1))/(double(GitL));
  }
    }
  }

  stopwatch::stopwatch Stopwatch(GitS);
  for(int i = 0; i<GitS;i++){
    cout << "i = " << i << ", eta: " << Stopwatch.do_step() << " sec." << endl;
    for(int j = 0;j<GitL;j++){
      double S[GitS];

    for(int sstep=0;sstep<GitS;sstep++){
          S[sstep]=Smax*(double)(sstep+1)/((double)(GitS));
```

```
        double Bon = Lgrosst(((Lmax * (double)(j+1))/((double)GitL)),
                (((Smax * (double)(i+1) * (double)GitL)/
                  (Lmax * (double)(j+1) * (double)GitS))
                 - (double)1),
                ((Smax * (double)(i+1))/((double)GitS)), S[sstep],
                deltaloc, yloc, gloc, xnullloc,
                alphaloc, zloc, aloc, bloc);

    for(int rstep=0;rstep<GitR;rstep++){


    bound[sstep][rstep] = Bon;

     }
    }


    int kn_r = GitR - 1;
    int kn_S = GitS - 1;

    DMatrix V0(_(0,kn_r),_(0,kn_S)),
            V1(_(0,kn_r),_(0,kn_S));
    for (int index_S = 0; index_S <= kn_S; index_S++) {
       for (int index_r = 0; index_r <= kn_r; index_r++) {
           V0(index_r,index_S) = bound[index_S][index_r];
       }
    }
    int kn_t = 100;
    if (solve(V0,V1,kn_r,kn_S,kn_t,Smax,1) == false) {
       cerr << "error!" <<endl;
       sleep(10);
    }


   for (int k=0;k<GitR;k++){
       Fbeg[i][j][k] = V1(k,i);

   }
    }
   }

   for(int tau=1;tau<Tloc;tau++){
     if (euram == 1){
   for(int i =0;i<GitS;i++){
     for(int j =0;j<GitL;j++){
       for(int k =0;k<GitR;k++){
         F[i][j][k] = Fbeg[i][j][k];        //European contract
       }
     }
   }
     }
     else if(euram == 2){                //non-European contract
   for(int i =0;i<GitS;i++){
     for(int j =0;j<GitL;j++){
       for(int k=0;k<GitR;k++){
         F[i][j][k] = max(Fbeg[i][j][k],(Lmax*(double)(j+1))/((double)GitL));
       }
     }
```

```
}
  }
  else{
cout << "This option is not defined." << endl;
exit(10);
  }
  if (tau==5) {
      Smax=0.5*Smax;
      Lmax=0.5*Lmax;
          }
  for(int i = 0; i<GitS;i++){
for(int j = 0;j<GitL;j++){
      double S[GitS];

   for (int sstep = 0;sstep<GitS;sstep++){
       S[sstep]=Smax*(double)(sstep+1)/((double)(GitS));

      double Bon = Lkleint(((Lmax * (double)(j+1))/((double)GitL)),
             (((Smax * (double)(i+1) * (double)GitL)/
               (Lmax * (double)(j+1) * (double)GitS))
              - (double)1),
             ((Smax * (double)(i+1))/((double)GitS)), S[sstep],
             deltaloc, yloc, gloc, xnullloc,
             alphaloc, zloc, aloc, bloc);
    double xzuhilf = xkleint(((Lmax * (double)(j+1))/((double)GitL)),
                  (((Smax * (double)(i+1) * (double)GitL)/
                    (Lmax * (double)(j+1) * (double)GitS))
                   - (double)1),
                  ((Smax * (double)(i+1))/((double)GitS)), S[sstep],
                  deltaloc, yloc, gloc, xnullloc,
                  alphaloc, zloc, aloc, bloc);
    double Div = Bon*(xzuhilf + (double)1);


    double indexBon = ((Bon * (double)GitL)/Lmax) - (double)1;
    double indexDiv = ((Div * (double)GitS)/Smax) - (double)1;

    double indexbelBon = floor(indexBon);
    double indexbelDiv = floor(indexDiv);

     for (int rstep = 0;rstep<GitR;rstep++){


    //Interpolation
     if (indexDiv < 0){
      if (indexBon < 0){
    bound[sstep][rstep] = (sqrt(0.5*((indexBon + 1)*(indexBon +1) +
                  (indexDiv + 1)*(indexDiv +1))))*
      F[0][0][rstep];
      }
      else if(indexBon < GitL-1){
    bound[sstep][rstep] = (indexDiv + 1)*
      (F[0][(int)indexbelBon][rstep] + (indexBon - indexbelBon)*
       (F[0][(int)indexbelBon +1][rstep] - F[0][(int)indexbelBon][rstep]));
      }
      else{
    bound[sstep][rstep] = (indexDiv + 1)*
      (F[0][GitL-2][rstep] + (indexBon - GitL + 2)*(F[0][GitL-1][rstep]
```

```
        - F[0][GitL-2][rstep]));
        }
      }
      else if(indexDiv < GitS-1){
        if (indexBon < 0){ //Dieser Fall kann nicht eintreten wegen Gar.
      bound[sstep][rstep] = (indexBon + 1)*
        (F[(int)indexbelDiv][0][rstep] + (indexDiv - indexbelDiv)*
         (F[(int)indexbelDiv + 1][0][rstep] - F[(int)indexbelDiv][0][rstep]));
        }
        else if(indexBon < GitL-1){

      bound[sstep][rstep] = (F[(int)indexbelDiv][(int)indexbelBon][rstep]+
          (indexDiv-indexbelDiv)*(F[(int)indexbelDiv+1][(int)
          indexbelBon][rstep]-F[(int)indexbelDiv][(int)indexbelBon][rstep]))
          +(indexBon-indexbelBon)*(F[(int)indexbelDiv][(int)indexbelBon+1]
          [rstep]+(indexDiv-indexbelDiv)*(F[(int)indexbelDiv+1][(int)
          indexbelBon+1][rstep]-F[(int)indexbelDiv][(int)indexbelBon+1][rstep])-
          (F[(int)indexbelDiv][(int)indexbelBon][rstep]+(indexDiv-indexbelDiv)*
          (F[(int)indexbelDiv+1][(int)indexbelBon][rstep]-F[(int)indexbelDiv]
          [(int)indexbelBon][rstep])));
        }
        else{
      bound[sstep][rstep]=(F[(int)indexbelDiv][GitL-2][rstep] +
          (indexDiv - indexbelDiv)*(F[(int)indexbelDiv +1][GitL-2][rstep] -
          F[(int)indexbelDiv][GitL-2][rstep])) + (indexBon - GitL +2)*
          ((F[(int)indexbelDiv][GitL-1][rstep] + (indexDiv - indexbelDiv)*
          (F[(int)indexbelDiv +1][GitL-1][rstep] - F[(int)indexbelDiv]
          [GitL-1][rstep])) - (F[(int)indexbelDiv][GitL-2][rstep] +
          (indexDiv - indexbelDiv)*(F[(int)indexbelDiv +1][GitL-2][rstep] -
          F[(int)indexbelDiv][GitL-2][rstep])));
        }
      }
      else{
        if (indexBon < 0){
      bound[sstep][rstep] =  (indexBon + 1)*
        (F[GitS-2][0][rstep] + (indexDiv - GitS + 2)*(F[GitS-1][0][rstep]
        - F[GitS-2][0][rstep]));
        }
        else if (indexBon < GitL-1){
      bound[sstep][rstep]=(F[GitS-2][(int)indexbelBon][rstep] +
          (indexBon - indexbelBon)*(F[GitS-2][(int)indexbelBon+1][rstep] -
          F[GitS-2][(int)indexbelBon][rstep])) + (indexDiv - GitS +2)*
          ((F[GitS-1][(int)indexbelBon][rstep] + (indexBon - indexbelBon)*
          (F[GitS-1][(int)indexbelBon +1][rstep] - F[GitS-1][(int)indexbelBon]
          [rstep])) - (F[GitS-2][(int)indexbelBon][rstep] +
          (indexBon - indexbelBon)*(F[GitS-2][(int)indexbelBon + 1][rstep] -
          F[GitS-2][(int)indexbelBon][rstep])));
        }
        else{
      bound[sstep][rstep] = F[GitS-2][GitL-2][rstep] + sqrt(0.5*(
          (indexDiv - GitS + 2)*(indexDiv - GitS + 2) +
          (indexBon - GitL + 2)*(indexBon - GitL + 2)))*
          (F[GitS-1][GitL-1][rstep] - F[GitS-2][GitL-2][rstep]);
        }
      }

    }
    }
```

```
    int kn_r = GitR - 1;
    int kn_S = GitS - 1;

    DMatrix V0(_(0,kn_r),_(0,kn_S)),
            V1(_(0,kn_r),_(0,kn_S));
    for (int index_S = 0; index_S <= kn_S; index_S++) {
       for (int index_r = 0; index_r <= kn_r; index_r++) {
           V0(index_r,index_S) = bound[index_S][index_r];
       }
    }

    int kn_t = 100;
    solve(V0,V1,kn_r,kn_S,kn_t,Smax,1);


   for (int k=0;k<GitR;k++){
       Fbeg[i][j][k] = V1(k,i);

   }

   }
}
}

     if (euram == 1){
   for(int i =0;i<GitS;i++){
     for(int j =0;j<GitL;j++){
       for(int k =0;k<GitR;k++){
         F[i][j][k] = Fbeg[i][j][k];        //European contract
       }
     }
   }
     }
     else if(euram == 2){                 //non-European contract
   for(int i =0;i<GitS;i++){
     for(int j =0;j<GitL;j++){
       for(int k=0;k<GitR;k++){
         F[i][j][k] = max(Fbeg[i][j][k],(Lmax*(double)(j+1))/((double)GitL));
       }
     }
   }
     }
     else{
   cout << "This option is not defined." << endl;
   exit(10);
     }

   if (logf){
     for(int i=(int)(floor(((Soutmin*(double)GitS)/Smax)-(double)1))
       ;i<(int)(ceil(((Soutmax*(double)GitS)/Smax)-(double)1));i++){
   for(int j=(int)(floor(((Loutmin*(double)GitL)/Lmax)-(double)1))
       ;j<(int)(ceil(((Loutmax*(double)GitL)/Lmax)-(double)1));j++){
     for(int k=(int)(floor(((routmin*(double)GitR)/(rmax-rmin))-(double)1))
       ;k<(int)(ceil(((routmax*(double)GitR)/(rmax-rmin))-(double)1));k++){

         *out << " S / L / R  : " <<((i+1)*Smax)/GitS <<" / " << ((j+1)*Lmax)/GitL
```

```
            <<" / "<<((k+1)*(rmax-rmin))/GitR
            << "...V_0 = " << F[i][j][k] << endl;
        }
    }
      }
    }
    else{
      for(int i=(int)(floor(((Soutmin*(double)GitS)/Smax)-(double)1))
        ;i<(int)(ceil(((Soutmax*(double)GitS)/Smax)-(double)1));i++){
    for(int j=(int)(floor(((Loutmin*(double)GitL)/Lmax)-(double)1))
        ;j<(int)(ceil(((Loutmax*(double)GitL)/Lmax)-(double)1));j++){
       for(int k=(int)(floor(((routmin*(double)GitR)/(rmax-rmin))-(double)1))
        ;k<(int)(ceil(((routmax*(double)GitR)/(rmax-rmin))-(double)1));k++){

      cout << " S / L / R : " <<((i+1)*Smax)/GitS <<" / " << ((j+1)*Lmax)/GitL
            <<" / "<<((k+1)*(rmax-rmin))/GitR
            << "...V_0 = " << F[i][j][k] << endl;

       }
    }
      }
    }
    cout << "Wait" << endl;
    sleep(5);
    cout << "continuing..." << endl;
  }
}
```

`pdesolver.*` contains the numerical approximation of the PDE with the help of FLENS:

```
#include <iomanip>
#include <iostream> #include <omp.h> #include "pdesolver.h"

using namespace std;

bool minmax (const DMatrix &V,int kn_r,int kn_S,double delta_r,
double delta_S) {
   double bmax = 60000; double bmin = 0;
   double max_temp = V(0,0); int max_temp_r = kn_r; int max_temp_S = 0;
   double min_temp = V(0,0); int min_temp_r = 0; int min_temp_S = 0;
   for (int index_r = 0; index_r <= kn_r; index_r++) {
       for (int index_S = 0; index_S <= kn_S; index_S++) {
           if (V(index_r,index_S) > max_temp) {
               max_temp = V(index_r,index_S);
               max_temp_r = index_r; max_temp_S = index_S;
           } else if (V(index_r,index_S) < min_temp) {
               min_temp = V(index_r,index_S);
               min_temp_r = index_r; min_temp_S = index_S;
           }
       }
   }
   if (max_temp > bmax) {
       cerr << "max at r=" << max_temp_r << " S=" << max_temp_S
```

```
            <<": " << max_temp << endl;
        return false;
    } else if (min_temp < bmin) {
        cerr << "min at r=" << min_temp_r << " S=" << min_temp_S
            <<": " << min_temp << endl;
        return false;
    }
    return true;
}

const double inline extrapol(const double barrierpoint, const
double innerpoint) {
    return (2 * barrierpoint - innerpoint);
}

bool solve(DMatrix &start_grid, DMatrix &end_grid, int kn_r, int
kn_S, int kn_t, double S_infinity, int num_of_threads) {
    omp_set_num_threads(num_of_threads);
    double t0 = 0; double T = 1;
    double r_infinity = 0.105;
    double delta_r = r_infinity / kn_r;
    double delta_S = S_infinity / kn_S;
    double delta_t = T / kn_t;
    double sigma_r = 0.01; double sigma_S = 0.075;
    double sigma_r_squared = pow(sigma_r,2);
    double sigma_S_squared = pow(sigma_S,2);
    double rho = 0.51; double a = 0.14; double b = 0.04;
    DMatrix Vj(_(0,kn_r),_(0,kn_S)),                    // V at tj
            Vjplus1(_(0,kn_r),_(0,kn_S)),               // Vj + delta_t * f
            Vjextrapol(_(-1,kn_r+1),_(-1,kn_S+1)), // Vj incl. extrapol. points,

            Vrr(_(0,kn_r),_(0,kn_S)),
            Vr(_(0,kn_r),_(0,kn_S)),
            Vss(_(0,kn_r),_(0,kn_S)),
            Vs(_(0,kn_r),_(0,kn_S)),
            Vsr(_(0,kn_r),_(0,kn_S)),                   // derivatives
            f(_(0,kn_r),_(0,kn_S));                     // rhs. of PDE
    stopwatch::stopwatch Stopwatch(kn_t);
    Vj = start_grid;
    for (double t = t0; t <= T; t+=delta_t) {
        Vjextrapol(_(0,kn_r),_(0,kn_S)) = Vj;
        int index_r, index_S;
        double r,S;
        #pragma omp parallel default(shared) private(index_r,index_S,r,S)
        {
            #pragma omp for schedule(static) nowait
            // north & south
            for (index_S = 0; index_S <= kn_S; index_S++) {
                Vjextrapol(0 - 1, index_S) =
                      extrapol(Vj(0, index_S),Vj(0 + 1, index_S));
                Vjextrapol(kn_r + 1, index_S) =
                      extrapol(Vj(kn_r, index_S),Vj(kn_r - 1, index_S));
            }
            #pragma omp for schedule(static)
            //east & west
            for (index_r = 0; index_r <= kn_r; index_r++) {
                Vjextrapol(index_r,-1)          =
                      extrapol(Vj(index_r,0),Vj(index_r,1));
```

```
            Vjextrapol(index_r,kn_S + 1)   =
                  extrapol(Vj(index_r,kn_S),Vj(index_r,kn_S - 1));
        }
        #pragma omp flush
        #pragma omp sections nowait
        {
            #pragma omp section // Vrr
            for (int index_r = 0; index_r <= kn_r; index_r++) {
                for (int index_S = 0; index_S <= kn_S; index_S++) {
                    Vrr(index_r,index_S) = (Vjextrapol(index_r + 1,index_S)
                                - 2 * Vjextrapol(index_r,index_S)
                                + Vjextrapol(index_r - 1,index_S))
                                            / (pow(delta_r,2));
                }
            }

            #pragma omp section // Vr
            for (int index_r = 0; index_r <= kn_r; index_r++) {
                for (int index_S = 0; index_S <= kn_S; index_S++) {
                    Vr(index_r,index_S) = (Vjextrapol(index_r + 1,index_S)
                                - Vjextrapol(index_r - 1,index_S))
                                            / (2 * delta_r);
                }
            }

            #pragma omp section // Vss
            for (int index_r = 0; index_r <= kn_r; index_r++) {
                for (int index_S = 0; index_S <= kn_S; index_S++) {
                    Vss(index_r,index_S) = (Vjextrapol(index_r,index_S + 1)
                                -  2 * Vjextrapol(index_r,index_S)
                                + Vjextrapol(index_r,index_S - 1))
                                            / (pow(delta_S,2));
                }
            }

            #pragma omp section // Vs
            for (int index_r = 0; index_r <= kn_r; index_r++) {
                for (int index_S = 0; index_S <= kn_S; index_S++) {
                    Vs(index_r,index_S) = (Vjextrapol(index_r,index_S + 1)
                                - Vjextrapol(index_r,index_S - 1))
                                            / (2 * delta_S);
                }
            }
        }
        #pragma omp flush
        #pragma omp for schedule(static) //Vsr
        for (int index_r = 0; index_r <= kn_r; index_r++) {
            for (int index_S = 0; index_S <= kn_S; index_S++) {
                if (index_r == 0) {
                    Vsr(index_r,index_S) =  (Vs(index_r + 1,index_S)
                     - extrapol(Vs(index_r,index_S),Vs(index_r + 1,index_S)))
                                                / (2 * delta_r);
                } else if (index_r == kn_r) {
                    Vsr(index_r,index_S) =
                    (extrapol(Vs(index_r,index_S),Vs(index_r - 1,index_S))
                     - Vs(index_r - 1,index_S))/(2 * delta_r);
                } else {
                    Vsr(index_r,index_S) =  (Vs(index_r + 1,index_S)
```

```
                        - Vs(index_r - 1,index_S))/(2 * delta_r);
                }
            }
        }
        #pragma omp flush
        #pragma omp for schedule(static)
        for (int index_r = 0; index_r <= kn_r; index_r++) {
            r = index_r * delta_r;
            for (int index_S = 0; index_S <= kn_S; index_S++) {
                S = index_S * delta_S;
                f(index_r,index_S) =  0.5 * sigma_S_squared
                            * pow(S,2) * Vss(index_r,index_S)
                            + rho * sigma_S * sigma_r * Vsr(index_r,index_S)
                            + 0.5 * sigma_r_squared * Vrr(index_r,index_S)
                            + r * S * Vs(index_r,index_S)
                            + a * (b - r) * Vr(index_r,index_S)
                            - r * Vj(index_r,index_S);
            }
        }
    }
    Vjplus1 = Vj + delta_t * f;          // step
    Vj = Vjplus1;
//      cout << "t: " << setw(8) << t;
    if (minmax(Vj,kn_r,kn_S,delta_r,delta_S) == false) {
        cerr << "error: barrier exceeded!" << endl;
        return false;
    }

//      cout << "\tETA: " << Stopwatch.do_step() << " sec.";
//      cout << "\ttps: " << Stopwatch.time_per_step() << " sec."
                        << "\n" << flush;
  }

  end_grid = Vj;
  return true;
}
```

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Ehrenwörtliche Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig angefertigt und nur die angegebenen Quellen benutzt habe.

Wörtlich oder inhaltlich übernommenes Gedankengut wurde nach bestem Wissen und Gewissen als solches kenntlich gemacht.

Diese Arbeit wurde bisher keinem anderen Prüfungsgremium vorgelegt und auch noch nicht veröffentlicht.

Ulm, den 26. Mai 2006 _____

(Katharina Zaglauer)