Some Results
on
Dynamic Risk Measures

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Contents

Abstract

This academic work contributes to the theory of dynamic risk measures. As bounded random variables and bounded discrete-time processes serve to model the discounted value of random future payments, it stands to reason that dynamic risk measures are commonly defined as families of certain functionals on the space of either one of them.

Such risk measures play a decisive role in the quantification of *target capital* as it is discussed within the Swiss Solvency Test. On the assessment of target capital, we provide some useful results which seem to advise against the current proposal. For this reason, we attempt an alternative approach towards the task of quantifying target capital by means of dynamic expected shortfall.

We introduce a notion of a conditional quantile and define conditional value at risk as the largest such quantile. Dynamic value at risk and dynamic expected shortfall then are constructed by iteration of conditional value at risk and conditional expected shortfall. This construction principle is advisable for dynamic consistency reasons. We provide characterizations of conditional expected shortfall by means of conditional quantiles and present similar results in the dynamic case.

Chapter 1

Introduction

In Europe's struggle for a risk-based solvency standard by means of Solvency II, the systematic treatment and discussion of risk measurement experiences not only a mathematical but also now a political and economical right of existence. Dating the latter on Herbert Lüthy's starting signal for the Swiss Solvency Test project in the spring of 2003, already six years prior to this did Artzner et al. in [1, 2] successfully attempt from a mathematical perspective the alteration of existing dogmas. In the arising dispute between many institutions and the insurance industry the academia's influence is increasing, as is the need for a theoretical view on this discussion.

A prominent concept of the current discussion surrounding minimum solvency requirements is *target capital*. Target capital is understood as the amount needed for an insurance company to be sure (in a sense to be specified) that the assets at the end of a year are sufficient to cover the liabilities. Sure means that even in an unlikely situation (say of a 1% probability) there is on average enough capital to allow the assets and liabilities to be transferred to a third party and in addition to this, to provide a capital endowment for that third party to cover its liabilities and future capital costs. Consequently, target capital (TC) is given by the sum of *1-year risk capital* (ES) which is the capital necessary for the risks emanating within a one year time horizon and the *risk margin* (M) which is defined as the minimal amount that allows a healthy insurer to take over the portfolio at no additional cost. In mathematical terms,

$$TC \ := \ ES + M.$$

Within the White Paper of the Swiss Solvency Test [21] the Swiss Federal Office of Private Insurance comes forward with a proposal of how to substantiate 1-year risk capital and risk margin in terms of *risk-bearing capital*. As risk-bearing capital (C) is given by the difference between a market consistent value of assets and a best estimate of liabilities a reasonable definition of 1-year risk capital is

$$ES \ := \ ES_r(C_1 - C_0) \ = \ C_0 + ES_r(C_1),$$

where ES_r denotes the expected shortfall at level $r \in (0,1)$. The confidence level $1-r$ is to be specified by the supervisor. The risk margin as well is quantified using the expected shortfall at level r

$$M \ := \ \alpha \sum_{s=2}^{T} ES_r(C_s - C_{s-1}),$$

where $T \geq 2$ is a finite time horizon and $\alpha > 0$ designates the spread between interest rates at which money can be borrowed and reinvested at no risk. As for the target capital, we arrive at

$$TC \ = \ C_0 + ES_r(C_1) + \alpha \sum_{s=2}^{T} ES_r(C_s - C_{s-1}) \ = \ C_0 + \Gamma_r(C),$$

where

$$\Gamma_r(C) \ := \ ES_r(C_1) + \alpha \sum_{s=2}^{T} ES_r(C_s - C_{s-1}).$$

We may view Γ_r as a risk measure on the discrete-time process C of risk-bearing capital and should inspect whether Γ_r satisfies some natural consistency properties: For instance, it seems reasonable to ask that $C_t \geq C_t^*$ with a 100% probability for all $t \in \{0, \ldots, T\}$ implies $\Gamma_r(C) \leq \Gamma_r(C^*)$ for two processes C and C^* of risk-bearing capital between which an insurer is free to choose. Unfortunately, it turns out that Γ_r fails to satisfy this inverse monotonicity property in general. Here is a counter-example.

Let $T = 2$ and consider a probabilistic model that consists only of two random future states ω_1 and ω_2 which both realize themselves with a 50% probability. The trajectories of C^* are zero irrespectively of what scenario occurs and the trajectory of C is zero if scenario ω_1 occurs. $C_0(\omega_2) = C_2(\omega_2) = 0$, whereas $C_1(\omega_2) = 1$. We have $C_t(\omega_i) \geq C_t^*(\omega_i)$ for $i \in \{1,2\}$ and $t \in \{0,1,2\}$. For arbitrary level $r \in (0,1)$ we have $\Gamma_r(C^*) = 0$ since $ES_r(0) = 0$. For all $r \in \left(0, \frac{1}{4}\right)$ we have $q_{C_1}(r) = 0$ and $q_{C_2 - C_1}(r) = -1$, where q_{C_1} and $q_{C_2 - C_1}$ respectively designate quantile functions of the random variables C_1 and $C_2 - C_1$. The expected shortfall at level $r = \frac{1}{4}$ of C_1 is given by

$$ES_{\frac{1}{4}}(C_1) \ = \ -\frac{1}{\frac{1}{4}} \int_0^{\frac{1}{4}} q_{C_1}(s) \, ds \ = \ 0$$

and of $C_2 - C_1$ it is given by

$$ES_{\frac{1}{4}}(C_2 - C_1) \ = \ -\frac{1}{\frac{1}{4}} \int_0^{\frac{1}{4}} q_{C_2 - C_1}(s) \, ds \ = \ 1.$$

Thus, $\Gamma_{\frac{1}{4}}(C) = \alpha > 0 = \Gamma_{\frac{1}{4}}(C^*)$.

This drawback motivated more detailed research on the risk measure Γ_r and on alternative approaches towards a quantification of target capital. It is the objective of the present thesis to provide a summary of the results we have collected so far and to point out possible directions of further research we suspect to be likely to contribute to a better handling of this complex on a day-to-day basis.

Outline. This thesis contributes to the theory of dynamic monetary risk measures in a discrete-time setup. As information is modeled by terms of filtration, such risk measures should no longer be static in the sense that only the information available at a single date is taken into account. Since risk assessments should be updated as new information is released, we should rather demand that the output of a dynamic monetary risk measure is to be a discrete-time process which is adapted to the underlying filtration. This is achieved by introducing dynamic monetary risk measures as families of conditional monetary risk measures at different times. A conditional monetary risk measure, say at time t, consequently is a mapping which assigns to a risky object a random variable (interpreted as the associated risk) which is measurable with respect to the information available up to time t. For this reason, the present thesis is roughly divided into two parts. Chapters 2, 3 and 4 discuss conditional monetary risk measures, whereas chapters 5 and 6 are devoted to dynamic monetary risk measures.

Throughout recent literature on dynamic monetary risk measures mainly two ideas of risky objects are taken into account. On the one hand, there are bounded random variables describing random future payments that materialize at a single future date and on the other hand, we have bounded discrete-time processes that are meant to display the entire evolution of an arbitrary financial position across time. Mathematically, monetary risk measures that deal with bounded discrete-time processes are more challenging and as such processes are capable of carrying more information about a risky object than a single bounded random variable is, a systematic treatment of this complex seems desirable from a sophisticated risk management viewpoint as well. This thesis contributes to both aspects, however the focus is on conditional and dynamic monetary risk measures for bounded random variables. It seems tempting to explore an extension of our results from bounded random variables to bounded discrete-time processes, yet the required technical effort of displaying this complex seems to lie beyond the scope of this thesis.

As this thesis is mainly based on literature which considers utility rather than its negative (utility $= -$risk), we set forth this convention as well as the preferred use of working with conditional and dynamic monetary utility functionals over conditional and dynamic monetary risk measures.

In chapter 2 we provide the setup, as well as the notation and we introduce conditional monetary utility functionals for bounded random variables and for bounded discrete-time processes. Chapter 3 is devoted to a discussion of the above introduced risk measure Γ_r on an axiomatic level. We consider functionals that satisfy the same set of axioms as Γ_r does and approximate such functionals by means of conditional coherent risk measures. We give necessary and sufficient conditions for the existence of a best (in a sense to be specified) approximation in terms of what we call monotone hulls. As a main result we explicitly construct the best approximation to Γ_r. Unfortunately, it turns out that this approximation is not capable of assessing the riskiness evolving from inter-temporal cash-flow streams. As we assume that this is not desirable, chapter 4 introduces the building blocks for our proposal of an alternative quantification of target capital in the spirit of expected shortfall. We present our notion of a distribution invariant conditional monetary utility functional and introduce conditional value at risk and conditional expected shortfall in terms of conditional quantiles. We provide some useful results on conditional quantiles and a generalization of the static case interaction between value at risk and expected shortfall. Chapter 5 then enters into the discussion of dynamic monetary utility functionals. We introduce basic definitions and in particular we define what we mean by time-consistency. We show how time-consistency relates to an iteration condition which yields a powerful construction principle: By backwards induction an arbitrary family of conditional monetary utility functionals serves as a construction kit for time-consistent dynamic monetary utility functionals. A recent duality result for time-consistent dynamic monetary utility functionals which are continuous in a mild sense is given. We introduce a concatenation operation for probability measures that are absolutely continuous with respect to some reference probability measure. This allows us to prove a modified version of this recent representation result by means of concatenated probability densities. In chapter 6 we are finally able to present dynamic value at risk and dynamic expected shortfall as time-consistent dynamic monetary utility functionals respectively constructed via iteration of conditional value at risk and conditional expected shortfall. We discuss time-consistency properties and present a characterization theorem of dynamic expected shortfall by combining the results of chapters 4 and 5.

The core of this thesis accumulates in the chapters 4 and 6. Although these chapters are a product of collective work, the key results are the earnings of Damir Filipović and Michael Kupper. The outlining and composition of chapter 4 mainly follows Michael Kupper's ideas. As apostle Paul, addressing the Romans, reminds us:

ἀπόδοτε πᾶσιν τὰς ὀφειλάς, [...] τῷ τὴν τιμὴν τὴν τιμήν. ΠΡΟΣ ΡΩΜΑΙΟΥΣ 13,7†

†Render therefore to all their dues: [...] honour to whom honour. Romans 13,7

Chapter 2

Conditional Monetary Utility Functionals

The two main objectives of this chapter are on the one hand, to present the setup and the notation as it is used throughout this thesis and on the other hand, to provide an axiomatic setup for risk measurement within a multi-period framework. We start off with a short reprise on the most striking features of stopping times and pass on to introducing one of the key notions of this thesis: a conditional monetary utility functional. In accordance with the introduction, conditional monetary utility functionals are separately introduced as functionals on the space of bounded random variables and on the space of bounded discrete-time processes. We translate the Swiss Solvency Test risk measure into our dynamic framework. By the end of this chapter we will have provided a sound foundation to start doing the mathematics.

2.1 Introduction

As evolving financial markets constantly provide more and more information, a prudent risk management should be equipped with a dynamic machinery that is capable of updating risk assessments across time. On this account, the recent literature on risk measurement comes forward with a variety of proposals to an extension of the celebrated static case axiomatics first presented in Artzner et al. [1, 2] to a dynamic temporal setting.

In this thesis we work in a discrete-time setup with finite time horizon T. Bounded discrete-time processes are meant to describe the evolution of discounted financial positions as time approaches T, whereas bounded random variables are to be understood as discounted future payoffs that are realized at the final date T. We simply call the former value processes and the latter final values. It is the aim of the present chapter to provide a sound axiomatic setup which ensures an adequate assessment of the riskiness arising from value processes as well as from final values.

Here and in the following, we order value processes and final values by almost sure dominance as it is done in Artzner et al. [4], Cheridito et al. [6, 7, 8], Cheridito and

Kupper [9] as well as in Kupper [19]. In a discrete-time setting we my always pass to increment processes and arrive at risk measurement of cash-flow streams, yet the order of almost sure dominance is not preserved under this transition. Riedel [20], Detlefsen and Scandolo [13] as well as Weber [23] for instance, order cash-flow streams by almost sure dominance and consequently propose a notion of a dynamic monetary risk measure that is monotone in terms of cash-flow streams.

Already in the static case Föllmer and Schied [15, 16] as well as Frittelli and Rosazza Gianin [14] established, by means of convex monetary risk measures, a more general axiomatic setting than the one originally presented in the seminal works Artzner et al. [1, 2] for finite and Delbaen [10, 11] for general probability spaces. For an excellent summary of the static case results we recommend the textbook Föllmer and Schied [17]. The mitigation of merely imposing convexity rather than coherence remains just as tempting in a dynamic temporal setting. Indeed, within a discrete-time setup it turns out that, under adequate adjustments, the static case duality results on convex monetary risk measures are still valid in a multi-period setting and can be proved by essentially the same techniques. Such conditional representation results in the case of convexity are first given in Detlefsen and Scandolo [13], whereas Riedel [20] follows Artzner et. al by studying dynamic coherent monetary risk measures. Within this thesis however we do not focus on any one of the two concepts as our main results are in fact valid in the case of convexity but are demonstrated by means of coherent risk measures.

The structure of this chapter is as follows: In the first section we present the setup and notation we choose to work with throughout this thesis. We present a short collection of some basic properties of stopping times and the associated σ-algebras. The main focus is on measurability. As information is modeled in terms of a filtration we then introduce in section 2.3 monetary utility functionals for bounded random variables conditioned on the information available up to stopping times. We present a few basic properties as well. The third section introduces our notion of a conditional monetary utility functional for bounded discrete-time processes. We conclude with a first discussion of the Swiss Solvency Test risk measure Γ_r which was introduced in chapter 1.

2.2 The Setup, Notation and Stopping Times

For the rest of this chapter, we fix a finite time horizon $T \in \mathbb{N}$ and shorten $\mathbb{T} = [0, T] \cap \mathbb{N}$. The stochastic basis is given by a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $(\mathcal{F}_t)_{t\in\mathbb{T}}$ such that $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F} = \mathcal{F}_T$. One should think of $\mathcal{F}_t$ as the information available up to time t.

For $t \in \mathbb{T}$ we denote by $L^0(\mathcal{F}_t) = L^0(\Omega, \mathcal{F}_t, P)$ the space of (equivalence classes of) $\mathcal{F}_t$-measurable random variables. For $X, Y \in L^0(\mathcal{F}_T)$, $X = Y$, a.s. P, by convention means that $X' = Y'$, a.s. P, for all pairings $(X', Y') \in X \times Y$. $L^\infty(\mathcal{F}_t) = L^\infty(\Omega, \mathcal{F}_t, P)$ denotes the subspace of P-almost surely bounded random variables, i.e.

$$L^\infty(\mathcal{F}_t) := \{X \in L^0(\mathcal{F}_t) \mid ||X||_{L^\infty} < \infty\},$$

where

$$||X||_{L^\infty} := \inf \{m \in \mathbb{R} \mid |X| \leq m, \text{ a.s. } P\}.$$

For $t = T$ we set $L^0 = L^0(\mathcal{F}) = L^0(\mathcal{F}_T)$ as well as $L^\infty = L^\infty(\mathcal{F}) = L^\infty(\mathcal{F}_T)$.

By $\mathcal{R}^0$ we denote the space of (equivalence classes[1] of) all adapted stochastic processes $(C_t)_{t\in\mathbb{T}}$ on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, P)$. Furthermore, the subspace $\mathcal{R}^\infty$ of $\mathcal{R}^0$ is given by

$$\mathcal{R}^\infty := \{C \in \mathcal{R}^0 \mid ||C||_{\mathcal{R}^\infty} < \infty\},$$

where

$$||C||_{\mathcal{R}^\infty} := \inf \{m \in \mathbb{R} \mid \sup_{t\in\mathbb{T}} |C_t| \leq m, \text{ a.s. } P\}.$$

Recall that an $\mathcal{F}$-measurable function $\tau : \Omega \to \mathbb{T} \cup \{+\infty\}$ is called $(\mathcal{F}_t)$-stopping time if for all $t \in \mathbb{T}$ the event $\{\tau \leq t\}$ is observable by time t. In other words, $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{T}$. Throughout this thesis we only consider finite $(\mathcal{F}_t)$-stopping times, i.e. an $(\mathcal{F}_t)$-stopping time τ is always assumed to satisfy $\tau(\omega) \leq T$, for all $\omega \in \Omega$. For two $(\mathcal{F}_t)$-stopping times τ and θ such that $\tau(\omega) \leq \theta(\omega)$, for all $\omega \in \Omega$, we define the projection $\pi_{\tau,\theta} : \mathcal{R}^0 \to \mathcal{R}^0$,

$$C \mapsto \pi_{\tau,\theta}(C)_t := 1_{\{\tau \leq t\}} C_{t\wedge\theta}, \qquad t \in \mathbb{T}.$$

Furthermore, we introduce the subspace

$$\mathcal{R}^\infty_{\tau,\theta} := \pi_{\tau,\theta}(\mathcal{R}^\infty)$$

of $\mathcal{R}^\infty$.

For an $(\mathcal{F}_t)$-stopping time τ we denote by $\mathcal{F}_\tau$ the σ-algebra of events determined prior to the stopping time τ, i.e.

$$\mathcal{F}_\tau := \sigma\{A \in \mathcal{F}_T \mid A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \in \mathbb{T}\}.$$

$L^\infty(\mathcal{F}_\tau) = L^\infty(\Omega, \mathcal{F}_\tau, P)$ is given as above.

Here is a collection of a few basic properties of stopping times and their associated σ-algebras:

- Since

  $$A^c \cap \{\tau \leq t\} = \big(A^c \cup \{\tau > t\}\big) \cap \{\tau \leq t\} = \big(A \cap \{\tau \leq t\}\big)^c \cap \{\tau \leq t\}$$

  as well as

  $$\{\tau \leq t\} \cap \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} \big(\{\tau \leq t\} \cap A_n\big),$$

  for $A, A_n \in \mathcal{F}_T, n \in \mathbb{N}$ and an $(\mathcal{F}_t)$-stopping time τ, we deduce that

  $$\mathcal{F}_\tau = \{A \in \mathcal{F}_T \mid A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \in \mathbb{T}\}. \qquad (2.2.1)$$

[1] By $C = D$, a.s. P, for two stochastic processes $C, D \in \mathcal{R}^0$ we mean that for P-almost all $\omega \in \Omega$, $C_t(\omega) = D_t(\omega)$ for all $t \in \mathbb{T}$.

- If $\tau(\omega) + n \leq T$, for all $\omega \in \Omega$, for an $(\mathcal{F}_t)$-stopping time τ and $n \in \mathbb{T}$, then $\tau + n : \Omega \to \mathbb{T}$, $\omega \mapsto \tau(\omega) + n$ is again an $(\mathcal{F}_t)$-stopping time. To see this first note that $\{\tau + n \leq t\} = \emptyset \in \mathcal{F}_t$ for $n > t$. Then observe that for $n \leq t$ we have

$$\{\tau + n \leq t\} = \{\tau \leq t - n\} \in \mathcal{F}_{t-n} \subset \mathcal{F}_t,$$

which verifies the statement.

- Consider an $(\mathcal{F}_t)$-stopping time τ, a process $C \in \mathcal{R}^\infty$ and $t \in \mathbb{T}$.

We have $\{\tau = t\} \in \mathcal{F}_\tau$. Hence, $\tau = \sum_{t \in \mathbb{T}} t 1_{\{\tau = t\}}$ is $\mathcal{F}_\tau$-measurable.
The $\mathcal{F}_t$-measurable functions $1_{\{\tau=t\}} C_t$ and $1_{\{\tau>t\}} C_t$ are $\mathcal{F}_\tau$-measurable. To prove this, it suffices to show that the events $\{1_{\{\tau \overset{(\geq)}{=} t\}} C_t \in B\} \cap \{\tau \leq s\}$ belong to the σ-algebra $\mathcal{F}_s$ for any Borel-set $B \in \mathcal{B}(\mathbb{R})$, for all $s \in \mathbb{T}$. To this end, observe that these events can also be written in the form

$$\Big(\big(\{C_t \in B\} \cap \{\tau \overset{(\geq)}{=} t\}\big) \cup \big(\{0 \in B\} \cap \{\tau \overset{(\leq)}{\neq} t\}\big) \Big) \cap \{\tau \leq s\}. \tag{2.2.2}$$

Note that all of these sets belong to either $\mathcal{F}_t$ or $\mathcal{F}_s$. Thus, we are done if $s \geq t$. Now let $s < t$ and consider the case where $0 \notin B$ first. Then (2.2.2) reduces to $\{C_t \in B\} \cap \{\tau \overset{(\geq)}{=} t\} \cap \{\tau \leq s\}$ which in both cases is the empty set and therefore both events belong to $\mathcal{F}_s$. For $0 \in B$ (2.2.2) reads $\{\tau \overset{(\leq)}{\neq} t\} \cap \{\tau \leq s\} = \{\tau \leq s\} \in \mathcal{F}_s$ and we are done for $s < t$ also.
We now derive that the $\mathcal{F}_t$-measurable functions

$$C_\tau := \sum_{t \in \mathbb{T}} 1_{\{\tau = t\}} C_t,$$

as well as

$$C_{t \wedge \tau} := 1_{\{\tau > t\}} C_t + \sum_{s=0}^{t} 1_{\{\tau = s\}} C_s,$$

are $\mathcal{F}_\tau$-measurable.

- Assume that an $(\mathcal{F}_t)$-stopping time τ satisfies $\tau(\omega) = s$, for all $\omega \in \Omega$ and some constant $s \in \mathbb{T}$. In this case $A \cap \{\tau \leq t\} = A \cap \{s \leq t\}$ is either the empty set or A itself. From (2.2.1) we thus deduce that $\mathcal{F}_\tau$ and $\mathcal{F}_s$ coincide.

- Consider two $(\mathcal{F}_t)$-stopping times τ and θ, such that $\tau(\omega) \leq \theta(\omega)$, for all $\omega \in \Omega$. We then have $\{\theta \leq t\} \subset \{\tau \leq t\}$, $t \in \mathbb{T}$ and hence

$$A \cap \{\theta \leq t\} = A \cap \{\tau \leq t\} \cap \{\theta \leq t\}.$$

Thus, $\mathcal{F}_\tau \subset \mathcal{F}_\theta$ and in turn $L^\infty(\mathcal{F}_\tau) \subset L^\infty(\mathcal{F}_\theta)$.

The above statements are still valid if equalities and inequalities between $(\mathcal{F}_t)$-stopping times as well as equalities and inclusions between sets are understood in the P-almost sure sense. For instance, if τ, θ is a pair of $(\mathcal{F}_t)$-stopping times such that $\tau \leq \theta$, a.s. P, then $\mathcal{F}_\tau \subset \mathcal{F}_\theta$ up to null-sets. However, in this case an $\mathcal{F}_\tau$-measurable function no longer has to be $\mathcal{F}_\theta$-measurable in turn. One may fix this problem by assuming that $\mathcal{F}_0$ already contains all null-sets in $\mathcal{F}$, yet such an approach is not attempted here.

2.3 Conditional Monetary Utility Functionals for Random Variables

The risky objects considered in this section are random variables of the vector space $L^\infty(\mathcal{F}_T)$. τ and θ are two $(\mathcal{F}_t)$-stopping times such that $\tau(\omega) \leq \theta(\omega)$ for all $\omega \in \Omega$.

Definition 2.3.1 *We call a functional $\phi_{\tau,\theta} : L^\infty(\mathcal{F}_\theta) \to L^\infty(\mathcal{F}_\tau)$ a conditional monetary utility functional (on $L^\infty(\mathcal{F}_\theta)$) if it satisfies the following properties:*

(**n**) **Normalization:** $\phi_{\tau,\theta}(0) = 0$, *a.s. P*

(**m**) **Monotonicity:** $\phi_{\tau,\theta}(X) \leq \phi_{\tau,\theta}(Y)$, *a.s. P, for all $X, Y \in L^\infty(\mathcal{F}_\theta)$ such that $X \leq Y$, a.s. P*

($\mathcal{F}_\tau$**-ti**) $\mathcal{F}_\tau$**-Translation Invariance:** $\phi_{\tau,\theta}(X + m) = \phi_{\tau,\theta}(X) + m$, *a.s. P, for all $X \in L^\infty(\mathcal{F}_\theta)$ and $m \in L^\infty(\mathcal{F}_\tau)$.*

We call a conditional monetary utility functional $\phi_{\tau,\theta}$ a conditional concave utility functional if it satisfies

($\mathcal{F}_\tau$**-c**) $\mathcal{F}_\tau$**-Concavity:** $\phi_{\tau,\theta}\big(\lambda X + (1-\lambda)Y\big) \geq \lambda\phi_{\tau,\theta}(X) + (1-\lambda)\phi_{\tau,\theta}(Y)$, *a.s. P, for all $X, Y \in L^\infty(\mathcal{F}_\theta)$ and $\lambda \in L^\infty(\mathcal{F}_\tau)$ such that $0 \leq \lambda \leq 1$, a.s. P.*

We call a conditional concave utility functional $\phi_{\tau,\theta}$ a conditional coherent utility functional if it satisfies

($\mathcal{F}_\tau$**-ph**) $\mathcal{F}_\tau$**-Positive Homogeneity:** $\phi_{\tau,\theta}(\lambda X) = \lambda\phi_{\tau,\theta}(X)$, *a.s. P, for all $X \in L^\infty(\mathcal{F}_\theta)$ and $\lambda \in L^\infty_+(\mathcal{F}_\tau) := \{f \in L^\infty(\mathcal{F}_\tau) \mid f \geq 0, \text{ a.s. } P\}$.*

$\rho_{\tau,\theta} : L^\infty(\mathcal{F}_\theta) \to L^\infty(\mathcal{F}_\tau)$ is a conditional monetary risk measure on $L^\infty(\mathcal{F}_\theta)$ if $-\rho_{\tau,\theta}$ is a conditional monetary utility functional on $L^\infty(\mathcal{F}_\theta)$. $\rho_{\tau,\theta}$ is a conditional convex risk measure if $-\rho_{\tau,\theta}$ is a conditional concave utility functional and $\rho_{\tau,\theta}$ is a conditional coherent risk measure if $-\rho_{\tau,\theta}$ is a conditional coherent utility functional.

Note that a conditional monetary utility functional (on $L^\infty(\mathcal{F}_\theta)$) is defined on equivalence classes of random variables that are equal P-almost surely. In this sense, conditional monetary utility functionals assign the same utility to random variables that coincide P-almost surely.

Here is a standard remark:

Remark 2.3.2 *A functional* $\phi_{\tau,\theta} : L^\infty(\mathcal{F}_\theta) \to L^\infty(\mathcal{F}_\tau)$ *is a conditional coherent utility functional if and only if it satisfies* (**m**), ($\mathcal{F}_\tau$-**ti**) *and* ($\mathcal{F}_\tau$-**ph**) *of the preceding definition together with*

(**sa**) **Superadditivity:** $\phi_{\tau,\theta}(X+Y) \geq \phi_{\tau,\theta}(X) + \phi_{\tau,\theta}(Y)$, *a.s.* P, *for all* $X, Y \in L^\infty(\mathcal{F}_\theta)$.

Any functional $\phi_{\tau,\theta} : L^\infty(\mathcal{F}_\theta) \to L^\infty(\mathcal{F}_\tau)$ *that satisfies* ($\mathcal{F}_\tau$-**ph**) *is normalized. Thus, necessity follows from*

$$\begin{aligned} \phi_{\tau,\theta}\big(\lambda X + (1-\lambda)Y\big) &\geq \phi_{\tau,\theta}(\lambda X) + \phi_{\tau,\theta}\big((1-\lambda)Y\big) \\ &= \lambda\phi_{\tau,\theta}(X) + (1-\lambda)\phi_{\tau,\theta}(Y), \quad a.s.\ P, \end{aligned}$$

for all $X, Y \in L^\infty(\mathcal{F}_\theta)$ *and* $\lambda \in L^\infty(\mathcal{F}_\tau)$ *such that* $0 \leq \lambda \leq 1$, *a.s.* P.

Conversely, let $\phi_{\tau,\theta}$ *be a conditional coherent utility functional. We then have*

$$\begin{aligned} \phi_{\tau,\theta}(X+Y) &\geq \lambda\phi_{\tau,\theta}\left(\frac{1}{\lambda}X\right) + (1-\lambda)\phi_{\tau,\theta}\left(\frac{1}{1-\lambda}Y\right) \\ &= \phi_{\tau,\theta}(X) + \phi_{\tau,\theta}(Y), \quad a.s.\ P, \end{aligned}$$

for all $X, Y \in L^\infty(\mathcal{F}_\theta)$ *and* $\lambda \in L^\infty(\mathcal{F}_\tau)$ *such that* $0 \leq \lambda \leq 1$, *a.s.* P.

Suppose that we are given a conditional monetary utility functional $\phi_{\tau,\theta} : L^\infty(\mathcal{F}_\theta) \to L^\infty(\mathcal{F}_\tau)$ and an $(\mathcal{F}_t)$-stopping time σ such that $\tau(\omega) \leq \sigma(\omega) \leq \theta(\omega)$, for all $\omega \in \Omega$. Recall that $L^\infty(\mathcal{F}_\sigma) \subset L^\infty(\mathcal{F}_\theta)$. Hence, $\phi_{\tau,\theta}$ induces a conditional monetary utility functional $\phi_{\tau,\sigma}$ (on $L^\infty(\mathcal{F}_\sigma)$) via

$$\phi_{\tau,\sigma}(X) := \phi_{\tau,\theta}(X), \quad X \in L^\infty(\mathcal{F}_\sigma). \tag{2.3.3}$$

Example 2.3.3 *Fix a discrete-time process* $C \in \mathcal{R}^\infty_{\tau,\theta}$, *a conditional monetary utility functional* $\phi_{\tau,\theta}$ *and* $t \in \mathbb{T} \setminus \{0\}$. *The* $\mathcal{F}_t$*-measurable increment* $\Delta C_t := C_t - C_{t-1} = C_{t\wedge\theta} - C_{(t-1)\wedge\theta}$ *is* $\mathcal{F}_\theta$*-measurable as the previous section tells us. Let us assume that* $\tau(\omega) + 1 \leq \theta(\omega)$, *for all* $\omega \in \Omega$. *Again in reference to the previous section we know that* $\tau+1$ *is an* $(\mathcal{F}_t)$*-stopping time and that* $C_{\tau+1}$ *is* $\mathcal{F}_{\tau+1}$*-measurable. Keeping* (2.3.3) *in mind we may thus define*

$$\Psi_{\tau,\theta}(C) := \phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha \sum_{\tau+1<s\leq T} \phi_{\tau,\theta}(\Delta C_s), \tag{2.3.4}$$

where $\alpha > 0$ *is some positive constant. From*

$$\sum_{\tau+1<s\leq T} \phi_{\tau,\theta}(\Delta C_s) = \sum_{t\in\mathbb{T}} 1_{\{\tau=t\}} \left(\sum_{t+1<s\leq T} \phi_{\tau,\theta}(\Delta C_s) \right),$$

we derive that $\Psi_{\tau,\theta}(C) \in L^\infty(\mathcal{F}_\tau)$.

If $\tau(\omega) = 0$ and $\theta(\omega) = T$ for all $\omega \in \Omega$ we may choose $-\phi_{0,T}$ as the expected shortfall ES_r at some level $r \in (0,1)$. After switching signs, (2.3.4) *then reads*

$$-\Psi_{0,T}(C) = ES_r(C_1) + \alpha \sum_{1<s\leq T} ES_r(\Delta C_s).$$

This is precisely the Swiss Solvency Test risk measure Γ_r which was introduced in chapter 1 already.

2.4 Conditional Monetary Utility Functionals for Discrete-Time Processes

The risky objects considered in this section are stochastic processes of the vector space $\mathcal{R}^\infty_{\tau,\theta}$. As in the previous section τ and θ are two $(\mathcal{F}_t)$-stopping times such that $\tau(\omega) \leq \theta(\omega)$ for all $\omega \in \Omega$.

Definition 2.4.1 *We call a functional $\Phi_{\tau,\theta} : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ a conditional monetary utility functional (on $\mathcal{R}^\infty_{\tau,\theta}$) if it satisfies the following properties:*

(**N**) **Normalization:** $\Phi_{\tau,\theta}(0) = 0$, *a.s. P*

(**M**) **Monotonicity:** $\Phi_{\tau,\theta}(C) \leq \Phi_{\tau,\theta}(D)$, *a.s. P, for all $C, D \in \mathcal{R}^\infty_{\tau,\theta}$ such that $C \leq D$, a.s. P*

($\mathcal{F}_\tau$-**TI**) $\mathcal{F}_\tau$-**Translation Invariance:** $\Phi_{\tau,\theta}(C + m1_{[\tau,T]}) = \Phi_{\tau,\theta}(C) + m$, *a.s. P, for all $C \in \mathcal{R}^\infty_{\tau,\theta}$ and $m \in L^\infty(\mathcal{F}_\tau)$.*

We call a conditional monetary utility functional $\Phi_{\tau,\theta}$ a conditional concave utility functional if it satisfies

($\mathcal{F}_\tau$-**C**) $\mathcal{F}_\tau$-**Concavity:** $\Phi_{\tau,\theta}\big(\lambda C + (1-\lambda)D\big) \geq \lambda\Phi_{\tau,\theta}(C) + (1-\lambda)\Phi_{\tau,\theta}(D)$, *a.s. P, for all $C, D \in \mathcal{R}^\infty_{\tau,\theta}$ and $\lambda \in L^\infty(\mathcal{F}_\tau)$ such that $0 \leq \lambda \leq 1$, a.s. P.*

We call a conditional concave utility functional $\Phi_{\tau,\theta}$ a conditional coherent utility functional if it satisfies

($\mathcal{F}_\tau$-**PH**) $\mathcal{F}_\tau$-**Positive Homogeneity:** $\Phi_{\tau,\theta}(\lambda C) = \lambda\Phi_{\tau,\theta}(C)$, *a.s. P, for all $C \in \mathcal{R}^\infty_{\tau,\theta}$ and $\lambda \in L^\infty_+(\mathcal{F}_\tau) = \{f \in L^\infty(\mathcal{F}_\tau) \mid f \geq 0,$ a.s. $P\}$.*

$\Gamma_{\tau,\theta} : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ is a conditional monetary risk measure on $\mathcal{R}^\infty_{\tau,\theta}$ if $-\Gamma_{\tau,\theta}$ is a conditional monetary utility functional on $\mathcal{R}^\infty_{\tau,\theta}$. $\Gamma_{\tau,\theta}$ is a conditional convex risk measure if $-\Gamma_{\tau,\theta}$ is a conditional concave utility functional and $\Gamma_{\tau,\theta}$ is a conditional coherent risk measure if $-\Gamma_{\tau,\theta}$ is a conditional coherent utility functional.

Note again that a conditional monetary utility functional (on $\mathcal{R}^\infty_{\tau,\theta}$) agrees in the P-almost sure sense on stochastic processes that are equal P-almost surely. Here and in the following, the difference between conditional monetary utility functionals on $L^\infty(\mathcal{F}_\theta)$ and on $\mathcal{R}^\infty_{\tau,\theta}$ is respectively indicated by lowercase and capital letters.

We may recast remark 2.3.2 as follows:

Remark 2.4.2 *A mapping* $\Phi_{\tau,\theta} : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ *is a conditional coherent utility functional if and only if it satisfies* (**M**), ($\mathcal{F}_\tau$-**TI**) *and* ($\mathcal{F}_\tau$-**PH**) *of the preceding definition together with*

(**SA**) **Superadditivity:** $\Phi_{\tau,\theta}(X+Y) \geq \Phi_{\tau,\theta}(X) + \Phi_{\tau,\theta}(Y)$, *a.s. P, for all* $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$.

This follows as in remark 2.3.2.

Again it is worth mentioning that a conditional monetary utility functional $\Phi_{\tau,\theta} : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ can be viewed as a conditional monetary utility functional $\Phi_{\tau,\sigma}$ (on $\mathcal{R}^\infty_{\tau,\sigma}$) via

$$\Phi_{\tau,\sigma}(C) := \Phi_{\tau,\theta}(C), \quad C \in \mathcal{R}^\infty_{\tau,\sigma}, \tag{2.4.5}$$

for an $(\mathcal{F}_t)$-stopping time σ such that $\tau(\omega) \leq \sigma(\omega) \leq \theta(\omega)$, for all $\omega \in \Omega$.

Example 2.4.3 *Let us continue the discussion of the functional* $\Psi_{\tau,\theta}$ *introduced in example* 2.3.3. *In addition we assume that* $\phi_{\tau,\theta}$ *is a conditional coherent utility functional (on* $L^\infty(\mathcal{F}_\theta)$*).*

Chapter 1 *presented an example which clarified that the Swiss Solvency Test risk measure* Γ_r *lacks monotonicity in general and so does the functional* $\Psi_{\tau,\theta} : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$*,*

$$C \mapsto \Psi_{\tau,\theta}(C) = \phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha \sum_{\tau+1<s\leq T} \phi_{\tau,\theta}(\Delta C_s),$$

in turn. $\Psi_{\tau,\theta}$ *does however satisfy* ($\mathcal{F}_\tau$-**PH**) *since the conditional coherent utility functional* $\phi_{\tau,\theta}$ *is* $\mathcal{F}_\tau$*-positive homogeneous. Furthermore, from*

$$\begin{aligned}
\Psi_{\tau,\theta}(C + m1_{[\tau,T]}) &= \phi_{\tau,\tau+1}(C_{\tau+1} + m) + \alpha \sum_{\tau+1<s\leq T} \phi_{\tau,\theta}(\Delta C_s) \\
&= m + \phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha \sum_{\tau+1<s\leq T} \phi_{\tau,\theta}(\Delta C_s) \\
&= m + \Psi_{\tau,\theta}(C)
\end{aligned}$$

and

$$\begin{aligned}
\Psi_{\tau,\theta}(C+D) &= \phi_{\tau,\tau+1}(C_{\tau+1} + D_{\tau+1}) + \alpha \sum_{\tau+1<s\leq T} \phi_{\tau,\theta}(\Delta C_s + \Delta D_s) \\
&\geq \phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha \sum_{\tau+1<s\leq T} \phi_{\tau,\theta}(\Delta C_s) \\
&\quad + \phi_{\tau,\tau+1}(D_{\tau+1}) + \alpha \sum_{\tau+1<s\leq T} \phi_{\tau,\theta}(\Delta D_s) \\
&= \Psi_{\tau,\theta}(C) + \Psi_{\tau,\theta}(D)
\end{aligned}$$

for all $C, D \in \mathcal{R}^\infty_{\tau,\theta}$ *and* $m \in L^\infty(\mathcal{F}_\tau)$ *it follows that* $\Psi_{\tau,\theta}$ *satisfies* ($\mathcal{F}_\tau$-**TI**) *and* (**SA**).

Section 3.4 is devoted to a further discussion of the functional $\Psi_{\tau,\theta}$.

Chapter 3

Monotone Hulls

In this chapter we provide some results on functionals $\Psi : \mathcal{R}^{\infty}_{\tau,\theta} \to L^{\infty}(\mathcal{F}_\tau)$ which satisfy the axioms of a conditional coherent utility functional on $\mathcal{R}^{\infty}_{\tau,\theta}$ except for (the economically highly reasonable) monotonicity. As a main result we give necessary and sufficient conditions for the existence of a smallest conditional monetary utility functional which majorizes such a Ψ in terms of monotone hulls. We then explicitly construct the greatest conditional coherent risk measure which is dominated by the Swiss Solvency Test risk measure Γ_r.

3.1 Introduction

Consider an insurance company which is enforced by the supervisor to determine its capital requirements by means of a certain risk measure, say Γ. Since Γ is exogenously specified, it may not appropriately reflect the actual risks run by the company. Moreover, Γ may not even satisfy the natural consistency axioms of definitions 2.3.1 and 2.4.1 of the preceding chapter. Such a situation is existent in the case of the Swiss Solvency Test risk measure Γ_r as the discussion of the two preceding chapters illustrates. There seems to be not much sense in applying Γ_r and it suggests itself to rather use a reasonable substitute of the mandatory risk measure even if it is of an in-house benefit only. As a first attempt, one may try to construct the largest conditional monetary risk measure which is majorized by Γ_r. This is precisely the objective of the present chapter.

In general, this task seems to be best approached by considering acceptance sets as the primary object. Therefore, section 3.2 starts with a brief repetition on acceptance sets and their beneficial features. We show how a conditional monetary utility functional can be recovered from its acceptance set and how such a set on its part may serve already as a notion of a conditional monetary utility functional. In section 3.3 we study the general case of approximating functionals $\Psi : \mathcal{R}^{\infty}_{\tau,\theta} \to L^{\infty}(\mathcal{F}_\tau)$ which satisfy the same set of axioms as $\Psi_{\tau,\theta}$ of example 2.3.3 does. The approximation is by means of conditional monetary utility functionals which majorize the given functional Ψ. We clarify what almost sure dominance means in terms of acceptance sets and define monotone hulls of certain sets as well as of certain functionals. The main result of this section is more or less a combination of the properties of acceptance sets. It states that there exists a smallest conditional monetary utility functional majorizing Ψ if and only if the monotone hull of Ψ is normalized. In this case, it is given by the monotone hull itself which automatically is conditional coherent. On the contrary, there is no (not necessarily smallest) conditional monetary utility functional majorizing Ψ at all, if the monotone hull is not normalized. In fact, normalization of the monotone hull is equivalent to the existence of any conditional monetary utility functional which majorizes Ψ. The last section of this chapter is devoted to the functional $\Psi_{\tau,\theta}$ of example 2.3.3. We explicitly construct the monotone hull of the functional $\Psi_{\tau,\theta}$.

Throughout this chapter we consider the setup of section 2.2 and let τ and θ be two $(\mathcal{F}_t)$-stopping times such that $\tau(\omega) \leq \theta(\omega)$ for all $\omega \in \Omega$.

3.2 Acceptance Sets

For all of this section $\phi_{\tau,\theta} : L^\infty(\mathcal{F}_\theta) \to L^\infty(\mathcal{F}_\tau)$ and $\Phi_{\tau,\theta} : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ are conditional monetary utility functionals on $L^\infty(\mathcal{F}_\theta)$ and on $\mathcal{R}^\infty_{\tau,\theta}$.

Definition 3.2.1 *The acceptance sets of $\phi_{\tau,\theta}$ and $\Phi_{\tau,\theta}$ are respectively given by*

$$\begin{aligned} \mathcal{A}^\phi_{\tau,\theta} &= \{X \in L^\infty(\mathcal{F}_\theta) \mid \phi_{\tau,\theta}(X) \geq 0, \ a.s.\ P\} \qquad \textit{and} \\ \mathcal{A}^\Phi_{\tau,\theta} &= \{C \in \mathcal{R}^\infty_{\tau,\theta} \mid \Phi_{\tau,\theta}(C) \geq 0, \ a.s.\ P\}. \end{aligned}$$

Remark 3.2.2 *Via the conventions* (2.3.3) *and* (2.4.5) *the sets*

$$\begin{aligned} \mathcal{A}^\phi_{\tau,\sigma} &= \{X \in L^\infty(\mathcal{F}_\sigma) \mid \phi_{\tau,\sigma}(X) \geq 0, \ a.s.\ P\} \qquad \textit{and} \\ \mathcal{A}^\Phi_{\tau,\sigma} &= \{C \in \mathcal{R}^\infty_{\tau,\sigma} \mid \Phi_{\tau,\sigma}(C) \geq 0, \ a.s.\ P\} \end{aligned}$$

are well defined for all $(\mathcal{F}_t)$-stopping times σ with $\tau(\omega) \leq \sigma(\omega) \leq \theta(\omega)$ for all $\omega \in \Omega$. We have

$$\begin{aligned} \mathcal{A}^\phi_{\tau,\sigma} &\subset \mathcal{A}^\phi_{\tau,\theta} \qquad \textit{and} \\ \mathcal{A}^\Phi_{\tau,\sigma} &\subset \mathcal{A}^\Phi_{\tau,\theta}. \end{aligned}$$

Definition 3.2.3 *For arbitrary $\mathcal{L} \subset L^\infty(\mathcal{F}_\theta)$ and $\mathcal{R} \subset \mathcal{R}^\infty_{\tau,\theta}$ and for all $X \in L^\infty(\mathcal{F}_\theta)$ and $C \in \mathcal{R}^\infty_{\tau,\theta}$ we define*

$$\begin{aligned} \phi_{\mathcal{L}}(X) &:= \text{ess.sup}\ \{m \in L^\infty(\mathcal{F}_\tau) \mid X - m \in \mathcal{L}\} \qquad \textit{and} \\ \Phi_{\mathcal{R}}(C) &:= \text{ess.sup}\ \{m \in L^\infty(\mathcal{F}_\tau) \mid C - m1_{[\tau,T]} \in \mathcal{R}\} \end{aligned}$$

with the convention

$$\text{ess.sup}\ \emptyset := -\infty.$$

The next three propositions demonstrate the usefulness of the concept of acceptance sets.

Proposition 3.2.4 *We have*

$$\begin{aligned}\phi_{\mathcal{A}^{\phi}_{\tau,\theta}} &= \phi_{\tau,\theta} \quad \textit{and} \\ \Phi_{\mathcal{A}^{\Phi}_{\tau,\theta}} &= \Phi_{\tau,\theta}.\end{aligned}$$

Proof. For all $X \in \mathcal{F}_\theta$,

$$\begin{aligned}\phi_{\mathcal{A}^{\phi}_{\tau,\theta}}(X) &= \text{ess.sup}\,\{m \in L^\infty(\mathcal{F}_\tau) \mid X - m \in \mathcal{A}^{\phi}_{\tau,\theta}\} \\ &= \text{ess.sup}\,\{m \in L^\infty(\mathcal{F}_\tau) \mid \phi_{\tau,\theta}(X-m) \geq 0, \text{ a.s. } P\} \\ &= \text{ess.sup}\,\{m \in L^\infty(\mathcal{F}_\tau) \mid \phi_{\tau,\theta}(X) \geq m, \text{ a.s. } P\} \\ &= \phi_{\tau,\theta}(X), \quad \text{a.s. } P.\end{aligned}$$

And for all $C \in \mathcal{R}^\infty_{\tau,\theta}$,

$$\begin{aligned}\Phi_{\mathcal{A}^{\Phi}_{\tau,\theta}}(C) &= \text{ess.sup}\,\{m \in L^\infty(\mathcal{F}_\tau) \mid C - m1_{[\tau,T]} \in \mathcal{A}^{\Phi}_{\tau,\theta}\} \\ &= \text{ess.sup}\,\{m \in L^\infty(\mathcal{F}_\tau) \mid \Phi_{\tau,\theta}(C - m1_{[\tau,T]}) \geq 0, \text{ a.s. } P\} \\ &= \text{ess.sup}\,\{m \in L^\infty(\mathcal{F}_\tau) \mid \Phi_{\tau,\theta}(C) \geq m, \text{ a.s. } P\} \\ &= \Phi_{\tau,\theta}(C), \quad \text{a.s. } P.\end{aligned}$$

□

Proposition 3.2.5 *The acceptance sets $\mathcal{A}^{\phi}_{\tau,\theta}$ and $\mathcal{A}^{\Phi}_{\tau,\theta}$ have the following properties:*

(**n**) **Normalization:** ess.inf $\{f \in L^\infty(\mathcal{F}_\tau) \mid f \in \mathcal{A}^{\phi}_{\tau,\theta}\} = 0$, *a.s.* P
(**N**) **Normalization:** ess.inf $\{f \in L^\infty(\mathcal{F}_\tau) \mid f1_{[\tau,T]} \in \mathcal{A}^{\Phi}_{\tau,\theta}\} = 0$, *a.s.* P
as well as
(**m**) **Monotonicity:** $X \in \mathcal{A}^{\phi}_{\tau,\theta}$, $Y \in L^\infty(\mathcal{F}_\theta)$, $X \leq Y$, *a.s.* $P \Rightarrow Y \in \mathcal{A}^{\phi}_{\tau,\theta}$
(**M**) **Monotonicity:** $C \in \mathcal{A}^{\Phi}_{\tau,\theta}$, $D \in \mathcal{R}^\infty_{\tau,\theta}$, $C \leq D$, *a.s.* $P \Rightarrow D \in \mathcal{A}^{\Phi}_{\tau,\theta}$.

If $\phi_{\tau,\theta}$ and $\Phi_{\tau,\theta}$ are conditional concave utility functionals, then $\mathcal{A}^{\phi}_{\tau,\theta}$ and $\mathcal{A}^{\Phi}_{\tau,\theta}$ satisfy

($\mathcal{F}_\tau$-**c**) $\mathcal{F}_\tau$-**Convexity:** $\lambda X + (1-\lambda)Y \in \mathcal{A}^{\phi}_{\tau,\theta}$ *for all* $X, Y \in \mathcal{A}^{\phi}_{\tau,\theta}$ *and* $\lambda \in L^\infty(\mathcal{F}_\tau)$ *such that* $0 \leq \lambda \leq 1$, *a.s.* P
($\mathcal{F}_\tau$-**C**) $\mathcal{F}_\tau$-**Convexity:** $\lambda C + (1-\lambda)D \in \mathcal{A}^{\Phi}_{\tau,\theta}$ *for all* $C, D \in \mathcal{A}^{\Phi}_{\tau,\theta}$ *and* $\lambda \in L^\infty(\mathcal{F}_\tau)$ *such that* $0 \leq \lambda \leq 1$, *a.s.* P.

If $\phi_{\tau,\theta}$ and $\Phi_{\tau,\theta}$ are conditional coherent utility functionals, then $\mathcal{A}^{\phi}_{\tau,\theta}$ and $\mathcal{A}^{\Phi}_{\tau,\theta}$ satisfy

($\mathcal{F}_\tau$-**ph**) $\mathcal{F}_\tau$-**Positive Homogeneity:** $\lambda X \in \mathcal{A}^{\phi}_{\tau,\theta}$ *for all* $X \in \mathcal{A}^{\phi}_{\tau,\theta}$ *and* $\lambda \in L^\infty_+(\mathcal{F}_\tau)$
($\mathcal{F}_\tau$-**PH**) $\mathcal{F}_\tau$-**Positive Homogeneity:** $\lambda C \in \mathcal{A}^{\Phi}_{\tau,\theta}$ *for all* $C \in \mathcal{A}^{\Phi}_{\tau,\theta}$ *and* $\lambda \in L^\infty_+(\mathcal{F}_\tau)$
as well as

(**sa**) **Superadditivity:** $X+Y \in \mathcal{A}^{\phi}_{\tau,\theta}$ *for all* $X, Y \in \mathcal{A}^{\phi}_{\tau,\theta}$
(**SA**) **Superadditivity:** $C+D \in \mathcal{A}^{\Phi}_{\tau,\theta}$ *for all* $C, D \in \mathcal{A}^{\Phi}_{\tau,\theta}$.

Proof. (**n**) and (**N**): From the definition of $\mathcal{A}^{\Phi}_{\tau,\theta}$, ($\mathcal{F}_\tau$-**TI**) and (**N**) of definition 2.4.1 it follows that

$$\begin{aligned} & \text{ess.inf}\,\{f \in L^\infty(\mathcal{F}_\tau) \mid f1_{[\tau,T]} \in \mathcal{A}^{\Phi}_{\tau,\theta}\} \\ = \; & \text{ess.inf}\,\{f \in L^\infty(\mathcal{F}_\tau) \mid \Phi_{\tau,\theta}(f1_{[\tau,T]}) \geq 0, \text{ a.s. } P\} \\ = \; & \text{ess.inf}\,\{f \in L^\infty(\mathcal{F}_\tau) \mid \Phi_{\tau,\theta}(0) + f \geq 0, \text{ a.s. } P\} \\ = \; & \text{ess.inf}\,\{f \in L^\infty(\mathcal{F}_\tau) \mid f \geq 0, \text{ a.s. } P\} = 0, \quad \text{a.s. } P. \end{aligned}$$

In the same way we derive from the definition of $\mathcal{A}^{\phi}_{\tau,\theta}$, ($\mathcal{F}_\tau$-**ti**) and (**n**) of definition 2.3.1 that

$$\begin{aligned} & \text{ess.inf}\,\{f \in L^\infty(\mathcal{F}_\tau) \mid f \in \mathcal{A}^{\phi}_{\tau,\theta}\} \\ = \; & \text{ess.inf}\,\{f \in L^\infty(\mathcal{F}_\tau) \mid f \geq 0, \text{ a.s. } P\} = 0, \quad \text{a.s. } P. \end{aligned}$$

(**m**) and (**M**) follow from (**m**) of definition 2.3.1 and (**M**) of definition 2.4.1. The remaining statements of the proposition also follow from the corresponding properties of $\phi_{\tau,\theta}$ and $\Phi_{\tau,\theta}$. □

Proposition 3.2.6 *Let* $\mathcal{L} \subset L^\infty(\mathcal{F}_\theta)$ *and* $\mathcal{R} \subset \mathcal{R}^\infty_{\tau,\theta}$. *If* $\mathcal{L}$ *and* $\mathcal{R}$ *respectively satisfy* (**n**), (**m**) *and* (**N**), (**M**) *of proposition* 3.2.5 *then* $\phi_{\mathcal{L}}$ *and* $\Phi_{\mathcal{R}}$ *are conditional monetary utility functionals (on* $L^\infty(\mathcal{F}_\theta)$ *and on* $\mathcal{R}^\infty_{\tau,\theta}$*).*

If $\mathcal{L}$ *and* $\mathcal{R}$ *respectively satisfy* (**n**), (**m**), ($\mathcal{F}_\tau$-**c**) *and* (**N**), (**M**), ($\mathcal{F}_\tau$-**C**) *of proposition* 3.2.5 *then* $\phi_{\mathcal{L}}$ *and* $\Phi_{\mathcal{R}}$ *are conditional concave utility functionals.*

If $\mathcal{L}$ *and* $\mathcal{R}$ *respectively satisfy* (**n**), (**m**), ($\mathcal{F}_\tau$-**c**), ($\mathcal{F}_\tau$-**ph**) *and* (**N**), (**M**), ($\mathcal{F}_\tau$-**C**), ($\mathcal{F}_\tau$-**PH**) *of proposition* 3.2.5 *then* $\phi_{\mathcal{L}}$ *and* $\Phi_{\mathcal{R}}$ *are conditional coherent utility functionals.*

Proof. (**n**), (**m**) of definition 2.3.1 and (**N**), (**M**) of definition 2.4.1 respectively follow from (**n**), (**m**) and (**N**), (**M**) of proposition 3.2.5. Normalization of $\mathcal{L}$ and $\mathcal{R}$ in particular guarantees that $\phi_{\mathcal{L}}$ and $\Phi_{\mathcal{R}}$ only take values in $L^\infty(\mathcal{F}_\tau)$. $\mathcal{F}_\tau$-translation invariance of $\phi_{\mathcal{L}}$ and $\Phi_{\mathcal{R}}$ in both cases follows from their definitions.
($\mathcal{F}_\tau$-**c**) and ($\mathcal{F}_\tau$-**C**): Take $X^1, X^2 \in L^\infty(\mathcal{F}_\theta)$, $C^1, C^2 \in \mathcal{R}^\infty_{\tau,\theta}$ and $m^1, m^2, M^1, M^2 \in L^\infty(\mathcal{F}_\tau)$ such that $X^i - m^i \in \mathcal{L}$ and $C^i - M^i 1_{[\tau,T]} \in \mathcal{R}$. From ($\mathcal{F}_\tau$-**c**) and ($\mathcal{F}_\tau$-**C**) of $\mathcal{L}$ and $\mathcal{R}$ it then follows that $\lambda(X^1 - m^1) + (1-\lambda)(X^2 - m^2) \in \mathcal{L}$ and $\lambda(C^1 - M^1 1_{[\tau,T]}) + (1-\lambda)(C^2 - M^2 1_{[\tau,T]}) \in \mathcal{R}$ for all $\lambda \in L^\infty(\mathcal{F}_\tau)$ such that $0 \leq \lambda \leq 1$, a.s. P. By ($\mathcal{F}_\tau$-**ti**) we get

$$\begin{aligned} 0 \; & \leq \; \phi_{\mathcal{L}}\big(\lambda(X^1 - m^1) + (1-\lambda)(X^2 - m^2)\big) \\ & = \; \phi_{\mathcal{L}}\big(\lambda X^1 + (1-\lambda) X^2\big) - \big(\lambda m^1 + (1-\lambda) m^2\big), \quad \text{a.s. } P, \end{aligned}$$

and by ($\mathcal{F}_\tau$-**TI**)

$$\begin{aligned} 0 \; & \leq \; \Phi_{\mathcal{R}}\big(\lambda(C^1 - M^1 1_{[\tau,T]}) + (1-\lambda)(C^2 - M^2 1_{[\tau,T]})\big) \\ & = \; \Phi_{\mathcal{R}}\big(\lambda C^1 + (1-\lambda) C^2\big) - \big(\lambda M^1 + (1-\lambda) M^2\big), \quad \text{a.s. } P. \end{aligned}$$

Taking the essential supremum yields ($\mathcal{F}_\tau$-**c**) and ($\mathcal{F}_\tau$-**C**).
($\mathcal{F}_\tau$-**ph**) and ($\mathcal{F}_\tau$-**PH**): As in the proof of concavity we derive that $\lambda\phi_{\mathcal{L}}(X) \leq \phi_{\mathcal{L}}(\lambda X)$, a.s. P, and that $\lambda\Phi_{\mathcal{R}}(C) \leq \Phi_{\mathcal{R}}(\lambda C)$, a.s. P, for $X \in L^\infty(\mathcal{F}_\theta)$, $C \in \mathcal{R}^\infty_{\tau,\theta}$ and $\lambda \in L^\infty_+(\mathcal{F}_\tau)$. To prove the reverse inequality, let $m, M \in L^\infty(\mathcal{F}_\tau)$ such that $m > \phi_{\mathcal{L}}(X)$, a.s. P, and $M > \Phi_{\mathcal{R}}(C)$, a.s. P, i.e. $X - m \notin \mathcal{L}$ and $C - M1_{[\tau,T]} \notin \mathcal{R}$. It follows from ($\mathcal{F}_\tau$-**ph**) of $\mathcal{L}$ and ($\mathcal{F}_\tau$-**PH**) of $\mathcal{R}$ that $\lambda X - \lambda m \notin \mathcal{L}$ and $\lambda C - \lambda M 1_{[\tau,T]} \notin \mathcal{R}$ and in turn $\lambda m > \phi_{\mathcal{L}}(\lambda X)$, a.s. P, and $\lambda M > \Phi_{\mathcal{R}}(\lambda C)$, a.s. P, for $\lambda \in L^\infty_+(\mathcal{F}_\tau)$. Hence, the assertion follows. □

Remark 3.2.7 *In view of remarks* 2.3.2 *and* 2.4.2 *we may recast the last statement of proposition* 3.2.6 *as follows: If $\mathcal{L}$ and $\mathcal{R}$ respectively satisfy* (**n**)*,* (**m**)*,* (**sa**)*,* ($\mathcal{F}_\tau$-**ph**) *and* (**N**)*,* (**M**)*,* (**SA**)*,* ($\mathcal{F}_\tau$-**PH**) *of proposition* 3.2.5 *then $\phi_{\mathcal{L}}$ and $\Phi_{\mathcal{R}}$ are conditional coherent utility functionals.*

Only (**sa**) *and* (**SA**) *remain to be proved. To this end, take $X^1, X^2 \in L^\infty(\mathcal{F}_\theta)$, $C^1, C^2 \in \mathcal{R}^\infty_{\tau,\theta}$ and $m^1, m^2, M^1, M^2 \in L^\infty(\mathcal{F}_\tau)$ such that $X^i - m^i \in \mathcal{L}$ and $C^i - 1_{[\tau,T]}M^i \in \mathcal{R}$ for $i \in \{0,1\}$. From* (**sa**) *and* (**SA**) *of $\mathcal{L}$ and $\mathcal{R}$ it then follows that $(X^1 - m^1) + (X^2 - m^2) \in \mathcal{L}$ and $(C^1 - 1_{[\tau,T]}M^1) + (C^2 - 1_{[\tau,T]}M^2) \in \mathcal{R}$. By* ($\mathcal{F}_\tau$-**ti**) *we get*

$$\begin{aligned} 0 &\leq \phi_{\mathcal{L}}\big((X^1 - m^1) + (X^2 - m^2)\big) \\ &= \phi_{\mathcal{L}}(X^1 + X^2) - (m^1 + m^2), \quad \text{a.s. } P, \end{aligned}$$

and by ($\mathcal{F}_\tau$-**TI**)

$$\begin{aligned} 0 &\leq \Phi_{\mathcal{R}}\big((C^1 - 1_{[\tau,T]}M^1) + (C^2 - 1_{[\tau,T]}M^2)\big) \\ &= \Phi_{\mathcal{R}}(C^1 + C^2) - (M^1 + M^2), \quad \text{a.s. } P. \end{aligned}$$

Taking the essential supremum yields (**sa**) *and* (**SA**).

3.3 The General Case

For all of this section $\Psi : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ denotes a functional that satisfies ($\mathcal{F}_\tau$-**TI**), ($\mathcal{F}_\tau$-**PH**) and (**SA**) of definition 2.4.1. As in remark 2.4.2 we derive that Ψ is normalized and $\mathcal{F}_\tau$-concave. Thus, the proofs of proposition 3.2.5 and remark 3.2.7 in particular tell us that the set

$$\mathcal{A} := \mathcal{A}(\Psi) := \{C \in \mathcal{R}^\infty_{\tau,\theta} \mid \Psi(C) \geq 0, \text{ a.s. } P\}$$

satisfies (**N**), ($\mathcal{F}_\tau$-**C**), ($\mathcal{F}_\tau$-**PH**) and (**SA**) of proposition 3.2.5. Moreover, $\mathcal{A}$ contains the origin.

Lemma 3.3.1 *For a conditional monetary utility functional $\Phi_{\tau,\theta}$ (on $\mathcal{R}^\infty_{\tau,\theta}$) we have*

$$\Psi(C) \leq \Phi_{\tau,\theta}(C), \text{ a.s. } P, \text{ for all } C \in \mathcal{R}^\infty_{\tau,\theta} \qquad \Leftrightarrow \qquad \mathcal{A} \subset \mathcal{A}^\Phi_{\tau,\theta}$$

Proof. To prove "$\Leftarrow$", note that in the proof of proposition 3.2.4 a conditional monetary utility functional only needed to satisfy ($\mathcal{F}_\tau$-**TI**) to be recovered from its acceptance set. Thus, we may as well recover Ψ from $\mathcal{A}$ by

$$\Psi(C) = \text{ess.sup}\,\{m \in L^\infty(\mathcal{F}_\tau) \mid C - m1_{[\tau,T]} \in \mathcal{A}\}, \text{ a.s. } P, \text{ for all } C \in \mathcal{R}^\infty_{\tau,\theta}.$$

It follows that for all $C \in \mathcal{R}^\infty_{\tau,\theta}$

$$\begin{aligned}\Psi(C) &= \text{ess.sup}\,\{m \in L^\infty(\mathcal{F}_\tau) \mid C - m1_{[\tau,T]} \in \mathcal{A}\} \\ &\leq \text{ess.sup}\,\{m \in L^\infty(\mathcal{F}_\tau) \mid C - m1_{[\tau,T]} \in \mathcal{A}^\Phi_{\tau,\theta}\} \\ &= \Phi_{\tau,\theta}(C), \quad \text{a.s. } P.\end{aligned}$$

Since $0 \leq \Psi(C) \leq \Phi_{\tau,\theta}(C)$, a.s. P, for all $C \in \mathcal{A}$ the reverse implication follows from the definition of acceptance sets. □

Remark 3.3.2 *As in the above lemma we derive that for two conditional monetary utility functionals $\Phi^1_{\tau,\theta}$ and $\Phi^2_{\tau,\theta}$ (on $\mathcal{R}^\infty_{\tau,\theta}$) we have*

$$\Phi^1_{\tau,\theta}(C) \leq \Phi^2_{\tau,\theta}(C), \text{ a.s. } P, \text{ for all } C \in \mathcal{R}^\infty_{\tau,\theta} \qquad \Leftrightarrow \qquad \mathcal{A}^{\Phi^1}_{\tau,\theta} \subset \mathcal{A}^{\Phi^2}_{\tau,\theta} \tag{3.3.1}$$

Under adequate adjustments (3.3.1) *is also valid for conditional monetary utility functionals on $L^\infty(\mathcal{F}_\theta)$.*

Definition 3.3.3 *The monotone hull of the subset $\mathcal{A}$ of $\mathcal{R}^\infty_{\tau,\theta}$ is given by*

$$\overline{\mathcal{A}}^M := \{D \in \mathcal{R}^\infty_{\tau,\theta} \mid \exists\, C \in \mathcal{A} : D \geq C, \text{ a.s. } P\}.$$

Note that $\mathcal{A} \subset \overline{\mathcal{A}}^M$. In particular, $\overline{\mathcal{A}}^M$ contains the origin and is thus nonempty.

The monotone hull of the functional $\Psi : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ is given by the functional $\Phi_{\overline{\mathcal{A}}^M} : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$,

$$C \mapsto \Phi_{\overline{\mathcal{A}}^M}(C) := \text{ess.sup}\,\{m \in L^\infty(\mathcal{F}_\tau) \mid C - m1_{[\tau,T]} \in \overline{\mathcal{A}}^M\}.$$

Lemma 3.3.4 *The monotone hull $\overline{\mathcal{A}}^M$ of $\mathcal{A} = \mathcal{A}(\Psi)$ is the smallest subset of $\mathcal{R}^\infty_{\tau,\theta}$ that contains $\mathcal{A}$ and satisfies* (**M**)*. Moreover, $\overline{\mathcal{A}}^M$ satisfies* (**SA**) *and* ($\mathcal{F}_\tau$-**PH**) *of proposition 3.2.5.*

Proof. First, observe that any subset of $\mathcal{R}^\infty_{\tau,\theta}$ that contains $\mathcal{A}$ and satisfies (**M**) has to contain $\overline{\mathcal{A}}^M$. From the definition of $\overline{\mathcal{A}}^M$ it follows that $\overline{\mathcal{A}}^M$ satisfies (**M**) which proves the first statement.
(**SA**): Let $C, D \in \overline{\mathcal{A}}^M$. Then we can find $\tilde{C}, \tilde{D} \in \mathcal{A}$ such that $\tilde{C} \leq C$, a.s. P, and $\tilde{D} \leq D$, a.s. P, which in turn yields $\tilde{C} + \tilde{D} \leq C + D$, a.s. P. But since $\tilde{C} + \tilde{D} \in \mathcal{A}$ due to (**SA**) of $\mathcal{A}$ we deduce that $\overline{\mathcal{A}}^M$ satisfies (**SA**) as well.
($\mathcal{F}_\tau$-**PH**): For $C \in \overline{\mathcal{A}}^M$ take $\tilde{C} \in \mathcal{A}$ such that $\tilde{C} \leq C$, a.s. P. Since $\mathcal{A}$ satisfies ($\mathcal{F}_\tau$-**PH**)

we have $\lambda\tilde{C} \in \mathcal{A}$ for all $\lambda \in L^\infty_+(\mathcal{F}_\tau)$. For such λ we also have $\lambda\tilde{C} \leq \lambda C$, a.s. P, and ($\mathcal{F}_\tau$-**PH**) of $\overline{\mathcal{A}}^M$ follows. □

The following theorem states that if there exists a conditional monetary utility functional which dominates Ψ then there also exists a smallest such functional. In this case, this functional is automatically conditional coherent and given by the monotone hull $\Phi_{\overline{\mathcal{A}}^M}$ of Ψ.

Theorem 3.3.5 *Assume that there exists a conditional monetary utility functional $\Phi_{\tau,\theta} : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty(\mathcal{F}_\tau)$ such that $\Psi(C) \leq \Phi_{\tau,\theta}(C)$, a.s. P, for all $C \in \mathcal{R}^\infty_{\tau,\theta}$. Then $\Phi_{\overline{\mathcal{A}}^M}$ is a conditional coherent utility functional and*

$$\Psi(C) \;\leq\; \Phi_{\overline{\mathcal{A}}^M}(C) \;\leq\; \Phi_{\tau,\theta}(C),$$

a.s. P, for all $C \in \mathcal{R}^\infty_{\tau,\theta}$.

Proof. By definition $\mathcal{A} \subset \overline{\mathcal{A}}^M$. From lemma 3.3.1 it follows that $\mathcal{A} \subset \mathcal{A}^\Phi_{\tau,\theta}$. Since $\Phi_{\tau,\theta}$ is a conditional monetary utility functional its acceptance set $\mathcal{A}^\Phi_{\tau,\theta}$ in particular satisfies (**M**) of proposition 3.2.5. Hence, from lemma 3.3.4 it follows that $\overline{\mathcal{A}}^M \subset \mathcal{A}^\Phi_{\tau,\theta}$. Altogether we have $\mathcal{A} \subset \overline{\mathcal{A}}^M \subset \mathcal{A}^\Phi_{\tau,\theta}$ and in turn for all $C \in \mathcal{R}^\infty_{\tau,\theta}$

$$\begin{aligned} & \text{ess.sup}\, \{m \in L^\infty(\mathcal{F}_\tau) \mid C - m1_{[\tau,T]} \in \mathcal{A}\} \\ \leq\; & \text{ess.sup}\, \{m \in L^\infty(\mathcal{F}_\tau) \mid C - m1_{[\tau,T]} \in \overline{\mathcal{A}}^M\} \\ \leq\; & \text{ess.sup}\, \{m \in L^\infty(\mathcal{F}_\tau) \mid C - m1_{[\tau,T]} \in \mathcal{A}^\Phi_{\tau,\theta}\}, \quad \text{a.s. } P, \end{aligned}$$

i.e. $\Psi(C) \leq \Phi_{\overline{\mathcal{A}}^M}(C) \leq \Phi_{\tau,\theta}(C)$, a.s. P, for all $C \in \mathcal{R}^\infty_{\tau,\theta}$. In particular, $\Phi_{\overline{\mathcal{A}}^M}$ takes only values in $L^\infty(\mathcal{F}_\tau)$ and is thus well defined. Since Ψ and $\Phi_{\tau,\theta}$ are normalized we have $0 \leq \Phi_{\overline{\mathcal{A}}^M}(0) \leq 0$ and hence, $\Phi_{\overline{\mathcal{A}}^M}$ is normalized as well. From the definition of $\Phi_{\overline{\mathcal{A}}^M}$ we derive that $\Phi_{\overline{\mathcal{A}}^M}$ satisfies ($\mathcal{F}_\tau$-**TI**). With lemma 3.3.4 we know that $\overline{\mathcal{A}}^M$ satisfies (**M**), (**SA**) and ($\mathcal{F}_\tau$-**PH**). As in proposition 3.2.6 and remark 3.2.7 we derive the corresponding properties for $\Phi_{\overline{\mathcal{A}}^M}$. By remark 2.4.2, $\Phi_{\overline{\mathcal{A}}^M}$ is a conditional coherent utility functional .□

Remark 3.3.6 *From the the proof of the above theorem we derive in particular that the monotone hull $\Phi_{\overline{\mathcal{A}}^M}$ of Ψ is normalized if there exists a conditional monetary utility functional majorizing Ψ. On the contrary, if the monotone hull $\Phi_{\overline{\mathcal{A}}^M}$ was assumed to be normalized, then itself would be a conditional coherent (in particular monetary) utility functional majorizing Ψ. Thus, we have equivalence between normalization of $\Phi_{\overline{\mathcal{A}}^M}$ and the existence of a conditional monetary utility functional majorizing Ψ.*

The remarkable consequence is: If $\Phi_{\overline{\mathcal{A}}^M}$ is not normalized, then there exists no conditional monetary utility functional which majorizes Ψ at all since otherwise normalization of $\Phi_{\overline{\mathcal{A}}^M}$ would follow which is impossible.

3.4 A Constructive Example

In this section we explicitly construct the monotone hull of the functional $\Psi_{\tau,T}$ of example 2.3.3 in the case where $\theta(\omega) = T$ for all $\omega \in \Omega$.

Lemma 3.4.1 *Let $\phi_{\tau,T}$ be a conditional coherent utility functional on L^∞, $\tau(\omega) \leq T-2$ for all $\omega \in \Omega$ (in particular $T \geq 2$) and $\alpha > 0$. If $\Phi_{\tau,T}$ is a conditional coherent utility functional on $\mathcal{R}^\infty_{\tau,T}$ with*

$$\Phi_{\tau,T}(C) \geq (1-\alpha)\phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha\big(\phi_{\tau,T-1}(C_{T-1}) + \phi_{\tau,T}(\Delta C_T)\big), \quad a.s.\ P, \tag{3.4.2}$$

for all $C \in \mathcal{R}^\infty_{\tau,T}$, then

$$\Phi_{\tau,T}(C) \geq (1-\alpha)\phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha\phi_{\tau,T}(C_T), \quad a.s.\ P,$$

for all $C \in \mathcal{R}^\infty_{\tau,T}$. Note that the case where $T = 2$ (and in turn $\tau(\omega) = 0$ for all $\omega \in \Omega$) is included.

Proof. Let $C \in \mathcal{R}^\infty_{\tau,T}$. The set $\{\tau = T-2\}$ belongs to the σ-algebra $\mathcal{F}_\tau$. Therefore, $\Phi_{\tau,T}(C)$ can also be written in the form $1_{\{\tau=T-2\}}\Phi_{\tau,T}(C) + 1_{\{\tau<T-2\}}\Phi_{\tau,T}(C)$. We prove the statement on the sets $\{\tau = T-2\}$ and $\{\tau < T-2\}$ separately.
$\{\tau = T-2\}$: Since $\Phi_{\tau,T}$ is superadditiv we deduce

$$\begin{aligned} 1_{\{\tau=T-2\}}\Phi_{\tau,T}(C) &= 1_{\{\tau=T-2\}}\Phi_{\tau,T}(C1_{[\tau,T-2]} + C_{T-1}1_{\{T-1\}} + C_T1_{\{T\}}) \\ &\geq 1_{\{\tau=T-2\}}\Big(\Phi_{\tau,T}(C1_{[\tau,T-2]}) \\ &\quad +\Phi_{\tau,T}(C_{T-1}1_{\{T-1\}}) + \Phi_{\tau,T}(C_T1_{\{T\}})\Big), \quad \text{a.s. } P. \end{aligned} \tag{3.4.3}$$

Applying (3.4.2) to all three summands in (3.4.3) gives

$$\begin{aligned} 1_{\{\tau=T-2\}}\Phi_{\tau,T}(C1_{[\tau,T-2]}) &\geq 0, \quad \text{a.s. } P, \\ 1_{\{\tau=T-2\}}\Phi_{\tau,T}(C_{T-1}1_{\{T-1\}}) &\geq 1_{\{\tau=T-2\}}\big((1-\alpha)\phi_{\tau,\tau+1}(C_{\tau+1})\big) \\ &\quad +1_{\{\tau=T-2\}}\Big(\alpha\big(\phi_{\tau,T-1}(C_{T-1}) + \phi_{\tau,T}(-C_{T-1})\big)\Big) \\ &= 1_{\{\tau=T-2\}}\big((1-\alpha)\phi_{\tau,\tau+1}(C_{\tau+1})\big), \quad \text{a.s. } P, \text{ and} \\ 1_{\{\tau=T-2\}}\Phi_{\tau,T}(C_T1_{\{T\}}) &\geq 1_{\{\tau=T-2\}}\alpha\phi_{\tau,T}(C_T), \quad \text{a.s. } P. \end{aligned}$$

Adding up yields the assertion on $\{\tau = T-2\}$.
$\{\tau < T-2\}$: (3.4.3) is valid on $\{\tau < T-2\}$ as well:

$$\begin{aligned} 1_{\{\tau<T-2\}}\Phi_{\tau,T}(C) &\geq 1_{\{\tau<T-2\}}\Big(\Phi_{\tau,T}(C1_{[\tau,T-2]}) \\ &\quad +\Phi_{\tau,T}(C_{T-1}1_{\{T-1\}}) + \Phi_{\tau,T}(C_T1_{\{T\}})\Big), \quad \text{a.s. } P. \end{aligned} \tag{3.4.4}$$

This time we apply (3.4.2) to the first and the last summand in (3.4.4):

$$\begin{aligned} 1_{\{\tau<T-2\}}\Phi_{\tau,T}(C1_{[\tau,T-2]}) &\geq 1_{\{\tau<T-2\}}\big((1-\alpha)\phi_{\tau,\tau+1}(C_{\tau+1})\big), \quad \text{a.s. } P, \\ 1_{\{\tau<T-2\}}\Phi_{\tau,T}(C_T 1_{\{T\}}) &\geq 1_{\{\tau<T-2\}}\alpha\phi_{\tau,T}(C_T), \quad \text{a.s. } P. \end{aligned}$$

The claim will now follow if we can show that $\Phi_{\tau,T}(C_{T-1}1_{\{T-1\}}) \geq 0$, a.s. P. To this end, note that $C \in \mathcal{R}^{\infty}_{\tau,T}$ is bounded. Thus, from monotonicity of $\Phi_{\tau,T}$, (3.4.2) and $\mathcal{F}_\tau$-translation invariance of $\phi_{\tau,T}$ we derive

$$\begin{aligned} \Phi_{\tau,T}(C_{T-1}1_{\{T-1\}}) &\geq \Phi_{\tau,T}\big((\text{ess.inf } C_{T-1})1_{\{T-1\}}\big) \\ &\geq \phi_{\tau,T-1}(\text{ess.inf } C_{T-1}) + \phi_{\tau,T}(-\text{ess.inf } C_{T-1}) \\ &= \text{ess.inf } C_{T-1} - \text{ess.inf } C_{T-1} = 0, \quad \text{a.s. } P. \end{aligned}$$

Hence, the assertion follows. □

Theorem 3.4.2 *Let $\phi_{\tau,T}$ be a conditional coherent utility functional on $L^\infty(\mathcal{F}_\tau)$ and $\alpha > 0$. As in example* 2.3.3 *we assume that $\tau(\omega) + 1 \leq \theta(\omega) = T$ for all $\omega \in \Omega$. If $\Phi_{\tau,T}$ is a conditional coherent utility functional on $\mathcal{R}^{\infty}_{\tau,T}$ with*

$$\Phi_{\tau,T}(C) \geq \Psi_{\tau,T}(C) = \phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha \sum_{\tau+1<s\leq T} \phi_{\tau,T}(\Delta C_s), \quad \textit{a.s. } P, \tag{3.4.5}$$

for all $C \in \mathcal{R}^{\infty}_{\tau,T}$, then

$$\Phi_{\tau,T}(C) \geq (1-\alpha)\phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha\phi_{\tau,T}(C_T), \quad \textit{a.s. } P, \tag{3.4.6}$$

for all $C \in \mathcal{R}^{\infty}_{\tau,T}$.

Proof. Fix $C \in \mathcal{R}^{\infty}_{\tau,T}$. The proof is by induction on T.
$T = 1$: In this case, we have $\tau(\omega) = 0$ for all $\omega \in \Omega$. The statement now follows from the observation that $\Psi_{0,1}(C)$ reduces to $\phi_{0,1}(C_1) = (1-\alpha)\phi_{0,1}(C_1) + \alpha\phi_{0,1}(C_1)$, a.s. P, which is the righthand side of (3.4.6).
Now suppose the assertion is true for some $T - 1 \geq 1$. Since $\{\tau + 1 = T\} \in \mathcal{F}_\tau$ we have $\Phi_{\tau,T}(C) = \Phi_{\tau,T}(C)1_{\{\tau+1=T\}} + \Phi_{\tau,T}(C)1_{\{\tau+1<T\}}$. By the assumption in (3.4.5),

$$\begin{aligned} \Phi_{\tau,T}(C)1_{\{\tau+1=T\}} &\geq \left(\phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha \sum_{\tau+1<s\leq T} \phi_{\tau,T}(\Delta C_s)\right) 1_{\{\tau+1=T\}} \\ &= \phi_{\tau,\tau+1}(C_{\tau+1})1_{\{\tau+1=T\}}, \quad \text{a.s. } P. \end{aligned}$$

Furthermore, since $1_{\{\tau+1=T\}}$ is $\mathcal{F}_\tau$-measurable and $\phi_{\tau,\tau+1}$ is $\mathcal{F}_\tau$-positive homogeneous, we have in view of (2.3.3)

$$\phi_{\tau,\tau+1}(C_{\tau+1})1_{\{\tau+1=T\}} = \phi_{\tau,\tau+1}(C_{\tau+1}1_{\{\tau+1=T\}}) = \phi_{\tau,T}(C_T 1_{\{\tau+1=T\}}) = \phi_{\tau,T}(C_T)1_{\{\tau+1=T\}},$$

a.s. P. Thus,

$$\big((1-\alpha)\phi_{\tau,\tau+1}(C_{\tau+1})+\alpha\phi_{\tau,T}(C_T)\big)1_{\{\tau+1=T\}}=\phi_{\tau,\tau+1}(C_{\tau+1})1_{\{\tau+1=T\}},\quad \text{a.s. } P,$$

and hence, the claim follows on $\{\tau+1=T\}$.
To prove the statement on $\{\tau+1<T\}$ we set $A=\{\tau+1<T\}$ and consider the probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P}):=(\Omega\cap A,\mathcal{F}\cap A,P|_{\mathcal{F}\cap A})$ endowed with the filtration $(\tilde{\mathcal{F}}_t)_{t\in\mathbb{T}}:=(\mathcal{F}_t\cap A)_{t\in\mathbb{T}}$. We denote by $\tilde{C}$ and $\tilde{\tau}$ the restrictions of C and τ to $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$. Since $\tilde{\tau}$ is an $(\tilde{\mathcal{F}}_t)$-stopping time we may set $\tilde{\mathcal{R}}^\infty_{\tilde{\tau},T}=\mathcal{R}^\infty_{\tilde{\tau},T}(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$. By

$$\begin{array}{rcl} X & \mapsto & \tilde{\phi}_{\tilde{\tau},T}(X):=\phi_{\tau,T}(X1_{\{\tau+1<T\}})|_{\tilde{\Omega}} \quad \text{and}\\ D & \mapsto & \tilde{\Phi}_{\tilde{\tau},T}(D):=\Phi_{\tau,T}(D1_{\{\tau+1<T\}})|_{\tilde{\Omega}} \end{array}$$

for all $X\in L^\infty(\tilde{\mathcal{F}}_T)$ and $D\in\tilde{\mathcal{R}}^\infty_{\tilde{\tau},T}$ we obtain conditional coherent utility functionals $\tilde{\phi}_{\tilde{\tau},T}$ on $L^\infty(\tilde{\mathcal{F}}_T)$ and $\tilde{\Phi}_{\tilde{\tau},T}$ on $\tilde{\mathcal{R}}^\infty_{\tilde{\tau},T}$. We may decompose $\tilde{C}=\tilde{\pi}_{\tilde{\tau},T-1}(\tilde{C})+\Delta\tilde{C}_T1_{\{T\}}$ and derive from (**SA**) of $\tilde{\Phi}_{\tilde{\tau},T}$

$$\tilde{\Phi}_{\tilde{\tau},T}(\tilde{C})\geq\tilde{\Phi}_{\tilde{\tau},T}\circ\tilde{\pi}_{\tilde{\tau},T-1}(\tilde{C})+\tilde{\Phi}_{\tilde{\tau},T}(\Delta\tilde{C}_T1_{\{T\}}),\quad \text{a.s. } \tilde{P}. \tag{3.4.7}$$

We may view $\tilde{\Phi}_{\tilde{\tau},T}\circ\tilde{\pi}_{\tilde{\tau},T-1}:\tilde{\mathcal{R}}^\infty_{\tilde{\tau},T-1}\to L^\infty(\tilde{\mathcal{F}}_{\tilde{\tau}})$ as a conditional coherent utility functional on $\tilde{\mathcal{R}}^\infty_{\tilde{\tau},T-1}$. For all $D\in\tilde{\mathcal{R}}^\infty_{\tilde{\tau},T-1}$ we have

$$\begin{array}{rcl} \tilde{\Phi}_{\tilde{\tau},T}\circ\tilde{\pi}_{\tilde{\tau},T-1}(D) & = & \tilde{\Phi}_{\tilde{\tau},T}(D)=\Phi_{\tau,T}(D1_{\{\tau+1<T\}})|_{\tilde{\Omega}}\\ & \geq & \Bigg(\phi_{\tau,\tau+1}(D_{\tau+1}1_{\{\tau+1<T\}})+\alpha\displaystyle\sum_{\tau+1<s\leq T-1}\phi_{\tau,T}(\Delta D_s1_{\{\tau+1<T\}})\\ & & +\phi_{\tau,T}\big((D_T-D_{T-1})1_{\{\tau+1<T\}}\big)\Bigg)|_{\tilde{\Omega}}\\ & = & \Bigg(\phi_{\tau,\tau+1}(D_{\tau+1}1_{\{\tau+1<T\}})+\alpha\displaystyle\sum_{\tau+1<s\leq T-1}\phi_{\tau,T}(\Delta D_s1_{\{\tau+1<T\}})\Bigg)|_{\tilde{\Omega}}\\ & = & \tilde{\phi}_{\tilde{\tau},\tilde{\tau}+1}(D_{\tilde{\tau}+1})+\alpha\displaystyle\sum_{\tilde{\tau}+1<s\leq T-1}\tilde{\phi}_{\tilde{\tau},T}(\Delta D_s)\\ & =: & \tilde{\Psi}_{\tilde{\tau},T-1}(D),\quad \text{a.s. } \tilde{P}, \end{array}$$

Hence, $\tilde{\Phi}_{\tilde{\tau},T}\circ\tilde{\pi}_{\tilde{\tau},T-1}$ is a conditional coherent utility functional on $\tilde{\mathcal{R}}^\infty_{\tilde{\tau},T-1}$ which dominates $\tilde{\Psi}_{\tilde{\tau},T-1}$. Moreover, $\tilde{\tau}(\omega)+1\leq T-1$ for all $\omega\in\tilde{\Omega}$ and thus induction hypotheses applies to $\tilde{\Phi}_{\tilde{\tau},T}\circ\tilde{\pi}_{\tilde{\tau},T-1}$, i.e

$$\tilde{\Phi}_{\tilde{\tau},T}\circ\tilde{\pi}_{\tilde{\tau},T-1}(\tilde{C})\geq(1-\alpha)\tilde{\phi}_{\tilde{\tau},\tilde{\tau}+1}(\tilde{C}_{\tilde{\tau}+1})+\alpha\tilde{\phi}_{\tilde{\tau},T-1}(\tilde{C}_{T-1}),\quad \text{a.s. } \tilde{P}.$$

Plugging this into (3.4.7) yields

$$\tilde{\Phi}_{\tilde{\tau},T}(\tilde{C})\geq(1-\alpha)\tilde{\phi}_{\tilde{\tau},\tilde{\tau}+1}(\tilde{C}_{\tilde{\tau}+1})+\alpha\tilde{\phi}_{\tilde{\tau},T-1}(\tilde{C}_{T-1})+\tilde{\Phi}_{\tilde{\tau},T}(\Delta\tilde{C}_T1_{\{T\}}),\quad \text{a.s. } \tilde{P}. \tag{3.4.8}$$

We may estimate the last summand in (3.4.8) by

$$\begin{aligned}\tilde{\Phi}_{\tilde{\tau},T}(\Delta\tilde{C}_T 1_{\{T\}}) &= \Phi_{\tau,T}\big(\Delta\tilde{C}_T 1_{\{T\}} 1_{\{\tau+1<T\}}\big)|_{\tilde{\Omega}} \\ &\geq \alpha\phi_{\tau,T}(\Delta\tilde{C}_T 1_{\{\tau+1<T\}})|_{\tilde{\Omega}} = \alpha\tilde{\phi}_{\tilde{\tau},T}(\Delta\tilde{C}_T), \quad \text{a.s. } \tilde{P},\end{aligned}$$

where the inequality follows from the assumption in (3.4.5). Plugging this into (3.4.8) yields

$$\tilde{\Phi}_{\tilde{\tau},T}(\tilde{C}) \geq (1-\alpha)\tilde{\phi}_{\tilde{\tau},\tilde{\tau}+1}(\tilde{C}_{\tilde{\tau}+1}) + \alpha\tilde{\phi}_{\tilde{\tau},T-1}(\tilde{C}_{T-1}) + \alpha\tilde{\phi}_{\tilde{\tau},T}(\Delta\tilde{C}_T), \quad \text{a.s. } \tilde{P}.$$

But since $\tilde{\tau}(\omega) \leq T-2$ for all $\omega \in \tilde{\Omega}$ we may now apply lemma 3.4.1 to $\tilde{\Phi}_{\tilde{\tau},T}$ and hence,

$$\tilde{\Phi}_{\tilde{\tau},T}(\tilde{C}) \geq (1-\alpha)\tilde{\phi}_{\tilde{\tau},\tilde{\tau}+1}(\tilde{C}_{\tilde{\tau}+1}) + \alpha\tilde{\phi}_{\tilde{\tau},T}(\tilde{C}_T), \quad \text{a.s. } \tilde{P}. \tag{3.4.9}$$

Finally, observe that

$$\begin{aligned}\tilde{\Phi}_{\tilde{\tau},T}(\tilde{C}) &= \Phi_{\tau,T}(\tilde{C}1_{\{\tau+1<T\}})|_{\tilde{\Omega}} \\ &= \Phi_{\tau,T}(C1_{\{\tau+1<T\}})|_{\tilde{\Omega}} \\ &= \big(\Phi_{\tau,T}(C)1_{\{\tau+1<T\}}\big)|_{\tilde{\Omega}} = \Phi_{\tau,T}(C)|_{\tilde{\Omega}}, \quad \text{a.s. } \tilde{P},\end{aligned}$$

as well as

$$\begin{aligned}\tilde{\phi}_{\tilde{\tau},T}(\tilde{C}_T) &= \phi_{\tau,T}(\tilde{C}_T 1_{\{\tau+1<T\}})|_{\tilde{\Omega}} \\ &= \phi_{\tau,T}(C_T 1_{\{\tau+1<T\}})|_{\tilde{\Omega}} \\ &= \big(\phi_{\tau,T}(C_T)1_{\{\tau+1<T\}}\big)|_{\tilde{\Omega}} = \phi_{\tau,T}(C_T)|_{\tilde{\Omega}}, \quad \text{a.s. } \tilde{P},\end{aligned} \tag{3.4.10}$$

where we have used that $\Phi_{\tau,T}$ and $\phi_{\tau,T}$ are $\mathcal{F}_\tau$-positive homogeneous. Note that in (3.4.10) we may as well replace $\tilde{\phi}_{\tilde{\tau},T}(\tilde{C}_T)$ (and $\phi_{\tau,T}(C_T)$) by $\tilde{\phi}_{\tilde{\tau},\tilde{\tau}+1}(\tilde{C}_{\tilde{\tau}+1})$ (and $\phi_{\tau,\tau+1}(C_{\tau+1})$). We now conclude

$$\begin{aligned}\Phi_{\tau,T}(C)1_{\{\tau+1<T\}} &= \tilde{\Phi}_{\tilde{\tau},T}(\tilde{C})1_{\{\tau+1<T\}} \\ &\geq \Big((1-\alpha)\tilde{\phi}_{\tilde{\tau},\tilde{\tau}+1}(\tilde{C}_{\tilde{\tau}+1}) + \alpha\tilde{\phi}_{\tilde{\tau},T}(\tilde{C}_T)\Big)1_{\{\tau+1<T\}} \\ &= \Big((1-\alpha)\phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha\phi_{\tau,T}(C_T)\Big)1_{\{\tau+1<T\}}, \quad \text{a.s. } P,\end{aligned}$$

where the inequality follows from (3.4.9). Hence, the assertion follows on $\{\tau+1<T\}$ and hence, on all of Ω. □

Remark 3.4.3 *Note that*

$$C \mapsto (1-\alpha)\phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha\phi_{\tau,T}(C_T)$$

is a conditional coherent utility functional on $\mathcal{R}^\infty_{\tau,T}$. Moreover, due to (**sa**) *of $\phi_{\tau,T}$ we have*

$$\begin{aligned}&\phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha\sum_{\tau+1<s\leq T}\phi_{\tau,T}(\Delta C_s) \\ =\; &(1-\alpha)\phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha\phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha\sum_{\tau+1<s\leq T}\phi_{\tau,T}(\Delta C_s) \\ \leq\; &(1-\alpha)\phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha\phi_{\tau,T}(C_T), \quad \textit{a.s. } P,\end{aligned}$$

for all $C \in \mathcal{R}^\infty_{\tau,T}$. *Thus, the above theorem proves that the monotone hull* $\Phi_{\overline{\mathcal{A}(\Psi_{\tau,T})}^M}$ *of* $\Psi_{\tau,T}$ *of example* 2.3.3 *is of the form*

$$\Phi_{\overline{\mathcal{A}(\Psi_{\tau,T})}^M}(C) = (1-\alpha)\phi_{\tau,\tau+1}(C_{\tau+1}) + \alpha\phi_{\tau,T}(C_T),$$

a.s. P, *for all* $C \in \mathcal{R}^\infty_{\tau,T}$.

We may apply this result to the case where $\tau(\omega) = 0$, $\theta(\omega) = T$, *for all* $\omega \in \Omega$, *and* $-\phi_{0,T} = ES_r$ *at some level* $r \in (0,1)$. *The above theorem states that in this case* $\overline{\Gamma_r}^M(C) := (1-\alpha)ES_r(C_1) + \alpha ES_r(C_T)$, $C \in \mathcal{R}^\infty_{0,T}$, *is the largest conditional monetary risk measure (on* $\mathcal{R}^\infty_{0,T}$*) that satisfies*

$$\overline{\Gamma_r}^M(C) \leq \Gamma_r(C) = ES_r(C_1) + \alpha \sum_{1<s\leq T} ES_r(\Delta C_s), \quad a.s.\ P,$$

for all $C \in \mathcal{R}^\infty_{0,T}$, *where* Γ_r *designates the Swiss Solvency Test risk measure. Note, that* $\overline{\Gamma_r}^M$ *is coherent and does not depend on* C_t *at the dates* $t \in \{2,\ldots,T-1\}$.

Chapter 4

Conditional Value at Risk and Conditional Expected Shortfall

As the aim of the remaining thesis is to explore the idea of "how bad is bad?" within a dynamic temporal setting, this chapter is to provide the building blocks for this task. In fact, we propose a notion of distribution invariant conditional monetary utility functionals as functionals defined on equivalence classes of regular conditional distributions. We then provide a discussion of our notion of conditional quantiles which we choose as a vehicle to enter into the world of conditional value at risk and conditional expected shortfall. The static case results on these well-understood risk measures are translated into our conditional framework via conditional quantiles.

4.1 Introduction

Historically, quantile-based risk measures such as value at risk have been among the most common risk measures for practitioners. However, when Artzner et al. attempted an axiomatic approach towards the task of risk assessment, the shortcomings of value at risk experienced a constant increase of presence: It is well known that value at risk rather penalizes than encourages diversification as it lacks convexity in general. In addition to this, value at risk does not account for the size of extremely large losses. This is when expected shortfall grew in prominence, as this risk measure, on the contrary, turns out to be coherent and gives an idea of "how bad is bad?". The objective of this chapter is to enter into a discussion of expected shortfall from a dynamic perspective, that is to present a notion of conditional expected shortfall and to provide characterizations of it so it may later serve as the foundation for the construction of a dynamic expected shortfall.

Value at risk and expected shortfall are to be numbered among the most extensively discussed examples of risk measures. The financial literature provides various characterizations of expected shortfall inter alia in terms of value at risk. Such characterizations allow for important interpretations. In fact, only risk measures that admit meaningful interpretations are of practical concern as there is no such thing as a universal risk measure that applies to all situations. The textbook Föllmer and Schied [17] dedicates the entire section 4.4 and partly section 4.5 to a discussion of value at risk and expected shortfall. In Artzner et al. [3, 4] a notion of conditional expected shortfall is introduced, there it is called tail value at risk. They discuss dynamic consistency properties and advise caution within the multi-period setting. A study of such properties within the range of this thesis is postponed to chapters 5 and 6.

In section 4.2 we introduce an equivalence relation on the space of regular conditional probabilities with respect to P-almost surely bounded random variables on a reference probability space $(\Omega, \mathcal{F}_\theta, P)$. The corresponding equivalence classes are constructed so that random variables which coincide P-almost surely induce the same equivalence class. It is then possible to introduce our notion of distribution invariant conditional monetary utility functionals as functionals defined on equivalence classes of regular conditional probabilities. In section 4.3 we present the key notion of this chapter: a conditional quantile. We start off by investigating certain measurability features of such quantiles and as a main result, we present the lemmas 4.3.5 and 4.3.7 which will play a crucial role in section 4.4. The statements of the two lemmas are quite intuitive in the case where we additionally impose certain structures, yet the general case requires some work. In section 4.4, we then introduce our notion of conditional value at risk as the largest conditional quantile. Due to lemma 4.3.7, conditional value at risk turns out to be a conditional monetary risk measure. As the static counterpart, conditional value at risk is not convex in general. We present conditional expected shortfall in its well-established form: we take essential supremum over linear functionals induced by certain probability densities. It turns out that we are able to characterize the probability measure for which the essential supremum is attained by explicitly constructing its associated density. The techniques we use are essentially the same as in the static case however, they require thorough preparatory work as conditional quantiles turn out to be not as manageable as "classical" quantile functions. We conclude the chapter by presenting characterizations of conditional expected shortfall which transfer the static case interpretations of the static expected shortfall into the dynamic setting. This chapter clarifies, in particular, how conditional value at risk and conditional expected shortfall fit into our context of distribution invariant dynamic monetary risk measures.

Throughout this chapter we consider the setup of section 2.2 and let τ and θ be two $(\mathcal{F}_t)$-stopping times such that $\tau(\omega) \leq \theta(\omega)$ for all $\omega \in \Omega$. Within this chapter we explicitly distinguish random variables on $(\Omega, \mathcal{F}_\theta, P)$ and the corresponding equivalence classes in $L^0(\mathcal{F}_\theta)$. Random variables are denoted by $X, Y, Z, \ldots$ and $\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \ldots$ respectively designate the corresponding equivalence classes. We denote by $\mathcal{B} := \mathcal{B}(\mathbb{R})$ the σ-algebra of Borel-sets on the real line. Furthermore, we assume that we are given a mapping

$$P^{\mathcal{F}_\tau} : \Omega \times \mathcal{F}_\theta \to [0, 1]$$

which satisfies the following properties:

(1) $\quad P^{\mathcal{F}_\tau}(\omega, .) : \mathcal{F}_\theta \to [0, 1]$ is a probability measure for all $\omega \in \Omega$

(2) $\quad P^{\mathcal{F}_\tau}(., A) : (\Omega, \mathcal{F}_\tau) \to [0, 1]$ is $\mathcal{F}_\tau$-measurable for all $A \in \mathcal{F}_\theta$

(3) $\quad \int_C P^{\mathcal{F}_\tau}(w, A)\ P(d\omega) = P(C \cap A)$ for all $C \in \mathcal{F}_\tau$ and $A \in \mathcal{F}_\theta$.

Since for all $A \in \mathcal{F}_\theta$

$$\int_C E[1_A \mid \mathcal{F}_\tau]\ dP = \int_C 1_A\ dP = P(C \cap A)$$

for all $C \in \mathcal{F}_\tau$ we deduce from (2) and (3) that for all $A \in \mathcal{F}_\theta$

$$P^{\mathcal{F}_\tau}(., A) : (\Omega, \mathcal{F}_\tau) \to [0,1], \quad \omega \mapsto P^{\mathcal{F}_\tau}(w, A)$$

is a version of $E[1_A \mid \mathcal{F}_\tau]$. Note that for $A \in \mathcal{F}_\theta$, $P^{\mathcal{F}_\tau}(., A)$ is defined for all $\omega \in \Omega$, whereas $E[1_A \mid \mathcal{F}_\tau]$ is only defined up to P-almost sure equality.

Since for a nullset $N \in \mathcal{F}_\theta$ we have $1_N = 0$, a.s. P, we derive

$$0 = E[1_N \mid \mathcal{F}_\tau] = P^{\mathcal{F}_\tau}(., N), \quad \text{a.s. } P. \tag{4.1.1}$$

In other words, for all nullsets N there exist nullsets $N^* = N^*(N)$ such that N is a $P^{\mathcal{F}_\tau}(\omega, .)$-nullset for all $\omega \in N^{*c}$.

4.2 Definitions and Notation

Theorem 4.2.1 *Under the assumption that $\Omega = (\Omega, \mathcal{T}_d)$ is a polish space and that the σ-algebra $\mathcal{F}_T = \sigma(\mathcal{T}_d)$ is generated by the open sets there exists a mapping*

$$P^{\mathcal{F}_\tau} : \Omega \times \mathcal{F}_\theta \to [0,1]$$

satisfying (1), (2) *and* (3).

Proof. Indeed, we may view the identity $id : (\Omega, \mathcal{F}_T) \to (\Omega, \mathcal{F}_T)$, $\omega \mapsto id(\omega) := \omega$ as an $\mathcal{F}_T$-measurable mapping taking values in the polish space Ω. Thus, 44.3 Satz in Bauer [5] yields the existence of a mapping

$$P^{\mathcal{F}_\tau} : \Omega \times \mathcal{F}_T \to [0,1]$$

which, for $\theta(\omega) = T$ for all $\omega \in \Omega$, satisfies the properties (1), (2) and (3). For general θ we may take its restriction to $\Omega \times \mathcal{F}_\theta$. □

Example 4.2.2 *Assume that the σ-algebra $\mathcal{F}_\tau$ is generated by a finite ($I = \{1, ..., n\}$) or countable ($I = \mathbb{N}$) partition*

$$\Omega = \bigcup_{i \in I} B_i,$$

with $B_i \in \mathcal{F}_\theta$, $P(B_i) > 0$ for all $i \in I$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. The probability measures $P_{B_i} : \mathcal{F}_\theta \to [0,1]$ on $\mathcal{F}_\theta$ are given by

$$P_{B_i}(A) := \frac{P(A \cap B_i)}{P(B_i)}$$

for all $A \in \mathcal{F}_\theta$ and $i \in I$. Consider the mapping

$$P^I : \Omega \times \mathcal{F}_\theta \to [0,1], \quad (\omega, A) \mapsto \sum_{i \in I} P_{B_i}(A) 1_{B_i}(\omega).$$

For all $A \in \mathcal{F}_\theta$ the mapping $P^I(.,A) : (\Omega, \mathcal{F}_\tau) \to [0,1]$ is $\mathcal{F}_\tau$-measurable . Since $B_i \cap B_j = \emptyset$ for $i \neq j$, $P^I(w,.) : \mathcal{F}_\theta \to [0,1]$ is a probability measure on $\mathcal{F}_\theta$ for all $\omega \in \Omega$. Moreover, for all $C \in \mathcal{F}_\tau$ we have

$$\begin{aligned}
\int_C P^I(w,A) P(d\omega) &= \sum_{i \in I} P_{B_i}(A) \int_C 1_{B_i}(\omega) P(d\omega) \\
&= \sum_{i \in I} \frac{P(A \cap B_i)}{P(B_i)} P(B_i \cap C) \\
&= \sum_{B_i \subset C} P(A \cap B_i) = P(A \cap C),
\end{aligned}$$

where the last two equalities follow from the fact that all events $C \in \mathcal{F}_\tau$ are of the form $C = \bigcup_{j \in J} B_j$ for a subset J of I. Hence, P^I satisfies the properties (1), (2) and (3). Since for all $i \in I$ the measures P_{B_i} are absolutely continuous with respect to P all nullsets N are $P^I(\omega,.)$-nullsets as well for all $\omega \in \Omega$. Note that this is much stronger than the statement in (4.1.1).

Definition 4.2.3 *For a random variable X on $(\Omega, \mathcal{F}_\theta, P)$ we call the mapping*

$$P_{X|\mathcal{F}_\tau} : \Omega \times \mathcal{B} \to [0,1], \quad (\omega, B) \mapsto P_{X|\mathcal{F}_\tau}(\omega, B) := P^{\mathcal{F}_\tau}\big(w, \{X \in B\}\big) \tag{4.2.2}$$

regular conditional distribution of X given $\mathcal{F}_\tau$. The mapping

$$F_{X|\mathcal{F}_\tau} : \Omega \times \mathbb{R} \to [0,1], \quad (\omega, x) \mapsto F_{X|\mathcal{F}_\tau}(\omega, x) := P_{X|\mathcal{F}_\tau}\big(\omega, \{X \leq x\}\big) \tag{4.2.3}$$

is called regular conditional distribution function of X given $\mathcal{F}_\tau$.

For two random variables X and Y on $(\Omega, \mathcal{F}_\theta, P)$ we say that $P_{X|\mathcal{F}_\tau}$ and $P_{Y|\mathcal{F}_\tau}$ coincide if there exists a nullset N such that $P_{X|\mathcal{F}_\tau}(\omega, B) = P_{Y|\mathcal{F}_\tau}(\omega, B)$ for all $\omega \in N^c$ and for all $B \in \mathcal{B}$. In this case we write

$$P_{X|\mathcal{F}_\tau} \sim P_{Y|\mathcal{F}_\tau}. \tag{4.2.4}$$

Theorem 4.2.4 *For two random variables X and Y on $(\Omega, \mathcal{F}_\theta, P)$ such that $X = Y$, a.s. P, we have $P_{X|\mathcal{F}_\tau} \sim P_{Y|\mathcal{F}_\tau}$.*

Proof. The mapping $P_{X|\mathcal{F}_\tau}$ given in (4.2.2) inherits the properties

(1*) $P_{X|\mathcal{F}_\tau}(\omega,.) : \mathcal{B} \to [0,1]$ is a probability measure for all $\omega \in \Omega$

(2*) $P_{X|\mathcal{F}_\tau}(.,B) : (\Omega, \mathcal{F}_\tau) \to [0,1]$ is $\mathcal{F}_\tau$-measurable for all $B \in \mathcal{B}$

(3*) $\int_C P_{X|\mathcal{F}_\tau}(w,B)\ P(d\omega) = P\big(C \cap \{X \in B\}\big)$ for all $C \in \mathcal{F}_\tau$ and $B \in \mathcal{B}$

from the corresponding ones of $P^{\mathcal{F}_\tau}$. (1*) and (2*) are also satisfied by $P_{Y|\mathcal{F}_\tau}$. Moreover, since for all $B \in \mathcal{B}$ the sets $\{X \in B\}$ and $\{Y \in B\}$ coincide up to a nullset we have

$$\int_C P_{Y|\mathcal{F}_\tau}(w, B)\ P(d\omega) = P\big(C \cap \{Y \in B\}\big) = P\big(C \cap \{X \in B\}\big)$$

for all $C \in \mathcal{F}_\tau$ and $B \in \mathcal{B}$ and hence, $P_{Y|\mathcal{F}_\tau}$ satisfies (3*) as well. Further, note that $\mathbb{R}$ is polish, that hence its topology has a countable basis and that in turn the σ-algebra of Borel-sets is countably generated. Thus, 44.2 Satz in Bauer [5] applies to the pairing $P_{X|\mathcal{F}_\tau}$, $P_{Y|\mathcal{F}_\tau}$ yielding a nullset N such that

$$P_{X|\mathcal{F}_\tau}(\omega, B) = P_{Y|\mathcal{F}_\tau}(\omega, B)$$

for all $\omega \in N^c$ and for all $B \in \mathcal{B}$. □

Remark 4.2.5 *From the properties* (2*) *and* (3*) *of the preceding proof we deduce as above that for a random variable X on $(\Omega, \mathcal{F}_\theta, P)$ and all $B \in \mathcal{B}$*

$$P_{X|\mathcal{F}_\tau}(., B) : (\Omega, \mathcal{F}_\tau) \to [0, 1], \quad \omega \mapsto P_{X|\mathcal{F}_\tau}(w, B)$$

is a version of $E[1_{\{X \in B\}} \mid \mathcal{F}_\tau]$ which again is defined for all $\omega \in \Omega$.

Consider the set

$$\big\{P_{X|\mathcal{F}_\tau} \mid X\ P\text{-almost surely bounded random variable on } (\Omega, \mathcal{F}_\theta, P)\big\} \tag{4.2.5}$$

of regular conditional distributions with respect to P-almost surely bounded random variables. For a random variable X on $(\Omega, \mathcal{F}_\theta, P)$ that is bounded P-almost surely there exists a nullset N such that $|X(\omega)| \le c < +\infty$ for all $\omega \in N^c$. In reference to (4.1.1) there exists a nullset $N^* = N^*(N)$ such that N is a $P_{X|\mathcal{F}_\tau}(\omega, .)$-nullset for all $\omega \in N^{*c}$. Thus, X is $P_{X|\mathcal{F}_\tau}(\omega, .)$-almost surely bounded for all ω in the complement of N^* and hence, for all such ω the probability measure $P_{X|\mathcal{F}_\tau}(\omega, .)$ has compact support on the real line.

$\sim$ in (4.2.4) defines an equivalence relation on the space given in (4.2.5). We denote the space of corresponding equivalence classes by $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$. For a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$, $\widetilde{P}_{X|\mathcal{F}_\tau}$ designates the induced element in $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$. Theorem 4.2.4 states that the mapping

$$L^\infty(\mathcal{F}_\theta) \to \mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau), \quad \widetilde{X} \mapsto \widetilde{P}_{X|\mathcal{F}_\tau}, \tag{4.2.6}$$

is well defined with respect to the choice of $X \in \widetilde{X}$.

We may now present our notion of distribution invariant conditional monetary utility functionals (resp. risk measures).

Definition 4.2.6 *We call a functional $\phi_{\tau,\theta} : \mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau) \to L^\infty(\mathcal{F}_\tau)$ conditional monetary (concave, coherent) utility functional on $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ if the induced functional $\phi^*_{\tau,\theta} : L^\infty(\mathcal{F}_\theta) \to L^\infty(\mathcal{F}_\tau)$ given by*

$$\widetilde{X} \mapsto \phi^*_{\tau,\theta}(\widetilde{X}) := \phi_{\tau,\theta}(\widetilde{P}_{X|\mathcal{F}_\tau}), \tag{4.2.7}$$

is a conditional monetary (concave, coherent) utility functional on $L^\infty(\mathcal{F}_\tau)$.

A conditional monetary (convex, coherent) risk measure $\rho_{\tau,\theta}$ on $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\theta)$ is a functional from $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ to $L^\infty(\mathcal{F}_\tau)$ such that $-\rho_{\tau,\theta}$ is a conditional monetary (concave, coherent) utility functional on $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$.

Note that the functional $\phi^*_{\tau,\theta}$ given in (4.2.7) of the above definition is well defined since the mapping given in (4.2.6) is so.

By definition, a conditional monetary utility functional (resp. risk measure) on $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ induces a conditional monetary utility functional (resp. risk measure) on $L^\infty(\mathcal{F}_\theta)$ which depends on the equivalence classes in $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ only. In this sense, conditional monetary utility functionals (resp. risk measures) on $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ admit an interpretation as distribution invariant "classical" conditional monetary utility functionals (resp. risk measures).

We introduce a partial order $\leq$ on $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$. $\widetilde{P}_{X|\mathcal{F}_\tau} \leq \widetilde{P}_{Y|\mathcal{F}_\tau}$ by definition means that for a pairing $(P_{X|\mathcal{F}_\tau}, P_{Y|\mathcal{F}_\tau}) \in \widetilde{P}_{X|\mathcal{F}_\tau} \times \widetilde{P}_{Y|\mathcal{F}_\tau}$ there exists a nullset N such that

$$P_{X|\mathcal{F}_\tau}\big(\omega, (-\infty, x]\big) \geq P_{Y|\mathcal{F}_\tau}\big(\omega, (-\infty, x]\big)$$

for all $\omega \in N^c$ and for all $x \in \mathbb{R}$.

Consider a random variable X on $(\Omega, \mathcal{F}_\theta, P)$ and a real constant $c \in \mathbb{R}$ such that $X(\omega) = c$ for all $\omega \in N^c$ with $P(N) = 0$. Then,

$$P_{X|\mathcal{F}_\tau}(\omega, B) = P^{\mathcal{F}_\tau}\big(\omega, \{X \in B\}\big) = \begin{cases} 1 & \text{if } c \in B \\ 0 & \text{if } c \notin B \end{cases} \tag{4.2.8}$$

for all $B \in \mathcal{B}$ and for all ω in the complement of a suitable nullset $N^* = N^*(N)$ with $P^{\mathcal{F}_\tau}(\omega, N) = 0$ for all $\omega \in N^{*c}$. Such N^* exists due to the statement in (4.1.1). The equivalence class in $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ induced by a regular conditional distribution such as in (4.2.8) is denoted by $\widetilde{\delta}_c$.

Proposition 4.2.7 *Consider a functional $\phi_{\tau,\theta} : \mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau) \to L^\infty(\mathcal{F}_\tau)$ which satisfies the following properties:*

(**n**) **Normalization:** $\phi_{\tau,\theta}(\widetilde{\delta}_0) = 0$, *a.s. P*

(**m**) **Monotonicity:** $\phi_{\tau,\theta}(\widetilde{P}_{X|\mathcal{F}_\tau}) \leq \phi_{\tau,\theta}(\widetilde{P}_{X|\mathcal{F}_\tau})$, *a.s. P, for all $\widetilde{P}_{X|\mathcal{F}_\tau}, \widetilde{P}_{Y|\mathcal{F}_\tau} \in \mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ such that $\widetilde{P}_{X|\mathcal{F}_\tau} \leq \widetilde{P}_{Y|\mathcal{F}_\tau}$*

($\mathcal{F}_\tau$**-ti**) $\mathcal{F}_\tau$**-Translation Invariance:** $\phi_{\tau,\theta}(\widetilde{P}_{X+m|\mathcal{F}_\tau}) = \phi_{\tau,\theta}(\widetilde{P}_{X|\mathcal{F}_\tau}) + \widetilde{m}$, *a.s. P, for all $\widetilde{P}_{X|\mathcal{F}_\tau} \in \mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ and $m \in \widetilde{m} \in L^\infty(\mathcal{F}_\tau)$.*

Then, $\phi_{\tau,\theta}$ is a conditional monetary utility functional on $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$.

Proof. We have to show that $\phi^*_{\tau,\theta} : L^\infty(\mathcal{F}_\theta) \to L^\infty(\mathcal{F}_\tau)$,

$$\widetilde{X} \mapsto \phi^*_{\tau,\theta}(\widetilde{X}) := \phi_{\tau,\theta}(\widetilde{P}_{X|\mathcal{F}_\tau}), \tag{4.2.9}$$

is a conditional monetary utility functional on $L^\infty(\mathcal{F}_\tau)$.
(**n**): Take $\widetilde{X} \in \mathcal{F}_\theta$ such that $X = 0$, a.s. P. As in (4.2.8) we derive $\widetilde{P}_{X|\mathcal{F}_\tau} = \widetilde{\delta}_0$ and hence, $\phi^*_{\tau,\theta}(0) = \phi_{\tau,\theta}(\widetilde{\delta}_0) = 0$, a.s. P.
(**m**): Take $\widetilde{X}, \widetilde{Y} \in \mathcal{F}_\theta$ such that $X \leq Y$, a.s. P, for $(X,Y) \in \widetilde{X} \times \widetilde{Y}$ and let N be a nullset such that $X(\omega) \leq Y(\omega)$ for all $\omega \in N^c$. Further, let $N^* = N^*(N)$ be a nullset such that $P^{\mathcal{F}_\tau}(\omega, N) = 0$ for all $\omega \in N^{*c}$ which exists due to the statement in (4.1.1). Then, $\{Y \leq x\} \setminus N \subset \{X \leq x\} \setminus N$ for all $x \in \mathbb{R}$ and hence,

$$\begin{aligned} P_{X|\mathcal{F}_\tau}\big(\omega, (-\infty, x]\big) &= P^{\mathcal{F}_\tau}\big(\omega, \{X \leq x\}\big) \\ &= P^{\mathcal{F}_\tau}\big(\omega, \{X \leq x\} \setminus N\big) \\ &\geq P^{\mathcal{F}_\tau}\big(\omega, \{Y \leq x\} \setminus N\big) \\ &\vdots \\ &= P_{Y|\mathcal{F}_\tau}\big(\omega, (-\infty, x]\big) \end{aligned}$$

for all $x \in \mathbb{R}$ and for all $\omega \in N^{*c}$, i.e. $\widetilde{P}_{X|\mathcal{F}_\tau} \leq \widetilde{P}_{Y|\mathcal{F}_\tau}$. Thus,

$$\phi^*_{\tau,\theta}(\widetilde{X}) = \phi_{\tau,\theta}(\widetilde{P}_{X|\mathcal{F}_\tau}) \leq \phi_{\tau,\theta}(\widetilde{P}_{Y|\mathcal{F}_\tau}) = \phi^*_{\tau,\theta}(\widetilde{Y}), \quad \text{a.s. } P.$$

($\mathcal{F}_\tau$**-ti**): For all $X \in \widetilde{X} \in L^\infty(\mathcal{F}_\theta)$ and for all $m \in \widetilde{m} \in L^\infty(\mathcal{F}_\tau)$,

$$\phi^*_{\tau,\theta}(\widetilde{X} + \widetilde{m}) = \phi_{\tau,\theta}(\widetilde{P}_{X+m|\mathcal{F}_\tau}) = \phi_{\tau,\theta}(\widetilde{P}_{X|\mathcal{F}_\tau}) + \widetilde{m} = \phi^*_{\tau,\theta}(\widetilde{X}) + \widetilde{m}, \quad \text{a.s. } P.$$

□

4.3 Conditional Quantiles

Definition 4.3.1 *For a random variable X on $(\Omega, \mathcal{F}_\theta, P)$ we consider the mapping $F_{X|\mathcal{F}_\tau} : \Omega \times \mathbb{R} \to [0,1]$ given in (4.2.3). We call a mapping*

$$q_{X|\mathcal{F}_\tau} : \Omega \times (0,1) \to \mathbb{R}, \quad (\omega, r) \mapsto q_{X|\mathcal{F}_\tau}(\omega, r),$$

conditional quantile (of X given $\mathcal{F}_\tau$) if for all $\omega \in \Omega$ the mapping

$$q_{X|\mathcal{F}_\tau}(\omega, .) : (0,1) \to \mathbb{R}, \quad r \mapsto q_{X|\mathcal{F}_\tau}(\omega, r),$$

is an inverse function of $F_{X|\mathcal{F}_\tau}(\omega, .) : \mathbb{R} \to [0,1]$. That is, $q_{X|\mathcal{F}_\tau}(\omega, .)$ is a function with

$$F_{X|\mathcal{F}_\tau}\big(\omega, q_{X|\mathcal{F}_\tau}(\omega, r)-\big) \leq r \leq F_{X|\mathcal{F}_\tau}\big(\omega, q_{X|\mathcal{F}_\tau}(\omega, r)\big)$$

for all $\omega \in \Omega$ and all $r \in (0,1)$, where $q_{X|\mathcal{F}_\tau}(\omega, r)- = \lim_{a \nearrow r} q_{X|\mathcal{F}_\tau}(\omega, a)$.

For a random variable X on $(\Omega, \mathcal{F}_\theta, P)$ and for all $\omega \in \Omega$ the functions

$$q^+_{X|\mathcal{F}_\tau}(\omega, .) : (0,1) \to \mathbb{R}, \qquad r \mapsto q^+_{X|\mathcal{F}_\tau}(\omega, r) := inf\ \{x \mid P_{X|\mathcal{F}_\tau}\big(\omega, \{X \le x\}\big) > r\}, \quad \text{and}$$
$$q^-_{X|\mathcal{F}_\tau}(\omega, .) : (0,1) \to \mathbb{R}, \qquad r \mapsto q^-_{X|\mathcal{F}_\tau}(\omega, r) := sup\ \{x \mid P_{X|\mathcal{F}_\tau}\big(\omega, \{X < x\}\big) < r\},$$

are right-continuous and left-continuous inverse functions of $F_{X|\mathcal{F}_\tau}(\omega, .) : \mathbb{R} \to [0,1]$. As a good reference on inverse functions we refer to the appendix A.3 of the textbook Föllmer and Schied [17].

Proposition 4.3.2 *Let $q_{X|\mathcal{F}_\tau}$ be a conditional quantile of a random variable X on $(\Omega, \mathcal{F}_\theta, P)$ given $\mathcal{F}_\tau$. For $r \in (0,1)$ the mapping*

$$q_{X|\mathcal{F}_\tau}(., r) : (\Omega, \mathcal{F}_\tau) \to \mathbb{R}, \quad \omega \mapsto q_{X|\mathcal{F}_\tau}(\omega, r)$$

is $\mathcal{F}_\tau$-measurable.

Proof. Let us fix $r \in (0,1)$. Recall that for a mapping $Y : \Omega \to \mathbb{R}$ and $\mathcal{G} \subset \mathcal{B}$ we have $\sigma\big(Y^{-1}(\mathcal{G})\big) = Y^{-1}\big(\sigma(\mathcal{G})\big)$. Thus, it suffices to show that for all $\eta \in \mathbb{R}$

$$q_{X|\mathcal{F}_\tau}(., r)^{-1}(-\infty, \eta] \in \mathcal{F}_\tau, \tag{4.3.10}$$

since $\big\{(-\infty, \eta] \mid \eta \in \mathbb{R}\big\}$ generates $\mathcal{B}$. To prove the statement in (4.3.10) let us fix $\eta \in \mathbb{R}$. We have

$$q_{X|\mathcal{F}_\tau}(., r)^{-1}(-\infty, \eta] = \{\omega \in \Omega \mid q_{X|\mathcal{F}_\tau}(\omega, r) \le \eta\} = \{\omega \in \Omega \mid F_{X|\mathcal{F}_\tau}(\omega, \eta) \ge r\}.$$

By definition, $F_{X|\mathcal{F}_\tau}(., \eta) : (\Omega, \mathcal{F}_\tau) \to [0,1]$ is $\mathcal{F}_\tau$-measurable and hence, the set $\{\omega \in \Omega \mid F_{X|\mathcal{F}_\tau}(\omega, \eta) \ge r\} \in \mathcal{F}_\tau$. Thus, (4.3.10) follows. □

For simplification we introduce the following notation. Consider a conditional quantile $q_{X|\mathcal{F}_\tau}$ of a random variable X on $(\Omega, \mathcal{F}_\theta, P)$. For $r \in (0,1)$ we shorten

$$q_{X|\mathcal{F}_\tau}(r) : \Omega \to \mathbb{R}, \quad \omega \mapsto q_{X|\mathcal{F}_\tau}(r)(\omega) := q_{X|\mathcal{F}_\tau}(\omega, r).$$

We call $q_{X|\mathcal{F}_\tau}(r)$ conditional r-quantile (of X given $\mathcal{F}_\tau$). In particular, $q^\pm_{X|\mathcal{F}_\tau}(r)$ denote the conditional r-quantiles

$$q^\pm_{X|\mathcal{F}_\tau}(r) : \Omega \to \mathbb{R}, \quad \omega \mapsto q^\pm_{X|\mathcal{F}_\tau}(r)(\omega) := q^\pm_{X|\mathcal{F}_\tau}(\omega, r).$$

Consider two random variables X and Y on $(\Omega, \mathcal{F}_\theta, P)$ such that $X = Y$, a.s. P. By theorem 4.2.4 there exists a nullset N such that $P_{X|\mathcal{F}_\tau}(\omega, B) = P_{Y|\mathcal{F}_\tau}(\omega, B)$ for all $\omega \in N^c$ and for all $B \in \mathcal{B}$. In turn, this means that $F_{X|\mathcal{F}_\tau}(\omega, x) = F_{Y|\mathcal{F}_\tau}(\omega, x)$ for all $\omega \in N^c$ and for all $x \in \mathbb{R}$. Hence,

$$q_{X|\mathcal{F}_\tau}(\omega, .) = q_{Y|\mathcal{F}_\tau}(\omega, .), \quad \text{a.s. } \lambda^1, \tag{4.3.11}$$

for all $\omega \in N^c$, where λ^1 designates the Lebesgue-measure. In particular,

$$q^\pm_{X|\mathcal{F}_\tau}(\omega, r) = q^\pm_{Y|\mathcal{F}_\tau}(\omega, r) \tag{4.3.12}$$

for all $\omega \in N^c$ and for all $r \in (0,1)$. For a fixed $r \in (0,1)$ and the conditional r-quantiles $q^{\pm}_{X|\mathcal{F}_\tau}(r)$ we denote the corresponding equivalence classes in $L^0(\mathcal{F}_\tau)$ by $\tilde{q}^{\pm}_{X|\mathcal{F}_\tau}(r)$. The statement in (4.3.12) tells us that for all $r \in (0,1)$, $\tilde{q}^{\pm}_{X|\mathcal{F}_\tau}(r)$ and $\tilde{q}^{\pm}_{Y|\mathcal{F}_\tau}(r)$ are the same elements in $L^0(\mathcal{F}_\tau)$.

Proposition 4.3.3 *For a random variable X on $(\Omega, \mathcal{F}_\theta, P)$ the mappings*

$$P^{\mathcal{F}_\tau}\big(., \{X \le .\}\big) : (\Omega \times \mathbb{R}, \mathcal{F}_\tau \otimes \mathcal{B}) \to [0,1], \quad (\omega, x) \mapsto P^{\mathcal{F}_\tau}\big(\omega, \{X \le x\}\big), \quad \textit{and}$$
$$P^{\mathcal{F}_\tau}\big(., \{X < .\}\big) : (\Omega \times \mathbb{R}, \mathcal{F}_\tau \otimes \mathcal{B}) \to [0,1], \quad (\omega, x) \mapsto P^{\mathcal{F}_\tau}\big(\omega, \{X < x\}\big),$$

are $\mathcal{F}_\tau \otimes \mathcal{B}$-measurable.

Proof. Since $P^{\mathcal{F}_\tau}(\omega, .) : \mathcal{F}_\theta \to [0,1]$ is a probability measure for all $\omega \in \Omega$ the function $P^{\mathcal{F}_\tau}\big(\omega, \{X \le .\}\big) : \mathbb{R} \to [0,1]$ is right-continuous for all $\omega \in \Omega$. Thus, for all $(\omega, x) \in \Omega \times \mathbb{R}$ we have

$$\sum_{k \in \mathbb{Z}} P^{\mathcal{F}_\tau}\left(\omega, \left\{X \le \frac{k+1}{N}\right\}\right) 1_{\left(\frac{k}{N}, \frac{k+1}{N}\right]}(x) \quad \longrightarrow \quad P^{\mathcal{F}_\tau}\big(\omega, \{X \le x\}\big), \tag{4.3.13}$$

as N tends to $+\infty$. For all $N \in \mathbb{N}$ the left-hand side of (4.3.13) is $\mathcal{F}_t \otimes \mathcal{B}$-measurable in (ω, x) since, by definition, $P^{\mathcal{F}_\tau}\left(., \left\{Y \le \frac{k+1}{N}\right\}\right)$ is $\mathcal{F}_\tau$-measurable and $1_{\left(\frac{k}{N}, \frac{k+1}{N}\right]}$ is $\mathcal{B}$-measurable. Hence, its point wise limit $P^{\mathcal{F}_\tau}\big(., \{X \le .\}\big)$ is $\mathcal{F}_\tau \otimes \mathcal{B}$-measurable as well.

From the fact that $P^{\mathcal{F}_\tau}(\omega, .) : \mathcal{F}_\theta \to [0,1]$ is a probability measure for all $\omega \in \Omega$ we derive that $P^{\mathcal{F}_\tau}\big(\omega, \{X < .\}\big) : \mathbb{R} \to [0,1]$ is left-continuous for all $\omega \in \Omega$. Thus, for all $(\omega, x) \in \Omega \times \mathbb{R}$ we have

$$\sum_{k \in \mathbb{Z}} P^{\mathcal{F}_\tau}\left(\omega, \left\{X < \frac{k}{N}\right\}\right) 1_{\left(\frac{k}{N}, \frac{k+1}{N}\right]}(x) \quad \longrightarrow \quad P^{\mathcal{F}_\tau}\big(\omega, \{X < x\}\big),$$

as N tends to $+\infty$, and $\mathcal{F}_\tau \otimes \mathcal{B}$-measurability of $P^{\mathcal{F}_\tau}\big(., \{X < .\}\big)$ follows as above. □

Remark 4.3.4 *From proposition* 4.3.3 *it follows that for two random variables X, Y on $(\Omega, \mathcal{F}_\theta, P)$ the mappings*

$$P^{\mathcal{F}_\tau}\big(., \{X \le Y(.)\}\big) : (\Omega, \mathcal{F}_\tau) \to [0,1], \qquad \omega \mapsto P^{\mathcal{F}_\tau}\big(\omega, \{X \le Y(\omega)\}\big), \quad \textit{and}$$
$$P^{\mathcal{F}_\tau}\big(., \{X < Y(.)\}\big) : (\Omega, \mathcal{F}_\tau) \to [0,1], \qquad \omega \mapsto P^{\mathcal{F}_\tau}\big(\omega, \{X < Y(\omega)\}\big),$$

are $\mathcal{F}_\theta$-measurable.

In particular, for an $\mathcal{F}_\tau$-measurable conditional r-quantile $q_{X|\mathcal{F}_\tau}(r)$, $r \in (0,1)$, of a random variable X on $(\Omega, \mathcal{F}_\theta, P)$ the mapping $P^{\mathcal{F}_\tau}\big(., \{X < q_{X|\mathcal{F}_\tau}(r)(.)\}\big)$ is $\mathcal{F}_\tau$-measurable. By definition of an inverse function we have

$$P^{\mathcal{F}_\tau}\big(\omega, \{X < q_{X|\mathcal{F}_\tau}(r)(\omega)\}\big) \;\le\; r \;\le\; P^{\mathcal{F}_\tau}\big(\omega, \{X \le q_{X|\mathcal{F}_\tau}(r)(\omega)\}\big),$$

for all $\omega \in \Omega$.

Lemma 4.3.5 *Fix $r \in (0,1)$. For a conditional r-quantile $q_{X|\mathcal{F}_\tau}(r)$ of a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$ we consider the sets $\{X \le q_{X|\mathcal{F}_\tau}(r)\}, \{X < q_{X|\mathcal{F}_\tau}(r)\} \in \mathcal{F}_\theta$. For the $\mathcal{F}_\tau$-measurable mappings*

$$\begin{aligned} &P^{\mathcal{F}_\tau}\big(.,\{X \le q_{X|\mathcal{F}_\tau}(r)\}\big) : (\Omega, \mathcal{F}_\tau) \to [0,1], \qquad \omega \mapsto P^{\mathcal{F}_\tau}\big(\omega, \{X \le q_{X|\mathcal{F}_\tau}(r)\}\big), \quad \textit{and} \\ &P^{\mathcal{F}_\tau}\big(.,\{X < q_{X|\mathcal{F}_\tau}(r)\}\big) : (\Omega, \mathcal{F}_\tau) \to [0,1], \qquad \omega \mapsto P^{\mathcal{F}_\tau}\big(\omega, \{X < q_{X|\mathcal{F}_\tau}(r)\}\big), \end{aligned}$$

we have

$$P^{\mathcal{F}_\tau}\big(.,\{X < q_{X|\mathcal{F}_\tau}(r)\}\big) \;\le\; r \;\le\; P^{\mathcal{F}_\tau}\big(.,\{X \le q_{X|\mathcal{F}_\tau}(r)\}\big), \quad \textit{a.s. } P.$$

Proof. For all of this proof we fix $r \in (0,1)$.
In a first step we assume that X is of the form $X = \sum_{i=1}^n \alpha_i 1_{A_i}$, $\alpha_i \in \mathbb{R}$, $A_i \in \mathcal{F}_\theta$ such that $A_i \cap A_j = \emptyset$ for $i \ne j$ and $n \in \mathbb{N}$. For such X the set $\{X \le q_{X|\mathcal{F}_\tau}(r)\}$ can also be written in the form

$$\{X \le q_{X|\mathcal{F}_\tau}(r)\} = \bigcup_{i=1}^n \big(\{\alpha_i \le q_{X|\mathcal{F}_\tau}(r)\} \cap A_i\big)$$

Thus, we derive from (1) that for all $\omega \in \Omega$ we have

$$P^{\mathcal{F}_\tau}\big(\omega, \{X \le q_{X|\mathcal{F}_\tau}(r)\}\big) = \sum_{i=1}^n P^{\mathcal{F}_\tau}\big(\omega, \{\alpha_i \le q_{X|\mathcal{F}_\tau}(r)\} \cap A_i\big).$$

There exists a null-set N_1 such that

$$\sum_{i=1}^n P^{\mathcal{F}_\tau}\big(\omega, \{\alpha_i \le q_{X|\mathcal{F}_\tau}(r)\} \cap A_i\big) = \sum_{i=1}^n E\left[1_{\{\alpha_i \le q_{X|\mathcal{F}_\tau}(r)\}} 1_{A_i} \mid \mathcal{F}_\tau\right](\omega)$$

for all $\omega \in N_1^c$. Since $q_{X|\mathcal{F}_\tau}(r)$ is $\mathcal{F}_\tau$-measurable so is $1_{\{\alpha_i \le q_{X|\mathcal{F}_\tau}(r)\}}$ and hence, we find a set N_2 of P-measure zero such that

$$\sum_{i=1}^n E\left[1_{\{\alpha_i \le q_{X|\mathcal{F}_\tau}(r)\}} 1_{A_i} \mid \mathcal{F}_\tau\right](\omega) = \sum_{i=1}^n 1_{\{\alpha_i \le q_{X|\mathcal{F}_\tau}(r)(\omega)\}} E[1_{A_i} \mid \mathcal{F}_\tau](\omega)$$

for all $\omega \in N_2^c$. And finally, for all $\omega \in N_3^c$ we have

$$\sum_{i=1}^n 1_{\{\alpha_i \le q_{X|\mathcal{F}_\tau}(r)(\omega)\}} E[1_{A_i} \mid \mathcal{F}_\tau](\omega) = \sum_{i=1}^n 1_{\{\alpha_i \le q_{X|\mathcal{F}_\tau}(r)(\omega)\}} P^{\mathcal{F}_\tau}(\omega, A_i),$$

where again $P(N_3) = 0$. For all $\omega \in (N_1 \cup N_2 \cup N_3)^c$ it follows that

$$P^{\mathcal{F}_\tau}\big(\omega, \{X \le q_{X|\mathcal{F}_\tau}(r)\}\big) = \sum_{i=1}^n 1_{\{\alpha_i \le q_{X|\mathcal{F}_\tau}(r)(\omega)\}} P^{\mathcal{F}_\tau}(\omega, A_i).$$

Take $\omega_0 \in (N_1 \cup N_2 \cup N_3)^c$. Then, by definition, $q_{X|\mathcal{F}_\tau}(r)(\omega_0)$ is an r-quantile of X with respect to the probability measure $P^{\mathcal{F}_\tau}(\omega_0, .)$ and hence

$$\sum_{i=1}^n 1_{\{\alpha_i \le q_{X|\mathcal{F}_\tau}(r)(\omega_0)\}} P^{\mathcal{F}_\tau}(\omega_0, A_i) = \sum_{i:\alpha_i \le q_{X|\mathcal{F}_\tau}(r)(\omega_0)} P^{\mathcal{F}_\tau}(\omega_0, A_i) \;\ge\; r.$$

In the same way as above we find a null-set N such that

$$P^{\mathcal{F}_\tau}\big(\omega, \{X < q_{X|\mathcal{F}_\tau}(r)\}\big) = \sum_{i=1}^n 1_{\{\alpha_i < q_{X|\mathcal{F}_\tau}(r)(\omega)\}} P^{\mathcal{F}_\tau}(\omega, A_i).$$

for all $\omega \in N^c$ and in turn

$$\sum_{i=1}^n 1_{\{\alpha_i < q_{X|\mathcal{F}_\tau}(r)(\omega)\}} P^{\mathcal{F}_\tau}(\omega, A_i) = \sum_{i:\alpha_i < q_{X|\mathcal{F}_\tau}(r)(\omega)} P^{\mathcal{F}_\tau}(\omega, A_i) \;\le\; r$$

for all $\omega \in N^c$. Hence, the assertion follows for $X = \sum_{i=1}^n \alpha_i 1_{A_i}$.

For general P-almost surely bounded X there exist $\mathcal{F}_\theta$-measurable step functions $(X_n)_{n\in\mathbb{N}}$ and a null-set N_1 such that $X_n(\omega) \searrow X(\omega)$ for all $\omega \in N_1^c$. Then $\{X_n \le x\} \setminus N_1 \subset \{X \le x\} \setminus N_1$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. By the statement in (4.1.1) there exists a null set $N_2 = N_2(N_1)$ such that

$$P^{\mathcal{F}_\tau}(\omega, N_1) = 0$$

for all $\omega \in N_2^c$ and hence,

$$\begin{aligned} P^{\mathcal{F}_\tau}\big(\omega, \{X_n \le x\}\big) &= P^{\mathcal{F}_\tau}\big(\omega, \{X_n \le x\} \setminus N_1\big) \\ &\le P^{\mathcal{F}_\tau}\big(\omega, \{X \le x\} \setminus N_1\big) \\ &= P^{\mathcal{F}_\tau}\big(\omega, \{X \le x\}\big) \end{aligned}$$

for all $\omega \in N_2^c$, for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. In turn we get

$$q^+_{X_n|\mathcal{F}_\tau}(r)(\omega) \;\ge\; q_{X|\mathcal{F}_\tau}(r)(\omega)$$

for all $\omega \in N_2^c$ and for all $n \in \mathbb{N}$. From this we derive

$$\big\{X_n < q_{X|\mathcal{F}_\tau}(r)\big\} \setminus N_2 \subset \big\{X_n < q^+_{X_n|\mathcal{F}_\tau}(r)\big\} \setminus N_2$$

for all $n \in \mathbb{N}$. Again, by (4.1.1) there exists $N_3 = N_3(N_2)$ with $P(N_3) = 0$ such that

$$\begin{aligned} P^{\mathcal{F}_\tau}\big(\omega, \{X_n < q_{X|\mathcal{F}_\tau}(r)\}\big) &= P^{\mathcal{F}_\tau}\big(\omega, \{X_n < q_{X|\mathcal{F}_\tau}(r)\} \setminus N_2\big) \\ &\le P^{\mathcal{F}_\tau}\Big(\omega, \Big\{X_n < q^+_{X_n|\mathcal{F}_\tau}(r)\Big\} \setminus N_2\Big) \\ &= P^{\mathcal{F}_\tau}\Big(\omega, \Big\{X_n < q^+_{X_n|\mathcal{F}_\tau}(r)\Big\}\Big) \;\le\; r \qquad (4.3.14) \end{aligned}$$

for all $\omega \in N_3^c$ and for all $n \in \mathbb{N}$, where the inequality in (4.3.14) follows from the first part of the proof. But now we deduce

$$\begin{aligned} P^{\mathcal{F}_\tau}\big(\omega, \{X < q_{X|\mathcal{F}_\tau}(r)\}\big) &= P^{\mathcal{F}_\tau}\big(\omega, \{X < q_{X|\mathcal{F}_\tau}(r)\} \setminus N_1\big) \\ &= P^{\mathcal{F}_\tau}\left(\omega, \bigcup_{n\in\mathbb{N}} \{X_n < q_{X|\mathcal{F}_\tau}(r)\} \setminus N_1\right) \\ &= \lim_{n\to\infty} P^{\mathcal{F}_\tau}\big(\omega, \{X_n < q_{X|\mathcal{F}_\tau}(r)\} \setminus N_1\big) \\ &= \lim_{n\to\infty} P^{\mathcal{F}_\tau}\big(\omega, \{X_n < q_{X|\mathcal{F}_\tau}(r)\}\big) \ \leq \ r, \end{aligned}$$

for all $\omega \in (N_2 \cup N_3)^c$.
To prove the upper inequality we take $\mathcal{F}_\theta$-measurable step functions $(X_n)_{n\in\mathbb{N}}$ and a null-set N_1 such that $X_n(\omega) \nearrow X(\omega)$ for all $\omega \in N_1^c$. This time we find a null-set $N_2 = N_2(N_1)$ such that

$$q^-_{X_n|\mathcal{F}_\tau}(r)(\omega) \ \leq \ q_{X|\mathcal{F}_\tau}(r)(\omega)$$

for all $\omega \in N_2^c$ and for all $n \in \mathbb{N}$. As above there exists $N_3 = N_3(N_2)$ with $P(N_3) = 0$ such that

$$r \ \leq \ P^{\mathcal{F}_\tau}\left(\omega, \left\{X_n \leq \ q^-_{X_n|\mathcal{F}_\tau}(r)\right\}\right) \ \leq \ P^{\mathcal{F}_\tau}\big(\omega, \{X_n \leq q_{X|\mathcal{F}_\tau}(r)\}\big),$$

for all $\omega \in N_3^c$ and for all $n \in \mathbb{N}$. We may let $n \to \infty$ and obtain

$$\begin{aligned} P^{\mathcal{F}_\tau}\big(\omega, \{X \leq q_{X|\mathcal{F}_\tau}(r)\}\big) &= P^{\mathcal{F}_\tau}\left(\omega, \bigcap_{n\in\mathbb{N}} \{X_n \leq q_{X|\mathcal{F}_\tau}(r)\}\right) \\ &= \lim_{n\to\infty} P^{\mathcal{F}_\tau}\big(\omega, \{X_n \leq q_{X|\mathcal{F}_\tau}(r)\}\big) \ \geq \ r \end{aligned}$$

for all $\omega \in (N_2 \cup N_3)^c$. □

An intuition of the above result is given in the following example.

Example 4.3.6 *Consider the mapping*

$$P^I : \Omega \times \mathcal{F}_\theta \to [0,1], \quad (\omega, A) \mapsto \sum_{i\in I} P_{B_i}(A) 1_{B_i}(\omega)$$

introduced in example 4.2.2 *and a* P*-almost surely bounded random variable* X *on* $(\Omega, \mathcal{F}_\theta, P)$*. In accordance with* (4.2.2) *and* (4.2.3) *we introduce the mappings*

$$P_{X|I} : \Omega \times \mathcal{B} \to [0,1], \quad (\omega, B) \mapsto \sum_{i\in I} P_{B_i}\{X \in B\} 1_{B_i}(\omega)$$

and

$$F_{X|I} : \Omega \times \mathbb{R} \to [0,1], \quad (\omega, x) \mapsto \sum_{i\in I} P_{B_i}\{X \leq x\} 1_{B_i}(\omega)$$

for a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$. The conditional quantiles (of X given $\mathcal{F}_\tau$) are of the form

$$q_{X|I} : \Omega \times (0,1) \to \mathbb{R}, \quad (\omega, r) \mapsto \sum_{i \in I} q_i(r) 1_{B_i}(\omega),$$

where $q_i : (0,1) \to \mathbb{R}$, $r \mapsto q_i(r)$ are inverse functions of $P_{B_i}\{X \leq .\} : \mathbb{R} \to [0,1]$, $x \mapsto P_{B_i}\{X \leq x\}$ for all $i \in I$. For $\omega \in B_i$ and $r \in (0,1)$ we have

$$P^I\big(\omega, \{X \overset{(<)}{\leq} q_{X|I}(r)\}\big) = P_{B_i}\{X \overset{(<)}{\leq} q_{X|I}(r)\} = P_{B_i}\{X \overset{(<)}{\leq} q_i(r)\}$$

and hence, the statement of lemma 4.3.5 follows from the definition of q_i as an inverse function of $P_{B_i}\{X \leq .\}$.

Lemma 4.3.7 *For a P-almost surely bounded random variable m on $(\Omega, \mathcal{F}_\tau, P)$ (i.e. m is $\mathcal{F}_\tau$-measurable) the mapping $P^{\mathcal{F}_\tau}\big(., \{m = m(.)\}\big) : (\Omega, \mathcal{F}_\tau) \to [0,1]$,*

$$\omega \mapsto P^{\mathcal{F}_\tau}\big(\omega, \{m = m(\omega)\}\big)$$

is $\mathcal{F}_\tau$-measurable. Moreover,

$$P^{\mathcal{F}_\tau}\big(., \{m = m(.)\}\big) \ = \ 1, \quad a.s.\ P. \tag{4.3.15}$$

Proof. In remark 4.3.4 we may choose $X(\omega) = Y(\omega) = m(\omega)$ for all $\omega \in \Omega$ and deduce that the mappings

$$\begin{aligned} &P^{\mathcal{F}_\tau}\big(., \{m \leq m(.)\}\big) : (\Omega, \mathcal{F}_\tau) \to [0,1], \qquad \omega \mapsto P^{\mathcal{F}_\tau}\big(\omega, \{m \leq m(\omega)\}\big), \quad \text{and} \\ &P^{\mathcal{F}_\tau}\big(., \{m < m(.)\}\big) : (\Omega, \mathcal{F}_\tau) \to [0,1], \qquad \omega \mapsto P^{\mathcal{F}_\tau}\big(\omega, \{m < m(\omega)\}\big), \end{aligned}$$

are $\mathcal{F}_\tau$-measurable. Since for all $\omega \in \Omega$ we have

$$P^{\mathcal{F}_\tau}\big(\omega, \{m \leq m(\omega)\}\big) - P^{\mathcal{F}_\tau}\big(\omega, \{m < m(\omega)\}\big) = P^{\mathcal{F}_\tau}\big(\omega, \{m = m(\omega)\}\big)$$

the first statement is proved.

We prove the second statement by contradiction. Assume that (4.3.15) is wrong. We then have

$$P \underbrace{\Big\{P^{\mathcal{F}_\tau}\big(., \{m = m(.)\}\big) \ < \ 1\Big\}}_{:=N} \ > \ 0$$

since $P^{\mathcal{F}_\tau}$ takes only values in $[0,1]$.
Next, we show that

$$N = \Big\{P^{\mathcal{F}_\tau}\big(., \{m \neq m(.)\}\big) \ > \ 0\Big\}. \tag{4.3.16}$$

To this end, we shorten $Z(\omega) = P^{\mathcal{F}_\tau}\big(\omega, \{m = m(\omega)\}\big)$ for all $\omega \in \Omega$. Then, $0 \leq Z(\omega) \leq 1$ for all $\omega \in \Omega$ and hence, $N = \{Z < 1\} = \{1 - Z > 0\}$. (4.3.16) now follows from the observation that

$$1 - Z(\omega) = P^{\mathcal{F}_\tau}\big(\omega, \{m \neq m(\omega)\}\big)$$

for all $\omega \in \Omega$.
For all $n \in \mathbb{N}$ and for all $\omega \in \Omega$ we have

$$\begin{aligned} & \left\{m \leq m(\omega) - \frac{1}{n}\right\} \cup \left\{m \geq m(\omega) + \frac{1}{n}\right\} \\ = \quad & \left\{m \notin \left(m(\omega) - \frac{1}{n}\ ,\ m(\omega) + \frac{1}{n}\right)\right\} \end{aligned}$$

and we may therefore derive that for all $n \in \mathbb{N}$ the mappings

$$\omega \mapsto P^{\mathcal{F}_\tau}\left(\omega, \left\{m \notin \left(m(\omega) - \frac{1}{n}\ ,\ m(\omega) + \frac{1}{n}\right)\right\}\right)$$

are $\mathcal{F}_\tau$-measurable. Thus, for all $n \in \mathbb{N}$ the events

$$N_n := \left\{P^{\mathcal{F}_\tau}\left(.,\left\{m \notin \left(m(.) - \frac{1}{n}\ ,\ m(.) + \frac{1}{n}\right)\right\}\right)\ \geq\ \frac{1}{n}\right\}$$

belong to the σ-algebra $\mathcal{F}_\tau$. In view of (4.3.16) we have $N = \bigcup_{n\in\mathbb{N}} N_n$ and since $N_n \subset N_{n+1}$ we deduce,

$$0 < P(N) = P\left(\bigcup_{n\in\mathbb{N}} N_n\right) = \lim_{n\to\infty} P(N_n).$$

Thus, there exits n_0 such that $P(N_{n_0}) > 0$. For $\varepsilon := \frac{1}{n_0}$ we now have for all $\omega \in N_{n_0}$

$$P^{\mathcal{F}_\tau}\left(\omega, \left\{m \notin \left(m(\omega) - \varepsilon\ ,\ m(\omega) + \varepsilon\right)\right\}\right)\ \geq\ \varepsilon. \tag{4.3.17}$$

Next we show that there exists $\omega^* \in N_{n_0}$ and $N^*_{n_0} \subset N_{n_0}$ with $N^*_{n_0} \in \mathcal{F}_\tau$ and $P(N^*_{n_0}) > 0$ such that for all $\omega \in N^*_{n_0}$ we have

$$|m(\omega) - m(\omega^*)| \leq\ \frac{\varepsilon}{4}.$$

To this end, recall that m is bounded P-almost surely and hence, $|m(\omega)| \leq c < +\infty$ for all $\omega \in M^c$ for a suitable nullset M. Since $P(N_{n_0} \setminus M^c) = P(N_{n_0}) > 0$ and since (4.3.17) is still valid for all $\omega \in N_{n_0} \setminus M^c$ we may update $N_{n_0} := N_{n_0} \setminus M^c$. But now we can find a finite subset $I \subset N_{n_0}$ such that

$$N_{n_0} = \bigcup_{\omega\in I}\Big(\big\{|m - m(\omega)| \leq \varepsilon\big\} \cap N_{n_0}\Big).$$

Since $P(N_{n_0}) > 0$ there exists $\omega_1 \in I$ such that the set $\big\{|m - m(\omega_1)| \leq \varepsilon\big\} \cap N_{n_0}$ has positive P-measure. For this ω_1 we define the following subset of N_{n_0}

$$N_1 := \left\{|m - m(\omega_1)| \leq\ \frac{\varepsilon}{4}\right\} \cap N_{n_0}.$$

If $P(N_1) > 0$ we set $N^*_{n_0} = N_1$ and $\omega^* = \omega_1$ and we are done. If not, take

$$\omega_2 \in \Big(\big\{|m - m(\omega_1)| \le \varepsilon\big\} \cap N_{n_0}\Big) \setminus N_1$$

and define

$$N_2 := \Big\{|m - m(\omega_2)| \le \frac{\varepsilon}{4}\Big\} \cap N_{n_0}.$$

Since this way we subsequently cover the whole set $\big\{|m - m(\omega_1)| \le \varepsilon\big\} \cap N_{n_0}$ (which has positive P-measure) in finitely many steps, we have to end up with some ω_i and a set

$$N_i := \Big\{|m - m(\omega_i)| \le \frac{\varepsilon}{4}\Big\} \cap N_{n_0}$$

such that $P(N_i) > 0$. $N^*_{n_0} := N_i$ and $\omega^* := \omega_i$ have the desired properties.
Finally, we deduce that

$$\begin{aligned}
0 &< \int_{N^*_{n_0}} P^{\mathcal{F}_\tau}\Big(\omega, \Big\{m \notin \Big(m(\omega) - \varepsilon\ ,\ m(\omega) + \varepsilon\Big)\Big\}\Big)\ P(d\omega) \\
&\le \int_{N^*_{n_0}} P^{\mathcal{F}_\tau}\Big(\omega, \Big\{m \notin \Big(m(\omega^*) - \frac{\varepsilon}{2}\ ,\ m(\omega^*) + \frac{\varepsilon}{2}\Big)\Big\}\Big)\ P(d\omega) \\
&= \int_{N^*_{n_0}} E\Big[1_{\{m \notin (m(\omega^*) - \frac{\varepsilon}{2}\ ,\ m(\omega^*) + \frac{\varepsilon}{2})\}} \mid \mathcal{F}_\tau\Big](\omega)\ P(d\omega) \\
&= P\Big(\Big\{m \notin \Big(m(\omega^*) - \frac{\varepsilon}{2}\ ,\ m(\omega^*) + \frac{\varepsilon}{2}\Big)\Big\}\ \cap\ N^*_{n_0}\Big) = 0
\end{aligned}$$

which is a contradiction. □

Remark 4.3.8 *Lemma* 4.3.7 *yields an alternative proof of lemma* 4.3.5 *as follows.*
For $r \in (0,1)$ consider a conditional r-quantile $q_{X|\mathcal{F}_\tau}(r)$ of a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$. Say N is a null set such that $|X(\omega)| \le c < +\infty$ for all $\omega \in N^c$. The statement in (4.1.1) *implies the existence of a nullset $N^* = N^*(N)$ such that N is a $P^{\mathcal{F}_\tau}(\omega, .)$-nullset for all $\omega \in N^{*c}$. Thus, X is $P^{\mathcal{F}_\tau}(\omega, .)$-almost surely bounded by c for all $\omega \in N^{*c}$ where it is to be considered that the constant c is independent from ω. Hence, for $x \le c$ we have $F_{X|\mathcal{F}_\tau}(\omega, x) = 0$ and for $x \ge c$ we have $F_{X|\mathcal{F}_\tau}(\omega, x) = 1$ for all $\omega \in N^{*c}$. Thus, $\big|q_{X|\mathcal{F}_\tau}(r)(\omega)\big| \le c < +\infty$ for all $\omega \in N^{*c}$, i.e. $q_{X|\mathcal{F}_\tau}(r)$ is bounded P-almost surely. Since $q_{X|\mathcal{F}_\tau}(r)$ is $\mathcal{F}_\tau$-measurable we may replace m in lemma* 4.3.7 *with $q_{X|\mathcal{F}_\tau}(r)$ and hence,* (4.3.15) *reads*

$$P^{\mathcal{F}_\tau}\big(., \{q_{X|\mathcal{F}_\tau}(r) = q_{X|\mathcal{F}_\tau}(r)(.)\}\big) = 1, \quad a.s.\ P.$$

We deduce

$$\begin{aligned}
& P^{\mathcal{F}_\tau}\big(\omega, \{X \le q_{X|\mathcal{F}_\tau}(r)\}\big) \\
=\ & P^{\mathcal{F}_\tau}\big(\omega, \{X \le q_{X|\mathcal{F}_\tau}(r)\} \cap \{q_{X|\mathcal{F}_\tau}(r) = q_{X|\mathcal{F}_\tau}(r)(\omega)\}\big) \\
=\ & P^{\mathcal{F}_\tau}\big(\omega, \{X \le q_{X|\mathcal{F}_\tau}(r)(\omega)\}\big)\ \ge\ r
\end{aligned}$$

for P-almost all $\omega \in \Omega$ and

$$\begin{aligned} & P^{\mathcal{F}_\tau}\big(\omega, \{X < q_{X|\mathcal{F}_\tau}(r)\}\big) \\ = \ & P^{\mathcal{F}_\tau}\big(\omega, \{X < q_{X|\mathcal{F}_\tau}(r)\} \cap \{q_{X|\mathcal{F}_\tau}(r) = q_{X|\mathcal{F}_\tau}(r)(\omega)\}\big) \\ = \ & P^{\mathcal{F}_\tau}\big(\omega, \{X < q_{X|\mathcal{F}_\tau}(r)(\omega)\}\big) \ \leq \ r \end{aligned}$$

for P-almost all $\omega \in \Omega$.

Remark 4.3.9 *Taking P-conditional expectation (given $\mathcal{F}_\tau$) can be viewed as integrating with respect to the random measure $P^{\mathcal{F}_\tau}$ in the following sense: For the $\mathcal{F}_\tau$-measurable mapping*

$$\omega \mapsto \int_\Omega X(\omega^*) \ P^{\mathcal{F}_\tau}(\omega, d\omega^*) \tag{4.3.18}$$

we have

$$E[X \mid \mathcal{F}_\tau](.) \ = \ \int_\Omega X(\omega^*) \ P^{\mathcal{F}_\tau}(., d\omega^*), \quad a.s.\ P, \tag{4.3.19}$$

where X is a P-almost surely bounded random variable on $(\Omega, \mathcal{F}_\theta, P)$.

For $X = 1_A$, $A \in \mathcal{F}_\theta$ the $\mathcal{F}_\tau$-measurability of (4.3.18) *follows from* (2) *and the equality in* (4.3.19) *follows from the fact that $P^{\mathcal{F}_\tau}(., A) = E[1_A \mid \mathcal{F}_\tau]$, a.s. P. Thus, the statements follow for $X = \sum_{i=1}^n \alpha_i 1_{A_i}$, $\alpha_i \in \mathbb{R}$, $A_i \in \mathcal{F}_\theta$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $n \in \mathbb{N}$. By monotone convergence the statement now follows for general X as well, where once more we have to use the statement in* (4.1.1).

Lemma 4.3.10 *Fix $\omega_0 \in \Omega$. Assume that U is a random variable on $(\Omega, \mathcal{F}_\theta, P)$ which is $P^{\mathcal{F}_\tau}(\omega_0, .)$-uniformly distributed on* $[0, 1]$*, i.e. for all $r \in [0, 1]$ we have*

$$P^{\mathcal{F}_\tau}\big(\omega_0, \{U \leq r\}\big) = r.$$

Then, for a conditional quantile $q_{X|\mathcal{F}_\tau}$ of a random variable X on $(\Omega, \mathcal{F}_\theta, P)$, the $\mathcal{F}_\theta$-measurable mapping

$$q_{X|\mathcal{F}_\tau}\big(\omega_0, U(.)\big) : (\Omega, \mathcal{F}_\theta) \to \mathbb{R}, \quad \omega \mapsto q_{X|\mathcal{F}_\tau}\big(\omega_0, U(\omega)\big)$$

has distribution function $F_{X|\mathcal{F}_\tau}(\omega_0, .)$.

Proof. The proof of this lemma requires some technical preparation which is why we refer to lemma A.19 in Föllmer and Schied [17]. □

Remark 4.3.11 *From remark* 4.3.9 *and lemma* 4.3.10 *we derive*

$$E[X \mid \mathcal{F}_\tau](.) \ = \ \int_\Omega X(\omega^*) \ P^{\mathcal{F}_\tau}(., d\omega^*) \ = \ \int_0^1 q_{X|\mathcal{F}_\tau}(., r) \ \lambda^1(dr), \quad a.s.\ P,$$

for a conditional quantile $q_{X|\mathcal{F}_\tau}$ of a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$.

4.4 Examples

4.4.1 Conditional Value at Risk

Definition 4.4.1 *Take* $r \in (0,1)$. *We call the mapping* $VaR_{\tau,\theta;r} : \mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau) \to L^\infty(\mathcal{F}_\tau)$,

$$\widetilde{P}_{X|\mathcal{F}_\tau} \mapsto VaR_{\tau,\theta;r}(\widetilde{P}_{X|\mathcal{F}_\tau}) := -\tilde{q}^+_{X|\mathcal{F}_\tau}(r),$$

conditional value at risk at level r *given* $\mathcal{F}_\tau$.

Due to the statement in (4.3.12), $VaR_{\tau,\theta;r}$ *is well defined for all* $r \in (0,1)$.

Remark 4.4.2 *Conditional value at risk at a level* $r \in (0,1)$ *is a conditional monetary risk measure on* $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$. *The induced conditional monetary risk measure on* $L^\infty(\mathcal{F}_\theta)$ *is denoted by*

$$VaR^*_{\tau,\theta;r} : L^\infty(\mathcal{F}_\theta) \to L^\infty(\mathcal{F}_\tau), \quad \widetilde{X} \mapsto VaR^*_{\tau,\theta;r}(\widetilde{X}) := VaR_{\tau,\theta;r}(\widetilde{P}_{X|\mathcal{F}_\tau}).$$

We have to verify (**n**), (**m**) *and* ($\mathcal{F}_\tau$-**ti**) *of proposition* 4.2.7 *for the mapping* $VaR_{\tau,\theta;r}$ *at arbitrary level* $r \in (0,1)$.
(**n**)*: Take a* P*-almost surely bounded random variable* X *on* $(\Omega, \mathcal{F}_\theta, P)$ *such that* $P_{X|\mathcal{F}_\tau} \in \widetilde{\delta}_0$. *Then there exists a nullset* N *such that* $P_{X|\mathcal{F}_\tau}(\omega, .)$ *are Dirac-measures concentrated in* $\{0\}$ *for all* $\omega \in N^c$. *Hence,* $-q^+_{X|\mathcal{F}_\tau}(r)(\omega) = 0$ *for all* $r \in (0,1)$ *and for all* $\omega \in N^c$.
(**m**)*: Let* $\widetilde{P}_{X|\mathcal{F}_\tau}, \widetilde{P}_{Y|\mathcal{F}_\tau} \in \mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ *such that* $\widetilde{P}_{X|\mathcal{F}_\tau} \leq \widetilde{P}_{Y|\mathcal{F}_\tau}$. *Then, for* $(P_{X|\mathcal{F}_\tau}, P_{Y|\mathcal{F}_\tau}) \in \widetilde{P}_{X|\mathcal{F}_\tau} \times \widetilde{P}_{Y|\mathcal{F}_\tau}$, *there exists a nullset* N *such that* $F_{X|\mathcal{F}_\tau}(\omega, x) \geq F_{Y|\mathcal{F}_\tau}(\omega, x)$ *for all* $\omega \in N^c$ *and for all* $x \in \mathbb{R}$. *But this means that* $-q^+_{X|\mathcal{F}_\tau}(r)(\omega) \geq -q^+_{Y|\mathcal{F}_\tau}(r)(\omega)$ *for all* $r \in (0,1)$ *and for all* $\omega \in N^c$.
($\mathcal{F}_\tau$-**ti**)*: Consider* $\widetilde{X} \in L^\infty(\mathcal{F}_\theta)$, *its associated equivalence class* $\widetilde{P}_{X|\mathcal{F}_\tau}$ *in* $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ *and* $\widetilde{m} \in L^\infty(\mathcal{F}_\tau)$. *Further, take arbitrary* $m \in \widetilde{m}$ *and a nullset* N *such that*

$$P^{\mathcal{F}_\tau}\big(\omega, \{m = m(\omega)\}\big) = 1$$

for all $\omega \in N^c$. *Such* N *exists due to lemma* 4.3.7. *For* $X \in \widetilde{X}$ *we derive,*

$$\begin{aligned} F_{X+m|\mathcal{F}_\tau}(\omega, x) &= P^{\mathcal{F}_\tau}\big(\omega, \{X + m \leq x\}\big) \\ &= P^{\mathcal{F}_\tau}\big(\omega, \{X + m \leq x\} \cap \{m = m(\omega)\}\big) \\ &= P^{\mathcal{F}_\tau}\big(\omega, \{X + m(\omega) \leq x\}\big) \\ &= F_{X|\mathcal{F}_\tau}\big(\omega, x - m(\omega)\big) \end{aligned}$$

for all $\omega \in N^c$ *and for all* $x \in \mathbb{R}$. *Hence,*

$$-q^+_{X+m|\mathcal{F}_\tau}(r)(\omega) = -\Big(q^+_{X|\mathcal{F}_\tau}(r)(\omega) + m(\omega)\Big) = -q^+_{X|\mathcal{F}_\tau}(r)(\omega) - m(\omega)$$

for all $\omega \in N^c$ *and for all* $r \in (0,1)$.

Recall that $\mathcal{F}_0$ is assumed to be of the form $\{\Omega, \emptyset\}$. Now, assume that $\tau(\omega) = 0$ for all $\omega \in \Omega$ and let $A \in \mathcal{F}_\theta$. Then it follows from property (2) of $P^{\mathcal{F}_0}$ that $P^{\mathcal{F}_0}(\omega, A) = c \in [0, 1]$ for all $\omega \in \Omega$. Since $\Omega \in \mathcal{F}_0$ we may further derive from property (3) of $P^{\mathcal{F}_0}$,

$$P^{\mathcal{F}_0}(\omega, A) = \int_\Omega P^{\mathcal{F}_0}(\omega^*, A)\ P(d\omega^*) = P(\Omega \cap A) = P(A)$$

for all $\omega \in \Omega$. Since A was arbitrary we arrive at

$$P^{\mathcal{F}_0}(\omega, A) = P(A)$$

for all $\omega \in \Omega$ and for all $A \in \mathcal{F}_\theta$.

Consider two regular conditional probabilities $P_{X|\mathcal{F}_0}$ and $P_{Y|\mathcal{F}_0}$ for P-almost surely bounded random variables X and Y on $(\Omega, \mathcal{F}_\theta, P)$. Since $\emptyset$ is the only nullset in $\mathcal{F}_0$ we have $P_{X|\mathcal{F}_0} \sim P_{Y|\mathcal{F}_0}$ if and only if $P_{X|\mathcal{F}_0}(\omega, B) = P_{Y|\mathcal{F}_0}(\omega, B)$ for all $\omega \in \Omega$ and for all $B \in \mathcal{B}$. In this sense, equivalence classes in $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_0)$ consist of one element only. Moreover, since regular conditional probabilities given $\mathcal{F}_0$ are independent from $\omega \in \Omega$, we may view $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ as a subset of $\mathcal{M}_{1,c}(\mathbb{R})$, where the latter denotes the set of all probability measures on the real line with compact support.

Remark 4.4.3 *If $(\Omega, \mathcal{F}_\theta, P)$ is an atomless probability space we may even identify $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_0)$ and $\mathcal{M}_{1,c}(\mathbb{R})$:*
Indeed, if $(\Omega, \mathcal{F}_\theta, P)$ is atomless there exists a random variable X on $(\Omega, \mathcal{F}_\theta, P)$ with continuous P-distribution function F_X and hence, $Z : \Omega \to [0, 1]$, $\omega \mapsto Z(\omega) := F_X\big(X(\omega)\big)$ is P-uniformly distributed on $[0, 1]$. Thus, for arbitrary $\mu \in \mathcal{M}_{1,c}(\mathbb{R})$ the random variable $q_\mu(Z) : \Omega \to \mathbb{R}$, $\omega \mapsto q_\mu\big(Z(\omega)\big)$ has P-distribution μ, where q_μ is an arbitrary inverse function of the function $x \mapsto \mu(-\infty, x]$ from $\mathbb{R}$ to $[0, 1]$. μ has compact support and hence, $q_\mu(Z)$ is bounded a.s. P. Further, since $P^{\mathcal{F}_0}(\omega, A) = P(A)$ for all $\omega \in \Omega$ and for all $A \in \mathcal{F}_\theta$, we deduce

$$P_{q_\mu(Z)|\mathcal{F}_0}(\omega, B) = P^{\mathcal{F}_0}\big(\omega, \{q_\mu(Z) \in B\}\big) = P\{q_\mu(Z) \in B\} = \mu(B)$$

for all $\omega \in \Omega$ and for all $B \in \mathcal{B}$.

For a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$ and a level $r \in (0, 1)$ take $q^+_{X|\mathcal{F}_0}(r) \in VaR_{0,\theta;r}(\widetilde{P}_{X|\mathcal{F}_0})$. In view of the above discussion and by definition of conditional quantiles we have

$$q^+_{X|\mathcal{F}_0}(r)(\omega) = -\inf\big\{x \mid \mu(-\infty, x] > r\big\},$$

for all $\omega \in \Omega$, where μ is the P-distribution of X. The good definition of the "classical" value at risk at level $r \in (0, 1)$ as a functional from $\mathcal{M}_{1,c}(\mathbb{R})$ to $\mathbb{R}$ reads

$$\mu \mapsto VaR_r(\mu) := -\inf\big\{x \mid \mu(-\infty, x] > r\big\}.$$

In this sense, $VaR_{0,\theta;r}$ and VaR_r essentially are the same if one identifies their domains $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_0)$ and $\mathcal{M}_{1,c}(\mathbb{R})$. Due to remark 4.4.3 this is possible if $(\Omega, \mathcal{F}_\theta, P)$ is atomless. In particular, we obtain that conditional value at risk is not convex in general.

Let us consider arbitrary τ again. For a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$ the measures $P_{X|\mathcal{F}_\tau}(\omega, .)$, $\omega \in N^c$, have compact support on the real line, where N is a suitable nullset whose existence is guaranteed by the statement in (4.1.1). For such X we have, by definition of conditional quantiles,

$$-q^+_{X|\mathcal{F}_\tau}(r)(\omega) = VaR_r\big(P_{X|\mathcal{F}_\tau}(\omega, .)\big)$$

for all $\omega \in N^c$ and for all $r \in (0,1)$. In particular, for all $r \in (0,1)$ the mapping $\omega \mapsto VaR_r\big(P_{X|\mathcal{F}_\tau}(\omega, .)\big)$ is P-almost everywhere well defined and $\mathcal{F}_\tau$-measurable. Hence, for all $r \in (0,1)$,

$$\omega \mapsto VaR_r\big(P_{X|\mathcal{F}_\tau}(\omega, .)\big) \;\in\; VaR_{\tau,\theta;r}(\widetilde{P}_{X|\mathcal{F}_\tau}) \tag{4.4.20}$$

for all $P_{X|\mathcal{F}_\tau} \in \widetilde{P}_{X|\mathcal{F}_\tau} \in \mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$.

The next example demonstrates the statement in (4.4.20)

Example 4.4.4 *Given the same situation as in example* 4.3.6 *we may define the right-continuous inverse functions*

$$q^+_i : (0,1) \to \mathbb{R}, \quad r \mapsto q^+_i(r) := \inf\big\{x \mid P_{B_i}\{X \le x\} > r\big\}$$

for all $i \in I$. As in example 4.3.6*,*

$$\sum_{i \in I} -q^+_i(r) 1_{B_i} \;\in\; -\tilde{q}^+_{X|\mathcal{F}_\tau}(r)$$

for all $r \in (0,1)$. Hence, for all $r \in (0,1)$,

$$\omega \mapsto \sum_{i \in I} VaR_r\big(P_{B_i}\{X \in .\}\big) 1_{B_i}(\omega) = \sum_{i \in I} -q^+_i(r) 1_{B_i}(\omega)$$

is an element of $VaR_{\tau,\theta;r}(\widetilde{P}_{X|I})$, where $\widetilde{P}_{X|I}$ designates the equivalence class in $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ induced by $P_{X|I}$.

4.4.2 Conditional Expected Shortfall

Definition 4.4.5 *Take $r \in (0,1)$. For a conditional r-quantile $q_{X|\mathcal{F}_\tau}(r)$ of a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$ we define the mappings $I_{X|\mathcal{F}_\tau}(r) : \Omega \to \mathbb{R}$,*

$$\omega \mapsto I_{X|\mathcal{F}_\tau}(r)(\omega) := \frac{1}{r}\Big(1_{\{X < q_{X|\mathcal{F}_\tau}(r)\}}(\omega) + \kappa_{X|\mathcal{F}_\tau}(\omega) 1_{\{X = q_{X|\mathcal{F}_\tau}(r)\}}(\omega)\Big)$$

and $\kappa_{X|\mathcal{F}_\tau} : \Omega \to \mathbb{R}$,

$$\omega \mapsto \kappa_{X|\mathcal{F}_\tau}(\omega) := \begin{cases} 0 & \text{on} \quad \big\{P^{\mathcal{F}_\tau}\big(., \{X = q_{X|\mathcal{F}_\tau}(r)\}\big) = 0\big\} \\ \frac{r - P^{\mathcal{F}_\tau}\left(\omega, \{X < q_{X|\mathcal{F}_\tau}(r)\}\right)}{P^{\mathcal{F}_\tau}\left(\omega, \{X = q_{X|\mathcal{F}_\tau}(r)\}\right)} & \text{on} \quad \big\{P^{\mathcal{F}_\tau}\big(., \{X = q_{X|\mathcal{F}_\tau}(r)\}\big) > 0\big\} \end{cases}.$$

Note that for r, X and $q_{X|\mathcal{F}_\tau}(r)$ as in the above definition the mappings $I_{X|\mathcal{F}_\tau}(r) : (\Omega, \mathcal{F}_\tau) \to \mathbb{R}$ and $\kappa_{X|\mathcal{F}_\tau} : (\Omega, \mathcal{F}_\tau) \to \mathbb{R}$ are $\mathcal{F}_\tau$-measurable.

Lemma 4.4.6 *For $r \in (0,1)$ take a conditional r-quantile $q_{X|\mathcal{F}_\tau}(r)$ of a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$. For the associated mapping $I_{X|\mathcal{F}_\tau}(r)$ we have*

$$E\left[I_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau\right] = 1, \quad a.s.\ P.$$

Moreover, $\kappa_{X|\mathcal{F}_\tau}$ takes only values between zero and one P-almost surely.

Proof. The sets $\left\{P^{\mathcal{F}_\tau}\left(., \{X = q_{X|\mathcal{F}_\tau}(r)\}\right) = 0\right\}$ and $\left\{P^{\mathcal{F}_\tau}\left(., \{X = q_{X|\mathcal{F}_\tau}(r)\}\right) > 0\right\}$ belong to the σ-algebra $\mathcal{F}_\tau$. Thus, we may prove the first claim on each of them separately.
On $\left\{P^{\mathcal{F}_\tau}\left(., \{X = q_{X|\mathcal{F}_\tau}(r)\}\right) > 0\right\}$ we have

$$\kappa_{X|\mathcal{F}_\tau}(.) = \frac{r - P^{\mathcal{F}_\tau}\left(., \left\{X < q_{X|\mathcal{F}_\tau}(r)\right\}\right)}{P^{\mathcal{F}_\tau}\left(., \left\{X = q_{X|\mathcal{F}_\tau}(r)\right\}\right)}$$

and hence, the claim follows from the fact that $P^{\mathcal{F}_\tau}(., A) = E[1_A \mid \mathcal{F}_\tau]$, a.s. P, for all $A \in \mathcal{F}_\theta$.
For $\omega \in \left\{P^{\mathcal{F}_\tau}\left(., \{X = q_{X|\mathcal{F}_\tau}(r)\}\right) = 0\right\}$ we have $\kappa_{X|\mathcal{F}_\tau}(\omega) = 0$. We deduce

$$E[I_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau] = \frac{1}{r} E\left[1_{\{X < q_{X|\mathcal{F}_\tau}(r)\}} \mid \mathcal{F}_\tau\right] = \frac{1}{r} P^{\mathcal{F}_\tau}\left(., \{X < q_{X|\mathcal{F}_\tau}(r)\}\right) \leq 1, \quad \text{a.s. } P,$$

as well as

$$\begin{aligned} E[I_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau] &= \frac{1}{r}\Big(P^{\mathcal{F}_\tau}\left(., \{X < q_{X|\mathcal{F}_\tau}(r)\}\right) + P^{\mathcal{F}_\tau}\left(., \{X = q_{X|\mathcal{F}_\tau}(r)\}\right)\Big) \\ &= \frac{1}{r} P^{\mathcal{F}_\tau}\left(., \{X \leq q_{X|\mathcal{F}_\tau}(r)\}\right) \geq 1, \quad \text{a.s. } P, \end{aligned}$$

on $\left\{P^{\mathcal{F}_\tau}\left(., \{X = q_{X|\mathcal{F}_\tau}(r)\}\right) = 0\right\}$, where each of the last inequalities follow from lemma 4.3.5. This proves the first assertion.
The second one follows from the observation

$$\begin{aligned} 0 &\leq r - P^{\mathcal{F}_\tau}\left(., \{X < q_{X|\mathcal{F}_\tau}(r)\}\right) \\ &\leq P^{\mathcal{F}_\tau}\left(., \{X \leq q_{X|\mathcal{F}_\tau}(r)\}\right) - P^{\mathcal{F}_\tau}\left(., \{X < q_{X|\mathcal{F}_\tau}(r)\}\right) \\ &= P^{\mathcal{F}_\tau}\left(., \{X = q_{X|\mathcal{F}_\tau}(r)\}\right), \quad \text{a.s. } P, \end{aligned}$$

where the first two inequalities are again consequences of lemma 4.3.5. □

Definition 4.4.7 *Take $r \in (0,1)$. We call the mapping $ES^*_{\tau,\theta;r} : L^\infty(\mathcal{F}_\theta) \to L^\infty(\mathcal{F}_\tau)$,*

$$\widetilde{X} \mapsto ES^*_{\tau,\theta;r}(\widetilde{X}) := \operatorname*{ess.sup}_{Q \in \mathcal{Q}_\tau(r)} E_Q[-\widetilde{X} \mid \mathcal{F}_\tau],$$

conditional expected shortfall at level r given $\mathcal{F}_\tau$, where $\mathcal{Q}_\tau(r)$ is given by the subset

$$\mathcal{Q}_\tau(r) := \left\{Q \ll P \mid \frac{\varphi}{E[\varphi \mid \mathcal{F}_\tau]} \leq \frac{1}{r},\ a.s.\ P,\ \varphi := \frac{dQ}{dP}\right\}$$

of all probability measures that are absolutely continuous with respect to P.

Remark 4.4.8 *For* $\widetilde{X} \in L^\infty(\mathcal{F}_\theta)$, $\text{ess.sup}_{Q\in\mathcal{Q}_\tau(r)}\ E_Q[-\widetilde{X} \mid \mathcal{F}_\tau]$ *is understood as the equivalence class of all* $\mathcal{F}_\tau$*-measurable random variables that are* P*-almost surely equal to* $\text{ess.sup}_{Q\in\mathcal{Q}_\tau(r)}\ E_Q[-X \mid \mathcal{F}_\tau]$ *for arbitrary* $X \in \widetilde{X}$*. Therefore, one should inspect that* $\text{ess.sup}_{Q\in\mathcal{Q}_\tau(r)}\ E_Q[-\widetilde{X} \mid \mathcal{F}_\tau]$ *is well defined with respect to the choice of* X*. To this end, take* $Y \in \widetilde{X}$*. Then,* $E_Q[-X \mid \mathcal{F}_\tau] = E_Q[-Y \mid \mathcal{F}_\tau]$*, a.s.* P*, for all* $Q \in \mathcal{Q}_\tau(r)$*. Hence,*

$$\underset{Q\in\mathcal{Q}_\tau(r)}{\text{ess.sup}}\ E_Q[-X \mid \mathcal{F}_\tau] \geq E_S[-X \mid \mathcal{F}_\tau] = E_S[-Y \mid \mathcal{F}_\tau], \quad a.s.\ P,$$

for all $S \in \mathcal{Q}_\tau(r)$*. Thus, by uniqueness of the essential supremum we have*

$$\underset{Q\in\mathcal{Q}_\tau(r)}{\text{ess.sup}}\ E_Q[-X \mid \mathcal{F}_\tau] \geq \underset{Q\in\mathcal{Q}_\tau(r)}{\text{ess.sup}}\ E_Q[-Y \mid \mathcal{F}_\tau], \quad a.s.\ P.$$

In the same way we derive,

$$\underset{Q\in\mathcal{Q}_\tau(r)}{\text{ess.sup}}\ E_Q[-Y \mid \mathcal{F}_\tau] \geq \underset{Q\in\mathcal{Q}_\tau(r)}{\text{ess.sup}}\ E_Q[-X \mid \mathcal{F}_\tau], \quad a.s.\ P.$$

Hence, $\text{ess.sup}_{Q\in\mathcal{Q}_\tau(r)}\ E_Q[-Y \mid \mathcal{F}_\tau] = \text{ess.sup}_{Q\in\mathcal{Q}_\tau(r)}\ E_Q[-X \mid \mathcal{F}_\tau]$*, a.s.* P*, and it follows that* $\text{ess.sup}_{Q\in\mathcal{Q}_\tau(r)}\ E_Q[-\widetilde{X} \mid \mathcal{F}_\tau]$ *is well defined.*

For all levels $r \in (0,1)$ we deduce from the properties of conditional expectation that $ES^*_{\tau,\theta;r} : L^\infty(\mathcal{F}_\theta) \to L^\infty(\mathcal{F}_\tau)$ is a conditional coherent risk measure on $L^\infty(\mathcal{F}_\theta)$.

Remark 4.4.9 *Take* $r \in (0,1)$ *and consider the mapping* $I_{X|\mathcal{F}_\tau}(r)$ *for a conditional* r*-quantile* $q_{X|\mathcal{F}_\tau}(r)$ *of a* P*-almost surely bounded random variable* X *on* $(\Omega, \mathcal{F}_\theta, P)$*. Then the probability measure* P^* *with density* $\frac{dP^*}{dP} := \varphi := I_{X|\mathcal{F}_\tau}(r)$ *is an element of set* $Q_\tau(r)$*. Indeed, from lemma* 4.4.6 *we derive*

$$E[\varphi] = E[I_{X|\mathcal{F}_\tau}(r)] = E\big[E[I_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau]\big] = 1.$$

Moreover, as a consequence of lemma 4.4.6 *and of the definition of* $I_{X|\mathcal{F}_\tau}(r)$ *we have* $\frac{\varphi}{E[\varphi|\mathcal{F}_\tau]} \leq \frac{1}{r}$*, a.s.* P*.*

Theorem 4.4.10 *For* $r \in (0,1)$ *take a conditional* r*-quantile* $q_{X|\mathcal{F}_\tau}(r)$ *of a* P*-almost surely bounded random variable* X *on* $(\Omega, \mathcal{F}_\theta, P)$ *and consider the associated measure* P^* *of remark* 4.4.9*. Then,*

$$\underset{Q\in\mathcal{Q}_\tau(r)}{\text{ess.sup}}\ E_Q[-X \mid \mathcal{F}_\tau] = E_{P^*}[-X \mid \mathcal{F}_\tau] = E[-X I_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau], \quad a.s.\ P. \tag{4.4.21}$$

In particular,

$$E[-X I_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau] \ \in\ ES^*_{\tau,\theta;r}(\widetilde{X}), \tag{4.4.22}$$

where $\widetilde{X}$ *denotes the equivalence class in* $L^\infty(\mathcal{F}_\theta)$ *induced by* X*.*

Proof. We have

$$E_{P^*}[-X \mid \mathcal{F}_\tau] = \frac{E[-XI_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau]}{E[I_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau]} = E[-XI_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau], \quad \text{a.s. } P,$$

which proves the second equality in (4.4.21). To verify the other one we first deduce from remark 4.4.9,

$$\operatorname*{ess.sup}_{Q\in\mathcal{Q}_\tau(r)} E_Q[-X \mid \mathcal{F}_\tau] \geq E_{P^*}[-X \mid \mathcal{F}_\tau], \quad \text{a.s. } P,$$

and hence, only the reverse inequality $\text{ess.sup}_{Q\in\mathcal{Q}_\tau(r)}\ E_Q[-X \mid \mathcal{F}_\tau] \leq E_{P^*}[-X \mid \mathcal{F}_\tau]$, a.s. P, remains to be proved. To this end, let $Q \in \mathcal{Q}_\tau(r)$ and observe that

$$E\left[\frac{\frac{dQ}{dP}}{E\left[\frac{dQ}{dP} \mid \mathcal{F}_\tau\right]} - I_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau\right] = 0, \quad \text{a.s. } P,$$

on $\left\{E\left[\frac{dQ}{dP} \mid \mathcal{F}_\tau\right] > 0\right\}$. Hence, on the same set we have

$$\begin{aligned}
E_{P^*}[-X \mid \mathcal{F}_\tau] - E_Q[-X \mid \mathcal{F}_\tau] &= E\left[\left(\frac{\frac{dQ}{dP}}{E\left[\frac{dQ}{dP} \mid \mathcal{F}_\tau\right]} - I_{X|\mathcal{F}_\tau}(r)\right) X \mid \mathcal{F}_\tau\right] \\
&= E\left[\left(\frac{\frac{dQ}{dP}}{E\left[\frac{dQ}{dP} \mid \mathcal{F}_\tau\right]} - I_{X|\mathcal{F}_\tau} r)\right) \big(X - q_{X|\mathcal{F}_\tau}(r)\big) \mid \mathcal{F}_\tau\right] \\
&\geq 0, \quad \text{a.s. } P.
\end{aligned}$$

In the above computation we may replace $\frac{\frac{dQ}{dP}}{E[\frac{dQ}{dP}|\mathcal{F}_\tau]}$ with 0 yielding the same result on $\left\{E\left[\frac{dQ}{dP} \mid \mathcal{F}_\tau\right] = 0\right\}$. Since the set $\left\{E\left[\frac{dQ}{dP} \mid \mathcal{F}_\tau\right] < 0\right\}$ has P-measure zero anyway the reverse inequality is proved for P-almost all $\omega \in \Omega$. □

Corollary 4.4.11 *For $r \in (0,1)$ take a conditional r-quantile $q_{X|\mathcal{F}_\tau}(r)$ of a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$. For the $\mathcal{F}_\tau$-measurable mappings*

$$\begin{aligned}
\omega &\mapsto \frac{1}{r}E\left[\big(q_{X|\mathcal{F}_\tau}(r) - X\big)^+ \mid \mathcal{F}_\tau\right](\omega) - q_{X|\mathcal{F}_\tau}(r)(\omega) \quad \textit{and} \\
\omega &\mapsto -\frac{1}{r}\int_0^r q_{X|\mathcal{F}_\tau}(s)(\omega)\ \lambda^1(ds)
\end{aligned}$$

we have

$$\begin{aligned}
E\left[-XI_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau\right](.) &= \frac{1}{r}E\left[\big(q_{X|\mathcal{F}_\tau}(r) - X\big)^+ \mid \mathcal{F}_\tau\right](.) - q_{X|\mathcal{F}_\tau}(r)(.) \quad (4.4.23) \\
&= -\frac{1}{r}\int_0^r q_{X|\mathcal{F}_\tau}(s)(.)\ \lambda^1(ds), \quad \textit{a.s. } P. \quad (4.4.24)
\end{aligned}$$

In particular,

$$\frac{1}{r}\int_0^r -q^+_{X|\mathcal{F}_\tau}(s)(.)\ \lambda^1(ds)\ \in\ ES^*_{\tau,\theta;r}(\widetilde{X}) \tag{4.4.25}$$

for all levels $r \in (0,1)$, where $\widetilde{X}$ denotes the equivalence class in $L^\infty(\mathcal{F}_\theta)$ induced by X.

Proof. We have

$$\begin{aligned}
&\frac{1}{r}E\left[\left(q_{X|\mathcal{F}_\tau}(r)-X\right)^+ \mid \mathcal{F}_\tau\right] - q_{X|\mathcal{F}_\tau}(r)\\
=\ &\frac{1}{r}\Big(E\left[\left(q_{X|\mathcal{F}_\tau}(r)-X\right)1_{\{X<q_{X|\mathcal{F}_\tau}(r)\}} \mid \mathcal{F}_\tau\right] - rq_{X|\mathcal{F}_\tau}(r)\Big)\\
=\ &\frac{1}{r}\Big(E\left[-X1_{\{X<q_{X|\mathcal{F}_\tau}(r)\}} \mid \mathcal{F}_\tau\right] + q_{X|\mathcal{F}_\tau}(r)E\left[1_{\{X<q_{X|\mathcal{F}_\tau}(r)\}} \mid \mathcal{F}_\tau\right] - rq_{X|\mathcal{F}_\tau}(r)\Big)\\
=\ &\frac{1}{r}\Big(E\left[-X1_{\{X<q_{X|\mathcal{F}_\tau}(r)\}} \mid \mathcal{F}_\tau\right] + q_{X|\mathcal{F}_\tau}(r)P^{\mathcal{F}_\tau}\big(.,\{X<q_{X|\mathcal{F}_\tau}(r)\}\big) - rq_{X|\mathcal{F}_\tau}(r)\Big),
\end{aligned} \tag{4.4.26}$$

a.s. P. On the set $\big\{P^{\mathcal{F}_\tau}\big(.,\{X=q_{X|\mathcal{F}_\tau}(r)\}\big)=0\big\}$ (4.4.26) can be written in the form

$$\begin{aligned}
&\frac{1}{r}\Big(E\left[-X1_{\{X<q_{X|\mathcal{F}_\tau}(r)\}} \mid \mathcal{F}_\tau\right] + q_{X|\mathcal{F}_\tau}(r)P^{\mathcal{F}_\tau}\big(.,\{X\le q_{X|\mathcal{F}_\tau}(r)\}\big) - rq_{X|\mathcal{F}_\tau}(r)\Big)\\
=\ &\frac{1}{r}E\left[-X1_{\{X<q_{X|\mathcal{F}_\tau}(r)\}} \mid \mathcal{F}_\tau\right], \quad \text{a.s. } P,
\end{aligned}$$

where the last equality follows from lemma 4.3.5. This yields the equality in (4.4.23) for P-almost all $\omega \in \big\{P^{\mathcal{F}_\tau}\big(.,\{X=q_{X|\mathcal{F}_\tau}(r)\}\big)=0\big\}$. On the set $\big\{P^{\mathcal{F}_\tau}\big(.,\{X=q_{X|\mathcal{F}_\tau}(r)\}\big)>0\big\}$ we have

$$\begin{aligned}
&q_{X|\mathcal{F}_\tau}(r)P^{\mathcal{F}_\tau}\big(.,\{X<q_{X|\mathcal{F}_\tau}(r)\}\big) - rq_{X|\mathcal{F}_\tau}(r)\\
=\ &\frac{\Big(q_{X|\mathcal{F}_\tau}(r)P^{\mathcal{F}_\tau}\big(.,\{X<q_{X|\mathcal{F}_\tau}(r)\}\big) - rq_{X|\mathcal{F}_\tau}(r)\Big)E\left[1_{\{X=q_{X|\mathcal{F}_\tau}(r)\}} \mid \mathcal{F}_\tau\right]}{P^{\mathcal{F}_\tau}\big(.,\{X=q_{X|\mathcal{F}_\tau}(r)\}\big)}\\
=\ &\frac{E\Big[\Big(q_{X|\mathcal{F}_\tau}(r)P^{\mathcal{F}_\tau}\big(.,\{X<q_{X|\mathcal{F}_\tau}(r)\}\big) - rq_{X|\mathcal{F}_\tau}(r)\Big)1_{\{X=q_{X|\mathcal{F}_\tau}(r)\}} \mid \mathcal{F}_\tau\Big]}{P^{\mathcal{F}_\tau}\big(.,\{X=q_{X|\mathcal{F}_\tau}(r)\}\big)}\\
=\ &E\left[-q_{X|\mathcal{F}_\tau}(r)\frac{r-P^{\mathcal{F}_\tau}\big(.,\{X<q_{X|\mathcal{F}_\tau}(r)\}\big)}{P^{\mathcal{F}_\tau}\big(.,\{X=q_{X|\mathcal{F}_\tau}(r)\}\big)}1_{\{X=q_{X|\mathcal{F}_\tau}(r)\}} \mid \mathcal{F}_\tau\right]\\
=\ &E\left[-X\kappa_{X|\mathcal{F}_\tau}1_{\{X=q_{X|\mathcal{F}_\tau}(r)\}} \mid \mathcal{F}_\tau\right], \quad \text{a.s. } P.
\end{aligned}$$

Plugging this into (4.4.26) yields the equality in (4.4.23) for P-almost all $\omega \in \big\{P^{\mathcal{F}_\tau}\big(.,\{X=q_{X|\mathcal{F}_\tau}(r)\}\big)>0\big\}$.

By remark 4.3.9 there exists a null-set N_1 such that

$$E\left[\left(q_{X|\mathcal{F}_\tau}(r)-X\right)^+ \mid \mathcal{F}_\tau\right](\omega) = \int_\Omega \left(q_{X|\mathcal{F}_\tau}(r)(\tilde\omega)-X(\tilde\omega)\right)^+\ P^{\mathcal{F}_\tau}(\omega,d\tilde\omega)$$

for all $\omega \in N_1^c$ and in turn

$$\begin{aligned} & \frac{1}{r} E\left[\left(q_{X|\mathcal{F}_\tau}(r) - X\right)^+ \mid \mathcal{F}_\tau\right](\omega) - q_{X|\mathcal{F}_\tau}(r)(\omega) \\ = \; & \frac{1}{r}\int_\Omega \left(q_{X|\mathcal{F}_\tau}(r)(\tilde{\omega}) - X(\tilde{\omega})\right)^+ \, P^{\mathcal{F}_\tau}(\omega, d\tilde{\omega}) - q_{X|\mathcal{F}_\tau}(r)(\omega) \end{aligned} \tag{4.4.27}$$

for all $\omega \in N_1^c$. Further, by lemma 4.3.7 there exits a null-set N_2 such that

$$P^{\mathcal{F}_\tau}\left(\omega, \{q_{X|\mathcal{F}_\tau}(r) \neq q_{X|\mathcal{F}_\tau}(r)(\omega)\}\right) = 0$$

for all $\omega \in N_2^c$. Thus, (4.4.27) reads

$$\frac{1}{r}\int_\Omega \left(q_{X|\mathcal{F}_\tau}(r)(\omega) - X(\tilde{\omega})\right)^+ \, P^{\mathcal{F}_\tau}(\omega, d\tilde{\omega}) - q_{X|\mathcal{F}_\tau}(r)(\omega) \tag{4.4.28}$$

for all $\omega \in (N_1 \cup N_2)^c$. In view of remark 4.3.11, we may write (4.4.28) in the form

$$\frac{1}{r}\int_0^1 \left(q_{X|\mathcal{F}_\tau}(r)(\omega) - q_{X|\mathcal{F}_\tau}(s)(\omega)\right)^+ \, \lambda^1(ds) - q_{X|\mathcal{F}_\tau}(r)(\omega)$$

for all $\omega \in (N_1 \cup N_2)^c$ (note that N_1 is the null-set of remark 4.3.11). We finally observe that for all $\omega \in \Omega$ (and in particular for all $\omega \in (N_1 \cup N_2)^c$),

$$\frac{1}{r}\int_0^1 \left(q_{X|\mathcal{F}_\tau}(r)(\omega) - q_{X|\mathcal{F}_\tau}(s)(\omega)\right)^+ \, \lambda^1(ds) - q_{X|\mathcal{F}_\tau}(r)(\omega) = -\frac{1}{r}\int_0^r q_{X|\mathcal{F}_\tau}(s)(\omega) \, \lambda^1(ds),$$

which proves the equality in (4.4.24). □

The statement in (4.4.25) reveals that conditional expected shortfall is a distribution invariant conditional monetary risk measure on $L^\infty(\mathcal{F}_\theta)$, i.e. for all $\widetilde{X}, \widetilde{Y} \in L^\infty(\mathcal{F}_\theta)$ and for all $r \in (0,1)$ we have

$$ES^*_{\tau,\theta;r}(\widetilde{X}) = ES^*_{\tau,\theta;r}(\widetilde{Y}), \quad \text{a.s. } P,$$

whenever the corresponding equivalence classes $\widetilde{P}_{X|\mathcal{F}_\tau}$ and $\widetilde{P}_{Y|\mathcal{F}_\tau}$ are the same. Thus, for all $r \in (0,1)$ the functional $ES_{\tau,\theta;r}$ from $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ to $L^\infty(\mathcal{F}_\tau)$ given by

$$\widetilde{P}_{X|\mathcal{F}_\tau} \mapsto ES_{\tau,\theta;r}(\widetilde{P}_{X|\mathcal{F}_\tau}) := ES^*_{\tau,\theta;r}(\widetilde{X}),$$

is well defined. By definition, $ES_{\tau,\theta;r}$ induces $ES^*_{\tau,\theta;r}$ and hence, $ES_{\tau,\theta;r}$ is a conditional coherent risk measure on $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ for all $r \in (0,1)$. We call $ES_{\tau,\theta;r}$ conditional expected shortfall at level $r \in (0,1)$ as well.

Consider a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$ and a level $r \in (0,1)$. We have

$$\frac{1}{r}\int_0^r -q^+_{X|\mathcal{F}_0}(s)(\omega) \, \lambda^1(ds) \;\in\; ES_{0,\theta;r}(\widetilde{P}_{X|\mathcal{F}_0})$$

Here $\tau(\omega) = 0$ for all $\omega \in \Omega$. In view of the discussion of the preceding subsection we have

$$\frac{1}{r}\int_0^r -q^+_{X|\mathcal{F}_0}(s)(\omega)\ \lambda^1(ds) = \frac{1}{r}\int_0^r VaR_s(\mu)\ \lambda^1(ds),$$

for all $\omega \in \Omega$, where μ is the P-distribution of X. Since the "classical" expected shortfall $ES_r : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ at level $r \in (0,1)$ is given by

$$\mu \mapsto ES_r(\mu) := \frac{1}{r}\int_0^r VaR_s(\mu)\ \lambda^1(ds),$$

again $ES_{0,\theta;r}$ and ES_r essentially are the same if one identifies their domains $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_0)$ and $\mathcal{M}_{1,c}(\mathbb{R})$. Recall that remark 4.4.3 clarifies this situation.

We consider general τ again. For a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$ there exists a nullset $N = N(X)$ such that $P_{X|\mathcal{F}_\tau}(\omega, .) \in \mathcal{M}_{1,c}(\mathbb{R})$ for all $\omega \in N^c$ (i.e. the probability measures $P_{X|\mathcal{F}_\tau}(\omega, .)$, $\omega \in N^c$, have compact support on the real line). By definition,

$$VaR_r\big(P_{X|\mathcal{F}_\tau}(\omega, .)\big) = -q^+_{X|\mathcal{F}_\tau}(r)(\omega)$$

for all $\omega \in N^c$ and all levels $r \in (0,1)$. Note that the nullset N indeed depends on X however it is independent from the levels $r \in (0,1)$. We derive

$$ES_r\big(P_{X|\mathcal{F}_\tau}(\omega, .)\big) = \frac{1}{r}\int_0^r VaR_s\big(P_{X|\mathcal{F}_\tau}(\omega, .)\big)\ \lambda^1(ds) = \frac{1}{r}\int_0^r -q^+_{X|\mathcal{F}_\tau}(s)(\omega)\ \lambda^1(ds)$$

for all $\omega \in N^c$ and for all $r \in (0,1)$. In particular, the mapping $\omega \mapsto ES_r\big(P_{X|\mathcal{F}_\tau}(\omega, .)\big)$ is P-almost everywhere well defined, $\mathcal{F}_\tau$-measurable and hence,

$$\omega \mapsto ES_r\big(P_{X|\mathcal{F}_\tau}(\omega, .)\big)\ \in\ ES_{\tau,\theta;r}(\widetilde{P}_{X|\mathcal{F}_\tau})$$

for all $P_{X|\mathcal{F}_\tau} \in \widetilde{P}_{X|\mathcal{F}_\tau} \in \mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$.

Again this statement is illustrated in the next example.

Example 4.4.12 *Let us fix a level* $r \in (0,1)$ *and consider once more the situation as in example* 4.3.6. *We have*

$$E[-XI_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau] = \sum_{i\in I} E_{B_i}[-XI_{X|\mathcal{F}_\tau}(r)]1_{B_i}, \quad a.s.\ P,$$

where

$$E_{B_i}[-XI_{X|\mathcal{F}_\tau}(r)] := \int_\Omega -XI_{X|\mathcal{F}_\tau}(r)\ dP_{B_i}.$$

Due to example 4.3.6 *we may choose a conditional* r*-quantile* $q_{X|\mathcal{F}_\tau}(r)$ *(for which* $I_{X|\mathcal{F}_\tau}(r)$ *is defined) of the form*

$$\omega \mapsto \sum_{i\in I} q_i(r)1_{B_i}(\omega),$$

where $q_i : (0,1) \to \mathbb{R}$ *designate suitable inverse functions for all* $i \in I$*. Then,*

$$\begin{aligned} & E_{B_i}\left[-X I_{X|\mathcal{F}_\tau}(r)\right] \\ = & \int_\Omega -X(\omega)\frac{1}{r}\Big(1_{\{X<q_{X|\mathcal{F}_\tau}(r)\}}(\omega)+\kappa_{X|\mathcal{F}_\tau}(\omega)1_{\{X=q_{X|\mathcal{F}_\tau}(r)\}}(\omega)\Big)\ P_{B_i}(d\omega) \\ = & \int_\Omega -X(\omega)\frac{1}{r}\Big(1_{\{X<q_i(r)\}}(\omega)+\kappa_i(\omega)1_{\{X=q_i(r)\}}(\omega)\Big)\ P_{B_i}(d\omega), \end{aligned}$$

where $\kappa_i : \Omega \to \mathbb{R}$ *is given by*

$$\omega \mapsto \kappa_i(\omega) := \begin{cases} 0 & if \quad P_{B_i}\{X = q_i(r)\} = 0 \\ \frac{r - P_{B_i}\{X<q_i(r)\}}{P_{B_i}\{X=q_i(r)\}} & if \quad P_{B_i}\{X = q_i(r)\} > 0 \end{cases}.$$

From remark 4.48 *in Föllmer and Schied* [17] *we can therefore deduce*

$$E_{B_i}[-X I_\tau(r)] = ES_r\big(P_{B_i}\{X \in .\}\big).$$

Hence, by theorem 4.4.10 *we have*

$$\omega \mapsto \sum_{i\in I} ES_r\big(P_{B_i}\{X \in .\}\big)1_{B_i}(\omega) \ \in \ ES_{\tau,\theta;r}(\widetilde{P}_{X|I}),$$

where $\widetilde{P}_{X|I}$ *designates the equivalence class in* $\mathcal{M}_{1,c}(\mathbb{R} \mid \mathcal{F}_\tau)$ *induced by* $P_{X|I}$*.*

Chapter 5

Dynamic Monetary Utility Functionals

This chapter mainly provides a brief collection of a few results on dynamic monetary utility functionals as they are found throughout the recent literature. We focus on dynamic consistency properties and its consequences for dynamic monetary utility functionals. In fact, we discuss a basic construction principle for time-consistent dynamic monetary utility functionals and provide recent representation results. In the last subsection we present, as a main result, a representation theorem in terms of concatenated probability densities.

5.1 Introduction

In chapter 2 we introduced conditional monetary utility functionals which depend on bounded random variables and on bounded discrete-time processes. Here, we present dynamic monetary utility functionals as families of conditional monetary utility functionals for bounded random variables at different times. An extension of the results presented in sections 5.2 and 5.3 to bounded discrete-time processes has already been established in Cheridito et al. [8] and Cheridito and Kupper [9]. However, a corresponding generalization of section 5.4 still is subject of ongoing research and therefore, the remainder of this thesis is devoted to the case of bounded random variables.

When risk assessments of final values are updated as new information is released, the associated capital requirements should not contradict one another across time. It is common sense to impose dynamic consistency conditions upon dynamic monetary utility functionals, cf. Wang [22], Delbaen [12], Artzner et al. [3, 4]. Characterizations and examples of dynamic monetary risk measures which are dynamic consistent are given in Riedel [20], Detlefsen and Scandolo [13] as well as in Weber [23]. The two latter address risk assessment of final values and Weber additionally focuses on distribution invariant dynamic risk measures. In [8], Cheridito et al. study a notion of dynamic consistency which can be characterized by means of a decomposition property of acceptance sets. As to my information, Delbaen first presented this useful characterization in [12] for coherent dynamic utility functionals that are defined for bounded random variables. In [9], Cheridito and Kupper work with the same concept of dynamic consistency and provide dual representations for dynamic consistent dynamic monetary concave utility functionals which are continuous in a mild sense and are defined for either bounded random variables or bounded discrete-time processes. As the results of the remaining thesis are strongly based on the ones given in [8] and [9] we follow their rather strong notion of dynamic consistency.

In section 5.2 we introduce dynamic monetary utility functionals for bounded random variables and present the notion of dynamic consistency which we choose to work with. We call it time-consistency and show how this concept relates to an iteration condition. More precisely, time-consistency means that the risk of a final value may be calculated directly, say at time t, or iteratively at time $t+1$ and then at time t. We show how this allows us to construct time-consistent dynamic monetary utility functionals out of arbitrary families of conditional monetary utility functionals. Section 5.3 provides a brief collection of a few recent duality results in the context of time-consistent dynamic monetary utility functionals. Time-consistent dynamic concave utility functionals which are continuous in a mild sense admit a representation in terms of what we call dynamic penalty function, where we take essential infimum over certain duals. As a main result, we provide in section 5.4 a characterization of those elements for which the essential infimum is attained. More precisely, if a time-consistent dynamic monetary utility functional is constructed via an arbitrary family of conditional monetary utility functionals and if for each of those conditional monetary utility functionals we know at which element the essential infimum is attained then we also know the element at which the essential infimum representing the time-consistent dynamic monetary utility functional is attained.

Throughout this chapter we consider the setup of section 2.2 and let τ and θ be two $(\mathcal{F}_t)$-stopping times such that $\tau(\omega) \leq \theta(\omega)$ for all $\omega \in \Omega$. We denote by $\mathcal{P}$ the set of probability measures that are absolutely continuous with respect to P. For convenience, we again identify random variables that are equal P-almost surely as it was done in chapters 2 and 3.

5.2 Definitions and Time Consistency

Definition 5.2.1 *Assume that for each $t \in \mathbb{T}$ we are given a conditional monetary (concave, coherent) utility functional $\phi_{t,T}$ (on $L^\infty(\mathcal{F}_T)$) with corresponding acceptance set $\mathcal{A}^\phi_{t,T}$. We call the family $(\phi_{t,T})_{t\in\mathbb{T}}$ a dynamic monetary (concave, coherent) utility functional (on $L^\infty(\mathcal{F}_T)$) and $(\mathcal{A}^\phi_{t,T})_{t\in\mathbb{T}}$ the family of corresponding acceptance sets.*

A dynamic monetary (convex, coherent) risk measure (on $L^\infty(\mathcal{F}_T)$) is given by a family $(\rho_{t,T})_{t\in\mathbb{T}}$ of conditional monetary (convex, coherent) risk measures (on $L^\infty(\mathcal{F}_T)$).

Remark 5.2.2 *For a dynamic monetary utility functional $(\phi_{t,T})_{t\in\mathbb{T}}$ we derive from* (**n**) *and* ($\mathcal{F}_T$-**ti**) *of $\phi_{T,T}$ that for all $X \in L^\infty(\mathcal{F}_T)$,*

$$\phi_{T,T}(X) = X, \quad \text{a.s. } P.$$

In particular, $\mathcal{A}^{\phi}_{T,T} = L^{\infty}_{+}(\mathcal{F}_T)$.

For a dynamic monetary (concave, coherent) utility functional $(\phi_{t,T})_{t\in\mathbb{T}}$ we define the mapping $\phi_{\tau,\theta} : L^{\infty}(\mathcal{F}_\theta) \to L^{\infty}(\mathcal{F}_\tau)$,

$$X \mapsto \phi_{\tau,\theta}(X) := \sum_{t\in\mathbb{T}} \phi_{t,T}(X) 1_{\{\tau=t\}}. \tag{5.2.1}$$

The functional $\phi_{\tau,\theta}$ inherits the the properties (**n**), (**m**) and (**sa**) from the corresponding ones of $\phi_{t,T}$, $t \in \mathbb{T}$. To obtain ($\mathcal{F}_\tau$-**ti**), ($\mathcal{F}_\tau$-**c**) and ($\mathcal{F}_\tau$-**ph**) consider the decomposition

$$\begin{aligned}\big\{\lambda 1_{\{\tau=t\}} \in B\big\} &= \big(\{\lambda \in B\} \cap \{\tau = t\}\big) \cup \big(\{0 \in B\} \cap \{\tau \neq t\}\big)\\ &= \big(\{\lambda \in B\} \cap \{\tau = t\} \cap \{\tau \leq t\}\big) \cup \big(\{0 \in B\} \cap \{\tau \neq t\}\big) \in \mathcal{F}_t\end{aligned}$$

valid for all $t \in \mathbb{T}$, for all $\lambda \in L^{\infty}(\mathcal{F}_\tau)$ and for all $B \in \mathcal{B}(\mathbb{R})$. Thus, $\lambda 1_{\{\tau=t\}}$ is $\mathcal{F}_t$-measurable for all $t \in \mathbb{T}$ and for all $\lambda \in L^{\infty}(\mathcal{F}_\tau)$ and the remaining properties are as well inherited from $\phi_{t,T}$, $t \in \mathbb{T}$.

The notion of dynamic consistency we work with in this thesis is given in the following definition.

Definition 5.2.3 *A dynamic monetary (concave, coherent) utility functional $(\phi_{t,T})_{t\in\mathbb{T}}$ (on $L^{\infty}(\mathcal{F}_T)$) is called time-consistent if for all $X, Y \in L^{\infty}(\mathcal{F}_T)$,*

$$\phi_{t+1,T}(X) \geq \phi_{t+1,T}(Y), \text{ a.s. } P, \quad \text{implies} \quad \phi_{t,T}(X) \geq \phi_{t,T}(Y), \text{ a.s. } P,$$

for all $t \in \{0, \ldots, T-1\}$. We call the family $(\phi_{t,t+1})_{t\in\{0,\ldots,T-1\}}$ the one-step transitions of $(\phi_{t,T})_{t\in\mathbb{T}}$, where $\phi_{t,t+1}$ denotes the restriction of $\phi_{t,T}$ to $L^{\infty}(\mathcal{F}_{t+1})$, $t \in \{0, \ldots, T-1\}$.

A dynamic monetary (convex, coherent) risk measure $(\rho_{t,T})_{t\in\mathbb{T}}$ (on $L^{\infty}(\mathcal{F}_T)$) is time-consistent if $(-\rho_{t,T})_{t\in\mathbb{T}}$ is a time-consistent dynamic monetary (concave, coherent) utility functional.

Proposition 5.2.4 *For a dynamic monetary utility functional $(\phi_{t,T})_{t\in\mathbb{T}}$ the following two conditions are equivalent:*

(1) $(\phi_{t,T})_{t\in\mathbb{T}}$ *is time-consistent*

(2) $\phi_{t,T}(X) = \phi_{t,T}\big(\phi_{t+1,T}(X)\big)$, *a.s. P, for all $X \in L^{\infty}(\mathcal{F}_T)$ and $t \in \{0, \ldots, T-1\}$.*

Proof. Fix $t \in \{0, \ldots, T-1\}$.
(1)⇒(2): For $X \in L^{\infty}(\mathcal{F}_T)$ define $Y := \phi_{t+1,T}(X)$. Then $\phi_{t+1,T}(X) = \phi_{t+1,T}(Y)$, a.s. P. Thus, $\phi_{t,T}(X) \leq \phi_{t,T}(Y)$, a.s. P, as well as $\phi_{t,T}(X) \geq \phi_{t,T}(Y)$, a.s. P, and hence,

$$\phi_{t,T}(X) = \phi_{t,T}(Y) = \phi_{t,T}\big(\phi_{t+1,T}(X)\big), \quad \text{a.s. } P.$$

(2)⇒(1): Let $X, Y \in L^{\infty}(\mathcal{F}_T)$ such that $\phi_{t+1,T}(X) \geq \phi_{t+1,T}(Y)$, a.s. P. Then

$$\phi_{t,T}(X) = \phi_{t,T}\big(\phi_{t+1,T}(X)\big) \geq \phi_{t,T}\big(\phi_{t+1,T}(Y)\big) = \phi_{t,T}(Y), \text{ a.s. } P.$$

□

As a consequence of the preceding proposition, a time-consistent dynamic monetary utility functional $(\phi_{t,T})_{t\in\mathbb{T}}$ is already fully determined by its one-step transitions $(\phi_{t,t+1})_{t\in\{0,\ldots,T-1\}}$. Indeed, given the one step transitions of $(\phi_{t,T})_{t\in\mathbb{T}}$ we may recover $\phi_{t,T}$ via

$$\phi_{t,T}(X) = \phi_{t,t+1}\big(\cdots \phi_{T-2,T-1}\big(\phi_{T-1,T}(X)\big)\cdots\big), \quad \text{a.s. } P,$$

for all $X \in L^\infty(\mathcal{F}_T)$ and for all $t \in \{0,\ldots,T-1\}$. For $t = T$ we have $\phi_{T,T}(X) = X$, a.s. P.

Moreover, we may even start with arbitrary conditional monetary utility functionals $\phi_{t,t+1}$ on $L^\infty(\mathcal{F}_{t+1})$ for all $t \in \{0,\ldots,T-1\}$. For $X \in L^\infty(\mathcal{F}_T)$ we then define by backwards induction:

$$\begin{aligned} \psi_{T,T}(X) &:= X \\ \psi_{t,T}(X) &:= \phi_{t,t+1}\big(\psi_{t+1,T}(X)\big) \quad \text{for all } t \in \{0,\ldots,T-1\} \end{aligned} \tag{5.2.2}$$

This yields a time-consistent dynamic monetary utility functional $(\psi_{t,T})_{t\in\mathbb{T}}$.

Time-consistency of a dynamic monetary utility functional $(\phi_{t,T})_{t\in\mathbb{T}}$ on $L^\infty(\mathcal{F}_T)$ in particular captures the following intuition. If a position $X \in L^\infty(\mathcal{F}_T)$ is accepted at date $t+1$, $t \in \{0,\ldots,T-1\}$, then it will be accepted at time t as well. This statement formalizes

$$\phi_{t+1,T}(X) \geq 0, \text{ a.s. } P, \quad \text{implies} \quad \phi_{t,T}(X) \geq 0, \text{ a.s. } P, \tag{5.2.3}$$

for all $X \in L^\infty(\mathcal{F}_T)$ and for all $t \in \{0,\ldots,T-1\}$. (5.2.3) is a consequence of the inequality

$$\phi_{t,T}(X) = \phi_{t,T}\big(\phi_{t+1,T}(X)\big) \geq \phi_{t,T}(0) \geq 0, \quad \text{a.s. } P,$$

which is valid for all $X \in L^\infty(\mathcal{F}_T)$ with $\phi_{t+1,T}(X) \geq 0$, a.s. P, $t \in \{0,\ldots,T-1\}$, provided that $(\phi_{t,T})_{t\in\mathbb{T}}$ is time-consistent. The discussion of subsection 6.2.1 in particular yields the insight that acceptance-consistency in the sense of (5.2.3) is indeed a weaker condition than time-consistency in the sense of definition 5.2.3.

5.3 Duality

For every $t \in \{1,\ldots,T\}$, we introduce the set of one-step transition densities

$$\mathcal{D}_t := \left\{\xi \in L^1_+(\mathcal{F}_t) \mid E[\xi \mid \mathcal{F}_{t-1}] = 1\right\}.$$

Every sequence $(\xi_{t+1},\ldots,\xi_T) \in \mathcal{D}_{t+1} \times \cdots \times \mathcal{D}_T$ induces a P-martingale $(M^\xi_r)_{r\in\mathbb{T}}$ by

$$M^\xi_r := \begin{cases} 1 & \text{for } r \in \{0,\ldots,t\} \\ \xi_{t+1}\cdot\cdots\cdot\xi_r & \text{for } r \in \{t+1,\ldots,T\} \end{cases}$$

and a probability measure Q^ξ in $\mathcal{P}$ with density

$$\frac{dQ^\xi}{dP} = M_T^\xi.$$

Indeed, for $r \in \{0, \ldots, t\}$ we have $E\left[M_{r+1}^\xi \mid \mathcal{F}_r\right] = 1 = M_r^\xi$, a.s.$P$, and for $r \in \{t+1, \ldots, T-1\}$,

$$E\left[M_{r+1}^\xi \mid \mathcal{F}_r\right] = \xi_{t+1} \cdot \cdots \cdot \xi_r \; E\left[\xi_{r+1} \mid \mathcal{F}_r\right] = \xi_{t+1} \cdot \cdots \cdot \xi_r = M_r^\xi, \quad \text{a.s. } P.$$

On the other hand, every probability measure Q in $\mathcal{P}$ induces a non-negative martingale

$$M_t^Q := E\left[\frac{dQ}{dP} \mid \mathcal{F}_t\right], \quad t \in \{0, \ldots, T\},$$

and since for $1 \leq t \leq T$,

$$E\left[M_{t+1}^Q 1_{\{M_t^Q = 0\}}\right] = E\left[E\left[M_{t+1}^Q \mid \mathcal{F}_t\right] 1_{\{M_t^Q = 0\}}\right] = E\left[M_t^Q 1_{\{M_t^Q = 0\}}\right] = 0,$$

we have

$$\left\{M_{t-1}^Q = 0\right\} \subset \left\{M_t^Q = 0\right\} \quad \text{for all } 1 \leq t \leq T. \tag{5.3.4}$$

The inclusion in (5.3.4) is understood in the P-almost sure sense. The sequence

$$\xi_t^Q := \begin{cases} \frac{M_t^Q}{M_{t-1}^Q} & \text{on } \left\{M_{t-1}^Q > 0\right\} \\ 1 & \text{on } \left\{M_{t-1}^Q = 0\right\} \end{cases} \quad \text{for } t = 1, \ldots, T,$$

is an element in $\mathcal{D}_1 \times \cdots \times \mathcal{D}_T$ that induces the measure Q. Indeed, on $\left\{M_{T-1}^Q = 0\right\}$ we have $\xi_1^Q \cdot \cdots \cdot \xi_T^Q = \xi_1^Q \cdot \cdots \cdot \xi_{T-1}^Q = 0 = M_T^Q$, a.s. P, and on $\left\{M_{T-1}^Q > 0\right\}$,

$$\frac{M_1^Q}{1} \frac{M_2^Q}{M_1^Q} \cdot \cdots \cdot \frac{M_T^Q}{M_{T-1}^Q} = M_T^Q, \quad \text{a.s. } P.$$

For all $X \in L^\infty(\mathcal{F}_T)$ and $t \in \{0, \ldots T-1\}$,

$$\begin{aligned} E_Q[X \mid \mathcal{F}_t] &= \frac{E\left[\xi_1^Q \cdot \cdots \cdot \xi_T^Q X \mid \mathcal{F}_t\right]}{E\left[\xi_1^Q \cdot \cdots \cdot \xi_T^Q \mid \mathcal{F}_t\right]} = \frac{\xi_1^Q \cdot \cdots \cdot \xi_t^Q E\left[\xi_{t+1}^Q \cdot \cdots \cdot \xi_T^Q X \mid \mathcal{F}_t\right]}{\xi_1^Q \cdot \cdots \cdot \xi_t^Q E\left[\xi_{t+1}^Q \cdot \cdots \cdot \xi_T^Q \mid \mathcal{F}_t\right]} \\ &= \frac{\xi_1^Q \cdot \cdots \cdot \xi_t^Q E\left[\xi_{t+1}^Q \cdot \cdots \cdot \xi_T^Q X \mid \mathcal{F}_t\right]}{\xi_1^Q \cdot \cdots \cdot \xi_t^Q}, \quad \text{a.s. } P. \end{aligned} \tag{5.3.5}$$

Since by (5.3.4) we have $\left\{\xi_1^Q \cdot \cdots \cdot \xi_t^Q = 0\right\} \subset \left\{\xi_{t+1}^Q \cdot \cdots \cdot \xi_T^Q = 0\right\}$, (5.3.5) reads

$$E\left[\xi_{t+1}^Q \cdot \cdots \cdot \xi_T^Q X \mid \mathcal{F}_t\right].$$

Thus, we may and will work with the convention

$$E_Q[X \mid \mathcal{F}_t] := E[\xi_{t+1}^Q \cdot \dots \cdot \xi_T^Q X \mid \mathcal{F}_t], \quad X \in L^\infty(\mathcal{F}_T), \quad t \in \{0, \dots T-1\}, \tag{5.3.6}$$

and $E[\xi_{t+1}^Q \cdot \dots \cdot \xi_T^Q X \mid \mathcal{F}_t]$ is a version of $E_Q[X \mid \mathcal{F}_t]$ that is defined up to P-almost sure equality, whereas $E_Q[X \mid \mathcal{F}_t]$ is only defined up to Q-almost sure equality.

The conditional expectation of a random variable X from Ω to $[0, \infty]$ is, as usual, understood as

$$E[X \mid \mathcal{F}_t] := \lim_{n \to \infty} E[X \wedge n \mid \mathcal{F}_t]. \tag{5.3.7}$$

Definition 5.3.1 *Let $t \in \mathbb{T}$. We call a conditional monetary utility functional $\phi_{t,T}$ (on $L^\infty(\mathcal{F}_T)$) continuous from above if*

$$\phi_{t,T}(X_n) \to \phi_{t,T}(X), \quad a.s.\ P,$$

for every sequence $(X_n)_{n \in \mathbb{N}}$ in $L^\infty(\mathcal{F}_T)$ that decreases P-almost surely to $X \in L^\infty(\mathcal{F}_T)$.

A dynamic monetary utility functional $(\phi_{t,T})_{t \in \mathbb{T}}$ is called continuous from above if all $\phi_{t,T}$, $t \in \mathbb{T}$, are so.

Definition 5.3.2 *Let $0 \le t < s \le T$. For a conditional concave utility functional $\phi_{t,s}$ (on $L^\infty(\mathcal{F}_s)$) that is continuous from above we define for all $Q \in \mathcal{P}$,*

$$\varphi_{t,s}^\phi(Q) := \operatorname*{ess.sup}_{X \in L^\infty(\mathcal{F}_s)} \left\{ E_Q[-X \mid \mathcal{F}_t] + \phi_{t,s}(X) \right\}$$

and call $\varphi_{t,s}^\phi$ the conditional penalty function of $\phi_{t,s}$.

For a time-consistent dynamic concave utility functional $(\phi_{t,T})_{t \in \mathbb{T}}$ we call the family $(\varphi_{t,t+1}^\phi)_{t \in \{0,\dots,T-1\}}$ the dynamic penalty function of $(\phi_{t,T})_{t \in \mathbb{T}}$.

Lemma 5.3.3 *Let $0 \le t < s \le T$. A conditional concave utility functional $\phi_{t,s}$ (on $L^\infty(\mathcal{F}_s)$) that is continuous from above admits a representation*

$$\phi_{t,s}(X) = \operatorname*{ess.inf}_{Q \in \mathcal{P}} \left\{ E_Q[X \mid \mathcal{F}_t] + \varphi_{t,s}^\phi(Q) \right\}, \quad a.s.\ P, \tag{5.3.8}$$

in terms of its conditional penalty function $\varphi_{t,s}^\phi$.

Proof. Let us consider the conditional convex risk measure $\rho_{t,s} := -\phi_{t,s}$. The statement in (5.3.8) now reads

$$\begin{aligned} \rho_{t,s}(X) &= -\operatorname*{ess.inf}_{Q \in \mathcal{P}} \left\{ E_Q[X \mid \mathcal{F}_t] + \varphi_{t,s}^\phi(Q) \right\} \\ &= \operatorname*{ess.sup}_{Q \in \mathcal{P}} \left\{ E_Q[-X \mid \mathcal{F}_t] - \varphi_{t,s}^\phi(Q) \right\}, \quad \text{a.s. } P, \end{aligned} \tag{5.3.9}$$

and the conditional penalty function $\varphi_{t,s}^\phi$ can be written in the form

$$\varphi_{t,s}^\phi(Q) = \operatorname*{ess.sup}_{X \in L^\infty(\mathcal{F}_s)} \left\{ E_Q[-X \mid \mathcal{F}_t] - \rho_{t,s}(X) \right\}, \quad \text{a.s. } P.$$

Now the statement in (5.3.9) can be viewed as a generalization of the static case duality result, a proof of which is given in Detlefsen and Scandolo [13], theorem 3.2. □

Lemma 5.3.4 *Let $t \in \{0, \ldots, T-1\}$. The conditional penalty function $\varphi^{\phi}_{t,t+1}$ of a conditional concave utility functional $\phi_{t,t+1}$ (on $L^{\infty}(\mathcal{F}_{t+1})$) that is continuous from above satisfies*

$$\varphi^{\phi}_{t,t+1}(1_A\xi + 1_{A^c}\xi') = 1_A\varphi^{\phi}_{t,t+1}(\xi) + 1_{A^c}\varphi^{\phi}_{t,t+1}(\xi'), \quad a.s.\ P,$$

for all $\xi, \xi' \in \mathcal{D}_{t+1}$ and $A \in \mathcal{F}_t$, with the convention

$$\varphi^{\phi}_{t,t+1}\left(\frac{dQ}{dP}\right) := \varphi^{\phi}_{t,t+1}(Q).$$

Proof. By convention we have

$$\varphi^{\phi}_{t,t+1}(1_A\xi + 1_{A^c}\xi') = \operatorname*{ess.sup}_{X \in L^{\infty}(\mathcal{F}_{t+1})} \big\{E_Q[(-X) \mid \mathcal{F}_t] + \phi_{t,s}(X)\big\},$$

where Q is given by $\frac{dQ}{dP} := 1_A\xi + 1_{A^c}\xi'$. Note that $E[1_A\xi + 1_{A^c}\xi'] = E\big[E[1_A\xi + 1_{A^c}\xi' \mid \mathcal{F}_t]\big] = 1$, a.s. P. Thus,

$$\varphi^{\phi}_{t,t+1}(1_A\xi + 1_{A^c}\xi') = \operatorname*{ess.sup}_{X \in L^{\infty}(\mathcal{F}_{t+1})} \big\{E[(1_A\xi + 1_{A^c}\xi')(-X) \mid \mathcal{F}_t] + \phi_{t,s}(X)\big\}, \qquad (5.3.10)$$

a.s. P, where we have in mind that $E[1_A\xi + 1_{A^c}\xi' \mid \mathcal{F}_t] = 1$, a.s. P. Since $A \in \mathcal{F}_t$ we may rewrite (5.3.10) in the form

$$\begin{aligned}
& \operatorname*{ess.sup}_{X \in L^{\infty}(\mathcal{F}_{t+1})} \Big\{1_A\big(E[\xi(-X) \mid \mathcal{F}_t] + \phi_{t,s}(X)\big) + 1_{A^c}\big(E[\xi'(-X) \mid \mathcal{F}_t] + \phi_{t,s}(X)\big)\Big\} \\
= & \operatorname*{ess.sup}_{X,Y \in L^{\infty}(\mathcal{F}_{t+1})} \Big\{1_A\big(E[\xi(-X) \mid \mathcal{F}_t] + \phi_{t,s}(X)\big) + 1_{A^c}\big(E[\xi'(-Y) \mid \mathcal{F}_t] + \phi_{t,s}(Y)\big)\Big\} \\
= & \ 1_A \operatorname*{ess.sup}_{X \in L^{\infty}(\mathcal{F}_{t+1})} \big\{E[\xi(-X) \mid \mathcal{F}_t] + \phi_{t,s}(X)\big\} \\
& \quad + 1_{A^c} \operatorname*{ess.sup}_{X \in L^{\infty}(\mathcal{F}_{t+1})} \big\{E[\xi'(-X) \mid \mathcal{F}_t] + \phi_{t,s}(X)\big\} \\
= & \ 1_A\varphi^{\phi}_{t,t+1}(\xi) + 1_{A^c}\varphi^{\phi}_{t,t+1}(\xi'), \quad \text{a.s. } P.
\end{aligned}$$

□

Theorem 5.3.5 *Let $(\phi_{t,T})_{t \in \mathbb{T}}$ be a time-consistent dynamic concave utility functional that is continuous from above. Then*

$$\phi_{t,s}(X) = \operatorname*{ess.inf}_{Q \in \mathcal{P}} E_Q\left[X + \sum_{j=t+1}^{s} \varphi^{\phi}_{j-1,j}(Q) \mid \mathcal{F}_t\right], \quad a.s.\ P,$$

for all $0 \le t < s \le T$ and $X \in L^{\infty}(\mathcal{F}_s)$.

Proof. Let $0 \le t < s \le T$ and define

$$V_{t,s}(X) = \underset{Q\in\mathcal{P}}{\text{ess.inf}}\ E_Q\left[X + \sum_{j=t+1}^{s} \varphi^{\phi}_{j-1,j}(Q) \mid \mathcal{F}_t\right]$$

for all $X \in L^\infty(\mathcal{F}_s)$. The proof is by induction over s. If $s = t+1$, then we obtain directly from (5.3.8) that

$$\phi_{t,t+1}(X) = \underset{Q\in\mathcal{P}}{\text{ess.inf}}\ E_Q[X + \varphi_{t,t+1}(Q) \mid \mathcal{F}_t] = V_{t,t+1}(X), \quad \text{a.s. } P,$$

for all $X \in L^\infty(\mathcal{F}_{t+1})$. Now, assume $s \ge t+2$ and $\phi_{t,s}(Y) = V_{t,s}(Y)$, a.s. P, for all $Y \in L^\infty(\mathcal{F}_{s-1})$. If $X \in L^\infty(\mathcal{F}_s)$, then $\phi_{s-1,s}(X) \in L^\infty(\mathcal{F}_{s-1})$, and we obtain by time consistency $\phi_{t,s}(X) = \phi_{t,s}\big(\phi_{s-1,s}(X)\big) = V_{t,s}\big(\phi_{s-1,s}(X)\big)$. Further,

$$\begin{aligned}
& V_{t,s}\big(\phi_{s-1,s}(X)\big) = \underset{Q\in\mathcal{P}}{\text{ess.inf}}\ E_Q\left[\phi_{s-1,s}(X) + \sum_{j=t+1}^{s} \varphi^{\phi}_{j-1,j}(Q) \mid \mathcal{F}_t\right] \\
= & \underset{(\xi_{t+1},\dots,\xi_T)\in\mathcal{D}_{t+1}\times\cdots\times\mathcal{D}_T}{\text{ess.inf}} E\left[(\xi_{t+1}\cdot\dots\cdot\xi_T)\left(\phi_{s-1,s}(X) + \sum_{j=t+1}^{s-1}\varphi_{j-1,j}(\xi_j)\right) \mid \mathcal{F}_t\right] \\
= & \underset{(\xi_{t+1},\dots,\xi_{s-1})\in\mathcal{D}_{t+1}\times\cdots\times\mathcal{D}_{s-1}}{\text{ess.inf}} E\Bigg[(\xi_{t+1}\cdot\dots\cdot\xi_{s-1}) \\
& \left(\underset{\xi_s\in\mathcal{D}_s}{\text{ess.inf}}\ E\left[\xi_s\big(X + \varphi_{s-1,s}(\xi_s)\big) \mid \mathcal{F}_{s-1}\right] + \sum_{j=t+1}^{s-1}\varphi_{j-1,j}(\xi_j)\right) \mid \mathcal{F}_t\Bigg],
\end{aligned} \tag{5.3.11}$$

a.s. P. The family

$$E[\xi_s X \mid \mathcal{F}_{s-1}] + \varphi_{s-1,s}(\xi_s), \quad \xi_s \in \mathcal{D}_s$$

is directed downwards. Indeed, take $\xi_s, \xi_s' \in \mathcal{D}_s$ and $A \in \mathcal{F}_{s-1}$ such that

$$\begin{aligned}
& E[\xi_s X \mid \mathcal{F}_{s-1}] + \varphi_{s-1,s}(\xi_s) \ \wedge\ E[\xi_s' X \mid \mathcal{F}_{s-1}] + \varphi_{s-1,s}(\xi_s') \\
= & 1_A\Big(E[\xi_s X \mid \mathcal{F}_{s-1}] + \varphi_{s-1,s}(\xi_s)\Big) + 1_{A^c}\Big(E[\xi_s' X \mid \mathcal{F}_{s-1}] + \varphi_{s-1,s}(\xi_s')\Big) \\
= & E[(1_A\xi_s + 1_{A^c})X \mid \mathcal{F}_{s-1}] + 1_A\varphi_{s-1,s}(\xi_s) + 1_{A^c}\varphi_{s-1,s}(\xi_s'), \quad \text{a.s. } P.
\end{aligned}$$

Lemma 5.3.4 now yields the assertion. But from this we derive that there exists a decreasing sequence

$$(\varphi^n)_{n\in\mathbb{N}} \ \subset\ \big\{E[\xi_s X \mid \mathcal{F}_{s-1}] + \varphi_{s-1,s}(\xi_s) \mid \xi_s \in \mathcal{D}_s\big\}$$

such that

$$\underset{\xi_s\in\mathcal{D}_s}{\text{ess.inf}}\ \big(E[\xi_s X \mid \mathcal{F}_{s-1}] + \varphi_{s-1,s}(\xi_s)\big) = \lim_{n\to\infty}\ \varphi^n, \quad \text{a.s. } P.$$

Therefore, since

$$\begin{aligned}
E\left[\xi_s\big(X + \varphi_{s-1,s}(\xi_s)\big) \mid \mathcal{F}_{s-1}\right] &= E\left[\xi_s X \mid \mathcal{F}_{s-1}\right] + E\left[\xi_s\varphi_{s-1,s}(\xi_s) \mid \mathcal{F}_{s-1}\right] \\
&= E\left[\xi_s X \mid \mathcal{F}_{s-1}\right] + \varphi_{s-1,s}(\xi_s), \quad \text{a.s. } P,
\end{aligned}$$

we can take, by monotone convergence, the essential infimum in (5.3.11) outside the conditional expectation and arrive at

$$
\begin{aligned}
& \operatorname*{ess.inf}_{(\xi_{t+1},\dots,\xi_{s-1})\in\mathcal{D}_{t+1}\times\dots\times\mathcal{D}_{s-1}} \ \operatorname*{ess.inf}_{\xi_s\in\mathcal{D}_s} \\
& E\left[(\xi_{t+1}\cdot\dots\cdot\xi_{s-1})\left(E[\xi_s\big(X+\varphi_{s-1,s}(\xi_s)\big)\mid\mathcal{F}_{s-1}]+\sum_{j=t+1}^{s-1}\varphi_{j-1,j}(\xi_j)\right)\mid\mathcal{F}_t\right] \\
= & \operatorname*{ess.inf}_{(\xi_{t+1},\dots,\xi_s)\in\mathcal{D}_{t+1}\times\dots\times\mathcal{D}_s} E\Big[(\xi_{t+1}\cdot\dots\cdot\xi_{s-1})\Big(E[\xi_s\big(X+\varphi_{s-1,s}(\xi_s)\big)\mid\mathcal{F}_{s-1}] \\
& \qquad +E\left[\xi_s\sum_{j=t+1}^{s-1}\varphi_{j-1,j}(\xi_j)\mid\mathcal{F}_{s-1}\right]\Big)\mid\mathcal{F}_t\Big] \\
= & \operatorname*{ess.inf}_{(\xi_{t+1},\dots,\xi_s)\in\mathcal{D}_{t+1}\times\dots\times\mathcal{D}_s} \\
& E\left[(\xi_{t+1}\cdot\dots\cdot\xi_{s-1})E\left[\xi_s\left(X+\varphi_{s-1,s}(\xi_s)+\sum_{j=t+1}^{s-1}\varphi_{j-1,j}(\xi_j)\right)\mid\mathcal{F}_{s-1}\right]\mid\mathcal{F}_t\right] \\
= & \operatorname*{ess.inf}_{(\xi_{t+1},\dots,\xi_s)\in\mathcal{D}_{t+1}\times\dots\times\mathcal{D}_s} E\left[(\xi_{t+1}\cdot\dots\cdot\xi_s)\left(X+\sum_{j=t+1}^{s}\varphi_{j-1}(\xi_j)\right)\mid\mathcal{F}_t\right]=V_{t,s}(X),
\end{aligned}
$$

a.s. P, which concludes the proof. □

Corollary 5.3.6 *Let $(\phi_{t,T})_{t\in\mathbb{T}}$ be a time-consistent dynamic concave utility functional that is continuous from above. Then*

$$\phi_{t,s}(X)=\operatorname*{ess.inf}_{Q\in\mathcal{P}} E_Q\left[X+\sum_{j=1}^{s}\varphi^{\phi}_{j-1,j}(Q)\mid\mathcal{F}_t\right],\quad a.s.\ P,$$

for all $0\le t<s\le T$ and $X\in L^\infty(\mathcal{F}_s)$.

Proof. By theorem 5.3.5 it suffices to show that

$$\operatorname*{ess.inf}_{Q\in\mathcal{P}} E_Q\left[X+\sum_{j=1}^{s}\varphi^{\phi}_{j-1,j}(Q)\mid\mathcal{F}_t\right] = \operatorname*{ess.inf}_{Q\in\mathcal{P}} E_Q\left[X+\sum_{j=t+1}^{s}\varphi^{\phi}_{j-1,j}(Q)\mid\mathcal{F}_t\right],\quad \text{a.s. } P,$$

for all $0\le t<s\le T$ and $X\in L^\infty(\mathcal{F}_s)$. To this end, observe that normalization of $\phi_{j-1,j}$, $j\in\{1,\dots,T\}$, implies

$$0=\phi_{j-1,j}(0)=\operatorname*{ess.inf}_{Q\in\mathcal{P}}\left\{E_Q[0\mid\mathcal{F}_{j-1}]+\varphi^{\phi}_{j-1,j}(Q)\right\}=\operatorname*{ess.inf}_{Q\in\mathcal{P}}\ \varphi^{\phi}_{j-1,j}(Q),\quad \text{a.s. } P,$$

where the second equality follows from the fact that $\varphi^{\phi}_{j-1,j}$ represents $\phi_{j-1,j}$ as in lemma 5.3.3. Moreover, for all $Q \in \mathcal{P}$ we have

$$\begin{aligned}
\varphi^{\phi}_{j-1,j}(Q) &= \operatorname*{ess.sup}_{X \in L^{\infty}(\mathcal{F}_j)} \{E_Q[-X \mid \mathcal{F}_{j-1}] + \phi_{j-1,j}(X)\} \\
&= \operatorname*{ess.sup}_{X \in L^{\infty}(\mathcal{F}_j)} \left\{E\left[\xi^Q_j \cdot \cdots \cdot \xi^Q_T(-X) \mid \mathcal{F}_{j-1}\right] + \phi_{j-1,j}(X)\right\} \\
&= \operatorname*{ess.sup}_{X \in L^{\infty}(\mathcal{F}_j)} \left\{E\left[\xi^Q_j(-X) \mid \mathcal{F}_{j-1}\right] + \phi_{j-1,j}(X)\right\}, \quad \text{a.s. } P,
\end{aligned}$$

and hence,

$$\operatorname*{ess.inf}_{\xi \in \mathcal{D}_j} \varphi^{\phi}_{j-1,j}(\xi) = \operatorname*{ess.inf}_{Q \in \mathcal{P}} \varphi^{\phi}_{j-1,j}(Q) = 0, \quad \text{a.s. } P, \tag{5.3.12}$$

for all $j \in \{1, \dots, T\}$. Now, let us fix $0 \le t < s \le T$, $X \in L^{\infty}(\mathcal{F}_s)$. We have

$$\begin{aligned}
&\operatorname*{ess.inf}_{Q \in \mathcal{P}} E_Q\left[X + \sum_{j=1}^{s} \varphi^{\phi}_{j-1,j}(Q) \mid \mathcal{F}_t\right] \\
=& \operatorname*{ess.inf}_{(\xi_1,\dots,\xi_T) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_T} E\left[\xi_{t+1} \cdot \cdots \cdot \xi_T \left(X + \sum_{j=1}^{s} \varphi^{\phi}_{j-1,j}(\xi_j)\right) \mid \mathcal{F}_t\right] \\
=& \operatorname*{ess.inf}_{(\xi_1,\dots,\xi_T) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_T} E\left[\xi_{t+1} \cdot \cdots \cdot \xi_T \left(X + \sum_{j=1}^{t} \varphi^{\phi}_{j-1,j}(\xi_j) + \sum_{j=t+1}^{s} \varphi^{\phi}_{j-1,j}(\xi_j)\right) \mid \mathcal{F}_t\right] \\
=& \operatorname*{ess.inf}_{(\xi_1,\dots,\xi_T) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_T} \left\{E\left[\xi_{t+1} \cdot \cdots \cdot \xi_T \sum_{j=1}^{t} \varphi^{\phi}_{j-1,j}(\xi_j) \mid \mathcal{F}_t\right]\right. \\
&\left. \qquad + E\left[\xi_{t+1} \cdot \cdots \cdot \xi_T \left(X + \sum_{j=t+1}^{s} \varphi^{\phi}_{j-1,j}(\xi_j)\right) \mid \mathcal{F}_t\right]\right\} \\
=& \operatorname*{ess.inf}_{(\xi_1,\dots,\xi_T) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_T} \qquad\qquad\qquad\qquad (5.3.13) \\
& \qquad \left\{\sum_{j=1}^{t} \varphi^{\phi}_{j-1,j}(\xi_j) + E\left[\xi_{t+1} \cdot \cdots \cdot \xi_T \left(X + \sum_{j=t+1}^{s} \varphi^{\phi}_{j-1,j}(\xi_j)\right) \mid \mathcal{F}_t\right]\right\} \\
=& \operatorname*{ess.inf}_{(\xi_1,\dots,\xi_t) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_t} \sum_{j=1}^{t} \varphi^{\phi}_{j-1,j}(\xi_j) \qquad\qquad\qquad\qquad (5.3.14) \\
& + \operatorname*{ess.inf}_{(\xi_{t+1},\dots,\xi_T) \in \mathcal{D}_{t+1} \times \cdots \times \mathcal{D}_T} E\left[\xi_{t+1} \cdot \cdots \cdot \xi_T \left(X + \sum_{j=t+1}^{s} \varphi^{\phi}_{j-1,j}(\xi_j)\right) \mid \mathcal{F}_t\right], \quad \text{a.s. } P. \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.3.15)
\end{aligned}$$

As a consequence of (5.3.12) the essential infimum in (5.3.14) equals zero and hence, (5.3.15) reduces to

$$
\begin{aligned}
& \underset{(\xi_{t+1},\dots,\xi_T)\in\mathcal{D}_{t+1}\times\cdots\times\mathcal{D}_T}{\text{ess.inf}} E\left[\xi_{t+1}\cdot\dots\cdot\xi_T\left(X+\sum_{j=t+1}^{s}\varphi^{\phi}_{j-1,j}(\xi_j)\right)\mid\mathcal{F}_t\right] \\
= \; & \underset{Q\in\mathcal{P}}{\text{ess.inf}}\; E_Q\left[X+\sum_{j=t+1}^{s}\varphi^{\phi}_{j-1,j}(Q)\mid\mathcal{F}_t\right], \quad \text{a.s. } P,
\end{aligned}
$$

which concludes the proof. □

5.4 Concatenation

Consider the densities q and s of two probability measures Q and S in $\mathcal{P}$. For $t\in\{1,\dots,T-1\}$ we define

$$
q\otimes_t s := \begin{cases} q & \text{on} \quad \{E[s\mid\mathcal{F}_t]=0\} \\ E[q\mid\mathcal{F}_t]\frac{s}{E[s\mid\mathcal{F}_t]} & \text{on} \quad \{E[s\mid\mathcal{F}_t]>0\} \end{cases}. \tag{5.4.16}
$$

Since

$$
\begin{aligned}
E[q\otimes_t s] &= E\left[E[q\mid\mathcal{F}_t]\frac{s}{E[s\mid\mathcal{F}_t]}1_{\{E[s|\mathcal{F}_t]>0\}}+q1_{\{E[s|\mathcal{F}_t]=0\}}\right] \\
&= E\left[E\left[E[q\mid\mathcal{F}_t]\frac{s}{E[s\mid\mathcal{F}_t]}1_{\{E[s|\mathcal{F}_t]>0\}}\mid\mathcal{F}_t\right]+E\left[q1_{\{E[s|\mathcal{F}_t]=0\}}\mid\mathcal{F}_t\right]\right] \\
&= E\left[E[q\mid\mathcal{F}_t]\frac{E[s\mid\mathcal{F}_t]}{E[s\mid\mathcal{F}_t]}1_{\{E[s|\mathcal{F}_t]>0\}}+E\left[q1_{\{E[s|\mathcal{F}_t]=0\}}\mid\mathcal{F}_t\right]\right] \\
&= E\big[E[q\mid\mathcal{F}_t]\big]=1
\end{aligned}
$$

and $q\otimes_t s\geq 0$, a.s. P, $q\otimes_t s$ induces a probability measure in $\mathcal{P}$. We denote it $Q\otimes_t S$ and call it concatenation of the measures Q and S.

For all of this section, we assume that we are given a dynamic concave utility functional $(\phi_{t,T})_{t\in\mathbb{T}}$ that is continuous from above. By lemma 5.3.3, we have for all $0\leq t<s\leq T$

$$
\phi_{t,s}(X)=\underset{Q\in\mathcal{P}}{\text{ess.inf}}\left\{E_Q[X\mid\mathcal{F}_t]+\varphi^{\phi}_{t,s}(Q)\right\}, \quad \text{a.s. } P, \tag{5.4.17}
$$

for all $X\in L^\infty(\mathcal{F}_s)$. The one-step transitions $(\phi_{t,t+1})_{t\in\{0,\dots,T-1\}}$ of $(\phi_{t,T})_{t\in\mathbb{T}}$ are concave and continuous from above. Thus, the time-consistent dynamic monetary utility functional $(\psi_{t,T})_{t\in\mathbb{T}}$ given as in (5.2.2) is concave and continuous from above in turn. By theorem

5.3.5, $(\psi_{t,T})_{t\in\mathbb{T}}$ admits for all $X \in \mathcal{F}_T$ a representation

$$\begin{aligned}\psi_{t,T}(X) &= \operatorname*{ess.inf}_{Q\in\mathcal{P}} E_Q\left[X + \sum_{j=t+1}^{T} \varphi^{\psi}_{j-1,j}(Q) \mid \mathcal{F}_t\right] \\ &= \operatorname*{ess.inf}_{Q\in\mathcal{P}} E_Q\left[X + \sum_{j=t+1}^{T} \varphi^{\phi}_{j-1,j}(Q) \mid \mathcal{F}_t\right], \quad \text{a.s. } P, \end{aligned} \tag{5.4.18}$$

in terms of the dynamic penalty function $(\varphi^{\phi}_{t,t+1})_{t\in\{0,\dots,T-1\}}$ of $(\phi_{t,T})_{t\in\mathbb{T}}$. Note that by construction, the dynamic penalty functions $(\varphi^{\psi}_{t,t+1})_{t\in\{0,\dots,T-1\}}$ and $(\varphi^{\phi}_{t,t+1})_{t\in\{0,\dots,T-1\}}$ coincide P-almost surely.

Let us assume that for all $t \in \{0,\dots,T-1\}$ and $X \in L^\infty(\mathcal{F}_T)$ there exists a probability measure $S_t = S_t(\psi_{t+1,T}(X)) \in \mathcal{P}$ such that the essential infimum in (5.4.17) is attained for the one-step transitions of $(\phi_{t,T})_{t\in\mathbb{T}}$, i.e.

$$\phi_{t,t+1}(\psi_{t+1,T}(X)) = E_{S_t}[\psi_{t+1,T}(X) \mid \mathcal{F}_t] + \varphi^{\phi}_{t,t+1}(S_t), \quad \text{a.s. } P, \tag{5.4.19}$$

for all $t \in \{0,\dots,T-1\}$. From this we derive for all $t \in \{0,\dots,T-1\}$,

$$\psi_{t,t+1}(\psi_{t+1,T}(X)) = \phi_{t,t+1}(\psi_{t+1,T}(X)) = E_{S_t}[\psi_{t+1,T}(X) \mid \mathcal{F}_t] + \varphi^{\phi}_{t,t+1}(S_t), \quad \text{a.s. } P. \tag{5.4.20}$$

For $t \in \{0,\dots,T-1\}$ we will show that, under a technical assumption, the essential infimum in (5.4.18) is attained by the concatenation of the probability measures $(S_s)_{s\in\{t,\dots,T-1\}}$ in (5.4.20). To this end, we set $s_t = \frac{dS_t}{dP}$ for all $t \in \{0,\dots,T-1\}$ and denote the induced non-negative martingales

$$M^{S_t}_s := E[s_t \mid \mathcal{F}_s], \quad s \in \mathbb{T},$$

for all $t \in \{0,\dots,T-1\}$. As in section 5.3 the corresponding sequences

$$\xi^{S_t}_s := \begin{cases} \frac{M^{S_t}_s}{M^{S_t}_{s-1}} & \text{on } \left\{M^{S_t}_{s-1} > 0\right\} \\ 1 & \text{on } \left\{M^{S_t}_{s-1} = 0\right\} \end{cases}, \quad s \in \{1,\dots,T\},$$

are elements in $\mathcal{D}_1 \times \cdots \times \mathcal{D}_T$ for all $t \in \{0,\dots,T-1\}$.

Remark 5.4.1 *Take $t \in \{0,\dots,T-1\}$ and assume (this is the above mentioned technical assumption) that*

$$M^{S_s}_s = E[s_s \mid \mathcal{F}_s] > 0, \quad \textit{a.s. } P, \textit{ for all } s \in \{t,\dots,T-1\}. \tag{5.4.21}$$

Then, by the definition in (5.4.16), *the density*

$$s_t \otimes_{t+1} \cdots \otimes_{T-1} s_{T-1}$$

is of the form

$$\begin{aligned} & E[s_t \mid \mathcal{F}_{t+1}] \frac{E[s_{t+1} \mid \mathcal{F}_{t+2}]}{E[s_{t+1} \mid \mathcal{F}_{t+1}]} \cdot \dots \cdot \frac{E[s_{T-2} \mid \mathcal{F}_{T-1}]}{E[s_{T-2} \mid \mathcal{F}_{T-2}]} \frac{s_{T-1}}{E[s_{T-1} \mid \mathcal{F}_{T-1}]} \\ = \;& E[s_t \mid \mathcal{F}_{t+1}] \; \xi_{t+2}^{S_{t+1}} \cdot \dots \cdot \xi_{T-1}^{S_{T-2}} \xi_T^{S_{T-1}}, \quad a.s.\ P. \end{aligned} \tag{5.4.22}$$

Thus,

$$\begin{aligned} E\Big[s_t \otimes_{t+1} \cdots \otimes_{T-1} s_{T-1} \mid \mathcal{F}_t\Big] &= E\left[E[s_t \mid \mathcal{F}_{t+1}] \; \xi_{t+2}^{S_{t+1}} \cdot \dots \cdot \xi_{T-1}^{S_{T-2}} \xi_T^{S_{T-1}} \mid \mathcal{F}_t\right] \\ &= E\left[E[s_t \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t\right] = E[s_t \mid \mathcal{F}_t], \quad a.s.\ P, \end{aligned}$$

where we have in mind that $\xi_{s+1}^{S_s}$ is $\mathcal{F}_{s+1}$-measurable and that $E[\xi_{s+1}^{S_s} \mid \mathcal{F}_s] = 1$, a.s. P, for all $s \in \{t, \dots, T-1\}$. From this we derive

$$\begin{aligned} E_{S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}}\Big[Y \mid \mathcal{F}_t\Big] &= \frac{E\left[E[s_t \mid \mathcal{F}_{t+1}] \; \xi_{t+2}^{S_{t+1}} \cdot \dots \cdot \xi_{T-1}^{S_{T-2}} \xi_T^{S_{T-1}} \; Y \mid \mathcal{F}_t\right]}{E[s_t \mid \mathcal{F}_t]} \\ &= E\left[\xi_{t+1}^{S_t} \xi_{t+2}^{S_{t+1}} \cdot \dots \cdot \xi_{T-1}^{S_{T-2}} \xi_T^{S_{T-1}} \; Y \mid \mathcal{F}_t\right], \end{aligned} \tag{5.4.23}$$

a.s. P, for all $Y \in L^1(\mathcal{F}_T)$.

Lemma 5.4.2 *Let $t \le s \le T-1$. Given that the assumption in* (5.4.21) *holds, we have*

$$\varphi_{s,s+1}^{\phi}(S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}) = \varphi_{s,s+1}^{\phi}(S_s), \quad a.s.\ P,$$

on the set $\left\{\xi_{t+1}^{S_t} \cdot \dots \cdot \xi_s^{S_{s-1}} > 0\right\} \in \mathcal{F}_s$.

Proof. By the assumption in (5.4.21) we have

$$\left\{\xi_{t+1}^{S_t} \cdot \dots \cdot \xi_s^{S_{s-1}} > 0\right\} = \left\{E[s_t \mid \mathcal{F}_{t+1}] \; \xi_{t+2}^{S_{t+1}} \cdot \dots \cdot \xi_s^{S_{s-1}} > 0\right\} \tag{5.4.24}$$

up to a null-set. By the definition of the dynamic penalty function it suffices to show that for all $Y \in L^\infty(\mathcal{F}_{s+1})$,

$$E_{S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}}[Y \mid \mathcal{F}_s] = E_{S_s}[Y \mid \mathcal{F}_s], \quad \text{a.s. } P.$$

To this end, let $Y \in L^\infty(\mathcal{F}_{s+1})$. We have

$$\begin{aligned} E_{S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}}[Y \mid \mathcal{F}_s] &= \frac{E\left[E[s_t \mid \mathcal{F}_{t+1}] \cdot \dots \cdot \xi_s^{S_{s-1}} \xi_{s+1}^{S_s} \cdot \dots \cdot \xi_T^{S_{T-1}} \; Y \mid \mathcal{F}_s\right]}{E\left[E[s_t \mid \mathcal{F}_{t+1}] \cdot \dots \cdot \xi_s^{S_{s-1}} \xi_{s+1}^{S_s} \cdot \dots \cdot \xi_T^{S_{T-1}} \mid \mathcal{F}_s\right]} \\ &= \frac{E[s_t \mid \mathcal{F}_{t+1}] \cdot \dots \cdot \xi_s^{S_{s-1}} E\left[\xi_{s+1}^{S_s} \cdot \dots \cdot \xi_T^{S_{T-1}} \; Y \mid \mathcal{F}_s\right]}{E[s_t \mid \mathcal{F}_{t+1}] \cdot \dots \cdot \xi_s^{S_{s-1}} E\left[\xi_{s+1}^{S_s} \cdot \dots \cdot \xi_T^{S_{T-1}} \mid \mathcal{F}_s\right]}, \end{aligned} \tag{5.4.25}$$

a.s. P, where the first equality follows from (5.4.22). In view of (5.4.24), (5.4.25) reads

$$\frac{E\Big[\xi_{s+1}^{S_s}\cdot\dots\cdot\xi_T^{S_{T-1}}\ Y\mid\mathcal{F}_s\Big]}{E\Big[\xi_{s+1}^{S_s}\cdot\dots\cdot\xi_T^{S_{T-1}}\mid\mathcal{F}_s\Big]}=E\Big[\xi_{s+1}^{S_s}\ Y\mid\mathcal{F}_s\Big]=E_{S_s}[Y\mid\mathcal{F}_s],\quad\text{a.s. } P,$$

where the first equality follows from the fact that Y is $\mathcal{F}_{s+1}$-measurable and that $E[\xi_{r+1}^{S_r}\mid\mathcal{F}_r]=1$, a.s. P, for all $r\in\{s,\dots,T-1\}$. □

Theorem 5.4.3 *Let $t\in\{0,\dots,T-1\}$. Given that the assumption in* (5.4.21) *holds, we have*

$$\begin{aligned}&\operatorname*{ess.inf}_{Q\in\mathcal{P}}\ E_Q\left[X+\sum_{j=t+1}^{T}\varphi_{j-1,j}^{\phi}(Q)\mid\mathcal{F}_t\right]\\ =\ &E_{S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1}}\left[X+\sum_{j=t+1}^{T}\varphi_{j-1,j}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\mid\mathcal{F}_t\right],\quad a.s.\ P,\end{aligned}$$

for all $X\in L^\infty(\mathcal{F}_T)$, where the $S_s=S_s\big(\psi_{s+1,T}(X)\big)\in\mathcal{P}$ are given as in (5.4.19) *for all $s\in\{t,\dots,T-1\}$.*

Proof. Take $X\in L^\infty(\mathcal{F}_T)$ and let us compute:

$$\begin{aligned}&E_{S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1}}\left[X+\sum_{j=t+1}^{T}\varphi_{j-1,j}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\mid\mathcal{F}_t\right]\\ =\ &E\left[\xi_{t+1}^{S_t}\cdot\dots\cdot\xi_{T-1}^{S_{T-2}}\xi_T^{S_{T-1}}\left(X+\sum_{j=t+1}^{T}\varphi_{j-1,j}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\right)\mid\mathcal{F}_t\right]\\ =\ &E\Bigg[\xi_{t+1}^{S_t}\cdot\dots\cdot\xi_{T-1}^{S_{T-2}}\\ &E\left[\xi_T^{S_{T-1}}\left(X+\sum_{j=t+1}^{T}\varphi_{j-1,j}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\right)\mid\mathcal{F}_{T-1}\right]\mid\mathcal{F}_t\Bigg],\end{aligned}\tag{5.4.26}$$

a.s. P, where the first equality follows from (5.4.23). Further,

$$\begin{aligned}
& E\left[\xi_T^{S_{T-1}}\left(X+\sum_{j=t+1}^{T}\varphi_{j-1,j}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\right)\mid\mathcal{F}_{T-1}\right]\\
= \;& E\Bigg[\xi_T^{S_{T-1}}\left(X+\varphi_{T-1,T}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\right)\\
& \qquad +\xi_T^{S_{T-1}}\sum_{j=t+1}^{T-1}\varphi_{j-1,j}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\mid\mathcal{F}_{T-1}\Bigg]\\
= \;& E\left[\xi_T^{S_{T-1}}\left(X+\varphi_{T-1,T}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\right)\mid\mathcal{F}_{T-1}\right]\\
& \qquad +\sum_{j=t+1}^{T-1}\varphi_{j-1,j}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1}),\quad \text{a.s.}P,
\end{aligned}$$

where the last equality follows from the fact that $E\left[\xi_T^{S_{T-1}}\mid\mathcal{F}_{T-1}\right]=1$, a.s. P. Plugging this into (5.4.26) yields

$$\begin{aligned}
E\Bigg[\xi_{t+1}^{S_t}\cdot\cdots\cdot\xi_{T-1}^{S_{T-2}}\Bigg(E\left[\xi_T^{S_{T-1}}\left(X+\varphi_{T-1,T}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\right)\mid\mathcal{F}_{T-1}\right]&\\
+\sum_{j=t+1}^{T-1}\varphi_{j-1,j}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\Bigg)\mid\mathcal{F}_t\Bigg],& \qquad (5.4.27)
\end{aligned}$$

a.s. P. We have

$$\begin{aligned}
& \xi_{t+1}^{S_t}\cdot\cdots\cdot\xi_{T-1}^{S_{T-2}}E\left[\xi_T^{S_{T-1}}\left(X+\varphi_{T-1,T}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\right)\mid\mathcal{F}_{T-1}\right]\\
= \;& \xi_{t+1}^{S_t}\cdot\cdots\cdot\xi_{T-1}^{S_{T-2}}\left(E_{S_{T-1}}\left[X\mid\mathcal{F}_{T-1}\right]+\varphi_{T-1,T}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\right)\\
= \;& \xi_{t+1}^{S_t}\cdot\cdots\cdot\xi_{T-1}^{S_{T-2}}\left(E_{S_{T-1}}\left[X\mid\mathcal{F}_{T-1}\right]+\varphi_{T-1,T}^{\phi}(S_{T-1})\right) \qquad (5.4.28)\\
= \;& \xi_{t+1}^{S_t}\cdot\cdots\cdot\xi_{T-1}^{S_{T-2}}\;\phi_{T-1,T}(X),\quad \text{a.s. }P,
\end{aligned}$$

where the equality in (5.4.28) is evident on the set $\left\{\xi_{t+1}^{S_t}\cdot\cdots\cdot\xi_{T-1}^{S_{T-2}}=0\right\}\in\mathcal{F}_{T-1}$ and on $\left\{\xi_{t+1}^{S_t}\cdot\cdots\cdot\xi_{T-1}^{S_{T-2}}>0\right\}$ it follows from lemma 5.4.2. Thus, we may write (5.4.27) in the form

$$
\begin{aligned}
& E\left[\xi_{t+1}^{S_t}\cdot\dots\cdot\xi_{T-1}^{S_{T-2}}\left(\phi_{T-1,T}(X)+\sum_{j=t+1}^{T-1}\varphi_{j-1,j}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\right)\mid\mathcal{F}_t\right]\\
= \;& E\Big[\xi_{t+1}^{S_t}\cdot\dots\\
& \cdots E\left[\xi_{T-1}^{S_{T-2}}\left(\phi_{T-1,T}(X)+\sum_{j=t+1}^{T-1}\varphi_{j-1,j}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\right)\mid\mathcal{F}_{T-2}\right]\mid\mathcal{F}_t\Bigg]\\
& \vdots\\
= \;& E\Big[\xi_{t+1}^{S_t}\cdot\dots\\
& \cdots\xi_{T-1}^{S_{T-2}}\left(\phi_{T-2,T-1}\big(\phi_{T-1,T}(X)\big)+\sum_{j=t+1}^{T-2}\varphi_{j-1,j}^{\phi}(S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1})\right)\mid\mathcal{F}_t\Bigg]\\
& \vdots\\
= \;& \phi_{t,t+1}\Big(\quad\cdots\quad\phi_{T-2,T-1}\big(\phi_{T-1,T}(X)\big)\quad\cdots\quad\Big),\quad \text{a.s. } P.
\end{aligned}
$$

But now we conclude

$$
\begin{aligned}
& \phi_{t,t+1}\Big(\quad\cdots\quad\phi_{T-2,T-1}\big(\phi_{T-1,T}(X)\big)\quad\cdots\quad\Big)\\
= \;& \psi_{t,t+1}\Big(\quad\cdots\quad\psi_{T-2,T-1}\big(\psi_{T-1,T}(X)\big)\quad\cdots\quad\Big)=\psi_{t,T}(X),\quad \text{a.s. } P,
\end{aligned}
$$

where the last equality follows from time-consistency of $(\psi_{t,T})_{t\in\mathbb{T}}$. $\square$

Chapter 6

Dynamic Value at Risk and Dynamic Expected Shortfall

This chapter is understood as an application of the results of chapter 5 to conditional value at risk and conditional expected shortfall: We define dynamic value at risk and dynamic expected shortfall by backwards induction, where conditional value at risk and conditional expected shortfall serve as corresponding one step transitions. We discuss time-consistency properties and present a characterization theorem of dynamic expected shortfall in terms of concatenated probability densities.

6.1 Introduction

In this chapter we finally propose a notion of dynamic value at risk and dynamic expected shortfall. The construction of these two dynamic monetary risk measures is performed by backwards induction as in (5.2.2) and therefore, dynamic value at risk and dynamic expected shortfall will be time-consistent. As a test of conditional value at risk and conditional expected shortfall at different times on time-consistency reveals severe drawbacks, such a construction seems advisable. To my information, this idea was first brought up in Cheridito and Kupper [9] in the context of expected shortfall, as Artzner et al. cautioned against the use of conditional expected shortfall at different times already in [3, 4].

In the following section we observe a useful martingale property. We then directly enter into the discussion of conditional value at risk at different times in subsection 6.2.1. This dynamic monetary risk measure turns out to be not time-consistent as counter-example 6.2.3 indicates. However, we are still able prove a weaker dynamic consistency property via the above mentioned martingale property. On the contrary, conditional expected shortfall at different times does not even satisfy this weaker dynamic consistency condition as the discussion in subsection 6.2.2 exemplifies. We therefore present a notion of a time-consistent dynamic expected shortfall by iterating one-step conditional expected shortfalls. A combination of the two theorems 5.3.5 and 5.4.3 yields a characterization of dynamic expected shortfall: As in the conditional case, we are able to explicitly construct the probability density for which the essential supremum which represents the dynamic expected shortfall is attained. Further, we compute the representing dynamic penalty function and arrive at a characterization of dynamic expected shortfall as an essential supremum over certain probability densities, similar to the definition 4.4.7 of conditional expected shortfall.

Throughout this chapter, we consider the setup of section 2.2 and let τ and θ be two $(\mathcal{F}_t)$-stopping times such that $\tau(\omega) \leq \theta(\omega)$ for all $\omega \in \Omega$. We denote by $\mathcal{P}$ the set of all probability measures that are absolutely continuous with respect to P. Within this chapter we again explicitly distinguish random variables on $(\Omega, \mathcal{F}_\theta, P)$ and the corresponding equivalence classes in $L^0(\mathcal{F}_\theta)$. As in chapter 4 random variables are denoted by $X, Y, Z, \ldots$ and $\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \ldots$ respectively designate the corresponding equivalence classes. $\mathcal{B} := \mathcal{B}(\mathbb{R})$ denotes the σ-algebra of Borel-sets on the reals.

For each $t \in \mathbb{T}$ we assume that we are given a mapping

$$P^{\mathcal{F}_t} : \Omega \times \mathcal{F}_T \to [0, 1]$$

which satisfies the well-known properties:

(1) $\quad P^{\mathcal{F}_t}(\omega, .) : \mathcal{F}_T \to [0, 1]$ is a probability measure for all $\omega \in \Omega$

(2) $\quad P^{\mathcal{F}_t}(., A) : (\Omega, \mathcal{F}_t) \to [0, 1]$ is $\mathcal{F}_t$-measurable for all $A \in \mathcal{F}_T$

(3) $\quad \int_C P^{\mathcal{F}_t}(w, A)\ P(d\omega) = P(C \cap A)$ for all $C \in \mathcal{F}_t$ and $A \in \mathcal{F}_T$.

The discussion of subsection 4.4.1 clarified that

$$P^{\mathcal{F}_0}(\omega, A) = P(A)$$

for all $\omega \in \Omega$ and for all $A \in \mathcal{F}_T$. Theorem 4.2.1 states that we can count on the existence of the family

$$\left(P^{\mathcal{F}_t}\right)_{t \in \mathbb{T}}$$

if Ω is endowed with a complete metric inducing a separable topology which generates the Borel σ-algebra $\mathcal{F}_T$.

6.2 Examples

Here is the martingale property which we gave notice of before:

Proposition 6.2.1 *For $A \in \mathcal{F}_T$ the discrete-time process $\left(P^{\mathcal{F}_t}(., A)\right)_{t \in \mathbb{T}}$ is a martingale.*

Proof. Integrability follows from the fact that for all $t \in \mathbb{T}$, $P^{\mathcal{F}_t}$ takes values in $[0, 1]$. Thus, for a fixed $t \in \{0, \ldots, T-1\}$ it remains to show that

$$E\left[P^{\mathcal{F}_{t+1}}(., A) \mid \mathcal{F}_t\right] = P^{\mathcal{F}_t}(., A), \quad \text{a.s. } P.$$

But since for all $s \in \mathbb{T}$, $P^{\mathcal{F}_s}(.,A) = E[1_A \mid \mathcal{F}_s]$, a.s. P, we deduce,

$$\begin{aligned} E\left[P^{\mathcal{F}_{t+1}}(.,A) \mid \mathcal{F}_t\right] &= E\left[E\left[1_A \mid \mathcal{F}_{t+1}\right] \mid \mathcal{F}_t\right] \\ &= E\left[1_A \mid \mathcal{F}_t\right] = P^{\mathcal{F}_t}(.,A), \quad \text{a.s. } P. \end{aligned}$$

□

As an immediate consequence of the preceding proposition we derive that for a fixed random variable X on $(\Omega, \mathcal{F}_T, P)$, for all $B \in \mathcal{B}$ and for all $x \in \mathbb{R}$ the discrete-time processes

$$\left(P_{X|\mathcal{F}_t}(.,B)\right)_{t\in\mathbb{T}} \quad \text{and} \quad \left(F_{X|\mathcal{F}_t}(.,x)\right)_{t\in\mathbb{T}}$$

are martingales as well. As the following example states, this property is not inherited by conditional quantiles.

Example 6.2.2 *Let $T = 1$, $\Omega = \{\omega_1, \omega_2\}$, P uniform on Ω (, i.e $P(\omega_i) = \frac{1}{2}$, $i \in \{1,2\}$), and $\mathcal{F}_1$ the power set. Consider the random variable $X : \Omega \to \mathbb{R}$, $\omega_i \mapsto X(\omega_i) := i$. For level $r = \frac{1}{2}$, example* 4.3.6 *tells us that*

$$q_{X|\mathcal{F}_0}\left(\frac{1}{2}\right)(\omega_1) = q_{X|\mathcal{F}_0}\left(\frac{1}{2}\right)(\omega_2) \in [1,2],$$

arbitrary, whereas

$$E[X] = \frac{3}{2}$$

and $q_{X|\mathcal{F}_1}\left(\frac{1}{2}\right)(\omega) = X(\omega)$ for all $\omega \in \Omega$.

6.2.1 Dynamic Value at Risk

Let us fix $r \in (0,1)$ and consider the dynamic monetary risk measure

$$(VaR^*_{t,T;r})_{t\in\mathbb{T}} \tag{6.2.1}$$

on $L^\infty(\mathcal{F}_T)$. For the conditional monetary risk measure $\sum_{t\in\mathbb{T}} VaR^*_{t,T;r} 1_{\{\tau=t\}}$ on $L^\infty(\mathcal{F}_\theta)$ we have

$$\sum_{t\in\mathbb{T}} VaR^*_{t,T;r}(\widetilde{X}) 1_{\{\tau=t\}} = VaR^*_{\tau,\theta;r}(\widetilde{X}), \quad \text{a.s. } P, \tag{6.2.2}$$

for all $\widetilde{X} \in L^\infty(\mathcal{F}_\theta)$. To verify the statement in (6.2.2) it suffices to show that for a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$ there exists a nullset $N = N(X)$ such that

$$\sum_{t\in\mathbb{T}} F_{X|\mathcal{F}_t}(\omega, x) 1_{\{\tau=t\}} = F_{X|\mathcal{F}_\tau}(\omega, x) \tag{6.2.3}$$

for all $\omega \in N^c$ and for all $x \in \mathbb{R}$. Then,

$$\sum_{t\in\mathbb{T}} q^+_{X|\mathcal{F}_t}(\omega, s) 1_{\{\tau=t\}} = q^+_{X|\mathcal{F}_\tau}(\omega, s)$$

for all $\omega \in N^c$ and for all $s \in (0,1)$. Hence, for r the equivalence classes in $L^\infty(\mathcal{F}_\tau)$ induced by $\sum_{t\in\mathbb{T}} q^+_{X|\mathcal{F}_t}(.,r)1_{\{\tau=t\}}$ and $q^+_{X|\mathcal{F}_\tau}(.,r)$ coincide, i.e. (6.2.2) is valid. In order to prove (6.2.3), fix a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$ and $t \in \mathbb{T}$. We show that there exists a nullset $N_t = N_t(X)$ such that

$$F_{X|\mathcal{F}_t}(\omega, x)1_{\{\tau=t\}} = F_{X|\mathcal{F}_\tau}(\omega, x)1_{\{\tau=t\}}$$

for all $\omega \in N_t^c$ and for all $x \in \mathbb{R}$. (6.2.3) will then follow as we set $N = \bigcup_{t\in\mathbb{T}} N_t$. We claim that for a P-almost surely bounded random variable Y on $(\Omega, \mathcal{F}_T, P)$ we have

$$E[Y \mid \mathcal{F}_t]1_{\{\tau=t\}} = E[Y \mid \mathcal{F}_\tau]1_{\{\tau=t\}}, \quad \text{a.s. } P. \tag{6.2.4}$$

We have already shown that the $\mathcal{F}_t$-measurable mapping $E[Y \mid \mathcal{F}_t]1_{\{\tau=t\}}$ is also $\mathcal{F}_\tau$-measurable since the decomposition in (2.2.2) is as well valid for $E[Y \mid \mathcal{F}_t]$. Thus, the assertion in (6.2.4) follows from the observation that for all events $C \in \mathcal{F}_\tau$ we have $\{\tau = t\} \cap C \in \mathcal{F}_t$ and hence,

$$\begin{aligned} E\big[E[Y \mid \mathcal{F}_t]1_{\{\tau=t\}}1_C\big] &= E\big[E[Y \mid \mathcal{F}_t]1_{\{\tau=t\}\cap C}\big] \\ &= E\big[Y1_{\{\tau=t\}\cap C}\big] = E\big[Y1_{\{\tau=t\}}1_C\big]. \end{aligned}$$

For all $x \in \mathbb{R}$ we may therefore derive

$$\begin{aligned} F_{X|\mathcal{F}_t}(., x)1_{\{\tau=t\}} &= P^{\mathcal{F}_t}\big(., \{X \le x\}\big)1_{\{\tau=t\}} \\ &= E\big[1_{\{X\le x\}} \mid \mathcal{F}_t\big]1_{\{\tau=t\}} \\ &= E\big[1_{\{X\le x\}} \mid \mathcal{F}_\tau\big]1_{\{\tau=t\}} \\ &= P^{\mathcal{F}_\tau}\big(., \{X \le x\}\big)1_{\{\tau=t\}} \\ &= F_{X|\mathcal{F}_\tau}(., x)1_{\{\tau=t\}}, \quad \text{a.s } P, \end{aligned}$$

which yields the existence of N_t and in turn verifies (6.2.2).

At first glance, the dynamic monetary risk measure $(VaR^*_{t,T;r})_{t\in\mathbb{T}}$ on $L^\infty(\mathcal{F}_T)$ for $r \in (0,1)$ seems to project the idea of the "classical" value at risk into our dynamic framework in a reasonable way as the identity given in (6.2.2) may suggest. However, a major drawback of $(VaR^*_{t,T;r})_{t\in\mathbb{T}}$ is that we do not find the iteration condition of proposition 5.2.4 (and equivalently time-consistency) to be satisfied as the following example states.

Example 6.2.3 *Let $T = 2$, $\Omega = \{\omega_1, \dots, \omega_4\}$, P uniform on Ω (, i.e $P(\omega_i) = \frac{1}{4}$, $i \in \{1, \dots, 4\}$), $\mathcal{F}_2$ the power set and $\mathcal{F}_1$ generated by $\{B_1, B_2\}$, where $B_1 = \{\omega_1, \omega_2\}$ and $B_2 = \{\omega_3, \omega_4\}$. Consider the random variable $X : \Omega \to \mathbb{R}$, $\omega_i \mapsto X(\omega_i) := i$. We fix the level $r = \frac{1}{2}$.In the notation of example* 4.3.6 *the probability measures P_{B_1} and P_{B_2} are given by*

$$P_{B_1}(\omega_1) = P_{B_1}(\omega_2) = \frac{1}{2}, \qquad P_{B_1}(\omega_3) = P_{B_1}(\omega_4) = 0, \quad \textit{and}$$
$$P_{B_2}(\omega_1) = P_{B_2}(\omega_2) = 0, \qquad P_{B_2}(\omega_3) = P_{B_2}(\omega_4) = \frac{1}{2}.$$

Example 4.3.6 *further tells us that*

$$q^+_{X|\mathcal{F}_1}\left(\frac{1}{2}\right) = 21_{B_1} + 41_{B_2}. \tag{6.2.5}$$

Since $\emptyset$ *is the only nullset,* $\tilde{q}^+_{X|\mathcal{F}_1}$ *consists of the element given in* (6.2.5) *only. We have*

$$q^+_{q^+_{X|\mathcal{F}_1}(\frac{1}{2})|\mathcal{F}_0}\left(\frac{1}{2}\right) = 4,$$

whereas

$$q^+_{X|\mathcal{F}_0}\left(\frac{1}{2}\right) = 3.$$

The preceding example clarifies that $(VaR^*_{t,T;r})_{t\in\mathbb{T}}$, $r \in (0,1)$, is not time-consistent in general, yet it turns out that $(VaR^*_{t,T;r})_{t\in\mathbb{T}}$ does satisfy a weaker dynamic consistency condition.

Lemma 6.2.4 *For all* $r \in (0,1)$, $(VaR^*_{t,T;r})_{t\in\mathbb{T}}$ *is acceptance-consistent in the sense of* (5.2.3), *i.e.*

$$VaR^*_{t+1,T;r}(\widetilde{X}) \geq 0, \text{ a.s. } P, \quad \text{implies} \quad VaR^*_{t,T;r}(\widetilde{X}) \geq 0, \text{ a.s. } P, \tag{6.2.6}$$

for all $\widetilde{X} \in L^\infty(\mathcal{F}_T)$ *and for all* $t \in \{0, \dots, T-1\}$.

Proof. For all of this proof we fix $\widetilde{X} \in L^\infty(\mathcal{F}_T)$ and $t \in \{0, \dots, T-1\}$.
In a first step we assume that there exists $X \in \widetilde{X}$ of the form $X = \sum_{i=1}^n \alpha_i 1_{A_i}$, $\alpha_i \in \mathbb{R}$, $A_i \in \mathcal{F}_T$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $n \in \mathbb{N}$. We may assume that $\alpha_1 < \ldots < \alpha_n$. From the definition of $q^+_{X|\mathcal{F}_t}(r)$,

$$\begin{aligned} q^+_{X|\mathcal{F}_t}(r)(\omega) &= \inf\left\{x \mid P_{X|\mathcal{F}_t}\big(\omega, \{X \leq x\}\big) > r\right\} \\ &= \min\left\{\alpha_i \mid F_{X|\mathcal{F}_t}(\omega, \alpha_i) > r\right\} \end{aligned}$$

for all $\omega \in \Omega$, we obtain

$$q^+_{X|\mathcal{F}_t}(r) = \sum_{i=1}^n \alpha_i 1_{\left\{F_{X|\mathcal{F}_t}(.,\alpha_{i-1}) \leq r < F_{X|\mathcal{F}_t}(.,\alpha_i)\right\}}.$$

Since we have already established that for all $x \in \mathbb{R}$, $\big(F_{X|\mathcal{F}_t}(.,x)\big)_{t\in\mathbb{T}}$ is a martingale, we may further derive

$$q^+_{X|\mathcal{F}_t}(r) = \sum_{i=1}^n \alpha_i 1_{\left\{E\left[F_{X|\mathcal{F}_{t+1}}(.,\alpha_{i-1})|\mathcal{F}_t\right] \leq r < E\left[F_{X|\mathcal{F}_{t+1}}(.,\alpha_i)|\mathcal{F}_t\right]\right\}}, \quad \text{a.s. } P.$$

Next, we show for arbitrary $j \in \{1, \ldots, n\}$ that if $q^+_{X|\mathcal{F}_t}(r)$ takes value α_j with positive probability, then $\left\{q^+_{X|\mathcal{F}_{t+1}}(r) \leq \alpha_j\right\}$ has positive P-measure. From this we deduce

$$q^+_{X|\mathcal{F}_t}(r) \geq \text{ess.inf}\ q^+_{X|\mathcal{F}_{t+1}}(r), \quad \text{a.s. } P,$$

which yields the assertion for finite step functions X. Let us fix $j \in \{1, \ldots, n\}$ and assume that

$$P\left\{E\left[F_{X|\mathcal{F}_{t+1}}(., \alpha_{j-1}) \mid \mathcal{F}_t\right] \leq r < E\left[F_{X|\mathcal{F}_{t+1}}(., \alpha_j) \mid \mathcal{F}_t\right]\right\} \ > \ 0,$$

i.e. $q^+_{X|\mathcal{F}_t}(r)$ takes value α_j with positive probability. In this case we have in particular

$$P\left\{r < E\left[F_{X|\mathcal{F}_{t+1}}(., \alpha_j) \mid \mathcal{F}_t\right]\right\} \ > \ 0, \tag{6.2.7}$$

and obtain

$$P\left\{r < F_{X|\mathcal{F}_{t+1}}(., \alpha_j)\right\} \ > \ 0,$$

since otherwise $F_{X|\mathcal{F}_{t+1}}(., \alpha_j) \leq r$, a.s. P, would imply that $E[F_{X|\mathcal{F}_{t+1}}(., \alpha_j) \mid \mathcal{F}_t] \leq r$, a.s. P, which in turn would contradict (6.2.7). Finally, for all $\omega \in \left\{r < F_{X|\mathcal{F}_{t+1}}(., \alpha_j)\right\}$ we have

$$q^+_{X|\mathcal{F}_{t+1}}(r)(\omega) = \min\left\{\alpha_i \mid F_{X|\mathcal{F}_{t+1}}(\omega, \alpha_i) > r\right\} \leq \alpha_j$$

and hence, $\left\{q^+_{X|\mathcal{F}_{t+1}}(r) \leq \alpha_j\right\}$ has positive P-measure.

As for general P-almost surely bounded X, we may assume in the usual manner that $N := \{X > ||X||_{L^\infty}\}$ is a $P^{\mathcal{F}_t}(\omega, .)$-nullset for all ω in the complement of a suitable nullset $N^* = N^*(N)$ and for all $t \in \mathbb{T}$. Note that indeed we may choose N^* independently from the fixed $t \in \mathbb{T}$ since there are only finitely many such t. We will show that

$$q^+_{X|\mathcal{F}_t}(r) \geq \text{ess.inf}\ q^+_{X|\mathcal{F}_{t+1}}(r), \quad \text{a.s. } P.$$

To this end, take $\mathcal{F}_T$-measurable step functions $(X_n)_{n\in\mathbb{N}}$ such that $X_n(\omega) \searrow X(\omega)$ for all $\omega \in N^c$. Then $\{X_n \leq x\} \setminus N \subset \{X \leq x\} \setminus N$ for all $x \in \mathbb{R}$, for all $n \in \mathbb{N}$ and hence,

$$F_{X_n|\mathcal{F}_t}(\omega, x) = P^{\mathcal{F}_t}\big(\omega, \{X_n \leq x\} \setminus N\big) \ \nearrow \ P^{\mathcal{F}_t}\big(\omega, \{X \leq x\} \setminus N\big) = F_{X|\mathcal{F}_t}(\omega, x) \tag{6.2.8}$$

for all $\omega \in N^{*c}$ and for all $x \in \mathbb{R}$. Since in (6.2.8) we may exchange t with $t+1$ we deduce

$$q^+_{X_n|\mathcal{F}_t}(r)(\omega) \ \searrow \ q^+_{X|\mathcal{F}_t}(r)(\omega) \quad \text{as well as} \tag{6.2.9}$$

$$q^+_{X_n|\mathcal{F}_{t+1}}(r)(\omega) \ \searrow \ q^+_{X|\mathcal{F}_{t+1}}(r)(\omega) \tag{6.2.10}$$

for all $\omega \in N^{*c}$. From the first part of the proof we know that

$$q^+_{X_n|\mathcal{F}_t}(r) \geq \text{ess.inf}\ q^+_{X_n|\mathcal{F}_{t+1}}(r), \quad \text{a.s. } P,$$

for all $n \in \mathbb{N}$ and hence, we derive from (6.2.10)

$$q^+_{X_n|\mathcal{F}_t}(r) \geq \text{ess.inf}\ q^+_{X|\mathcal{F}_{t+1}}(r), \quad \text{a.s. } P,$$

for all $n \in \mathbb{N}$. Since the same is true for the limit given in (6.2.9) we conclude

$$q^{+}_{X|\mathcal{F}_t}(r) \geq \text{ess.inf } q^{+}_{X|\mathcal{F}_{t+1}}(r), \quad \text{a.s. } P.$$

□

Definition 6.2.5 *For every $t \in \{0, \ldots, T-1\}$ let $r_t \in (0,1)$. As in* (5.2.2) *the one-step transitions*

$$\phi_{t,t+1}(\widetilde{X}) := \tilde{q}^{+}_{X|\mathcal{F}_t}(r_t),$$

$\widetilde{X} \in L^\infty(\mathcal{F}_{t+1})$, $t \in \{0, \ldots, T-1\}$, induce a time-consistent dynamic monetary utility functional $(\psi_{t,T})_{t\in\mathbb{T}}$. For all $t \in \{0, \ldots, T-1\}$, $-\phi_{t,t+1}$ is a conditional value at risk at level r_t and consequently we call $(-\psi_{t,T})_{t\in\mathbb{T}}$ dynamic value at risk at dynamic level $(r_0, \ldots, r_{T-1})$.

6.2.2 Dynamic Expected Shortfall

Let us fix $r \in (0,1)$ and consider the dynamic coherent risk measure

$$(ES^*_{t,T;r})_{t\in\mathbb{T}} \tag{6.2.11}$$

on $L^\infty(\mathcal{F}_T)$. For the conditional coherent risk measure $\sum_{t\in\mathbb{T}} ES^*_{t,T;r} 1_{\{\tau=t\}}$ on $L^\infty(\mathcal{F}_\theta)$ we have for all P-almost surely bounded random variables X on $(\Omega, \mathcal{F}_\theta, P)$ and corresponding equivalence class $\widetilde{X}$ in $L^\infty(\mathcal{F}_\theta)$

$$\begin{aligned}
\sum_{t\in\mathbb{T}} E[-XI_{X|\mathcal{F}_t}(r) \mid \mathcal{F}_t] 1_{\{\tau=t\}} &\in \sum_{t\in\mathbb{T}} ES^*_{t,T;r}(\widetilde{X}) 1_{\{\tau=t\}}, \quad \text{as well as} \\
\sum_{t\in\mathbb{T}} E[-XI_{X|\mathcal{F}_t}(r) \mid \mathcal{F}_t] 1_{\{\tau=t\}} &\in ES^*_{\tau,\theta;r}(\widetilde{X}), \qquad (6.2.12)
\end{aligned}$$

in other words,

$$\sum_{t\in\mathbb{T}} ES^*_{t,T;r}(\widetilde{X}) 1_{\{\tau=t\}} = ES^*_{\tau,\theta;r}(\widetilde{X}), \quad \text{a.s. } P, \tag{6.2.13}$$

for all $\widetilde{X} \in L^\infty(\mathcal{F}_\theta)$. In view of (4.4.22) only the statement in (6.2.12) is to be verified. To this end, fix a P-almost surely bounded random variable X on $(\Omega, \mathcal{F}_\theta, P)$ and $t \in \mathbb{T}$. It suffices to show that,

$$E[-XI_{X|\mathcal{F}_t}(r) \mid \mathcal{F}_\tau] 1_{\{\tau=t\}} = E[-XI_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau] 1_{\{\tau=t\}}, \quad \text{a.s. } P, \tag{6.2.14}$$

since then we derive from (6.2.4)

$$\begin{aligned}
\sum_{t\in\mathbb{T}} E[-XI_{X|\mathcal{F}_t}(r) \mid \mathcal{F}_t] 1_{\{\tau=t\}} &= \sum_{t\in\mathbb{T}} E[-XI_{X|\mathcal{F}_t}(r) \mid \mathcal{F}_\tau] 1_{\{\tau=t\}} \\
&= \sum_{t\in\mathbb{T}} E[-XI_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau] 1_{\{\tau=t\}} \\
&= E[-XI_{X|\mathcal{F}_\tau}(r) \mid \mathcal{F}_\tau], \quad \text{a.s. } P,
\end{aligned}$$

which again in view of (4.4.22) yields the statement in (6.2.12). We may assume that $I_{X|\mathcal{F}_t}(r)$ and $I_{X|\mathcal{F}_\tau}(r)$ are constructed with respect to the conditional quantiles $q^+_{X|\mathcal{F}_t}(.,r)$ and $q^+_{X|\mathcal{F}_\tau}(.,r)$. Take a null-set $N = N(X)$ such that

$$F_{X|\mathcal{F}_t}(\omega, x) = F_{X|\mathcal{F}_\tau}(\omega, x)$$

for all $\omega \in N^c \cap \{\tau = t\}$ and for all $x \in \mathbb{R}$. Such N exists due to the statement in (6.2.3). From this we may deduce $q^+_{X|\mathcal{F}_t}(\omega, r) = q^+_{X|\mathcal{F}_\tau}(\omega, r)$ for all $\omega \in N^c \cap \{\tau = t\}$ and in turn

$$I_{X|\mathcal{F}_t}(r)1_{\{\tau=t\}} = I_{X|\mathcal{F}_\tau}(r)1_{\{\tau=t\}}, \quad \text{a.s. } P,$$

which eventually yields (6.2.14).

Again, the identity given in (6.2.13) may at first glance support the viewpoint that the dynamic coherent risk measure $(ES^*_{t,T;r})_{t\in\mathbb{T}}$, $r \in (0,1)$, provides a reasonable idea of "how bad is bad?" within our dynamic temporal setting. However, care must be taken in general as $(ES^*_{t,T;r})_{t\in\mathbb{T}}$ fails to even satisfy acceptance-consistency in the sense of (5.2.3). Here is the counterexample taken from Artzner et al. [4]:

Example 6.2.6 *Let $T = 2$, $\Omega = \{\omega_1, \ldots, \omega_6\}$, P uniform on Ω (, i.e $P(\omega_i) = \frac{1}{6}$, $i \in \{1, \ldots, 6\}$), $\mathcal{F}_2$ the power set and $\mathcal{F}_1$ generated by $\{B_1, B_2\}$, where $B_1 = \{\omega_1, \omega_2, \omega_3\}$ and $B_2 = \{\omega_4, \omega_5, \omega_6\}$. Consider the random variable $X : \Omega \to \mathbb{R}$, $X(\omega_1) := -10$, $X(\omega_2) := 12$, $X(\omega_3) := 14$, $X(\omega_4) := -20$, $X(\omega_5) := 22$, $X(\omega_6) := 22$. We compute $ES^*_{0,2;\frac{2}{3}}(\widetilde{X})$ and $ES^*_{1,2;\frac{2}{3}}(\widetilde{X})$. Note, that equivalence classes in $L^\infty(\mathcal{F}_t)$, $t \in \{0,1,2\}$ consist of one element only since $\emptyset$ is the only nullset occurring in this setup. In the notation of example* 4.3.6 *the probability measures P_{B_1} and P_{B_2} are given by*

$$P_{B_1}(\omega_1) = P_{B_1}(\omega_2) = P_{B_1}(\omega_3) = \frac{1}{3}, \qquad P_{B_1}(\omega_4) = P_{B_1}(\omega_5) = P_{B_1}(\omega_6) = 0, \quad \textit{and}$$
$$P_{B_2}(\omega_1) = P_{B_2}(\omega_2) = P_{B_2}(\omega_3) = 0, \qquad P_{B_2}(\omega_4) = P_{B_2}(\omega_5) = P_{B_2}(\omega_6) = \frac{1}{3}.$$

Example 4.3.6 *further tells us that*

$$q^+_{X|\mathcal{F}_1}(s) = -10 1_{B_1} + -20 1_{B_2}.$$

for all $s \in (0, \frac{1}{3})$ and

$$q^+_{X|\mathcal{F}_1}(s) = 12 1_{B_1} + 22 1_{B_2}.$$

for all $s \in [\frac{1}{3}, \frac{2}{3})$. Thus, by example 4.4.12

$$-\frac{3}{2}\int_0^{\frac{2}{3}} q^+_{X|\mathcal{F}_1}(s)(.)\ \lambda^1(ds) = -\frac{3}{2}\left(\frac{1}{3}(-10+12)1_{B_1} + \frac{1}{3}(-20+22)1_{B_2}\right) = -1_\Omega$$

is the only element in $ES^*_{1,2;\frac{2}{3}}(\widetilde{X})$*, whereas*

$$q^+_{X|\mathcal{F}_0}(s) = -20,\ \textit{for all } s \in \left(0, \frac{1}{6}\right), \qquad q^+_{X|\mathcal{F}_0}(s) = -10,\ \textit{for all } s \in \left[\frac{1}{6}, \frac{2}{6}\right)$$
$$q^+_{X|\mathcal{F}_0}(s) = 12,\ \textit{for all } s \in \left[\frac{2}{6}, \frac{3}{6}\right), \qquad q^+_{X|\mathcal{F}_0}(s) = 14,\ \textit{for all } s \in \left[\frac{3}{6}, \frac{4}{6}\right)$$

implies that

$$-\frac{3}{2}\int_0^{\frac{2}{3}} q^+_{X|\mathcal{F}_0}(s)(.)\ \lambda^1(ds) = -\frac{3}{2}\left(\frac{1}{6}(-20-10+12+14)1_\Omega\right) = 1_\Omega$$

is the only element in $ES^*_{0,2;\frac{2}{3}}(\widetilde{X})$*. To sum up,* $\widetilde{X}$ *is accepted at time* $t = 1$*, yet rejected at date* 0*.*

For every $t \in \{0, \ldots, T-1\}$ we let $r_t \in (0,1)$ and shorten $r = (r_0, \ldots, r_{T-1})$. Consider the one-step transitions

$$\phi_{t,t+1}(\widetilde{X}) := \operatorname*{ess.inf}_{Q \in \mathcal{Q}_t(r_t)} E_Q[\widetilde{X} \mid \mathcal{F}_t]$$

for all $\widetilde{X} \in L^\infty(\mathcal{F}_{t+1})$. Then, for all $t \in \{0, \ldots, T-1\}$, $-\phi_{t,t+1}$ is a conditional expected shortfall $ES^*_{t,t+1;r_t}$ on $L^\infty(\mathcal{F}_{t+1})$ at level r_t. As in (5.2.2) the one-step transitions $(\phi_{t,t+1})_{t\in\{0,\ldots,T-1\}}$ induce a time-consistent dynamic coherent utility functional $(\psi_{t,T})_{t\in\mathbb{T}}$.

Lemma 6.2.7 *In the above notation we have*

$$\begin{aligned}
\psi_{t,T}(\widetilde{X}) &= \operatorname*{ess.inf}_{Q\in\mathcal{P}} E_Q\left[\widetilde{X} + \sum_{j=1}^{T} \varphi^{\psi}_{j-1,j}(Q) \mid \mathcal{F}_t\right] && (6.2.15)\\
&= \operatorname*{ess.inf}_{Q\in\mathcal{P}} E_Q\left[\widetilde{X} + \sum_{j=t}^{T} \varphi^{\psi}_{j-1,j}(Q) \mid \mathcal{F}_t\right] && (6.2.16)\\
&= E_{S_t\otimes_{t+1}\cdots\otimes_{T-1}S_{T-1}}\left[\widetilde{X} + \sum_{j=t}^{T} \varphi^{\psi}_{j-1,j}(S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}) \mid \mathcal{F}_t\right], \quad \textit{a.s. } P, && (6.2.17)
\end{aligned}$$

for all $t \in \{0, \ldots, T-1\}$ *and for all* $\widetilde{X} \in L^\infty(\mathcal{F}_T)$*, where the probability measures* S_t*,* $t \in \{0, \ldots, T-1\}$*, are given by the densities* $I_{X_t|\mathcal{F}_t}(r_t)$ *of definition* 4.4.5 *and*

$$X_t := \psi_{t+1,T}(X) \quad \textit{for } t \in \{0, \ldots, T-1\}.$$

Proof. (6.2.15) and (6.2.16) are immediate consequences of theorem 5.3.5 and its corollary 5.3.6. (6.2.17) follows from theorem 5.4.3 together with theorem 4.4.10. □

Theorem 6.2.8 *In the notation of the above lemma* 6.2.7 *we have for all* $t \in \{0, \dots, T-1\}$ *and for all* $\widetilde{X} \in L^\infty(\mathcal{F}_T)$,

$$
\begin{aligned}
& E_{S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}} \left[\widetilde{X} + \sum_{j=t}^{T} \varphi^{\psi}_{j-1,j}(S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}) \mid \mathcal{F}_t \right] \\
= \; & E\Big[E\left[I_{X_t|\mathcal{F}_t}(r_t) \mid \mathcal{F}_{t+1} \right] E\left[I_{X_{t+1}|\mathcal{F}_{t+1}}(r_{t+1}) \mid \mathcal{F}_{t+2} \right] \cdot \cdots \\
& \cdots \cdot E\left[I_{X_{T-2}|\mathcal{F}_{T-2}}(r_{T-2}) \mid \mathcal{F}_{T-1} \right] I_{X_{T-1}|\mathcal{F}_{T-1}}(r_{T-1}) \widetilde{X} \mid \mathcal{F}_t \Big] \qquad (6.2.18) \\
= \; & \operatorname*{ess.inf}_{Q \in \mathcal{Q}(r)} E_Q \left[\widetilde{X} \mid \mathcal{F}_t \right], \quad a.s. \ P, \qquad (6.2.19)
\end{aligned}
$$

where

$$\mathcal{Q}(r) := \left\{ Q \in \mathcal{P} \mid Q \in \mathcal{Q}_s(r_s) \text{ for all } s \in \{t, \dots, T-1\} \right\}$$

and the $\mathcal{Q}_s(r_s)$, $s \in \{t, \dots, T-1\}$, *are given as in definition* 4.4.7. *Note that* $\mathcal{Q}(r)$ *depends on* t.

For the proof of theorem 6.2.8 we will need the following lemma

Lemma 6.2.9 *In the notation of the above theorem* 6.2.8 *we have for all* $t \in \{0, \dots, T-1\}$ *and for all* $\widetilde{X} \in L^\infty(\mathcal{F}_T)$

$$S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1} \in \mathcal{Q}(r). \qquad (6.2.20)$$

Recall that $\mathcal{Q}(r)$ *depends on* t *and that the measure* $S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}$ *depends on* $\widetilde{X}$.

Proof. For all of this proof, we fix $t \in \{0, \dots, T-1\}$ and $\widetilde{X} \in L^\infty(\mathcal{F}_T)$. For $Q \notin \mathcal{Q}(r)$ there exists $s \in \{t, \dots, T-1\}$ such that $Q \notin \mathcal{Q}_s(r_s)$. We show that for such s there exists $M \in \mathcal{F}_s$ with $P(A) > 0$ such that

$$\varphi^{\psi}_{s,s+1}(Q) = +\infty \qquad (6.2.21)$$

P-almost surely on M.
To this end, let us consider arbitrary $Q \notin \mathcal{Q}(r)$ and $s \in \{t, \dots, T-1\}$ such that $Q \notin \mathcal{Q}_s(r_s)$. We set

$$\xi = \begin{cases} \frac{\frac{dQ}{dP}}{E\left[\frac{dQ}{dP} | \mathcal{F}_s\right]} & \text{on } \left\{ E\left[\frac{dQ}{dP} \mid \mathcal{F}_s\right] > 0 \right\} \\ 1 & \text{else} \end{cases}$$

and claim that for $r' \in (0, r_s)$,

$$P^{\mathcal{F}_s}\left(., \left\{ \xi \geq \frac{1}{r'} \right\} \right) = E\left[1_{\{\xi \geq \frac{1}{r'}\}} \mid \mathcal{F}_s \right] \leq \frac{1}{r'}, \quad \text{a.s. } P. \qquad (6.2.22)$$

Let us assume that (6.2.22) is wrong. Then there exists $A \in \mathcal{F}_s$ with $P(A) > 0$ such that

$$E\left[1_{\{\xi \geq \frac{1}{r'}\}} \mid \mathcal{F}_s\right] > \frac{1}{r'}$$

P-almost surely on A. Thus,

$$\begin{aligned} E\left[\xi 1_{\{\xi \geq \frac{1}{r'}\}} \mid \mathcal{F}_s\right] &\geq E\left[\frac{1}{r'} 1_{\{\xi \geq \frac{1}{r'}\}} \mid \mathcal{F}_s\right] \\ &= \frac{1}{r'} E\left[1_{\{\xi \geq \frac{1}{r'}\}} \mid \mathcal{F}_s\right] \\ &> r' \frac{1}{r'} = 1 \end{aligned}$$

P-almost surely on A. But now we derive

$$1 = E[\xi \mid \mathcal{F}_s] = \underbrace{E\left[\xi 1_{\{\xi \geq \frac{1}{r'}\}} \mid \mathcal{F}_s\right]}_{> 1, \text{ a.s. } P \text{ on } A} + \underbrace{E\left[\xi 1_{\{\xi < \frac{1}{r'}\}} \mid \mathcal{F}_s\right]}_{\geq 0, \text{ a.s. } P}, \quad \text{a.s. } P,$$

which is a contradiction and hence, the statement in (6.2.22) is valid. Next, we define for real constants $c, k > 0$ and for $r' \in (0, r_s)$ the P-almost surely bounded random variable

$$X^{(c)} := -c(\xi \wedge k) 1_{\{\xi \geq \frac{1}{r'}\}}$$

and derive from the statement in (6.2.22), from that fact that $\frac{1}{r'} < \frac{1}{r_s}$ and the observation $\{X^{(c)} < 0\} = \{\xi \geq \frac{1}{r'}\}$ that

$$q^{+}_{X^{(c)}|\mathcal{F}_s}(r_s) = 0, \quad \text{a.s. } P.$$

Hence, the statement in (4.4.23) of corollary 4.4.11 yields

$$-\frac{c}{r_s} E\left[(\xi \wedge k) 1_{\{\xi \geq \frac{1}{r'}\}} \mid \mathcal{F}_s\right] \in \phi_{s,s+1}(\widetilde{X}^{(c)}),$$

where $\widetilde{X}^{(c)}$ denotes the equivalence class in $L^\infty(\mathcal{F}_{t+1})$ induced by $X^{(c)}$. Further,

$$E_Q\left[-X^{(c)} \mid \mathcal{F}_s\right] = c\, E\left[\xi(\xi \wedge k) 1_{\{\xi \geq \frac{1}{r'}\}} \mid \mathcal{F}_s\right] \geq \frac{c}{r'}\, E\left[(\xi \wedge k) 1_{\{\xi \geq \frac{1}{r'}\}} \mid \mathcal{F}_s\right] \quad \text{a.s. } P.$$

Since $Q \notin \mathcal{Q}_s(r_s)$ there exist $r' \in (0, r_s)$ and a real constant k such that

$$P\left((\xi \wedge k) \geq \frac{1}{r'}\right) > 0. \tag{6.2.23}$$

From this we derive that

$$M := \left\{E\left[(\xi \wedge k) 1_{\{\xi \geq \frac{1}{r'}\}} \mid \mathcal{F}_s\right] > 0\right\} \in \mathcal{F}_s \tag{6.2.24}$$

has positive P-measure since otherwise

$$0 = E\left[E\left[(\xi \wedge k)1_{\{\xi \geq \frac{1}{r'}\}} \mid \mathcal{F}_s\right]\right] = E\left[(\xi \wedge k)1_{\{\xi \geq \frac{1}{r'}\}}\right]$$

would contradict the statement in (6.2.23). But now we derive that the difference

$$E\left[(\xi \wedge k)1_{\{\xi \geq \frac{1}{r'}\}} \mid \mathcal{F}_s\right] c\left(\frac{1}{r'} - \frac{1}{r_s}\right) \in E_Q[-\widetilde{X} \mid \mathcal{F}_t] + \phi_{t,t+1}(\widetilde{X})$$

becomes arbitrarily large P-almost surely on M as c tends to infinity. Hence,

$$\varphi^{\psi}_{s,s+1}(Q) = \operatorname*{ess.sup}_{\widetilde{X} \in L^\infty(\mathcal{F}_{s+1})} \left\{E_Q[-\widetilde{X} \mid \mathcal{F}_s] + \phi_{s,s+1}(\widetilde{X})\right\} = +\infty$$

P-almost surely on M and the statement in (6.2.21) is obtained.

We conclude as follows: If we assumed that $S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1} \notin \mathcal{Q}(r)$ then there would exist $s \in \{t, \ldots, T-1\}$ and $M \in \mathcal{F}_s$ with $P(M) > 0$ such that

$$\varphi^{\psi}_{s,s+1}(S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}) = +\infty$$

P-almost surely on M. But this would imply

$$\sum_{j=t}^{T} \varphi^{\psi}_{j-1,j}(S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}) = +\infty$$

P-almost surely on M which in turn would contradict the fact that

$$\begin{aligned} L^\infty(\mathcal{F}_t) \quad &\ni \quad \operatorname*{ess.inf}_{Q \in \mathcal{P}} E_Q\left[\widetilde{X} + \sum_{j=t}^{T} \varphi^{\psi}_{j-1,j}(Q) \mid \mathcal{F}_t\right] \\ &= \quad E_{S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}}\left[\widetilde{X} + \sum_{j=t}^{T} \varphi^{\psi}_{j-1,j}(S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}) \mid \mathcal{F}_t\right], \quad \text{a.s. } P, \end{aligned}$$

where we have in mind the convention in (5.3.7) and where it remains to show that $S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}(M) > 0$. But this follows as

$$\left\{E\left[(\xi \wedge k)1_{\{\xi \geq \frac{1}{r'}\}} \mid \mathcal{F}_s\right] = 0\right\} \subset \left\{(\xi \wedge k)1_{\{\xi \geq \frac{1}{r'}\}} = 0\right\}$$

except for a P-nullset and hence, by construction of M in (6.2.24) we have

$$M \supset \left\{(\xi \wedge k)1_{\{\xi \geq \frac{1}{r'}\}} > 0\right\}$$

except for a P-nullset, where this time ξ is given by

$$\xi = \begin{cases} \frac{\frac{dS_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}}{dP}}{E\left[\frac{dS_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}}{dP} \mid \mathcal{F}_s\right]} & \text{on } \left\{E\left[\frac{dS_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}}{dP} \mid \mathcal{F}_s\right] > 0\right\} \\ 1 & \text{else} \end{cases}.$$

□

Here is the proof of theorem 6.2.8:

Proof. We first show that $\frac{dS_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}}{dP}$ P-almost surely equals

$$E\left[I_{X_t|\mathcal{F}_t}(r_t) \mid \mathcal{F}_{t+1}\right] E\left[I_{X_{t+1}|\mathcal{F}_{t+1}}(r_{t+1}) \mid \mathcal{F}_{t+2}\right] \cdot \cdots \\ \cdots \cdot E\left[I_{X_{T-2}|\mathcal{F}_{T-2}}(r_{T-2}) \mid \mathcal{F}_{T-1}\right] I_{X_{T-1}|\mathcal{F}_{T-1}}(r_{T-1}). \quad (6.2.25)$$

Recall, that by the definition in (5.4.16) $\frac{dS_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}}{dP}$ P-almost surely equals

$$E\left[I_{X_t|\mathcal{F}_t}(r_t) \mid \mathcal{F}_{t+1}\right] \frac{E\left[I_{X_{t+1}|\mathcal{F}_{t+1}}(r_{t+1}) \mid \mathcal{F}_{t+2}\right]}{E\left[I_{X_{t+1}|\mathcal{F}_{t+1}}(r_{t+1}) \mid \mathcal{F}_{t+1}\right]} \cdot \cdots \\ \cdots \cdot \frac{E\left[I_{X_{T-2}|\mathcal{F}_{T-2}}(r_{T-2}) \mid \mathcal{F}_{T-1}\right]}{E\left[I_{X_{T-2}|\mathcal{F}_{T-2}}(r_{T-2}) \mid \mathcal{F}_{T-2}\right]} \frac{I_{X_{T-1}|\mathcal{F}_{T-1}}(r_{T-1})}{E\left[I_{X_{T-1}|\mathcal{F}_{T-1}}(r_{T-1}) \mid \mathcal{F}_{T-1}\right]}.$$

By lemma 4.4.6 we have for all $s \in \{t+1, \ldots, T-1\}$

$$E\left[I_{X_s|\mathcal{F}_s}(r_s) \mid \mathcal{F}_s\right] = 1, \quad \text{a.s. } P, \quad (6.2.26)$$

and hence, the statement in (6.2.25) follows. (6.2.26) also yields

$$E\Bigg[E\left[I_{X_t|\mathcal{F}_t}(r_t) \mid \mathcal{F}_{t+1}\right] E\left[I_{X_{t+1}|\mathcal{F}_{t+1}}(r_{t+1}) \mid \mathcal{F}_{t+2}\right] \cdot \cdots \\ \cdots \cdot E\left[I_{X_{T-2}|\mathcal{F}_{T-2}}(r_{T-2}) \mid \mathcal{F}_{T-1}\right] I_{X_{T-1}|\mathcal{F}_{T-1}}(r_{T-1}) \mid \mathcal{F}_t\Bigg] = 1, \quad \text{a.s. } P,$$

and hence, the identity in (6.2.18) follows if we can show that

$$\sum_{j=t}^{T} \varphi^{\psi}_{j-1,j}(S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}) = 0, \quad \text{a.s. } P. \quad (6.2.27)$$

Since by lemma 6.2.9 $S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1} \in \mathcal{Q}(r)$, it suffices to show that for all $Q \in \mathcal{Q}(r)$

$$\sum_{j=t}^{T} \varphi^{\psi}_{j-1,j}(Q) = 0, \quad \text{a.s. } P. \quad (6.2.28)$$

To this end, recall, that for $s \in \{t, \ldots, T-1\}$ we have by definition

$$\begin{aligned} \varphi^{\psi}_{s,s+1}(Q) &= \underset{\widetilde{X} \in L^\infty(\mathcal{F}_{s+1})}{\text{ess.sup}} \left\{E_Q[-\widetilde{X} \mid \mathcal{F}_s] + \psi_{s,s+1}(\widetilde{X})\right\} \\ &= \underset{\widetilde{X} \in L^\infty(\mathcal{F}_{s+1})}{\text{ess.sup}} \left\{E_Q[-\widetilde{X} \mid \mathcal{F}_s] + \phi_{s,s+1}(\widetilde{X})\right\}, \quad \text{a.s. } P. \end{aligned}$$

Let us consider $s \in \{t, \ldots, T-1\}$ and $Q \in \mathcal{Q}(r)$. Then, $Q \in \mathcal{Q}_s(r_s)$ and for $\widetilde{X} \in L^\infty(\mathcal{F}_{s+1})$ we have

$$\begin{aligned} E_Q[-\widetilde{X} \mid \mathcal{F}_s] + \phi_{s,s+1}(\widetilde{X}) &= \operatorname*{ess.inf}_{S \in \mathcal{Q}_s(s_t)} E_S[\widetilde{X} \mid \mathcal{F}_s] - E_Q[\widetilde{X} \mid \mathcal{F}_s] \\ &\leq 0, \quad \text{a.s. } P, \end{aligned} \tag{6.2.29}$$

and hence,

$$\begin{aligned} \varphi^\psi_{s,s+1}(Q) &= \operatorname*{ess.sup}_{\widetilde{X} \in L^\infty(\mathcal{F}_{s+1})} \left\{ E_Q[-\widetilde{X} \mid \mathcal{F}_s] + \phi_{s,s+1}(\widetilde{X}) \right\} \\ &= \operatorname*{ess.sup}_{\widetilde{X} \in L^\infty(\mathcal{F}_{s+1})} \left\{ E_Q[-\widetilde{X} \mid \mathcal{F}_s] + \operatorname*{ess.inf}_{S \in \mathcal{Q}_s(r_s)} E_S[\widetilde{X} \mid \mathcal{F}_s] \right\} \\ &= 0, \quad \text{a.s. } P, \end{aligned}$$

where the last equality follows from (6.2.29) together with the fact that $0 \in L^\infty(\mathcal{F}_{s+1})$. Hence, the statement in (6.2.28) is verified and the identity given in (6.2.18) is proved in turn.

It remains to prove the equality given in (6.2.19). But lemma 6.2.7 states that the essential infimum

$$\operatorname*{ess.inf}_{Q \in \mathcal{P}} E_Q \left[\widetilde{X} + \sum_{j=t}^{T} \varphi^\psi_{j-1,j}(Q) \mid \mathcal{F}_t \right]$$

is attained by the measure $S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1}$ and since $S_t \otimes_{t+1} \cdots \otimes_{T-1} S_{T-1} \in \mathcal{Q}(r)$, it suffices to take the essential infimum over all Q in $\mathcal{Q}(r)$. Thus, we arrive at

$$\begin{aligned} &\operatorname*{ess.inf}_{Q \in \mathcal{P}} E_Q \left[\widetilde{X} + \sum_{j=t}^{T} \varphi^\psi_{j-1,j}(Q) \mid \mathcal{F}_t \right] \\ = &\operatorname*{ess.inf}_{Q \in \mathcal{Q}(r)} E_Q \left[\widetilde{X} + \sum_{j=t}^{T} \varphi^\psi_{j-1,j}(Q) \mid \mathcal{F}_t \right] = \operatorname*{ess.inf}_{Q \in \mathcal{Q}(r)} E_Q \left[\widetilde{X} \mid \mathcal{F}_t \right], \end{aligned} \tag{6.2.30}$$

a.s. P, where the second identity in (6.2.30) follows from the statement in (6.2.28) and the proof is concluded. □

Definition 6.2.10 *For every $t \in \{0, \ldots, T-1\}$ we take $r_t \in (0,1)$, shorten $r = (r_0, \ldots, r_{T-1})$ and consider the the mappings $DES^*_{t,T;r} : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$,*

$$\widetilde{X} \mapsto DES^*_{t,T;r}(\widetilde{X}) := \operatorname*{ess.sup}_{Q \in \mathcal{Q}(r)} E_Q \left[-\widetilde{X} \mid \mathcal{F}_t \right],$$

where again we call attention to the fact that $\mathcal{Q}(r)$ depends on t. We call

$$\left(DES^*_{t,T;r} \right)_{t \in \mathbb{T}}$$

dynamic expected shortfall at dynamic level $r = (r_0, \ldots, r_{T-1})$, where $DES^*_{T,T;r}$ is as usual understood as the identity on $L^\infty(\mathcal{F}_T)$.

Chapter 7

Outlook

Coming to the end of this thesis, we see at least two major fields of possible further research which we outline in the following.

As to the first one, we have to admit that we rather presented results on conditional and dynamic monetary risk measures for bounded random variables than for bounded discrete-time processes, although the latter would certainly be richer in mathematical ideas as well as of more practical concern. When we explicitly constructed the monotone hull of the Swiss Solvency Test risk measure Γ_r and found it not to be capable of taking into account the riskiness evolving from inter-temporal cash-flow streams, we attempted to come forward with an alternative way of quantifying target capital. It was our aim to construct a time-consistent dynamic monetary risk measure which translates the idea of expected shortfall, that is "how bad is bad?", into our dynamic temporal setting. We presented dynamic expected shortfall as such a risk measure and, indeed, it seems to be capable of being a possible substitute for Γ_r. However, dynamic expected shortfall is up to date defined as a functional on the space of bounded random variables, whereas Γ_r depends on bounded discrete-time processes.

Time-consistency and the equivalent iteration condition given in proposition 5.2.4 have already been established by Cheridito et al. in the context of dynamic monetary utility functionals for bounded discrete-time processes, cf. definition 4.2 together with proposition 4.4. in [8]. Theorems 3.16 and 3.18 in [8] generalize the duality result of lemma 5.3.3 by providing representations of dynamic concave and coherent utility functionals for bounded discrete-time processes that are continuous in a mild sense. Moreover, in [9], Cheridito and Kupper prove a representation for such utility functionals which is similar to the representations of theorem 5.3.5 and corollary 5.3.6. This theoretical background seems encouraging for further research on a construction of a time-consistent dynamic expected shortfall that depends on bounded discrete-time processes by means of a conditional expected shortfall which, in turn should depend on bounded discrete-time processes as well.

It seems suggestive to attempt a generalization of the results of chapter 4 yielding a notion of a conditional expected shortfall for bounded discrete-time processes. Within this generalized setting, the probability measures over which the essential supremum is taken in theorem 4.4.10 are likely to be replaced by adapted increasing processes of integrable variation as dual representations in the form of theorems 3.16 and 3.18 in [8] are based on linear functionals induced by such processes. The boundedness conditions of definition 4.4.7 imposed upon probability densities may in turn by translated into boundedness conditions for adapted increasing processes of integrable variation. This might yield the desired notion of conditional expected shortfall for bounded discrete-time processes by taking essential supremum over adapted increasing processes of integrable variation which satisfy the boundedness conditions. Moreover, one may even hope to to be able to provide a result similar to the one of theorem 4.4.10 as one might be able to construct an adapted increasing process of integrable variation for which the essential supremum is attained. As soon as conditional expected shortfall for bounded discrete-time processes is well understood, one may define dynamic expected shortfall for bounded discrete-time processes again by backwards induction. We shall then be able to generalize the results of chapter 6 as well.

Above, we briefly outlined how we believe that further research is likely to blossom into a notion of dynamic expected shortfall for bounded discrete-time processes which may serve as a possible substitute for the Swiss Solvency Test risk measure Γ_r. However, such a dynamic risk measure will be of practitioners concern only if it admits reasonable interpretations as well as efficient computation algorithms. Characterizations and representation theorems may contribute to this and we therefore present the following as our second proposed field of further research.

Recall the representations

$$\omega \mapsto VaR_{r_t}\big(P_{X|\mathcal{F}_t}(\omega,.)\big) \;\in\; VaR_{t,T;r_t}(\widetilde{P}_{X|\mathcal{F}_t}) \quad \text{for all } t \in \mathbb{T}$$

and

$$\omega \mapsto ES_{r_t}\big(P_{X|\mathcal{F}_t}(\omega,.)\big) \;\in\; ES_{t,T;r_t}(\widetilde{P}_{X|\mathcal{F}_t}) \quad \text{for all } t \in \mathbb{T}$$

in terms of the static risk measures VaR_{r_t} and ES_{r_t} which were derived in subsections 4.4.1 and 4.4.2. In a slightly different context, Weber proves such a representation for general distribution invariant dynamic risk measures in terms of static risk measures in his recent paper [23]. Moreover, if the distribution invariant dynamic risk measure satisfies certain dynamic consistency properties, then the representing static risk measures are independent from date $t \in \mathbb{T}$. We suspect that such a representation is valid in our context as well, however we propose a different approach to the task of a proof. By means of conditional quantiles we believe that we are able to generalize the static case representation results for distribution invariant convex risk measures that are continuous in a mild sense, as they are presented by Föllmer and Schied in section 4.5 of [17]. In particular, this would yield a characterization of the represented distribution invariant dynamic risk measure in terms of conditional expected shortfall. More precisely, the conjecture is a representation of the form

$$\omega \mapsto \sup_{\mu \in \mathcal{M}_1(0,1)} \left(\int_{(0,1)} ES_s\big(P_{X|\mathcal{F}_t}(\omega,.)\big)\; \mu(ds) - \beta_t(\mu) \right) \;\in\; \rho_{t,T}(\widetilde{P}_{X|\mathcal{F}_t}) \quad \text{for all } t \in \mathbb{T}$$

for an arbitrary distribution invariant dynamic convex risk measure $(\rho_{t,T})_{t\in\mathbb{T}}$ that is continuous in a mild sense in terms of suitable penalty functions β_t, $t \in \mathbb{T}$, where $\mathcal{M}_1(0,1)$ denotes the space of all probability measures on $(0,1)$. It stands to reason that time-consistency of the distribution invariant dynamic convex risk measure has strong consequences on the representing static risk measures

$$\sup_{\mu\in\mathcal{M}_1(0,1)} \left(\int_{(0,1)} ES_s(.)\ \mu(ds) - \beta_t(\mu) \right), \quad t \in \mathbb{T}.$$

However, since Weber is working with different dynamic consistency properties than we are, it is not clear whether the representing static risk measures will be independent from $t \in \mathbb{T}$ if the represented dynamic convex risk measure is time-consistent. Still, one may hope for a strong representation result for distribution invariant dynamic convex risk measures that are continuous in a mild sense.

And then, when such representations are established for bounded random variables, it seems positively challenging to explore corresponding results in the context of dynamic monetary utility functionals for bounded discrete-time processes.

Bibliography

[1] Artzner, Ph., Delbaen, F., Eber, J.M., Heath, D., (1997). Thinking coherently. *RISK*, **10**, November, 68–71.

[2] Artzner, Ph., Delbaen, F., Eber, J.M., Heath, D., (1999). Coherent Risk Measures. *Mathematical Finance*, **9(3)**, 203–228.

[3] Artzner, Ph., Delbaen, F., Eber, J.M., Heath, D., Ku, H., (2002). Coherent Multiperiod Risk Measurement. Preprint.

[4] Artzner, Ph., Delbaen, F., Eber, J.M., Heath, D., Ku, H., (2004). Coherent multiperiod risk adjusted values and Bellman's principle. Preprint.

[5] Bauer, H., (2002). Wahrscheinlichkeitstheorie. de Gruyter Lehrbuch.

[6] Cheridito, P., Delbaen F., Kupper M. (2004a). Coherent and convex risk measures for bounded càdlàg processes, *Stoch. Proc. Appl.* **112**(1), 1–22.

[7] Cheridito, P., Delbaen F., Kupper M. (2004b). Coherent and convex risk measures for unbounded càdlàg processes. *Finance and Stochastics*, **9(3)**, 369–387.

[8] Cheridito, P., Delbaen F., Kupper M. (2004). Dynamic monetary risk measures for bounded discrete-time processes. to appear in *Electronic Journal of Probability.*

[9] Cheridito, P., Kupper M. (2006). Composition of time-consistent dynamic monetary utility functions in discrete time. Preprint.

[10] Delbaen, F., (2001). Coherent Risk Measures. Lecture Notes, Scuola Normale Superiore di Pisa.

[11] Delbaen, F., (2002). Coherent risk measures on general probability spaces. *Advances in Finance and Stochastics*, 1–37, Springer-Verlag, Berlin.

[12] Delbaen, F., (2003). The structure of m-stable sets and in particular of the set of risk neutral measures. Preprint.

[13] Detlefsen, K and G. Scandolo (2005). Conditional and dynamic convex risk measures. *Finance and Stochastics*, **9(4)**, 539–561.

[14] Frittelli, M., Gianin, E.R. (2002). Putting order in risk measures. *Journal of Banking and Finance* **26(7)**, 1473-1486

[15] Föllmer, H., Schied, A., (2002a). Convex measures of risk and trading constraints. *Finance and Stochastics*, **6(4)**, 429–447.

[16] Föllmer, H., Schied, A. (2002b). Robust preferences and convex measures of risk. *Advances in Finance and Stochastics*, 39–56, Springer-Verlag, Berlin.

[17] Föllmer, H., Schied, A., (2004). Stochastic Finance, An Introduction in Discrete Time. de Gruyter Studies in Mathematics **27**.

[18] Karatzas, I., Shreve, S.E., (2000). Brownian Motion and Stochastic Calculus. Graduate Texts in Mathematics **113**.

[19] Kupper, M., (2005). Monetary Risk Measures for Stochastic Processes. Ph.D. Thesis, Eidgenössische Technische Hochschule Zürich.

[20] Riedel, F. (2004). Dynamic coherent risk measures. *Stoch. Proc. Appl.*, **112**(2), 185–200.

[21] Swiss Federal Office of Private Insurance. (2004). White Paper of the Swiss Solvency Test.

[22] Wang, T. (2003). Conditional preferences and updating. *J. Econom Theory*, **108(2)**, 286–321.

[23] Weber, S. (2006). Distribution-Invariant Risk Measures, Information, And Dynamic Consitency. *Mathematical Finance*, **16(2)**, 419–441.

Solemn Declaration:

I hereby declare that this thesis was written independently and that no other sources have been availed than those indicated. Hitherto, this thesis has not been presented to any further examination authority nor has it been previously published.

Munich, June 23rd, 2006