N° d'ordre NNT : 2016LYSE1161

THÈSE DE DOCTORAT DE L'UNIVERSITÉ DE LYON
opérée au sein de
l'Université Claude Bernard Lyon 1

École Doctorale ED 486
Sciences Économiques et de Gestion

Spécialité de doctorat : Sciences de Gestion
Discipline : Sciences actuarielles

Soutenue publiquementle 20/09/2016, par :
Julien Vedani

Conceptualisation et Mise en Œuvre du processus Own Risk and Solvency Assessment pour l'Assurance Vie

---

Devant le jury composé de :

Nicole EL KAROUI, Professeure à l'Université Pierre et Marie CURIE, Paris 6	Présidente
Pierre DEVOLDER, Professeur à l'Université Catholique de Louvain	Rapporteur
Jean-Paul LAURENT, Professeur à l'Université Paris 1 Panthéon Sorbonne	Rapporteur
Christian ROBERT, Professeur à l'Université Lyon 1	Examinateur
Stéphane LOISEL, Professeur à l'Université Lyon 1	Directeur de thèse
Jean-Luc PRIGENT, Professeur à l'Université de Cergy-Pontoise	Co-directeur de thèse
Laurent DEVINEAU, Directeur scientifique, Milliman	Tuteur de thèse CIFRE

Remerciements

La thèse de doctorat représente un travail s'inscrivant dans la durée, et pour cette raison, constitue le fil conducteur d'une tranche de vie de son auteur, parfois au crépuscule de la candeur étudiante, et souvent à l'aube de la maturité scientifique. De nombreuses personnes se retrouvent ainsi de manière fortuite ou non, pour le pire ou le meilleur, entre le doctorant et son doctorat. Ce sont certaines de ces personnes que j'aimerais mettre en avant dans ces remerciements.

Je remercie chaleureusement toutes les personnes qui m'ont aidé pendant l'élaboration de ma thèse à commencer par mes directeur et co-directeur, Stéphane Loisel et Jean-Luc Prigent, pour leur intérêt et leur soutien, leur disponibilité (comme dirait Einstein, tout est relatif) et leurs très nombreux conseils quant à la préséance dans les dîners de gala colombiens et, plus important, durant la rédaction de mes articles et de ma thèse. Merci aussi à Nicole El Karoui pour toutes nos heures de travail, plus ou moins fructueuses mais toujours extrêmement intéressantes, à Jussieu comme ailleurs. Vous m'avez tous les trois nourrit avec tant de "food for thought", et je compte sur vous pour que cela continue.

Ce travail n'aurait pas été possible sans le soutien et la confiance de Milliman Paris, à commencer par Jérôme Nebout et Eric Serant, sans oublier évidemment Laurent Devineau, à qui je dois le plus grand de mes connaissances pratiques actuarielles et que j'ai connu comme professeur, comme boss et comme ami. Merci aussi à tous les autres, tous ceux que j'ai côtoyés, comme associés, directeurs, managers, consultants et stagiaire. Vous m'avez tous beaucoup appris.

Je me dois de remercier l'Université Claude Bernard Lyon 1, l'Ecole Doctorale de Sciences économique et de Gestion et l'ANRT. Ils ont simplement permis à un rêve de mon esprit de se concrétiser. Je connais aujourd'hui la complexité de l'inscription en études doctorale (pour la 5^{me} année je ne vous raconte pas) alors je ne peux qu'imaginer celle nécessaire pour gérer tant de dossiers et je vous remercie humblement.

Ce travail n'aurait pu être mené à bien sans la disponibilité et l'accueil chaleureux que m'ont témoignés les membres du laboratoire SAF. Merci pour cette ambiance et cette émulation entre chercheurs qui m'ont tant apporté. Merci à tous mes co-thésards, les anciens comme les nouveaux, Yahia, Christophe, Xavier, Ibra, Alex, Pierro, Jean de Dieu [...] et les plus belles, Anissa, Iness, Claire, Gwladys [...] et tous les autres que j'oublie dans ces lignes mais qui partagent avec moi des souvenirs souvent mémorables et toujours funky. Ah et un grand merci à LoLitA pour m'avoir dévoilé ses charmes rares...

Au terme de ce parcours, je remercie enfin celles et ceux qui me sont chers et que j'ai quelque peu délaissés ces derniers mois (années ?) pour achever cette thèse. Leurs attentions et encouragements m'ont accompagné tout au long de ces années. Je suis mille fois redevable à ma mère, Martine, pour son soutien moral et matériel et je remercie aussi fortement mes grands-parents, Jean-Pierre et Renée Amary, les premiers à m'avoir encouragé dans mon travail de thèse. Enfin, j'ai une pensée particulièrement forte pour mon père, Thierry, absent aujourd'hui et à qui je veux dédier ce travail.

Pour finir, je veux remercier celle qui partage ma vie depuis bientôt deux ans. Merci Sophie.

À Martine et Thierry

Résumé

La directive Solvabilité II, soumise par la Commission Européenne en 2009, est rentrée en application en janvier 2016. L'objectif du texte est de formaliser un cadre de solvabilité réglementaire pour le marché européen de l'assurance et de la réassurance, menant à une définition du capital requis plus adaptée aux spécificités des entreprises, à une meilleur compréhension des risques des assureurs et à une diminution du risque systémique en Europe. Il corrige, en particulier, un certain nombre de limites de la directive précédente, Solvabilité I (2002). Il se base sur trois piliers. Le premier pilier traite des obligations quantitatives liées au calcul du capital de solvabilité requis. Le second pilier, plus qualitatif, traite de la gouvernance des risques. Le troisième pilier concerne les documents et informations requis, la discipline de marché. Pour l'assurance vie, les obligations quantitatives (pilier I et une partie du pilier II) introduisent un haut niveau de complexité. En effet, pour créer un dispositif adapté aux spécificités des entreprises, la directive a introduit un cadre de valorisation du bilan des assureurs très particulier, la valorisation économique, basée sur la notion de *juste valeur*. Ce concept, déjà présent au cœur des normes comptables internationales IFRS, est particulièrement difficile à appliquer aux passifs d'assurances vie. Du fait de cette complexité, la plupart des assureurs vie européens ont, durant leurs premières années passées à implémenter la directive (2004 à 2010), choisi de se focaliser sur le pilier I en sachant que le calcul de l'exigence en capital serait une part essentielle du dispositif, et en supposant que les deux autres piliers seraient plus aisés à mettre en œuvre.

Dans cette thèse, j'ai choisi de concentrer mon travail sur le second pilier de la directive et plus précisément sur le processus Own Risk and Solvency Assessment (ORSA). Cet outil réglementaire est en fait la seconde source de complexité majeure de Solvabilité II. L'ORSA est un processus de gestion des risques totalement intégré à l'entreprise dont l'objectif est de mener les assureurs à une meilleure compréhension de leurs risques, d'un point de vue plus entrepreneurial et stratégique. Mais il est, toutefois, toujours intimement lié à la valorisation économique, ce qui implique, dès lors, une grande complexité. De plus, les textes réglementaires ne fournissaient que des directions très générales et des concepts imprécis aux actuaires, telles que les notions de conformité permanente et de Besoin Global de Solvabilité. Ces termes, complexes à appréhender, laissaient de nombreuses interrogations den suspens pour les praticiens, et ne donnaient guère de réponses.

Au cours de mon travail, j'ai cherché à conceptualiser et à proposer des mises en œuvre opérationnelles pour répondre aux problématiques induites par l'ORSA. J'ai formalisé les concepts de conformité permanente et de Besoin Global de Solvabilité et proposé des outils pragmatiques pour ces obligations réglementaires, toujours en gardant à l'esprit les objectifs business et stratégiques de l'ORSA. Afin d'aider les actuaires dans leur gestion des lourdes problématiques opérationnelles liées à l'utilisation du cadre de valorisation économique, que ce soit dans le calcul du capital requis, ou de l'implémentation de l'ORSA, j'ai travaillé à l'étude d'un algorithme d'accélération basé sur la notion de Valeur Actuelle Nette de marges *forward*. Sa mise en œuvre a menée à une division de la complexité liée au calcul du Capital de Solvabilité Requis par 100 (comparée à l'algorithme simulatoire standard). J'ai aussi étudié des méthodes d'amélioration et de correction de l'erreur liée à l'utilisation d'approches approximées pour la projection de bilans économiques. Enfin, au travers d'un travail commun avec N. El Karoui, S. Loisel et J.-L . Prigent, nous avons analysé et exemplifié certains des dangers majeurs induits par la valorisation économique. Nous avons notamment montré, au travers de plusieurs exemples, l'instabilité et la manipulabilité de ce cadre de valorisation. Ces analyses ont mené à la proposition de plusieurs alternatives pour la mise en place d'un système de régulation de l'assurance vie en Europe plus sûr et rigoureux.

Mots-clés: Solvabilité II, assurance vie, ORSA, conformité permanente, Besoin Global de Solvabilité, proxies, valorisation économique

Abstract

The Solvency II directive issued in 2009 by the European Commission has been put into action in January 2016. The objective of the text is to formalize a solvency framework for the whole European insurance and reinsurance market, leading to more entity-adapted regulatory capital requirements, to a better understanding of the undertakings risks and to a reduction of the European systemic risk. In particular, it fixes a certain number of limits induced by the former Solvency I directive (2002). It is based on three pillars. The first pillar addresses the quantitative requirements to assess the solvency capital needs. The second pillar, more qualitative, addresses the risks governance. The third pillar addresses the required disclosures. For life insurance, the quantitative requirements (pillar I and a part of pillar II) have introduced a high level of complexity. Indeed, to create an entity-adapted scheme, the directive has developed a very specific process to evaluate the insurance balance sheets, namely the economic valuation, based on the notion of *fair value*. This concept, fully embedded in the International Financial Reporting Standards (IFRS) accounting norms is especially difficult to apply to life insurance liabilities. Considering this complexity, most European life insurance undertakings have chosen to focus on pillar I, at the beginning of the implementation of the directive (from 2004 to 2010), knowing the regulatory capital assessment is an essential part of the solvency scheme and assuming the two other pillars would be easier to apply.

In this thesis I focus my work on the second pillar of the directive and more precisely on the Own Risk and Solvency Assessment (ORSA) process. This regulatory tool is the second major source of complexity when implementing the directive. The ORSA is a complete undertaking-embedded risk management process which aims to deepen the insurance knowledge of its risks in a more busyness and strategic light. But it is still associated to economic valuation, which leads to great operational complexity. In addition, regulatory documents only supplied actuaries with general directions and broadly defined concepts, such as continuous compliance and Overall Solvency Needs. These terms, complex to apprehend, led practitioners to numerous interrogations and little answers.

In my work I have tried to conceptualize and propose operational implementations to answer the ORSA issues. I have formalized the continuous compliance and Overall Solvency Needs concepts and proposed pragmatic implementation tool for these regulatory requirements, still keeping in mind the business and strategic objectives of the ORSA process. To help actuaries dealing with the high operational complexities induced by the economic valuation scheme, be it for regulatory capital assessment or ORSA implementations, I have studied an acceleration algorithm based on the so-called *forward* Net Present Value of margins. Its implementation leads to a reduction of the Solvency Capital Requirement assessment complexity by 100 times. I have also studied possibilities to improve and correct approximation methods to accelerate the projection of economic balance sheets. Finally, through a joint work with N. El Karoui, S. Loisel and J.-L. Prigent, we have underlined, analyzed and exemplified some of the major hazard sources induced by the economic valuation. We have for example shown, through various examples, the instability and manipulability of this valuation scheme. These analysis have led us to propose different alternatives for a safer and cleaner European life insurance regulatory framework.

Keywords: Solvency II, life insurance, ORSA, Continuous compliance, Overall Solvency Need, proxies, economic valuation

Table des matières

Introduction

Les travaux présentés ici ont été réalisés dans le cadre d'une thèse CIFRE réalisée chez Milliman Paris et au laboratoire de Sciences Actuarielle et Financière de l'Université Lyon 1, débutée en mars 2012.

Dans cette introduction j'explicite tout d'abord le contexte de mon étude. Puis, après avoir donné de premiers éléments de notation, je replace mes travaux dans la littérature existante. Enfin, je présente les différents articles et principaux résultats de cette thèse.

Mise en contexte de mes travaux

L'épargne est une des sources principales de financement de l'économie française, en fournissant aux organismes financiers (banques, assurances) des ressources qu'ils peuvent réinvestir sur les marchés financiers, en échange d'un rendement fourni au souscripteur. L'assurance vie est un des produits qui jouent un rôle clé dans la constitution de cette épargne : l'ensemble des contrats d'assurance vie constitue en effet 57% de l'épargne longue des ménages en 2011 (voir Fédération Française des Sociétés d'Assurance (2011)), soit environ 1500 milliards d'euros, ce qui représente environ 78% du PIB français (de l'ordre de 1900 milliards d'euros, source : INSEE). Les contrats d'assurance vie sont principalement partagés entre contrats en Unités de Compte (UC) et contrats en euros (voir Bonnin *et al.* (2014), Petauton (2002)). Dans le cadre de cette thèse, je m'intéresse aux contrats en euros, qui sont les seuls comportant un risque pour l'assureur (voir par exemple Albizzati et Geman (1994), Bouychou et Attal (2010)).

Toutefois, les contrats d'épargne en euros sont des produits complexes, dont la valorisation constitue un enjeu important dans l'optique d'assurer la pérennité de la compagnie d'assurance. En effet, il existe de nombreux types de contrats, avec des clauses spécifiques (Garanties plancher, Taux Minimum Garanti... voir Palerm (2006), Tosetti *et al.* (2003) pour une définition plus fine de ces sources d'optionalité) qui font qu'il est extrêmement difficile de définir un cadre d'étude unique. Toutefois, ils présentent quelques caractéristiques communes. D'une part les intérêts versés sur l'épargne doivent être au moins égaux à 60% du taux moyen des emprunts d'état, lorsqu'il n'y a pas de taux minimal garanti par le contrat. D'autre part, l'assureur est tenu de reverser une partie de ses bénéfices aux assurés, au titre de la Participation aux Bénéfices.

La réversion de cette provision peut passer par la constitution d'une Provision pour Participation aux

Excédents, dans laquelle l'assureur met une partie de ses bénéfices financiers et techniques, destinée à lisser les rendements et à pallier d'éventuels mauvais résultats. Cette provision doit être reversée dans les huit ans suivant sa constitution.

De leur côté, les assurés ont la possibilité de racheter leur épargne à tout moment de la vie du contrat ; le montant d'épargne présent sur le contrat peut également être reversé suite à certains incidents (décès du client assuré, par exemple). On distingue ainsi deux types de rachat de l'épargne : le rachat structurel, qui a lieu indépendamment des conditions économiques, et le rachat conjoncturel, lié aux conditions économiques.

L'assureur utilise l'épargne collectée auprès de sa clientèle pour constituer un portefeuille d'actifs destiné à lui permettre de verser les flux associés à son portefeuille d'épargne.

A l'heure de l'homogénéisation des réglementations internationales comptables et prudentielles bancaires (normes IFRS, voir Raffournier (2005), Bâle II, voir de Bâle (1988)), il est rapidement apparu que la grande complexité des produits d'assurance nécessitait une réglementation prudentielle particulièrement pointue et efficace.

Ainsi, dans la lignée du monde bancaire et des accords de Bâle II (voir Schubert et Grießmann (2004), Gatzert et Wesker (2012)), le Parlement Européen a voté en avril 2009, deux ans après les premiers travaux de la Commission, une réforme réglementaire du monde de l'assurance visant à obtenir une meilleure adéquation entre les fonds propres détenus par les compagnies d'assurance et les risques liés à leur activité, la directive Solvabilité II (European Commission (2009)). Sur le modèle de Bâle II, la réglementation comporte trois piliers. Le pilier I concerne les exigences quantitatives (fonds propres et provisions techniques). Le pilier II concerne les exigences qualitatives (gestion des risques et gouvernance). Enfin, le pilier III concerne la transparence de l'entreprise.

La directive, Solvabilité II, est entrée en application en janvier 2016. Dans ce nouveau cadre, l'assureur doit (premier pilier) associer à son portefeuille de clients un montant de fonds propres lui permettant de garantir la pérennité de son activité à un horizon d'un an dans 99.5% des cas (une probabilité de ruine de 0.5%).

La principale complexité induite par cette contrainte régulatoire tient au cadre de valorisation des postes du bilan de l'assureur, sur lesquels la notion de ruine se définit. De fait, dans le cadre de la réglementation, un assureur est tenu d'évaluer ses engagements à leur valeur de marché. Il s'agit pour cela d'estimer les flux futurs probables actualisés sur le passif. Ce calcul met en œuvre un processus assez lourd, tant en termes d'outils que de données. Il est effectué à l'inventaire. On obtient ainsi une évaluation de la valeur des engagements appelée *best estimate*. De son côté, la valorisation de l'actif économique se rapproche de la juste valeur (voir Grosen et Jørgensen (2000), qui évoque directement la notion de juste valeur de passif d'assurance) introduite par les normes comptables internationales (IAS, IFRS). Les fonds propres économiques, la variable d'intérêt du cadre Solvabilité II, permettant de définir la contrainte de solvabilité présentée ci-dessus, s'obtiennent par différence entre l'actif et le *best estimate*.

La directive introduit ainsi la notion de valeur économique, lui permettant de définir le concept de bilan économique, distinct du bilan comptable plus usuel. Dans la vision économique, les différents postes du bilan de l'assureur (actif, passif et fonds propres), doivent être valorisés à l'aide d'une approche simulatoire appliquée dans un univers de probabilité où les actifs évoluent de manière risque neutre (voir Babbel et Merrill (1998), Kemp (2009), Wüthrich *et al.* (2008), Moehr (2011) et Pelsser (2010) pour la notion associée de *market-consistency*). En pratique, il s'agira de réaliser des valorisations de chaque poste du bilan par méthode de Monte-Carlo (voir par exemple Rubinstein (1981), Glasserman (2003), et Korn *et al.* (2010) pour des exemples liés à l'assurance) en utilisant des tables de scénarios économiques, dites risque neutre.

Une fois le calcul du bilan économique effectué, l'organisme d'assurance calcule le montant de capital de solvabilité (*SCR*, Solvancy Capital Requirement) requis par le régulateur. Ces capitaux peuvent en pratique être obtenus selon plusieurs types de procédures (voir Devineau et Loisel (2009b) pour une étude poussée des deux premières approches, Börger (2010) et Gatzert et Wesker (2012) pour une comparaison de ces approches du point vue de sources de risque spécifiques).

Une approche simplifiée, la Formule Standard.

La Formule Standard permet une détermination du *SCR* en se basant sur des calculs de chocs immédiats. Elle permet une approche analytique basée sur les données historiques de l'entreprise afin de déterminer de manière cohérente les risques les plus consommateurs en capital pour la société. Le sous-capital nécessaire pour chaque risque est déterminé comme écart entre deux valeurs de fonds propres économiques, centrale et choquée. La valorisation de ces fonds propres économiques nécessite généralement de disposer d'un modèle de gestion actif-passif simple. Les sous-capitaux obtenus pour chaque risque sont ensuite agrégés, par module de risques, puis entre modules afin d'obtenir un *SCR* global. Les corrélations utilisées pour l'agrégation sont soit des paramètres standards proposés par le superviseur (EIOPA : European Insurance and Occupational Pensions Authority, voir EIOPA (2010a)), soit des coefficients déterminés et justifiables par l'entité. Cette méthode repose sur un certain nombre d'hypothèses simplificatrices. En particulier, elle suppose que l'évolution historique de la couverture de l'assureur restera la même à l'avenir. Toutefois, lorsque le temps s'écoule, l'environnement économique et financier ainsi que le portefeuille (d'actif et de passif) se déforment, de même que les *SCR* par module de risque. Sa simplicité a entraîné son adoption par un grand nombre de petits / moyens acteurs du marché. Cependant ses hypothèses et approximations en font un outil moins fiable que les méthodes mettant en œuvre un modèle interne. En particulier, elle donne une vision imparfaite du profil de risque de l'assureur (voir par exemple Pfeifer et Strassburger (2008) ou EIOPA (2010b)).

Une méthode prospective totalement spécifique, le modèle interne.

L'utilisation d'un modèle interne permet de répondre à certaines lacunes de la Formule Standard. Son objectif est de pouvoir faire évoluer les différents risques de la société sur un an afin de déterminer un profil exhaustif de risque. C'est donc une approche simulatoire qui est considérée, permettant de projeter le bilan de la société dans le futur. L'entreprise peut en effet déterminer des réalisations de valeurs économiques de l'actif, du passif et des fonds propres, dans 1 an, associées à chacun de ses produits. La projection du bilan économique associé à un produit s'appuie sur une modélisation stochastique des différents risques financiers auxquels il est soumis. La plateforme de simulations à créer doit donc être capable de modéliser les corrélations entre ces risques et d'introduire des contraintes et des procédures de couverture (atténuation des risques) reflétant le comportement de chaque produit. Le traitement des interactions actif-passif est assuré par un modèle de gestion spécifique. Si la société d'assurance utilise un modèle interne, l'application de la directive Solvabilité II revient à déterminer le quantile à 0,5% des fonds propres de l'entreprise dans 1 an et à s'assurer que ceux-ci sont positifs ou, dans le cas contraire, à rajouter un montant dans les fonds propres actuels ($t = 0$) suffisant pour que ce quantile soit au moins égal à 0. La mise en place d'un modèle interne et son calibrage permettent d'obtenir une valeur du besoin en capital plus fiable et généralement plus faible que celle obtenue par la Formule Standard. Sa grande complexité, comparée à une approche analytique, en fait un outil plus aisément accessible aux grands groupes. Cependant, l'utilisation d'un modèle interne complet et flexible constitue, à terme, un objectif stratégique quelle que soit la taille de l'acteur.

Une approche mixte, le modèle interne partiel.

Le modèle interne partiel est une approche mixte permettant de déterminer les *SCR* associés à certains modules de risque spécifiques par approche simulatoire à un an puis d'agréger ces capitaux par une mise en œuvre de type Formule Standard (voir Planchet *et al.* (2010) pour plus d'informations sur cette approche alternative).

Dans le cadre du second pilier de la directive est défini un système intégré de gestion stratégique du risque porté par le portefeuille de l'entreprise d'assurance vie, le processus ORSA (Own Risk and Solvency Assessment). Ce processus introduit deux définitions particulièrement difficiles à appréhender et à mettre en œuvre par les entreprises d'assurance vie.

Le Besoin Global de Solvabilité.

Dans un premier temps, l'ORSA introduit la notion de Besoin Global de Solvabilité (BGS), un capital nécessaire pour la couverture de la solvabilité de l'assureur sur un horizon de temps pluriannuel. Ce capital de solvabilité alternatif doit être afférent à une contrainte de solvabilité de moyen terme (souvent un horizon de 3 à 10 ans), et ce sur une variable propre à l'entreprise (pas nécessairement la variable fonds propres économiques, contrairement au cadre défini par la contrainte réglementaire). L'assureur doit définir cette contrainte de manière stratégique et propre à ses limites de tolérance au risque. Ici, la problématique est double. D'un côté il s'agit de définir le concept de solvabilité pluriannuelle, induisant de nombreuses questions, allant de l'horizon considéré au seuil de probabilité choisi, et passant par le cadre de valorisation (comptable, économique,...). D'autre part, il s'agit de proposer des outils efficaces de calcul du BGS permettant de mettre en œuvre des projections de la variable d'intérêt sur plusieurs années, tandis que le cadre usuel de valorisation régulatoire, déjà complexe à conceptualiser et à implémenter, ne s'applique que sur une contrainte à un an.

La conformité permanente.

De plus, l'entreprise d'assurance est tenue de pouvoir justifier à tout moment au régulateur qu'elle satisfait aux contraintes de marge de solvabilité ; on parle alors de conformité permanente. Mais elle ne peut pas raisonnablement refaire l'intégralité du calcul de *best estimate*. Il est donc nécessaire de mettre en place des outils permettant de mesurer la déformation du profil de risque des *SCR* de chaque module pour en déduire la variation du *SCR* global lorsque les hypothèses et données du calcul évoluent.

La conceptualisation et la mise en place du processus ORSA sont particulièrement stratégiques pour les assureurs vie européens puisqu'il leur permet de faire le lien entre les exigences quantitatives régulatoires du pilier I, la gestion des risques internes et le besoin d'informations tant qualitatives que quantitatives des investisseurs. Il convient alors de rappeler qu'en 2012 (année de début de ma thèse), tous les acteurs du marché étaient majoritairement occupés au développement interne du pilier I. En effet, la mise en œuvre de celui-ci représentait déjà une source de difficulté majeure, tant conceptuelle que d'implémentation, pour des acteurs ayant découvert la notion de valeur économique en 2009. La mise en œuvre de l'ORSA représentait donc une problématique sérieuse, mise jusqu'alors de côté afin de pallier aux difficultés liées aux exigences réglementaires plus directes.

Cette thèse a été réalisée sous une convention CIFRE, dans un cabinet de conseil en actuariat, ce qui m'a permis une analyse comparée efficace des diverses pratiques opérationnelles quant à la mise en œuvre des aspects quantitatifs de la directive Solvabilité II. D'autre part, le travail dans le cadre du laboratoire de Sciences Actuarielle et Financière m'a permis de développer une vision plus théorique sur les aspects quantitatifs induits par la régulation, tant concernant l'ORSA que pour les obligations liées au calcul de *SCR*.

Éléments de notation

Avant de développer plus avant les articles et résultats présentés dans cette thèse, il convient d'introduire différentes notions essentielles de comptabilité économique, nécessaires à la bonne compréhension des problématiques, majoritairement simulatoires, induites par la bonne application de la directive Solvabilité II dans le cadre de l'assurance vie européenne. Avant toute chose, considérons les objets sur lesquels sont définis les aspects simulatoires de la directive. Le bilan économique tout d'abord, est un cadre standard de comptabilité introduit par la régulation.

Le concept de valorisation économique.

Tout comme le bilan comptable, le bilan économique est séparé en un actif et un passif, mais tous deux sont estimés en juste valeur dite *market-consistent*. Pour l'actif il s'agit de valoriser les différents postes à leur valeur de marché (financier). Pour le passif, décomposé en des fonds propres économiques et la valeur économique des engagements de l'assureur (le *best estimate*, ou simplement la valeur économique de passif), la valorisation s'avère plus complexe. L'idée est d'estimer le *best estimate* comme étant la valeur estimée qu'auraient ces engagements sur un marché complet et liquide. En pratique cette valeur est obtenue comme espérance de la valeur actuelle nette des flux futurs de passif économique obtenus sous une probabilité risque neutre. Seuls les risques financiers ("hedgeables") sont pris en compte ce qui simplifie les choses. De plus, on suppose qu'il n'y aura aucun contrat dans le futur. Les assurés existants à la date initiale sont donc progressivement amenés à racheter leurs contrats, à mourir, ou à arriver au terme de leurs contrats, en un temps maximal donné T. Il convient dès lors de disposer d'une table de N (N élevé) scénarios prospectifs financiers, évoluant sous une bonne probabilité risque neutre entre la date de valorisation et l'horizon d'extinction des contrats (T). Le bilan comptable initial est projeté sur cette table, les cash-flows de passif, d'actif, et par soustraction (actif-passif) de marges (ou résultats) sont ainsi estimés pour chaque scénario et chaque date future. En notant δ_t^n (resp. L_t^n, R_t^n) le facteur d'actualisation (resp. cash-flow de passif, de marges), en date $t \in [\![1;T]\!]$ et simulation $n \in [\![1;N]\!]$, les valeurs économiques complexes sont estimées par approche Monte-Carlo, pour le *best estimate*,

$$BE_0 = \frac{1}{N} \sum_{n=1}^{N} \sum_{t=1}^{T} \delta_t^n L_t^n,$$

et pour les fonds propres,

$$FP_0 = \frac{1}{N} \sum_{n=1}^{N} \sum_{t=1}^{T} \delta_t^n R_t^n = \frac{1}{N} \sum_{n=1}^{N} VAN_0^n,$$

en notant VAN_0^n la Valeur Actuelle Nette de marges, vue en $t = 0$, associée au scénario n

Si l'état de l'économie a été préalablement projeté entre la date 0 et une date $t \geq 0$, il est alors possible de réutiliser une table de scénarios économiques, conditionnée par le niveau de l'économie en date t afin d'estimer FP_t et / ou BE_t.

Calcul du capital de solvabilité requis - approche modèle interne.

Revenons maintenant au cadre de l'estimation du *SCR* en approche modèle interne. Le principe est, pour l'assureur, d'estimer le niveau de fonds propres économiques nécessaire aujourd'hui afin de se couvrir contre la ruine économique dans un an au seuil de 99.5%, si les conditions économiques ont évolué de manière réaliste sur cette première année. Notons donc $\mathscr{P}$ la probabilité historique sous laquelle évoluent les conditions économiques. La contrainte réglementaire est donc,

$$\mathbb{P}^{\mathscr{P}} \left[FP_1 \geq 0 \right] = 99.5\%.$$

Le capital est, ensuite, souvent estimé par les assureurs à l'aide de l'approximation suivante,*

$$SCR_0 = FP_0 + q_{0.5\%} \left(\delta_1 FP_1 \right),$$

ou encore, par souci de simplicité,

$$SCR_0 = FP_0 + P(0;1) q_{0.5\%} \left(FP_1 \right).$$

En notant $P(t;m)$ le prix zéro coupon en date $t \geq 0$, pour la maturité $m \geq 0$.

Il est dès lors aisé de comprendre toute la complexité d'une telle mise en œuvre. La difficulté principale vient en effet de l'estimation de $q_{0.5\%} \left(\delta_1 FP_1 \right)$ (ou de $q_{0.5\%} \left(FP_1 \right)$), qui nécessite une estimation fine de la queue de distribution de la variable aléatoire $\delta_1 FP_1$ (ou de FP_1).

En pratique, la méthodologie la plus précise pour l'estimation du quantile susnommé est une approche par Monte-Carlo imbriqué, les simulations dans les simulations (SdS, voir Broadie *et al.* (2011)). Cette méthodologie consiste en un tirage aléatoire d'un grand nombre P de scénarios financiers "primaires" sous $\mathscr{P}$, puis, conditionnellement à chaque état de l'économie simulé à un an, de lancer une valorisation de Monte-Carlo sur une table de N scénarios "secondaires" risque neutre (voir illustration en Figure 1).

$*$. Cette approximation est vérifiée si le capital ajouté aux fonds propres initiaux est uniquement investi en actif non-risqué.

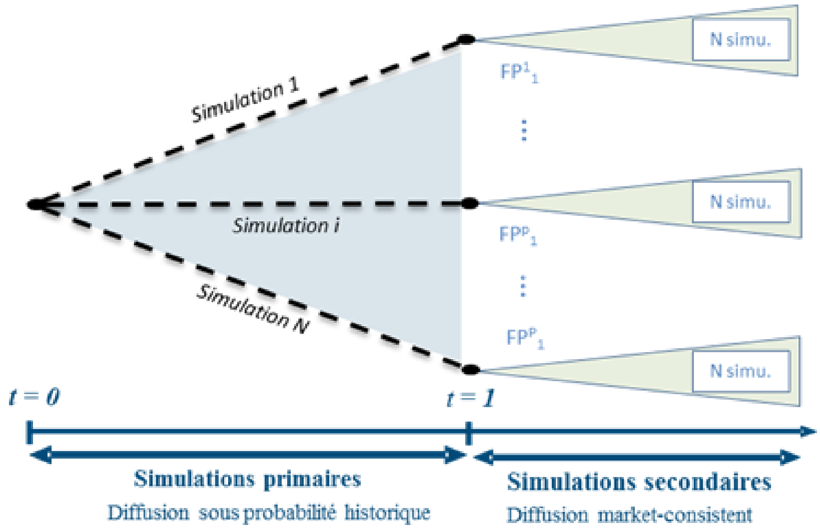

FIGURE 1 – Illustration de complexité algorithmique - Approche SdS pour le calcul du *SCR* modèle interne

Ce type d'approche Nested Monte-Carlo est très utilisé en simulation, voir par exemple Mak (1992). Le terme SdS est une pure adaptation au cadre assurantiel. Cette approche permet de disposer d'une distribution empirique approchée de la variable $\delta_1 FP_1$. De manière à estimer efficacement le quantile à 0.5% de cette variable, il s'agit de choisir P et N très élevés. En effet, la variable VAN_1 (variable aléatoire Valeur Actuelle Nette de marges en $t = 1$) est très volatile et les estimateurs de FP_1 nécessiteront de considérer N élevé afin d'éviter des erreurs de convergence trop élevées (en particulier dans la queue de distribution à gauche).

D'autre part, il s'agira de considérer P élevé afin de bien estimer le quantile très adverse recherché.

Ce type d'approche totalement simulatoire est donc à la fois complexe à mettre en œuvre et extrêmement coûteux en ce qui concerne le temps de calcul (complexité en $P \times N$). En pratique, en France, aucun assureur vie n'a souhaité s'attaquer à cette méthode. Les assureurs utilisant des approches par modèle interne, complet ou partiel, utilisent des approches approximées.

Calcul du capital de solvabilité requis - approche Formule Standard.

L'approche simplifiée *Formule Standard* peut aussi être utilisée pour calculer une valeur de SCR_0 sans nécessiter une telle complexité algorithmique.

L'idée de la Formule Standard est, dans un premier temps, d'estimer, pour la plupart des risques de l'entreprise, un niveau de capital marginal égal à la perte qui serait observée suite à un choc instantané équivalant au choc à $99,5\%$ sur un an, du risque considéré. Puis que ces capitaux marginaux puissent être agrégés de manière à reconstituer le *SCR* global.

À la date de calcul du SCR_0, notons FP_0 la valeur des fonds propres non choqués, FP_0^r la valeur des fonds propres économiques obtenus après application du choc instantané équivalent, fourni par le régulateur, au titre du risque r. La Formule Standard regroupe les risques par module de risque. Notons donc M le nombre de modules, et à un module $m \in [\![1;M]\!]$ donné, R_m le nombre de risques considérés.

Pour tout module $m \in [\![1;M]\!]$ et tout risque $r_m \in [\![1;R_m]\!]$. Le SCR marginal au titre du risque r_m est $SCR_0^{m,r_m} = FP_0 - FP_0^{r_m}$.
Le SCR du module m est $SCR_0^m = \sqrt{\sum_{s,r \in [\![1;R_m]\!]} \rho_m^{s,r} SCR^{s,m} SCR^{r,m}}$ (agrégation intra-modulaire). Le SCR global est $SCR_0 = \sqrt{\sum_{m,n \in [\![1;M]\!]} \rho^{m,n} SCR^m SCR^n}$ (agrégation inter-modulaire).

Les coefficients de corrélation inter- $((\rho^{s,r})_{s,r \in [\![1;R_m]\!]}, \forall m \in [\![1;M]\!])$ et intra- $((\rho^{m,n})_{m,n \in [\![1;M]\!]})$ modulaire de cette approche simplifiée sont fournis par le régulateur.

Cette approche ne requiert pas de simulation historique (simulations primaires de l'approche SdS pour le calcul du SCR modèle interne, voir illustration en Figure 2) et est majoritairement privilégiée par les acteurs du marché français. Notons R le nombre de risque total intégrant le cas "sans risque" = central, la complexité de cette approche est en $R \times N$ (avec $R \ll P$ en pratique).

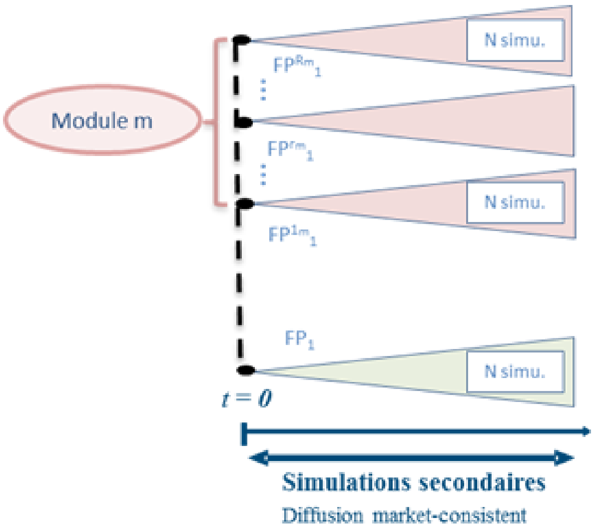

FIGURE 2 – Illustration de complexité algorithmique - Calcul du SCR Formule Standard (module de risque m)

Avant de présenter les différents articles et principaux résultats de ma thèse, la partie suivante a pour objectif de replacer mon travail dans la littérature académique et opérationnelle existante sur les sujets considérés.

Revue de littérature

La solvabilité dans l'ORSA - approche pluriannuelle et temps continu.

Les problématiques induites par la mise en œuvre de l'ORSA en assurance vie sont nouvelles pour les opérationnels et les académiques, peu habitués au suivi de la solvabilité dans le temps. Toutefois ces concepts ne sont pas nouveaux dans les sciences actuarielles et en particulier la théorie de la ruine qui s'intéresse depuis Lundberg (1903) et Cramér (1930) au suivi des capitaux d'une entreprise d'assurance non-vie stylisée, soumise aux sinistres de ses assurés, en temps continu.

Dans ces travaux, loin de l'approche statique du calcul de capital réglementaire sous Solvabilité II qui revient à estimer la probabilité de ruine (économique) dans 1 an, la solvabilité de la compagnie est suivie de manière continue (voir Gerber (1974), Gerber (1979), Embrechts et Veraverbeke (1982), Asmussen et Albrecher (2010), Albrecher *et al.* (2011)). Cependant, certains travaux étudient la probabilité de ruine en temps fini ce qui se rapproche du cadre de la directive (voir Picard et Lefèvre (1997), Ignatov *et al.* (2001)) ou encore à date d'inventaire (voir Rulliere et Loisel (2004)) ce qui se rapproche du cadre pluriannuel de l'ORSA.

Si ces modèles peuvent sembler trop théoriques, il est toutefois clair que les rapprochements avec le nouveau contexte régulatoire européen fournit des pistes de recherche pertinentes (voir Gerber et Loisel (2012) sur ce thème).

L'approche Simulations dans les Simulations.

L'approche SdS est considérée comme la méthode de projection la plus exacte théoriquement pour l'obtention de distributions prospectives de fonds propres économiques.

Si son utilisation en assurance vie est récente, sa théorie est plus largement développée dans les domaines nécessitant des simulations imbriquées, impliquant des calculs prospectifs d'espérance et de variance conditionnelles. Elle est, par exemple, utilisée pour des mesures de risque de portefeuille statiques (voir par exemple Gordy et Juneja (2010), Juneja et Ramprasath (2009), Lan *et al.* (2007), Sun *et al.* (2011)), ou en ingénierie simulatoire (méthode multilevel Monte-Carlo introduite par Heinrich (2001) pour le calcul d'intégrales paramétriques et développée par Giles (2008) pour le contrôle de l'erreur quadratique moyenne commise lors d'une mise en œuvre Monte-Carlo standard). Pour des applications au calcul d'équations différentielles stochastiques et partielles voir aussi Barth *et al.* (2011), Dereich (2011), Charrier *et al.* (2013). D'autre part, la littérature est fournie pour ce qui est de l'estimation optimisée d'un calcul de quantile lorsque la complexité d'estimation de la fonction d'intérêt est raisonnable (voir Embrechts *et al.* (1997) pour l'application à la théorie des valeurs extrêmes, Egloff *et al.* (2010) qui travaille sur l'inversion de la fonction de répartition empirique. Voir aussi Bardou *et al.* (2009), Arouna (2004)) mais ce n'est pas le cas ici où la fonction FP_1 pose plus de problèmes à approcher.

Toutefois, de nombreux travaux ont cherché à améliorer l'approche standard du Monte-Carlo imbriqué qui impose une complexité algorithmique trop lourde pour les assureurs, par une réduction directe des scénarios utilisés (Devineau et Loisel (2009a), Broadie *et al.* (2011)), par l'utilisation de méta-modèles (voir Liu et Staum (2010), Chen *et al.* (2012) pour l'application du krigeage stochastique), ou par l'utilisation de méthodologies de simulation avancées (voir en particulier les travaux récents de Lemaire *et al.* (2014) et Lemaire *et al.* (2015)).

Les Générateurs de Scénarios Économiques.

La valorisation des bilans assurantiels dans le cadre Solvabilité II requiert l'utilisation de tables de scénarios économiques. Pour leur obtention il s'agit de simuler des indices financiers (action, taux, crédit,...) dans un univers de probabilité risque neutre. Les outils utilisés sont appelés GSE, Générateurs de Scénarios Économiques.

Les modèles utilisés sont des modèles de marché. La littérature sur ces derniers est très riche, en particulier pour la modélisation action (Black et Scholes (1973), Merton (1973), Carr et Wu (2004), Heston (1993), Craine *et al.* (2000) - voir aussi Chernov *et al.* (2003)), taux (voir Brigo et Mercurio (2007), Black et Karasinski (1991), Harris (1995), Pelsser (2000), Hull et White (2000), Hull *et al.* (1993), Jamshidian (1989)). Il existe aussi des modèles de crédits (voir Jarrow *et al.* (1997), Longstaff *et al.* (2005)) pertinents pour les critères actuariels de risque neutralité.

Les Générateurs de Scénarios Économiques disposent de plus d'une littérature spécifique, agrégée (voir Wilkie (1984), Plomp (2013), Baldvinsdóttir et Palmborg (2011), Ahlgrim *et al.* (2004)).

D'autres part, d'autres types de scénarios peuvent être pertinents, des scénarios en univers *monde réel*. Les critères actuariels pour valider un modèle produisant de tels scénarios sont plus *statistiques*. C'est un critère de réalisme, les simulations produites doivent permettre de reproduire efficacement les trajectoires observées sur des historiques de données. La littérature est donc distincte et plus économétrique (voir Mills et Markellos (2008), Hamilton et Susmel (1994), Hamilton et Lin (1996)). Des modèles à changements de régime peuvent aussi donner des résultats particulièrement efficaces (voir Harris (1999), Hamilton (1990), Hamilton (2005), Hardy (2001)).

La valorisation* market-consistent *en assurance.

En comptabilité, les normes internationales International Financial Reporting Standards (IFRS) requièrent des compagnies cotées une valorisation complète de leurs états financiers en juste valeur, introduite comme "le montant lu pour lequel un actif pourrait être échangé, ou un passif éteint, entre parties bien informées, consentantes, et agissant dans des conditions de concurrence normale". Cette notion a été introduite par Hodges et Neuberger (1989) (voir aussi Davis (1997) qui construit cette notion à partir de la valeur marginale) et étendue par Barrieu et El Karoui (2007) pour des mesures de risque dans un cadre général, en présence de coûts de transaction (voir aussi Delbaen *et al.* (2002), Rouge et El Karoui (2000) ou Henderson et Hobson (2004)).

En économie, la juste valeur d'échange d'un bien est bien connue dans le cadre de la théorie de l'équilibre général (introduit par Walras (1874) et développée entre autre par Arrow et Debreu (1954)) où les bonnes propriétés du marché permettent de garantir l'existence et l'unicité de la probabilité et du prix d'équilibre.

Dans un cadre financier, les dérivés d'Arrow-Debreu, rapportant une unité du numéraire à une date future, peuvent être considérés comme la brique atomique des options financières dont le cash-flow dépend du temps. Ils déterminent ainsi le système de prix unique "risque neutre" (voir Becherer et Davis (2010)). Ainsi, de manière naturelle, la notion de prix d'équilibre, pour des produits financiers et dans le cadre des hypothèses du marché complet, dispose d'une littérature fournie (voir par exemple Cochrane (2001), Duffie (1996), Ingersoll (1987), Tsanakas et Christofides (2006), Föllmer et Schied (2011)).

Le cadre théorique du marché incomplet pose des problèmes de multiplicité des mesures de valorisation et des prix mais ces difficultés ont été étudiées dans la littérature financière (voir par exemple El Karoui et Quenez (1995), Carr *et al.* (2001)). Ainsi, par extension, la juste valeur d'un actif financier, telle qu'introduite par les normes IFRS, devient la valeur fournie par le marché de son sous-jacent.

La directive Solvabilité II va s'inspirer des nouvelles normes comptables pour introduire un système de valorisation spécifique, la valorisation économique. Dans ce cadre, tous les postes du bilan de l'assureur doivent être estimés à leur juste valeur.

Ce dispositif est toutefois particulièrement délicat à mettre en œuvre puisqu'il n'existe aucun marché pour les risques spécifiques des contrats d'assurance (rachats, mortalité en assurance vie, risques de catastrophe naturelle en assurance non-vie,...). Ainsi, si la juste valeur (économique) des actifs d'un assureur est fournie par les marchés financiers, celle d'un passif d'assurance pose de difficiles problèmes de conceptualisation et d'estimation. Il existe bien des marchés de rétrocession sur lesquels s'échangent des cat-bonds ("obligation catastrophe", de mortalité ou de catastrophes naturelles, voir Horst et Müller (2007), Muermann (2003), Schmock (1999)), mais leur manque de liquidité pose un réel problème d'objectivité et ne permet pas d'en déduire un système de valorisation complet, adapté à la régulation européenne.

En pratique, un cadre plus général sera donc développé autour de la notion de *market-consistency*. Ce concept permet d'étendre la valorisation de marché dans un cadre incomplet. Kemp (2009) en donne la définition suivante :

> "A market-consistent value of an asset or liability is its market value, if it is readily traded on a market at the point in time that the valuation is struck, and, for any other asset or liability, a reasoned best estimate of what its market value would have been had it been readily traded at the relevant valuation point."

Cette définition générale est au cœur de l'application de la directive Solvabilité II mais conserve une grande part de subjectivité. Tout un pan de la littérature actuarielle académique et opérationnelle s'est intéressé à sa mise en œuvre pour la valorisation de passifs d'assurance (voir en particulier Malamud *et al.* (2008), Sheldon et Smith (2004), Grosen et Jørgensen (2000), Wüthrich *et al.* (2008)).

Présentation des différentes parties de cette thèse et principaux résultats

Au travers des articles rédigés au cours de ma thèse, j'ai cherché à développer des méthodologies alternatives aux approches totalement simulatoires, afin de répondre aux problèmes majeurs posés par l'ORSA. Les deux premiers articles de cette thèse présentent les approches développées pour apporter une solution à la conceptualisation et à l'implémentation pratique de l'ORSA. Dans les troisième et quatrième parties de ma thèse je me suis attaché à trouver des pistes d'accélération des méthodes simulatoires, par l'utilisation d'algorithmes de calcul accélérés et l'amélioration de méthodologies par approximation. De manière générale, les résultats obtenus sont applicables opérationnellement tant aux mises en œuvre de type ORSA que pour des problématiques liées au calcul de *SCR*. L'article présenté en cinquième partie de cette thèse s'attache à analyser la légitimité et l'impact du cadre de valorisation économique sur la solvabilité du marché de l'assurance vie européen. Il conclut mes années d'analyse opérationnelle et théorique de la mise en œuvre de la directive Solvabilité II.

Chapitre 1.

L'article présenté en première partie du mémoire a été réalisé afin de traiter les problématiques induites par le calcul du BGS (article [a]).

Le BGS se présente comme une nouvelle forme de capital réglementaire, tel que le *SCR*, introduit dans le premier pilier de la directive. Toutefois, positionné au cœur du processus ORSA, il convient de le définir d'un point de vue plus entrepreneurial tenant compte de la stratégie et des objectifs de prise de risque des compagnies. De fait, si le cadre défini par la contrainte de solvabilité réglementaire, à court terme (1 an) et sur les fonds propres économiques uniquement, est suffisant pour assurer un suivi de la solvabilité année après année, il ne permet pas de tenir compte de la stratégie de l'entreprise, de ses prises de risque et de ses enjeux qui n'ont d'effet qu'à moyen/long terme. Le choix de la contrainte de solvabilité à utiliser était donc particulièrement important.

Définition de la contrainte de solvabilité adaptée au calcul du BGS.

Nous avons choisi de conserver le cadre de valorisation économique de la directive en considérant différentes alternatives de formulation pour proposer un panel complet des contraintes de solvabilité pluriannuelle.

Tout d'abord, rappelons la contrainte de solvabilité réglementaire. Elle requiert de disposer, à date de calcul $t = 0$, une valeur de FP_0 suffisante pour vérifier,

$\mathbb{P}(FP_1 \geq 0) \geq 99.5\%$, (CS0)

Pour la contrainte de solvabilité associée au calcul du BGS, il conviendra tout d'abord de choisir un horizon pluriannuel d'étude des capacités de l'entreprise, T (en pratique c'est souvent l'horizon du *business plan* de la compagnie qui est considéré).

Contraintes sur fonds propres économiques.

Une première possibilité consiste simplement à adapter cette contrainte à $T > 1$ années. Dans ce cas, deux alternatives sont envisageables selon que la *path-dependency* (le chemin suivi par les trajectoires) est prise en compte ou non :

$\forall t \in [\![1;T]\!], \mathbb{P}(FP_t \geq 0) \geq p$. (CS1)

Avec ce type de contrainte, les trajectoires ne sont pas prises en compte d'un seul tenant, seules les valeurs inférieures à 0 chaque année sont associées à un état d'insolvabilité. Le BGS est obtenu par la formule suivante,

$\text{BGS}_{(\text{SC1})} = FP_0 + K\ /: K = -min_{0<t\leq T}\left[q_{1-p}\left(\delta_t FP_t\right)\right].$ *

La prise en compte de la *path-dependency* permet d'obtenir une telle contrainte,

$\mathbb{P}\left(\cap_{t=1}^{T}\{FP_t \geq 0\}\right) \geq p.$ (CS2)

Avec ce type de contrainte, toute valeur de fonds propres économiques négatifs implique l'insolvabilité de l'intégralité de sa trajectoire. Le BGS est obtenu par la formule suivante,

$\text{BGS}_{(\text{SC2})} = FP_0 + K\ /: K = Argmin_X\left[\mathbb{P}\left(\cap_{t=1}^{T}\{FP_t + \frac{X}{\delta_t} \geq 0\}\right) = p\right].$

Dans ces deux contraintes, la probabilité p est à choisir par l'assureur de manière à traduire efficacement son appétence au risque (typiquement, $p < 99.5$).

Contraintes sur ratios de solvabilité

Une alternative au choix de contraintes sur fonds propres est de considérer, comme variable d'intérêt des contraintes, le ratio de solvabilité (fonds propres divisés par le capital réglementaire). Ce ratio traduit plus efficacement l'aspect stratégique de la contrainte car il permet d'intégrer un aspect rentabilité risque particulièrement intéressant pour les investisseurs et les gestionnaires des risques. On a donc de manière similaire, les deux contraintes suivantes,

$\forall t \in [\![1;T]\!], \mathbb{P}\left(\frac{FP_t}{SCR_t} \geq \alpha\right) \geq p$ (CS3)

et $\mathbb{P}\left(\cap_{t=1}^{T}\{\frac{FP_t}{SCR_t} \geq \alpha\}\right) \geq p.$ (CS4)

Nous obtenons deux contraintes tenant compte (CS4), ou non (CS3), de l'aspect *path-dependent* des trajectoires. Le BGS se calcule respectivement par, †

$\text{BGS}_{(\text{CS3})} = FP_0 + K\ /: K = Argmin_X\left[min_{0<t\leq T}\left[q_{1-p}\left(\frac{FP_t + \frac{X}{\delta_t}}{SCR_t(X)}\right)\right] \geq \alpha\right]$

et $\text{BGS}_{(\text{CS4})} = FP_0 + K\ /: K = Argmin_X\left[\mathbb{P}\left(\cap_{t=1}^{T}\{\frac{FP_t + \frac{X}{\delta_t}}{SCR_t(X)} \geq \alpha\}\right) \geq p\right].$

Ici, deux paramètres sont à choisir par la compagnie, p et α. Il est tout à fait possible de considérer $\alpha > 100\%$ de manière à respecter une valeur du ratio de solvabilité minimale dans l'objectif par exemple de conserver un *rating* minimal. De même, il est possible de considérer $\alpha < 100\%$ pour avoir une limite d'intervention proche du Minimum Capital Requirement (*MCR*), le niveau minimum de fonds propres en dessous duquel l'intervention de l'autorité de contrôle est automatique ($MCR < SCR$). Ces paramètres doivent être choisis en ligne avec les limites de tolérance au risque préalablement choisies par l'entreprise.

∗. Nous faisons ici, et dans les calculs de BGS présentés ci-après, l'hypothèse, standard opérationnellement, que le capital ajouté X est investi en actif sans risque

†. Le *SCR* dépend théoriquement du capital ajouté X, ce qui explique les notations $SCR(X)$.

Contraintes multi-déterministes

Une dernière approche plus simple peut consister en un choix de J scénarios pluriannuels et la vérification du respect d'un niveau α de couverture du SCR sur l'horizon considéré,

$$\forall j \in [\![1;J]\!], \forall t \in [\![1;T]\!], \frac{NAV_t^j}{SCR_t^j} \geq \alpha. \text{ (CS5)}$$

Ce qui permet d'obtenir la valeur du BGS suivante,

$$\text{BGS}_{(\text{SC5})} = FP_0 + K \,/ : K = -min_{0<t\leq T,\, 1\leq j\leq J}\left[\delta_t^j\left(FP_t^j - \alpha \times SCR_t^j\right)\right].$$

Problématiques simulatoires et implémentation proposée.

La mise en œuvre de la contrainte (CS5) est plus simple (en pratique J est suffisamment petit pour ne pas poser de problème d'implémentation). Toutefois les contraintes (CS1) à (CS4) posent de sérieux problèmes de mise en œuvre. En effet, elles requièrent de disposer de distributions empiriques de fonds propres économiques à chaque date de l'horizon $[\![1;T]\!]$, et pour les contraintes (CS3) et (CS4), des distributions de SCR associées. Rappelons qu'en approche modèle interne, deux niveaux de simulations (primaires et secondaires) sont nécessaires pour l'estimation de la valeur de SCR_0. L'approche SdS le permet mais elle est déjà particulièrement consommatrice en temps de calcul. Si la dimension temporelle est ajoutée (passage du calcul du SCR_0 à celui de distributions de $SCR_1,...,SCR_T$), la complexité algorithmique rend l'approche inenvisageable (imbrication de scénarios risque neutre dans un système de scénarios monde réel déjà imbriqués, voir Figure 3).

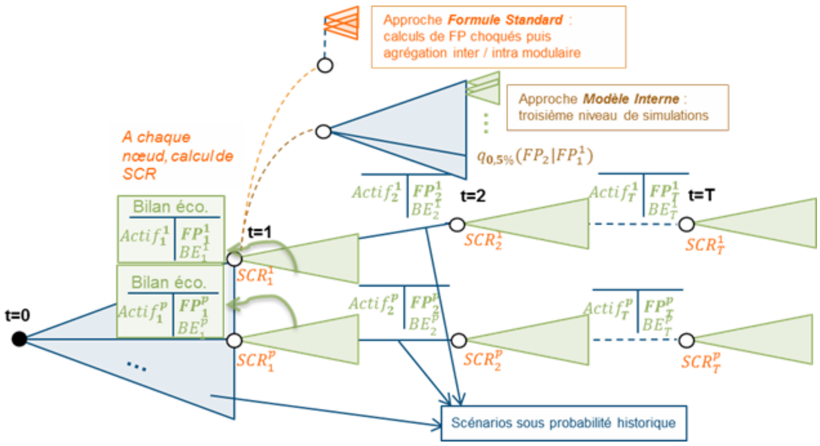

FIGURE 3 – Illustration de complexité algorithmique - Calcul de distributions jointes (FP,SCR) en pluriannuel, Formule Standard vs. Modèle interne

Pour la mise en œuvre des estimations de BGS associés à ces contraintes, nous avons donc choisi de considérer des calculs simplifiés par approche Formule Standard afin de réduire la complexité algorithmique.

Toutefois, cette complexité est toujours trop élevée puisque une telle mise en œuvre nécessiterait une approche SdS par date de l'horizon $[\![1;T]\!]$, et par risque de la Formule Standard (pour R risques projetés - intégrant le cas central -, on a une complexité en $P \times N \times T \times R$). Pour la réduire, nous

nous sommes intéressés à des approches de type *proxies*, le Least Squares Monte-Carlo (LSMC) et le Curve Fitting (CF). Ces méthodes, inspirées des techniques d'interpolation financières (voir en particulier Longstaff et Schwartz (2001)), sont utilisées en assurance pour accélérer le SdS, pour le calcul d'une distribution de fonds propres économiques à 1 an (voir Bauer *et al.* (2010), Devineau et Chauvigny (2011)). Elles consistent toutes les deux à traduire l'information financière observée dans les simulations primaires en objets simples mais complets, les "facteurs de risque" (je noterai dans la suite x le vecteur des facteurs de risque). Dans un second temps, elle consiste à tenter de répliquer la fonction $FP_1(x)$ par un polynôme en ces facteurs. La différence entre ces deux approches vient de la variable sur laquelle est calibré le polynôme. Dans le LSMC, le *proxy* est calibré par moindres carrés ordinaires (MCO) sur un très grand nombre P^{LSMC} de réalisations indépendantes de VAN_1. Dans le CF, le *proxy* est calibré par MCO sur un nombre limité P^{CF} de réalisations indépendantes de FP_1 (elles-mêmes estimées sur N réalisations de VAN_1). *. Les bonnes propriétés des MCO, sous une hypothèse d'exogénéité forte, assurent que ces deux mises en œuvre convergent asymptotiquement vers le même jeu de paramètres optimaux et permettent d'approcher la fonction $FP_1(x)$, du fait de l'égalité $FP_1(x) = \mathbb{E}^{\mathscr{Q}}[VAN_1|x]$. La Figure 4 illustre l'implémentatrion de ces deux approches

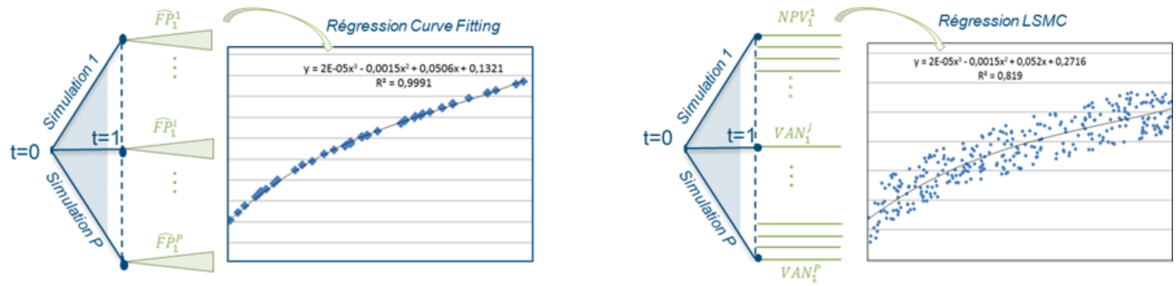

FIGURE 4 – Illustration Curve Fitting vs. Least Squares Monte Carlo (pour x scalaire)

Les *proxies* présentés sont imparfaits et basés sur des hypothèses relativement fortes. Ils supposent en particulier que $\sigma(x) = \mathscr{F}_1$ et que FP_1 est bien de la forme polynômiale estimée. Si la première hypothèse est souvent facile à vérifier, la seconde est impossible à tester, la connaissance de FP_1 n'étant qu'imparfaite et nécessitant une procédure de type Monte Carlo. D'autre part, l'hypothèse d'exogénéité forte dans les modèles multi-linéaires est impossible à tester. Cependant, l'utilisation de telles approches donne souvent de bons résultats empiriques.

La problématique induite par leur extension à $t > 1$ an est que le nombre de régresseurs des polynômes augmente très rapidement. Si x est de dimension d pour $t = 1$, impliquant $O(d^n)$ monômes / régresseurs à l'ordre n, ils peuvent facilement monter à $O(t^n \times d^n)$ en date t si autant de facteurs de risque sont extraits au cours de chaque période successive d'1 an.

Nous avons donc proposé et implémenté dans cet article une méthodologie extrapolant les *proxies* sur 5 ans. Pour réduire fortement le nombre de régresseurs lors des calibrages à $t > 1$ nous avons considéré une mise en œuvre par apprentissage, en intégrant, comme régresseur en date $t > 1$, la valeur des fonds propres économiques approchée par le *proxy* calibré en date $t-1$. Nous avons de plus calibré ces *proxies* en "central" et en appliquant chaque choc Formule Standard. Nous obtenons donc, par date de projection (entre $t = 1$ et $t = 5$), un *proxy* de fonds propres économiques centraux et un *proxy* de fonds propres choqués pour chaque choc. Par agrégation nous obtenons donc un *proxy* de

*. Avec en général $P^{LSMC} \approx P^{CF} \times N$.

FP_t et un *proxy* du SCR_t, donc un *proxy* du ratio de solvabilité, $RS_t = \frac{FP_t}{SCR_t}$ à chaque date.

Implémentation.

Nous avons appliqué nos méthodes (LSMC, CF et une approche totalement simulatoire - type SdS pluriannuel - pour comparer les résultats à de vraies valeurs), sur un produit d'épargne standard, en considérant $P = 5000$, $N = 500$, $P^{LSMC} = 50000$ et $P^{CF} = 100$ *. Nous obtenons les QQ-plots et résultats suivants (voir Figure 5 et Table 1).

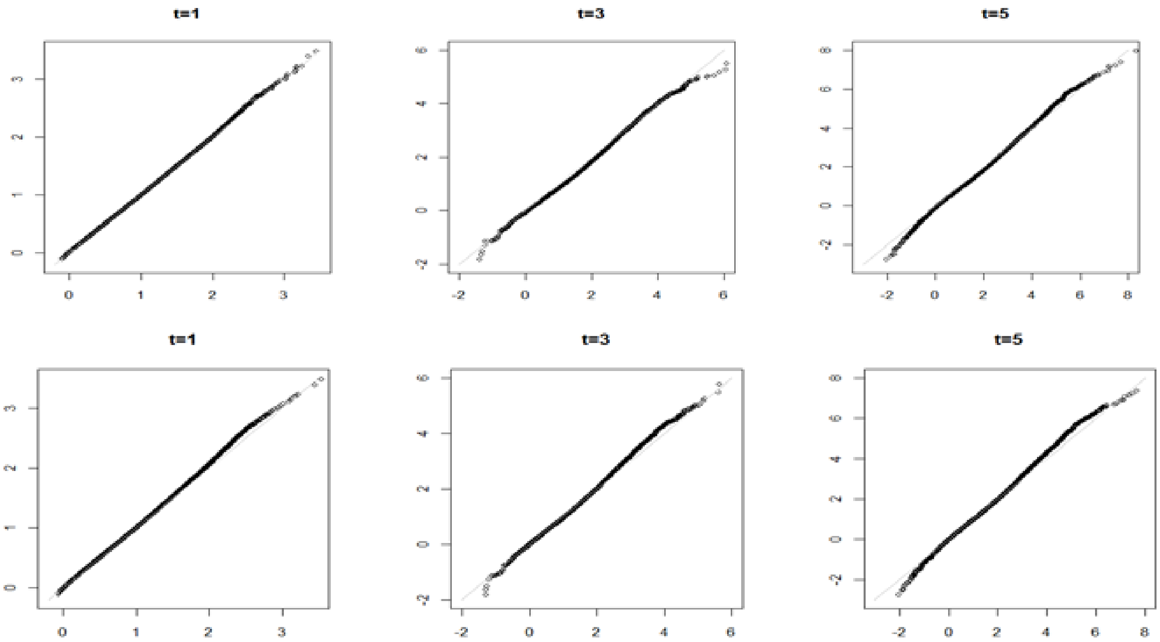

FIGURE 5 – QQ-plots sur ratios de solvabilité (proxy vs. approche totalement simulatoire) pour $t = 1$, $t = 3$, $t = 5$ - CF (graphes du haut) et LSMC (graphes du bas)

TABLE 1 – Erreur relative observée sur les BGS estimés par *proxy* vs. approche totalement simulatoire

Contrainte de solvabilité pluriannuelle	LSMC	Curve Fitting
(CS3) : $\forall t \in [\![1;5]\!], \mathbb{P}\left(\frac{FP_t}{SCR_t} \geq 110\%\right) \geq 85\%$	7.9%	6.7%
(CS4) : $\mathbb{P}\left(\cap_{t=1}^{5} \frac{FP_t}{SCR_t} \geq 110\%\right) \geq 85\%$	9.3%	11.9%

Comparaison asymptotique des approches LSMC et CF.

Les deux approches *proxies* envisagées montrent leur efficacité pour le calcul du BGS, même dans le cas des plus complexes contraintes de solvabilité pluriannuelle (de type (CS4) et (CS5)). Nous avons voulu comparer l'efficacité de ces approches dans un cadre plus théorique.

Soit $\widehat{FP}_t$ la variable aléatoire fonds propres économiques approchée par Monte Carlo (c'est celle

*. En particulier, on a bien $P^{CF} \times N = P^{LSMC}$

obtenue par approche SdS et utilisée pour le calibrage du CF), en date $t \in \mathbb{N}^*$ et X_t le vecteur des régresseurs utilisés dans les régressions linéaires (monômes simples et croisés en les facteurs de risque). Nous considérons trois modèles dans cette étude,

$$\begin{cases} \widehat{FP}_t = X_t.^1\beta_t + u_t \text{ Modèle CF,} \\ VAN_t = X_t.^2\beta_t + v_t \text{ Modèle LSMC,} \\ FP_t = X_t.^3\beta_t + w_t \text{ Modèle } \textit{théorique} \end{cases}$$

Sous les hypothèses suivantes,

$$\begin{cases} \mathscr{H}1 : X_t^{'}X_t \text{est inversible,} \\ \mathscr{H}2 : \mathbb{E}\left[\widehat{FP}_t|X_t\right] = X_t.^1\beta_t, \mathbb{E}\left[VAN_t|X_t\right] = X_t.^2\beta_t, \mathbb{E}\left[FP_t|X_t\right] = X_t.^3\beta_t \end{cases}$$

,

alors les estimateurs des MCO des trois modèles existent et convergent vers ${}^1\beta_t = {}^2\beta_t = {}^3\beta_t = \beta_t$.

Et si de plus,

$$\mathscr{H}3 : \mathbb{V}\left[u_t|X_t\right] = {}^u\sigma^2, \mathbb{V}\left[v_t|X_t\right] = {}^v\sigma^2,$$

alors, en notant $\mathbb{V}[w_t] = {}^w\sigma^2$, $\mathbb{V}[VAN] = {}^{VAN}\sigma^2$ et en égalisant les vitesses de convergences asymptotiques des estimateurs LSMC et CF, on obtient la formule suivante,

$$P^{LSMC} = P^{CF} \times N \times \left(\frac{1 + \frac{{}^w\sigma_t^2}{{}^{VAN}\sigma_t^2}}{1 + N \times \frac{{}^w\sigma_t^2}{{}^{VAN}\sigma_t^2}} \right) \leq P^{CF} \times N$$

Cette formule montre que la mise en œuvre LSMC est asymptotiquement au moins aussi efficace que la mise en œuvre CF. Notons que l'égalité est obtenue si et seulement si $N = 1$ ou ${}^w\sigma = 0$. Le premier cas est celui d'un CF spécifique pour lequel une seule simulation secondaire est tirée, par simulation primaire, c'est donc une mise en œuvre LSMC et nous vérifions bien l'égalité des deux méthodes. Le second cas est atteint, sous nos hypothèses, si $\sigma(X_t) = \mathscr{F}_t$, ce qui est en général le cas pour $t = 1$ mais pas toujours pour $t > 1$ car $\mathscr{F}_t$ devient trop dimensionnée à mesure que t augmente (d'où notre utilisation de l'estimateur de FP_{t-1} comme régresseur pour estimer FP_t).

Dans cet article nous avons réfléchi à la définition de la notion de solvabilité pluriannuelle et proposé une méthodologie permettant d'estimer la Besoin Global de Solvabilité afférent aux différents cadres proposés. Les approches *proxies* utilisées permettent d'assurer une opérationnalisation de nos propositions de manière rapide et avec une précision acceptable pour l'ORSA. Pour finir, nous avons présenté les résultats de notre étude asymptotique des méthodes Curve Fitting et LSMC afin de disposer d'un élément de justification théorique de la préférence d'une approche par rapport à l'autre. Sous les hypothèses considérées, le LSMC s'avère au moins aussi efficace que le Curve Fitting.

Le chapitre suivant propose une formalisation et une résolution de la problématique de conformité permanente.

Chapitre 2.

Dans le second chapitre de ma thèse, je présente un article dans lequel, avec mes coauteurs, nous nous sommes appliqués à proposer une solution pour le suivi de la conformité permanente (article [b]).

Problématique de conformité permanente.

Cette problématique posait de nombreuses difficultés de conceptualisation aux assureurs européens. En particulier, la définition proposée par la directive, tirée de l'article introduisant l'ORSA, est peu explicite à ce sujet :

> As part of its risk-management system every insurance undertaking and reinsurance undertaking shall conduct its own risk and solvency assessment. That assessment shall include at least the following : (...) the compliance, on a continuous basis, with the capital requirements.

D'une part le suivi continu de l'exigence de capital sur un an pose des problèmes simulatoires et opérationnels lourds, d'autant plus que la mise à jour des données d'actif et de passif nécessitait à elle seule plusieurs jours. D'autre part, la conformité permanente reste peu évoquée dans la littérature académique (hormis en théorie de la ruine) ou opérationnelle actuarielle.

Dans cet article, nous nous sommes donc tout d'abord attachés à proposer une formalisation plus précise de la notion de conformité permanente, en termes d'historique et de textes dans la régulation et de complexité d'implémentation.

Approche de suivi de la conformité permanente proposée.

Nous avons développé une approche par *proxy*, basée sur une adaptation du LSMC, pour permettre le suivi continu du niveau d'exigence en capital (SCR estimé en approche Formule Standard) ainsi que du niveau de capital disponible pour l'entreprise, dans le temps.

La méthodologie proposée consiste à mettre en place le dispositif suivant.
- Etape 1 : Choix des SCR marginaux à monitorer.
- Etape 2 : Choix des indicateurs économiques, ayant un impact sur le bilan économique. Ce sont ces indicateurs qui seront suivis pour évaluer l'évolution des niveaux de SCR et de capital éligible. Typiquement une compagnie investissant en France suivra le niveau du CAC / de l'Eurostoxx, des taux swap européens,...
- Etape 3 : Calibrage des polynômes LSMC expliquant les niveaux de FP central et choqués en fonction des indicateurs choisis.
(- Etape 4 : Validation des polynômes.)

Puis, à chaque date de suivi (voir Figure 6).
- Etape 1 : Récupération des nouvelles valeurs des indicateurs.
- Etape 2 : Estimation des FP central et choqués associés par les poynômes LSMC.
- Etape 3 : Reconstruction des SCR marginaux et agrégation pour obtenir le SCR global. Certains SCR marginaux, très peu volatiles (immobilier, mortalité,...) et non suivis (pas de LSMC dédié), seront supposés invariants lors de cette étape.
- Etape 4 : Obtention du ratio de solvabilité.

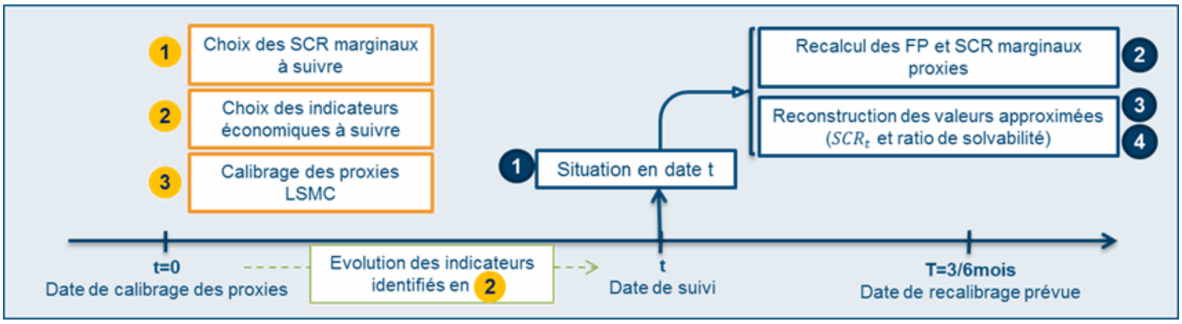

FIGURE 6 – Illustration - Dispositif de suivi de la conformité permanente

Implémentation.

Nous proposons dans l'article une implémentation de cette approche (4 indicateurs suivis – voir Table 2 – et 4 SCR marginaux – action, taux, spread et illiquidité) et illustrons différentes possibilités fournies par notre outil pouvant aider les opérationnels, gestionnaires des risques.

TABLE 2 – Risques sélectionnés et indicateurs associés

Risques sélectionnés	Indicateurs composites
Action (risque de niveau)	EUROSTOXX50
Taux (risque de niveau)	Courbe Euroswap (evolution du niveau moyen de la courbe)
Spread (gouvernemental)	Spread moyen obligations Fr rate vs. Taux Euroswap
Spread (corporate)	valeur de l'*iTraxx Europe Generic 10Y Corporate*

Cette approche permet bien entendu de suivre une valeur approchée de son ratio de solvabilité dans le temps, mais aussi d'estimer des sensibilités simples et croisées du ratio de solvabilité approché aux valeurs prises par les indicateurs suivis (voir Figure 7). Elle permet, encore, d'estimer l'impact, indicateur par indicateur / source de risque par source de risque, d'une évolution de l'économie sur le ratio de solvabilité approché (voir Figure 8).

Toutefois, cette mise en œuvre a plusieurs limites. Tout d'abord, elle repose sur l'hypothèse que le portefeuille d'actif et de passif de l'assureur n'a pas trop évolué entre la date de calibrage et celle de suivi puisque les polynômes obtenus sont conditionnés par les données ALM initiales. Un recalibrage régulier de l'outil est donc nécessaire. D'autre part, les *proxies* perdent leur efficacité en cas de trop forte déviation des indicateurs suivis.

Il convient de noter l'aspect approché de cette mise en œuvre et l'existence d'une zone de validité in-

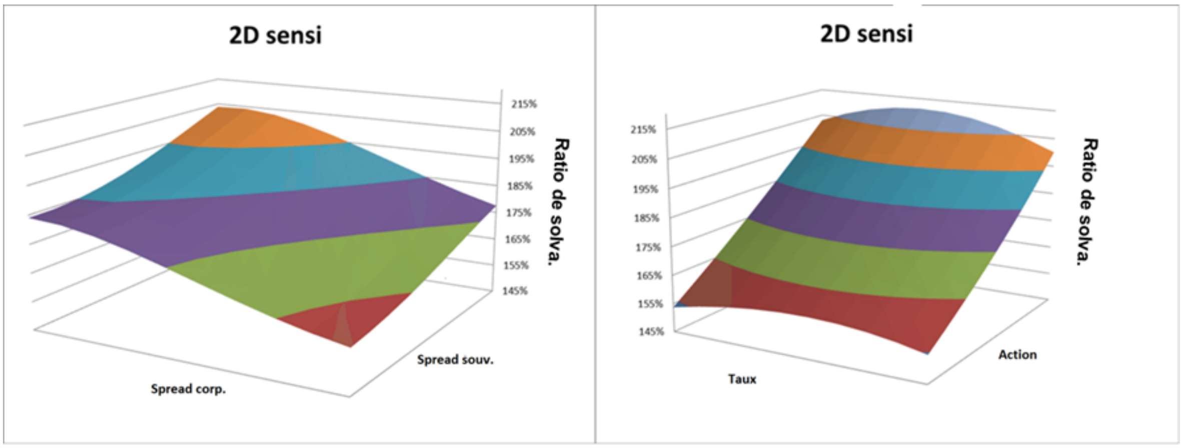

FIGURE 7 – Illustration - Sensibilités croisées du ratio de solvabilité aux indicateurs de suivi

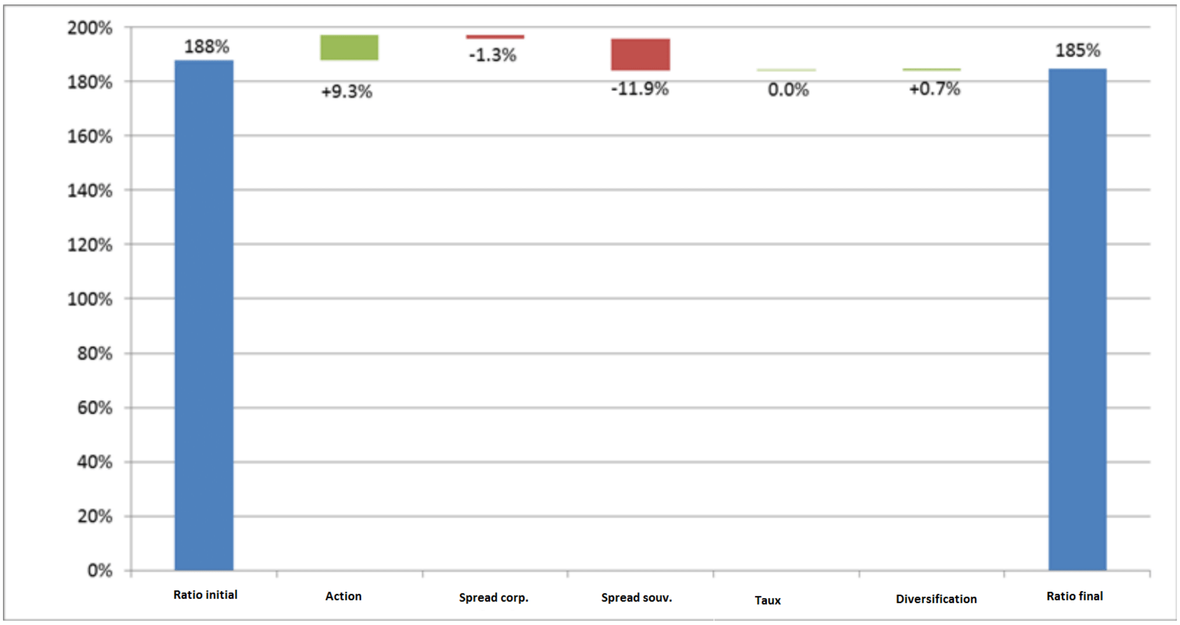

FIGURE 8 – Illustration - Etude des impacts marginaux d'une modification du ratio de solvabilité approché entre deux date de suivi

extensible. En particulier, en cas de sinistre extrême (crise sur les marchés financiers, remonté subite des taux directeurs, actuellement très bas, vague de rachat massif, pandémie,...), il est probable que le *proxy* ne soit plus valide. Or, c'est justement à cette période que l'autorité de contrôle (en France, l'Autorité de Contrôle Prudentiel et de Régulation) peut faire valoir son droit au contrôle de la conformité permanente. L'outil proposé ne permettant plus de bien estimer le ratio de solvabilité de l'entité, elle devra remettre l'intégralité de ses modèles à jour et recalculer proprement son ratio.

Estimation théorique de la capacité à respecter la conformité permanente.

Afin de conforter l'entreprise d'assurance quant à sa capacité à tenir ses engagements régulatoires nous avons pour objectif (travail en cours), dans une dernière partie, en se basant sur des résultats usuels de théorie probabiliste, des outils d'évaluation du risque de non-conformité dans l'année.

Finalement cet article nous permet de formaliser une première solution complète de suivi de la conformité permanente et d'estimation *a priori* de la capacité de l'entreprise à supporter les chocs économiques exogènes du point de vue de sa solvabilité réglementaire. Enfin, nous avons implémenté nos outils et montré leur utilité pratique. La problématique liée à l'utilisation de méthodes Curve Fitting et LSMC reste toutefois l'erreur de *proxy*, couplant erreur d'estimation et erreur de spécification. Particulièrement délicate à contrôler, je propose, dans l'article suivant, plusieurs méthodes pour la réduire, tout en assurant l'aspect conservateur de leur mise en œuvre.

Chapitre 3.

La troisième partie de ma thèse présente une étude (article [c]) axée sur les différentes approches *proxy* permettant d'accélérer les approches simulatoires utilisées afin d'implémenter la directive Solvabilité II, tant pour répondre aux exigences quantitatives du pilier I que pour la mise en œuvre de l'ORSA (voir Martial et Garnier (2013) pour une présentation des différentes approches *proxies* utilisées par les opérationnels).

***Cadre général des approches* proxies.**

Les *proxies* sont majoritairement estimés par régressions multilinéaires et approximent les valeurs économiques dans le futur, soit par un polynôme en des facteurs de risque synthétisant l'information contenue dans les trajectoires historiques (LSMC, CF), soit par un portefeuille d'actifs de marché simples (Replicating Portfolio). L'article présenté s'attache dans un premier temps à proposer une formalisation théorique homogène des approches *proxy* les plus souvent utilisées en pratique (voir Table 3).

Si ces approches paramétriques ont toutes l'avantage de permettre d'accélérer fortement les calculs, elles introduisent deux sources d'erreurs distinctes dans les estimations de valeurs économiques futures ; une erreur liée à la non-convergence de leurs estimateurs (à distance finie), mais aussi une erreur difficilement compressible liée à leurs hypothèses de spécification.

***Méthodes de gestion de l'erreur de* proxy.**

En se focalisant, sans perte de généralité, sur l'approche LSMC dans le cadre du calcul du *SCR* (nécessitant le calibrage d'un *proxy* de FP_1), dans une mise en œuvre de type modèle interne, l'article propose tout d'abord plusieurs mises en œuvre aisément opérationnalisables permettant d'accélérer la convergence de l'estimateur LSMC. Ces outils simples, inspirés des techniques usuelles de réduction de variance (variables antithétiques et variables de contrôle), sont pourtant rarement utilisées en pratique et permettent de réduire une première source d'erreur (non-convergence des estimateurs).

Toutefois, ces techniques ne permettent pas de contrôler efficacement l'erreur de *proxy* puisque une erreur de spécification est toujours possible et que la convergence des estimateurs ne reste qu'asymptotique. Ces erreurs posent des problèmes lors de la validation des *proxies*. C'est en effet la difficulté

TABLE 3 – Synthèse des différentes approches *proxy* (paramétriques) utilisées par les compagnies d'assurance vie

Approche *proxy*	Modèle de régression associé	Hyp. de spécification
Curve Fitting	$\widehat{FP}_t(x_t) = x_t.^{CF}\beta + {}^{CF}u$ où les ${}^ix_t$ $(i \in [\|1,K\|])$ sont des monômes en les *facteurs de risque*.	$\mathbb{E}^{\mathscr{Q}_t}\left[\widehat{FP}_t(x_t)\|x_t\right] = x_t.^{CF}\beta$
LSMC	$NPV_t(x_t) = x_t.^{LSMC}\beta + {}^{LSMC}u$ où les ${}^ix_t$ $(i \in [\|1,K\|])$ sont des monômes en les *facteurs de risque*.	$\mathbb{E}^{\mathscr{Q}_t}[NPV_t(x_t)\|x_t] = x_t.^{LSMC}\beta$
Replicating Portfolio (v1)	$\widehat{NAV}_t(x_t) = x_t.^{RP1}\beta + {}^{RP1}u$ où les ${}^ix_t$ $(i \in [\|1,K\|])$ sont des prix d'actifs simple.	$\mathbb{E}^{\mathscr{Q}_t}\left[\widehat{NAV}_t(x_t)\|x_t\right] = x_t.^{RP1}\beta$
Replicating Portfolio (v2)	$NPV_t(x_t) = x_t.^{RP2}\beta + {}^{RP2}u$ où les ${}^ix_t$ $(i \in [\|1,K\|]$ sont des prix ou cash-flows d'actifs simples.	$\mathbb{E}^{\mathscr{Q}_t}[NPV_t(x_t)\|x_t] = x_t.^{RP2}\beta$

majeure de ces outils. Ce processus consiste généralement à comparer, sur quelques scénarios non utilisés pour le calibrage, des valeurs approchées à celles calculées par approche totalement simulatoire. Cette mise en œuvre est particulièrement délicate puisqu'il s'agit de valider ou d'invalider les *proxies* en se basant sur un nombre limité de valeurs. D'autre part, si les erreurs sont trop grandes la question reste de savoir si l'augmentation du nombre de scénarios de calibrage permettra réellement d'améliorer les résultats ou si l'erreur est majoritairement liée à un problème de spécification.

Afin de proposer une alternative plus robuste dans l'objectif de réduire cette difficulté, nous présentons une approche de méta-modélisation semi-paramétrique permettant de modéliser l'erreur de modélisation.

Nous considérons, à chaque scénario primaire, représenté par son jeu de facteurs de risque x, la fonction résidu $Res(x) = FP_t(x) - A(x)$, en notant A le *proxy* de FP_1 calibré par LSMC (et amélioré à l'aide des méthodes de réduction de variance présentées).

L'hypothèse de base consiste à supposer,

$$\mathscr{H} : Res(x)|A(x) \sim \mathscr{N}\left(\mu(A(x)), \sigma^2(A(x))\right).$$

Cette hypothèse peut être testée *a posteriori*, une fois le méta-modèle calibré. Toutefois nous montrons, dans notre implémentation de la méthode, que le méta-modèle apporte de l'information sur la majeure partie de la distribution de FP_1 et que l'estimation du quantile à 0.5% est améliorée, même si la contrainte n'est pas respectée.

La mise en place du méta-modèle consiste donc à calibrer les fonctions $\mu(A(x)) = \mathbb{E}[Res(x)|A(x)]$ et $\sigma(A(x)) = \mathbb{V}[Res(x)|A(x)]$, pour cela nous proposons de nouvelles estimations paramétriques et travaillons actuellement sur des mises en œuvre non-paramétriques. D'autre part, la méthodologie proposée peut être appliquée sans nécessiter de simulation supplémentaire.

Implémentation de l'approche proposée.

Nous implémentons la méthodologie proposée sur une modélisation simplifiée nous permettant de connaitre la vraie valeur de la fonction FP_1. Nous obtenons les QQ-plots comparatifs (A vs. FP_1 - Figure 9 - et A+méta-modèle vs. FP_1 - Figure 10 -) et résultats sur l'estimation du quantile extrême (voir Table 3).

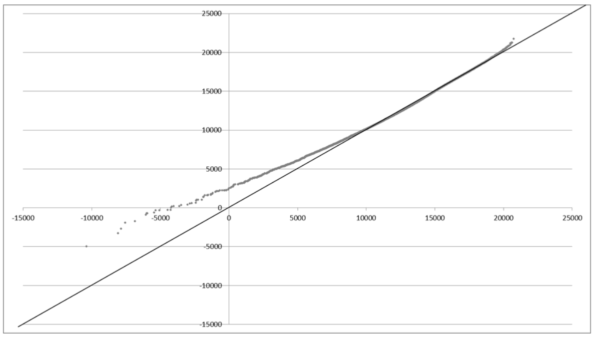

FIGURE 9 – QQ-plot *proxy* A vs. vraie distribution de FP_1 (100'000 réalisations)

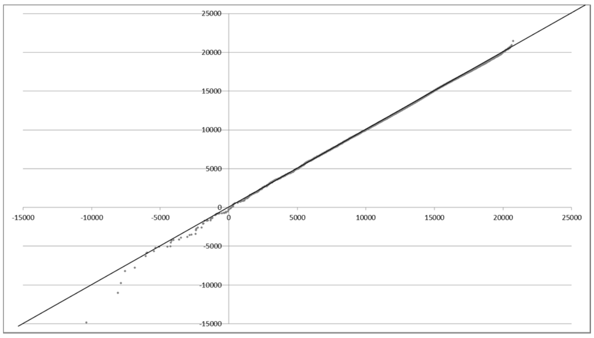

FIGURE 10 – QQ-plot *proxy* + méta-modèle vs. vraie distribution de $FP_1(x)$

TABLE 4 – Comparaison des quantiles à 0.5% estimés avec sa vraie valeur

	Proxy seul	**Proxy+méta-modèle**	FP_1
Quantile à 0.5%	6504	5697	5533
Écart relatif	17.6%	2.9%	/

L'hypothèse initiale de Gaussiannité des résidus n'est respectée que très localement autour du quantile à 0.5% de la distribution de FP_1. Malgré cela le QQ-plot obtenu montre que notre approche a, sur ce produit, efficacement redressé la distribution de FP_1 approchée avec le *proxy* uniquement. Cette mise en œuvre fournit de plus un résultat conservateur. Toutefois, pour s'assurer de ce caractère, nous conseillons aux utilisateurs de telles approches de ne prendre en compte le quantile corrigé que lorsqu'il fournit un SCR_0 supérieur au capital estimé par le *proxy* seul.

Dans une dernière partie (Section 5), nous avons souhaité étendre notre analyse à l'une des problématiques liées au calibrage des *proxies*, le choix de la distribution à considérer pour simuler l'échantillon des facteurs de risques de calibrage.

***Problématique liée à la distribution des simulations de calibrage des* proxies.**

Deux approches sont utilisées en pratique, soit celle qui utilise la vraie distribution soit celle utilisant une distribution de type uniforme (pseudo- ou quasi-aléatoire), sur la majeure partie du support de la vraie distribution. L'idée sous-jacente à l'approche uniforme est que les quantiles extrêmes de la véritable distribution sont sous-densifiés, alors que ce sont généralement les scénarios associés à des quantiles extrêmes de la variable FP_1, ceux qui, justement, intéressent les actuaires. En équipondérant une plage large du support de la vraie loi des facteurs de risque, l'utilisateur espère sur-densifier les zones extrêmes pour améliorer la capacité de prédiction du *proxy* dans celles-ci. D'autre part, les zones plus centrales de la vraie densité des facteurs de risque sont aussi équiprobabilisées, de manière à assurer une couverture acceptable de cette partie plus centrale.

Dans notre article, nous montrons tout d'abord que, sous l'hypothèse d'exogénéité forte, les deux approches convergent vers le même estimateur. En pratique, pour calibrer de manière optimale le *proxy*, il faudrait considérer des scénarios pour lesquels la variance du résidu, conditionnellement à x, est minimale. Sans *a priori* sur cette fonctionnelle [*], le choix de la distribution uniforme assure, au minimum, que les scénarios de variance conditionnelle maximale ne seront pas sur-probabilisés.

Sous une hypothèse d'exogénéité faible, une hypothèse rarement supposée empiriquement mais pourtant plus probable [†], au contraire, les estimateurs ne convergeront pas vers la même valeur. Dans ce cas tout est plus complexe et si l'utilisateur souhaite une efficacité globalement similaire du *proxy* sur toute la véritable distribution sans avoir d'*a priori* sur la variance conditionnelle des résidus, il est pertinent de considérer plutôt une distribution uniforme sur une large portion du support de la vraie loi (voir exemple de la Figure 11).

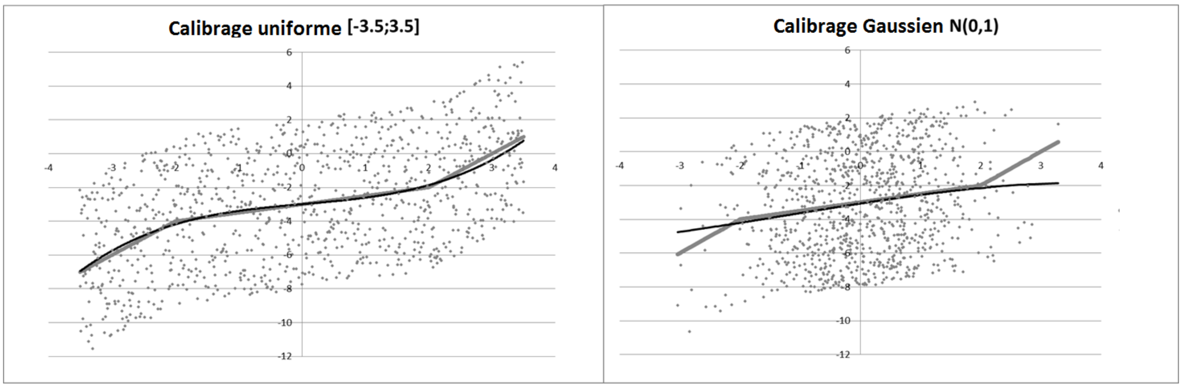

FIGURE 11 – Exemple où x est scalaire, cadre homoscédastique sous exogénéité faible - calibrage du *proxy* Gaussien (vraie distribution) vs. calibrage uniforme sur [-3,5 ;3,5] (*VAN* : points gris, vraie fonction FP : courbe grise épaisse, *proxy* : ligne noire)

Cet article nous a permis de présenter et d'implémenter différentes méthodologies permettant de ré-

∗. Ce qui est généralement le cas.

†. Et qui peut être testée par un test d'Hausman par exemple (voir Hausman (1978)).

duire l'erreur de *proxy* (convergence et spécification). D'autre part, il nous a permis d'étudier la problématique opérationnelle des simulations utilisées pour le calibrage des *proxies*. Dans le cadre de cette dernière analyse, nous avons soulevé la question de l'hypothèse de spécification utilisée, hypothèse qui n'était jusqu'ici pas remise en question.

Bien que les méthodes *proxy* permettent de pallier à la forte complexité des approches totalement simulatoires, il est certain que l'erreur spécifique à leur utilisation (convergence et spécificité) ne pourra jamais être réduite à zéro. Dans le papier présenté en chapitre 4, je me suis intéressé de plus près à une approche permettant d'accélérer l'estimation de quantiles extrêmes de la variable aléatoire FP_1 par approche SdS (typiquement, le quantile à 0.5% de FP_1 pour l'estimation du *SCR*).

Chapitre 4.

Dans la quatrième partie de ma thèse, je présente un article (article [d]), très opérationnel, dont les résultats présentés ont été initiés au travers de l'écriture d'un mémoire d'actuariat dont j'ai soumis le sujet, et que j'ai encadré sur l'année scolaire 2013-2014. Les auteurs de ce mémoire (Charlotte Belin et Rémi Gerboud) ont souhaité poursuivre ce travail et nous avons co-écrit ce papier, sur la notion de *VAN forward*.

L'idée initiale de ces travaux était de proposer une approche permettant d'accélérer l'estimation du quantile à 0.5% de FP_1 par approche SdS. Je travaillais alors dans l'objectif de développer un outil synthétique permettant d'apporter une information, non pas sur la valeur de FP_1 associée à un scénario primaire, mais sur le degré d'adversité des scénarios. Ainsi, si l'outil permet, par co-monotonie avec FP_1 en particulier, de trier les scénarios par ordre de valeur de FP_1 sans avoir à les calculer, il est possible de localiser le scénario menant au quantile et d'éviter des calculs par Monte Carlo imbriqués inutiles.

L'accélérateur SdS.

Cette idée de disposer d'un *a priori* sur l'adversité des scénarios primaires est déjà considérée dans Devineau et Loisel (2009a). Les co-auteurs utilisent alors une norme sur les facteurs de risque extraits des scénarios primaires. Notons $x = \left({}^1\varepsilon, ..., {}^K\varepsilon\right)$ le vecteur des facteurs de risque primaires associés à un scénario x est supposé Gaussien. S'il ne vérifie pas cette propriété, l'utilisateur peut l'imposer par méthode de la transformée inverse. Devineau et Loisel considèrent, comme *a priori*, la norme *standard* (Euclidienne),

$$\|x\|^{std} = \sqrt{\sum_{k=1}^{K} {}^k\varepsilon}$$

La problématique avec ce choix d'*a priori* est que les scénarios extrêmes au sens de cette norme incluent effectivement les scénarios les plus adverses mais aussi les plus favorables.

Devineau et Loisel utilisent donc un algorithme itératif spécifique pour écarter les scénarios favorables et capter uniquement les scénarios de queue de distribution (pour $P = 5000$, il s'agit de localiser les 25 pires).

Itération 1 : Localisation des $M \geq P \times 0.5\%$ pires scénarios au sens de l'*a priori* choisi. Calcul des M valeurs de FP_1 associées.

Itération N : N.1 : Localisation des M pires scénarios suivants. Calcul des M valeurs de FP_1 associées. N.2 : Comparaison des $P \times 0.5\%$ pires scénarios du jeu de $(N-1) \times M$ scénarios obtenus en agrégeant les valeurs calculées aux itérations 1 à $N-1$, aux $P \times 0.5\%$ du jeu de $N \times M$ scénarios, agrégation des étapes 1 à N. - S'ils sont similaires : Arrêt de l'algorithme. - Sinon, nouvelle itération de l'algorithme

La convergence, vers les bonnes valeurs, de cet algorithme (appelé *accélérateur SdS*) n'est pas assurée. Toutefois il a été testé sur de très nombreux produits d'assurance sans jamais avoir été pris au piège pour $M = 2$ à $4 \times N$. Toutefois il peut cependant converger lentement (4-5 itérations) et l'*a priori* « norme standard » parait perfectible. Une première possibilité est proposée par les auteurs, la norme ajustée (ou norme *sensibilité*), qui tient compte des sensibilités du produit aux risques associés aux facteurs de risque utilisés.

La norme s'obtient en deux étapes. Tout d'abord les facteurs de risque sont décorrélés entre eux en utilisant la matrice de Cholesky inverse de la matrice de variance du vecteur x. Soit $\dot{x} = \left({}^1\dot{\varepsilon}, ..., {}^K\dot{\varepsilon}\right)$ le nouveau vecteur et $(s_k)_{k \in [\![1;K]\!]}$ les sensibilités du produit aux différents facteurs de risque. Ces sensibilités doivent être estimées avant le calcul de la norme, toutefois elles ne nécessitent pas de calculs lourds*.

La norme *sensibilité* est alors donnée par,

$$\|\dot{x}\|^{sensi} = \sqrt{\sum_{k=1}^{K} {}^k\dot{\varepsilon}}.$$

La convergence de l'algorithme est généralement accélérée lorsque cette norme est utilisée.

Notion de *VAN* **forward.**

Dans l'article, nous étudions les caractéristiques de la *VAN* de marge afférente au scénario secondaire déterministe dans lequel tous les indicateurs économiques des trajectoires secondaires (action, taux, indice immobilier, inflation, etc...) évoluent à leurs valeurs espérées (*forward*), vues en $t = 1$. Ainsi, en notant ε le jeu d'indicateurs issu de la trajectoire secondaire et $\mathscr{F}_1$ la filtration induite par l'évolution primaire (entre $t = 0$ et $t = 1$) de la situation économique, on a

$$FP_1(x) = \mathbb{E}\left[VAN_1(x)|\mathscr{F}_1\right].$$

La *VAN* forward, elle, s'obtient par,

$$VANfwd_1 = VAN_1\left(\mathbb{E}\left[x|\mathscr{F}_1\right]\right).$$

Nous considérons un premier produit d'épargne réel et mettons tout d'abord en évidence la pertinence de la *VAN forward* pour localiser les scénarios associés à la queue de distribution de la variable FP_1 (voir Figure 12).

*. En particulier elles ont souvent été déjà calculées lors de précédentes mises en œuvre (gestion des risques,...).

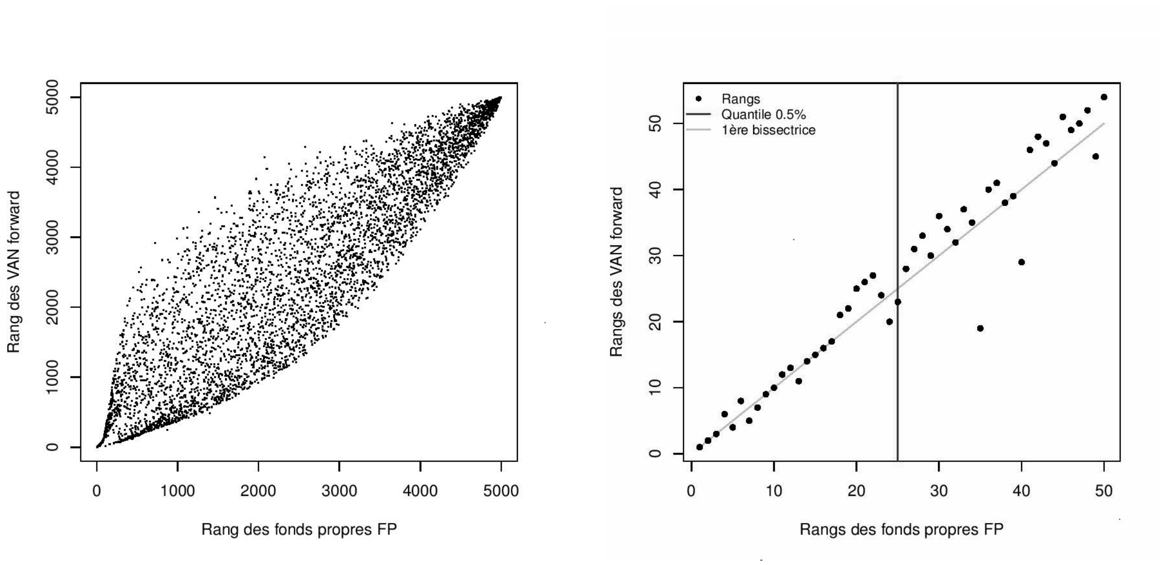

FIGURE 12 – Rangs des *VAN forward* en fonction des rangs des fonds propres (à gauche) et zoom sur la queue basse (à droite).

La *VAN forward* apparaît pertinente pour disposer d'un *a priori* efficace sur le degré d'adversité (en termes de valeur de fonds propres économiques en $t = 1$) d'un scénario primaire. Ce caractère est particulièrement efficace pour la localisation des scénarios de queue de distribution. De plus, son estimation est très rapide puisqu'il ne nécessite de calculer qu'une unique valeur de *VAN* par scénario primaire de l'approche SdS considérée. Toutefois, elle n'assure pas une détection *a priori* totalement efficace du scénario quantile. Nous utilisons finalement la *VAN forward* comme *a priori* pour le lancement de l'accélérateur SdS.

***Couplage de l'Accélérateur SdS et de la** VAN* **forward.**

Nous obtenons le graphe de la Figure 13 quant aux convergences comparées des algorithmes, appliqués à la norme euclidienne, à la norme *sensibilité* et à la *VAN forward*, et en fonction du choix de M (le nombre de scénarios considérés à chaque itération). L'algorithme est ici lancé avec $p = 5000$ *.

Dans notre mise en œuvre, tous les lancements de l'algorithme ont convergé (plus ou moins lentement selon l'*a priori* utilisé). Si la *VAN forward* améliore effectivement la vitesse de convergence de l'accélérateur, comparée aux 2 normes proposées dans Devineau et Loisel (2009a), on observe que le nombre d'itérations de l'algorithme étant au moins égal à 2, le nombre de valeurs de FP_1 estimées ne pourra jamais passer sous $2 \times M$.

***Couplage de la méthode du recuit simulé et de la** VAN* **forward.**

Nous avons donc testé une approche Bayesienne de type recuit simulé (voir Davis (1987), Aarts et Korst (1988), Hwang (1988)). Cette approche permet d'utiliser de manière différente les caractéris-

*. Ce chiffre a été choisi de manière à correspondre à l'ordre de grandeur considéré par les opérationnels. En toute objectivité, il ne permet pas d'obtenir une estimation robuste du quantile extrême à 0.5%. Cette problématique n'est pas traitée dans l'article

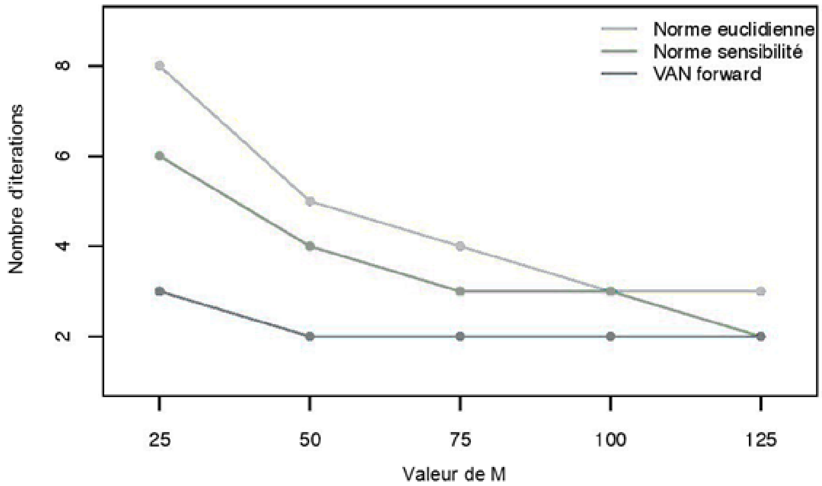

FIGURE 13 – Vitesses des convergences (en nombre d'itérations de l'algorithme par valeur de M fixée), pour différents critères d'adversité.

tiques structurelles de l'*a priori* considéré (et de manière plus optimale lorsque l'*a priori* est très bon, comme dans le cas de la *VAN forward*). La problématique de cette approche est qu'elle peut aboutir à une estimation erronée du 25^{me} pire scénario ($P \times 0.5\%$ pour $P = 5000$). Ainsi, la rapidité de convergence est estimée en moyenne, et à cette moyenne sont associés un taux de réussite (car l'algorithme peut ne pas converger vers la bonne valeur) et un intervalle de confiance du nombre de valeurs de FP_1 calculées. Les résultats que nous avons obtenus sont les suivants :

a priori considéré	Nombre moyen de valeur de FP_1 calculées	$IC_{95\%}$	Taux de réussite
Norme standard	927	$[600, 1\,459]$	60,3 %
Norme *sensibilité*	425	$[283, 660]$	95,8 %
VAN *forward*	62	$[44, 76]$	99,7 %

TABLE 5 – Nombre de fonds propres calculés au cours de l'algorithme du recuit simulé, pour différents *a priori*.

Nous avons donc démontré l'intérêt de la *VAN forward* comme outil de repérage *a priori* des scénarios primaires adverses. De plus, nous avons proposé et implémenté deux algorithmes opérationnels. Notre application a montré que cet outil pouvait mener à une détection du scénario quantile 2 à 3 fois plus rapide, dans le cas de l'accélérateur SdS, et 5 à 10 fois plus rapide, dans le cas du recuit simulé, que lorsque les normes proposées par Devineau et Loisel (2009a) étaient utilisées pour jauger l'adversité des scénarios primaires.

Chapitre 5.

La cinquième partie de ma thèse s'attache à revenir aux bases de la valorisation économique, notion inspirée des pratiques de valorisation risque neutre issues de la finance de marché, autour de laquelle l'architecture quantitative de la directive Solvabilité II s'est développée.

Nombreux sont les papiers et autres documents évoquant le manque de comparabilité et de réalisme des valeurs obtenues grâce à cette approche de valorisation (voir par exemple Planchet (2006), Chaire Management de la Modélisation (2014), Horton *et al.* (2007) ou O'Brien (2009) dont l'étude présente les différences de pratiques en Angleterre), permettant d'obtenir une juste valeur des passifs et fonds propres d'assurance grâce à une probabilisation risque neutre des scénarios économiques utilisés pour les estimations par approche Monte-Carlo.

L'article présenté dans cette partie (article [e]) est le premier qui s'attache à comparer en profondeur l'utilisation faite de la valorisation risque neutre en finance à celle appliquée en assurance. Au travers d'une étude historique (voir les travaux de Black, Scholes Black et Scholes (1973) et Merton Merton (1973) pour les bases historiques du risque neutre en finance de marché), théorique et pratique, nous présentons, mes co-auteurs et moi-même, nos conclusions sur les différences majeures entre les deux approches. Nous soulevons la problématique du manque de légitimité du cadre de valorisation imposé par la régulation assurantielle, dans l'objectif de sensibiliser les acteurs du marché aux différents problèmes pouvant découler de son utilisation.

Le cadre de valorisation risque neutre actuariel - présentation et analyse.

En pratique, l'utilisation du risque neutre en assurance se heurte à de nombreuses problématiques. Le marché de l'assurance vie est très incomplet. Les risques techniques sont impossibles à *hedger*, le *hedging* étant pourtant un prérequis essentiel pour tout instrument dont la valorisation risque neutre est légitime. D'autre part, même en mettant ces risques de côté, comment assurer l'objectivité et la comparabilité de valeurs estimées sous une probabilité risque neutre alors que ces probabilités, et donc ces valeurs, sont multiples ? En finance cette problématique est solutionnée par la présence d'un marché régulé où la loi de l'offre et de la demande assure une forme d'équilibre et de rationalité. En assurance ce processus n'existe pas et une contrainte a donc été ajoutée afin d'objectiver la mesure de probabilité à utiliser pour les valorisations économiques : elle doit pouvoir assurer un critère spécifique dit de *market-consistency*. Toutefois, il apparait que ce critère est source de nombreuses inhomogénéités entre les pratiques assurantielles et ne permet pas d'assurer un choix robuste et pertinent de la mesure à utiliser pour la valorisation.

Plusieurs points d'attention ont donc été soulevés, concernant en particulier la gestion du risque de taux, un risque essentiel pour les assureurs. Ceux-ci sont en effet soumis à une double adversité à la hausse, menant à une augmentation des rachats, et à la baisse des taux, ce qui entraine des difficultés pour servir à leurs assurés un taux garanti. Or, ce risque est particulièrement mal pris en compte dans la valorisation économique telle qu'elle est implémentée aujourd'hui.

Avant de présenter les résultats majeurs de cette étude, commençons par expliquer le processus mis en œuvre pour obtenir la valeur économique d'un poste du bilan d'un assureur vie. En pratique cette approche consiste tout d'abord à ne se concentrer que sur les risques financiers des assureurs, les risques *hedgeable*. La valeur économique est l'espérance sous une probabilité risque neutre, de la somme des cash-flows futurs actualisés (cash-flows de résultat pour obtenir une valeur de fonds propres économiques, de passif pour un *best estimate*). Les liens entre actif et passif étant particulièrement complexes (taux minimums garantis, options de rachat...), cette espérance est très complexe à estimer. Ainsi, par approche Monte-Carlo, le prix est donc approximé par la moyenne de la somme des cash-flows futurs actualisés obtenus par projection, par les praticiens, de leur bilan suivant un grand nombre de scénarios financiers (évolution des prix Zéro-Coupon, d'indices actions, immobilier, des spreads de crédit,...), projetés sur 30 à 60 ans.

Le choix de la probabilité de diffusion des scénarios financiers est réalisé de manière à ce qu'elle permette de reproduire des données de marché spécifiques (pour les taux, par exemple, il s'agit généralement de répliquer une matrice de volatilités implicites de swaptions receveuses à la monnaie). Ainsi, une fois le modèle de taux choisi, et connaissant la courbe des taux Zéro-Coupon initiale (fournie par le régulateur Européen, l'European Insurance and Occupational Pensions Authority), il s'agit de déterminer les paramètres permettant de reproduire au mieux la matrice de volatilités de swaptions à la date de valorisation.

En pratique, les résultats obtenus sont très difficilement comparables entre les compagnies. Celles-ci ont le choix du modèle à utiliser, des maturités/tenors de la matrice de volatilité des swaptions qu'elles souhaitent répliquer, de la méthodologie de calibrage des modèles,... D'autre part les calculs sont souvent réalisés sur des données de marché au 31 décembre (date de clôture comptable). Cette date est particulièrement dangereuse du fait de l'effet bien connu du "turn-of-the-year" sur les marchés. Pour finir, il convient de noter que la courbe de taux fournie par le régulateur est très particulière, peu réaliste, non adaptée aux volatilités implicites de swaption au 31/12. Cette inadéquation est théoriquement inconcevable puisqu'elle peut mener à l'apparition d'opportunités d'arbitrage, inconcevables dans un cadre risque neutre.

Problématiques du cadre de valorisation actuariel et premières alternatives.

Ces analyses ont été étoffées de plusieurs exemples et les résultats obtenus sont les suivants.

Nous avons tout d'abord considéré la courbe de taux fournie par le régulateur. Dans la Figure 14 on observe clairement la méthodologie d'interpolation utilisée par le régulateur pour faire converger les taux forwards 1 an vers une valeur "ultime" spécifique, l'Ultimate Forward Rate (UFR), valant actuellement 4,2 %. Plus étonnant, l'algorithme d'interpolation (Smith-Wilson) utilisé est censé avoir convergé entre les maturités 20 et 40 ans. Or, après 40 ans on voit clairement l'instabilité de l'algorithme. C'est un effet bien connu mais les utilisateurs de Smith-Wilson ont généralement tendance à lisser la courbe obtenue pour assurer une convergence exacte et ne pas prendre en compte cette instabilité artificielle. Le régulateur a choisi de conserver la courbe telle quelle.

Partant de cette courbe nous avons donc simulé 1000 trajectoires d'un modèle assurantiel Libor-Market-Model, calibré de manière *market-consistent* au 31 décembre 2014 et étudié l'évolution des courbes obtenues. On observe que, rapidement, un grand nombre de courbes explosent ou prennent des formes et valeurs fortement conditionnées par la courbe des taux forward. Par exemple, la Figure 15 représente 20% des courbes simulées à 10 ans et la courbe forward.

Outre le manque de réalisme de ces courbes, il semble que le modèle assurantiel utilisé, pourtant d'une

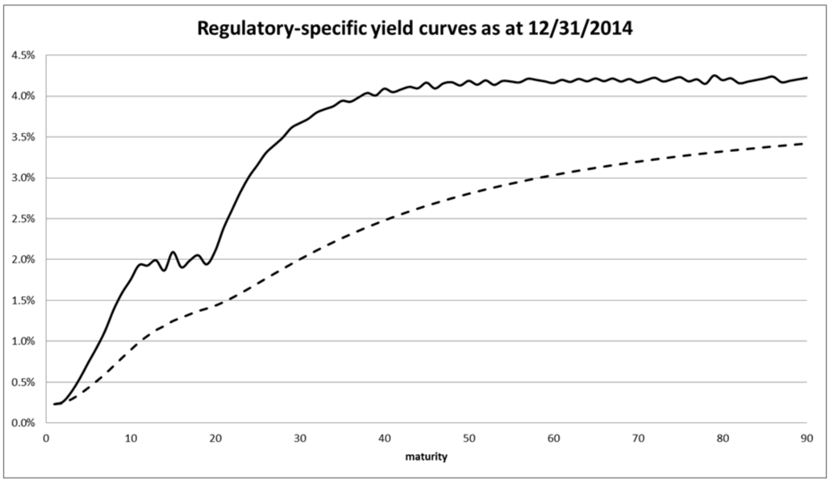

FIGURE 14 – Courbe régulatoire en $t = 0$: courbe des taux ZC (pointillés) et courbe des taux forwards 1 an (noir)

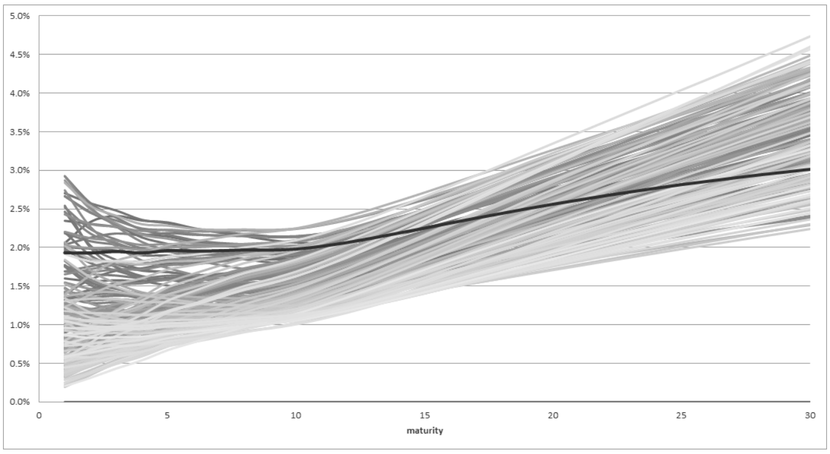

FIGURE 15 – t=10 ans - 20% (choisis) des courbes simulées (nuances de gris) + courbe *forward* vue en t=0 (noir)

complexité particulièrement élevée (de calibrage / simulation) parmi les modèles de taux utilisés, a tendance à fournir des résultats particulièrement simples et condensés, mais aussi très fortement dépendants de la courbe initiale, comme, de manière générale, tous les modèles risque neutre. Enfin, ces modèles sont connus pour fournir des courbes soit nulles, soit explosives à l'infini (voir El Karoui *et al.* (1997), Dybvig *et al.* (1996)). Il apparait dès lors particulièrement étonnant que de tels outils soient utilisés pour diffuser des comportements d'assurés. Dans un cadre de taux explosif y aurait-il 100% de rachats ? Et dans un cadre de taux nuls, la perte pour l'assureur serait-elle utile à quantifier ? La réponse la plus pertinente à ces questions, dans l'état actuel des choses serait de revoir les modèles utilisés pour intégrer plus de réalisme, d'utiliser par exemple des variantes beaucoup plus simples et réalistes de Nelson-Siegel.

Dans un second temps, nous avons cherché à étudier la robustesse des valorisations réalisées au 31 décembre 2014. Nous avons pour cela considéré l'évolution de la volatilité implicite de swaption de ténor 5 ans et maturité 5 ans (une swaption particulièrement liquide). Nous obtenons le graphe présenté en Figure 16

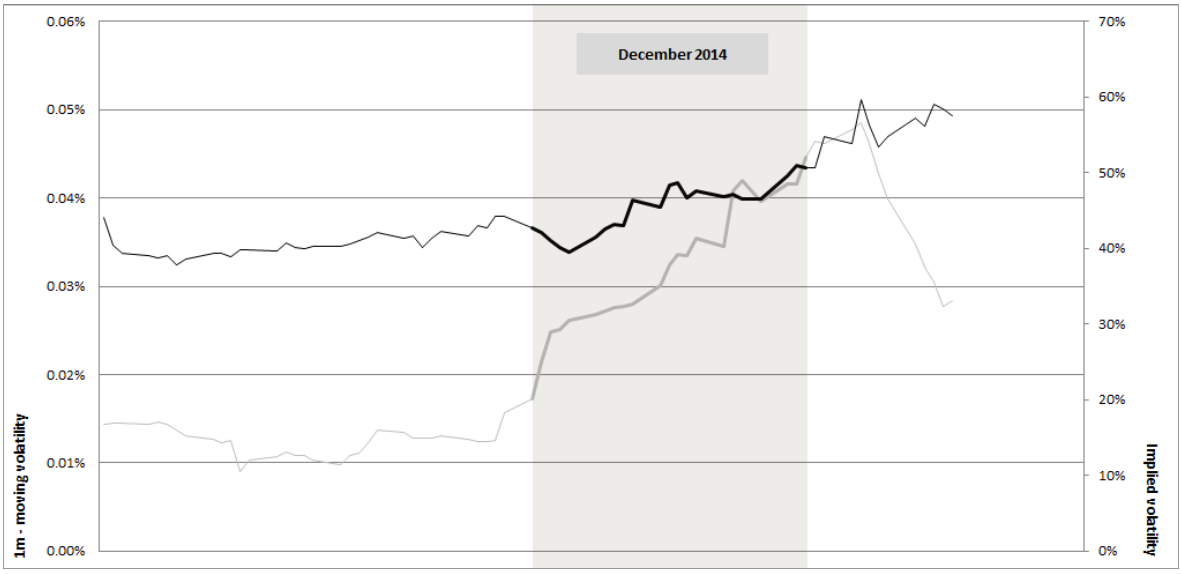

FIGURE 16 – Statistiques observées sur la volatilité implicite de swaption maturité 5 / ténor 5 - / noir : volatilité mobile 1 mois entre le 15/10/14 et le 15/01/15 - gris : volatilité implicite entre le 15/10/14 et le 15/01/15

On voit très nettement le *turn-of-the-year effect* bien connu : décembre est le mois où les grandes entreprises cherchent à optimiser leur bilan comptable et achètent et vendent de grosses quantités d'actions et de dérivés, ce qui vient ajouter de la volatilié artificielle et fausser les données de marché.

Dans un second temps nous avons cherché un moyen plus robuste de calibrer les modèles de taux pour le calcul des fonds propres économiques au 31 décembre 2014. Une méthode simple pour rajouter de l'inertie dans le calibrage effectué était, selon nous, de considérer non des données au 31 décembre uniquement, mais plutôt des données moyennées sur 1 ou 2 mois entiers. Nous avons donc choisi deux approches de valorisation *market-consistent* standard. L'approche v1 consiste à calibrer le modèle de taux sur un jeu de volatilités implicites de swaptions de maturité 10 ans / ténors 1 à 10 ans mais aussi de ténor 10 ans / maturité 1 à 10 ans. L'approche v2 consiste à calibrer le modèle de taux sur un jeu de volatilités implicites de swaptions de maturité 5 ans / ténors 1 à 10 ans mais aussi de ténor 5 ans / maturité 1 à 10 ans. Dans un second temps, nous avons considéré deux approches où le modèle de taux a été calibré sur la croix 5/5 (approche type v2) mais moyennée sur les données d'octobre et d'octobre-novembre 2014. Nous avons donc calibré 4 tables de scénarios et réalisé 4 valorisations de fonds propres économiques sur 3 produits d'assurance vie différents. Les résultats sont présentés dans la Table 6.

Ces résultats permettent de souligner différents points importants concernant la valorisation actuarielle risque neutre. Tout d'abord il apparait qu'une mise en œuvre sur des volatilités moyennées donne des résultats relativement plus robustes, stables et significativement différents des approches sur calibrage au 31 décembre.

De plus, ces résultats soulèvent une problématique très importante pour ce type de mise en œuvre, la

TABLE 6 – Comparaison entre les fonds propres obtenus avec les 4 approches considérées

	Octobre 2014	Oct. & Nov. 2014	31/12/14 v1	31/12/14 v2
Produit 1	16'898	15'614	7'046	10'000
Produit 2	12'826	12'283	9'517	10'000
Produit 3	12'553	12'073	6'050	10'000

manipulabilité des chiffres obtenus. En effet, deux approches de type *standard* et *market-consistent* (approches v1 et v2) donnent des résultats pouvant se révéler très différents. En pratique, le choix d'utiliser une approche de type v1 ou v2 (ou même une v3 encore plus favorable) pour le calibrage, pourra se justifier de diverses manières. L'approche v2 est centrée sur des options plus liquides, alors que l'approche v1 est centrée sur des options de plus long terme. D'autre part, pourquoi ne pas considérer une approche de calibrage dont la matrice sous-jacente est centrée sur les swaptions dont la duration est équivalente à celle du passif d'assurance valorisé ou de l'actif associé ? D'autre part une grande entreprise maîtrisant efficacement sa valorisation pourrait aussi, à terme, acheter ou vendre certaines swaptions en grande quantité afin d'influer sur leurs cours et modifier ainsi la structure de la matrice de volatilités implicites dans le sens le moins adverse, au 31 décembre. De telles idées peuvent paraître grossières mais le contrôle des valeurs fournies est extrêmement complexe et la méthode à employer est peu encadrée.

Afin d'objectiver avec plus de précision le choix de la probabilité utilisée lors des valorisations risque neutre actuarielles, nous avons enfin cherché à restreindre la contrainte de *market-consistency* en essayant d'intégrer au calibrage des modèles une forte dépendance aux caractéristiques propres du produit d'assurance valorisé. Nous avons considéré des produits d'assurance dont l'actif associé intègre des dérivés de taux (des swaptions) utilisés pour protéger le produit contre ce risque financier. Ce portefeuille "de protection" n'est pas, à proprement parler, un portefeuille de *hedging*, mais il est ce qui s'en rapproche le plus dans notre cadre. Nous avons donc choisi, pour notre calibrage du modèle de taux, les volatilités implicites des swaptions utilisés, pondérées par leur part dans le portefeuille *de protection*. L'approche développée, dite *Local Market Concensus* ou LMC, fournie les résultats de la Table 7.

TABLE 7 – Comparaison entre les fonds propres économiques obtenus

Valeur économique	Valorisation standard	Approche LMC	Ecart relatif
fonds propres	2'069	2'113	2.1%

Les résultats sont significativement différents bien que leur écart soit moins élevé.

En pratique, ce type de portefeuille de protection ne se retrouve pas sur tous les produits d'assurance vie. Notre méthodologie n'est donc pas totalement généralisable. Elle présente, toutefois, une première possibilité d'implémentation objective pour le calibrage des modèles de taux. Des développements futurs pourront affiner cette approche, notre objectif était ici de sensibiliser les opérationnels et académiques à cette problématique.

Articles présentés dans cette thèse

[a] Julien VEDANI et Laurent DEVINEAU (2013). Solvency assessment within the ORSA framework : issues and quantitative methodologies. *Bulletin Français d'Actuariat*, 13(25) :35–71.

[b] Julien VEDANI, Stéphane LOISEL et Fabien RAMAHAROBANDRO (2015). Continuous compliance : a *proxy*-based monitoring framework. *En cours d'écriture*.

[c] Julien VEDANI, Fabien CONNEAU et Laurent DEVINEAU (2016).Economic balance sheet *proxies* improvement : variance reduction and metamodel approach. *Working paper*.

[d] Charlotte BELIN, Rémy GERBOUD et Julien VEDANI (2015). Propriétés structurelles de la *VAN forward* - Application à l'optimisation de l'approche Simulations dans les Simulations en assurance vie.*soumis au Bulletin Français d'Actuariat*.

[e] Nicole EL KAROUI, Stéphane LOISEL, Jean-Luc PRIGENT et Julien VEDANI (2015). Market inconsistencies of the *market-consistent* European life insurance economic valuations : pitfalls and practical solutions. *soumis à l'Européan Actuarial Journal*.

Chapitre 1

Solvency assessment within the ORSA framework: issues and quantitative methodologies

Abstract

The implementation of the Own Risk and Solvency Assessment is a critical issue raised by Pillar II of Solvency II framework. In particular the Overall Solvency Needs calculation left the Insurance companies to define an optimal entity-specific solvency constraint on a multi-year time horizon. In a life insurance society framework, the intuitive approaches to answer this problem can sometimes lead to new implementation issues linked to the highly stochastic nature of the methodologies used to project a company Net Asset Value over several years. One alternative approach can be the use of polynomial proxies to replicate the outcomes of this variable throughout the time horizon.

Polynomial functions are already considered as efficient replication methodologies for the Net Asset Value over 1 year. The Curve Fitting and Least Squares Monte-Carlo procedures are the best-known examples of such procedures. In this article we introduce a possibility of adaptation for these methodologies to be used on a multi-year time horizon, in order to assess the Overall Solvency Needs.

Keywords : *Own Risk and Solvency Assessment, ORSA, Overall Solvency Needs, Solvency II, multi-year solvency, solvency ratio, Net Asset Value, polynomial proxy, Nested Simulations, Curve Fitting, Least Squares Monte-Carlo, Standard Formula..*

Introduction

At the heart of the Solvency II directive, Article 45 introduces the guidelines of the Own Risk and Solvency Assessment (ORSA) process. The ORSA framework leads companies towards a better understanding and an optimal management of their risk profiles, consistent with their strategic choices. Its implementation requires a thorough analysis of both short and long term risks. One major challenge of the introduced framework is the definition of the Overall Solvency Needs, which corresponds to the capital level required in order to satisfy a solvency constraint resulting from the firm's strategic choices and generally speaking from the firm's risk appetite. Its practical assessment should be made through a prospective analysis of the Net Asset Value (NAV) funding need. This implies the choice of a relevant time horizon, in order to take into account the overall strategic business plan of the undertaking, and the choice of an approach of the multi-year solvency. This latest notion is particularly difficult to apprehend. The solvency concept is indeed well-known in its regulatory sense, related to the first pillar of the directive and to a 1-year time horizon. But it must be generalized to a more economic framework in order to optimally fit the strategic issues of insurance companies.

The notion of multi-year ruin has already been studied theoretically through various ruin models, as presented in Asmussen and Albrecher (2010), or in Rulliere and Loisel (2004). However, the various theoretical assumptions leading to semi-closed formulas do not fit the Overall Solvency Needs simulatory framework and the great complexity of general life insurance products. The first objective of this paper, developed through Section 1.1, is to propose a practical formalization of the multi-year solvency concept. We present three main approaches of this notion, depending on the underlying goals and on the considered risk variable (NAV or Solvency Ratio). These approaches can be related to three kinds of metrics leading to different implementation constraints.

The use of the most complicated constraints presented in Section 1.1 introduces great implementation issues. Indeed, the prospective analysis of the *NAV* and of the Solvency Ratio requires the user to be able to project values of these underlying variables through a multi-year time horizon. As presented in Section 1.2, the Nested Simulations may be a relevant methodology to achieve the necessary projections. However, the highly stochastic nature of the methodology leads to very complex and time-consuming implementations. Several operational studies have been proposed in order to accelerate the estimation of extreme quantile values of the *NAV*, for a single-period time horizon, see for example Devineau and Loisel (2009*a*) or Nteukam and Planchet (2012). In the multi-year framework, a first approach based on closed formulas has been studied by Bonnin et al. (2014). To our knowledge, no acceleration methodology has been adapted to the Overall Solvency Needs assessment framework without needing relatively strong model assumptions. In order to enable the use of the proposed metrics, we develop implementation alternatives based on multi-year adaptations of the polynomial proxies procedures already known as Curve Fitting or Least Squares Monte-Carlo, as presented respectively in Algorithmics (2011) and Barrie & Hibbert (2011).

In Section 1.3, we formalize the mathematical basics of these proxy methodologies. This enables us to prove the convergence of both procedures and to propose a formula in order to compare the asymptotic efficiency of both Curve Fitting and Least Squares Monte-Carlo through a multi-year horizon. In the last section, we present the results obtained after having implemented the methodologies proposed in Section 3, on a standardized life insurance product.

1.1 Single-period and multi-year solvency

1.1.1 Single-period solvency

The notion of 1-year horizon solvency, in its regulatory sense, is defined by the directive's first pillar. It is based on the $VaR_{99.5\%}$ risk measure. Being solvent means having enough own funds to be able to avoid economic bankruptcy over 1 year with a 99.5% threshold.

Let NAV_t be the NAV at time $t \geq 0$, SCR_0 be the initial 1-year regulatory capital, the Solvency Capital Requirement (SCR), and δ_1 be the discount factor at the end of the first period.

The regulatory solvency constraint is denoted: $\mathbb{P}(NAV_1 \geq 0) \geq 99.5\%$. (SC0)

And we have $NAV_0 \geq SCR_0 \Leftrightarrow$ CurrentSolvencyRatio $= \frac{FP_0}{SCR_0} \geq 100\%$.

The required capital can be calculated on the basis of the formula:

$SCR_0 = NAV_0 + K$ /: $K = -VaR_{0.5\%}(\delta_1.NAV_1) = -q_{0.5\%}(\delta_1.NAV_1)$.

This formula is true under the assumption that the additional own funds are invested in risk-free assets (the additional capital is capitalized, from $t = 0$ on, at the risk free rate). For a more complete analysis of this formula and of the underlying assumptions needed the reader may consult Devineau and Loisel (2009*b*).

This single-period solvency notion corresponds to the regulatory definition of solvency. The multi-year solvency concept introduced by the ORSA framework leads to several conceptualization issues.

1.1.2 Multi-year solvency

The multi-year framework is particularly efficient to make the undertaking's strategic choices objective. Indeed, the strategic planning time horizon is generally of three to five years. Therefore, the Overall Solvency Needs assessment is particularly relevant for Enterprise Risk Management.

For the company, the major issue is to define its risk limits, coherently with its own strategy planning. In parallel, the firm must define a multi-year solvency constraint in order to assess a required capital level. Basically, a relevant approach requires an in-depth analysis of the questions it is supposed to answer.

A first possibility for this solvency constraint can be to adapt the single-period regulatory constraint. The multi-year solvency notion is then related to owning enough capital today to be able to avoid economic bankruptcy over the whole time horizon with a p threshold. This framework can lead to several implementation issues. One can notice that it is therefore not the type of constraint that has been chosen by most of the French insurance companies.

The adjusted constraint can be framed as:

$$\mathbb{P}\left(\cap_{t=1}^{T}\{NAV_t \geq 0\}\right) \geq p \Leftrightarrow \mathbb{P}\left(min_{t\in[\![1;T]\!]}\left[NAV_t \geq 0\right]\right) \geq p.$$

More generally, it should be possible to consider alternative risk measures (such as Expected Shortfall) or alternative underlying risky variable such as the Solvency Ratio or the Return on Risk Adjusted Capital, as defined in Decupère (2011).

Considering the presented constraint, the ORSA framework raises much thinking about. First, the p probability threshold should typically be chosen rather lower than the 99.5% threshold. Indeed, considering such a constraint enables to develop an overall risk running on both short and long term and in a realistic environment. In this framework it is clearly not appropriate to consider a too restrictive level of solvency. In a broader sense, the p probability threshold can be indexed by the time period in order to relax the constraint over the considered horizon. In addition, the underlying scope considered here is an enlargement of the SCR definition. In particular, as an economic approach, it seems necessary to consider the new business underwritten through the time horizon.

In section 1.1.3 we formalize different possible approaches of the multi-year solvency constraint. In particular, we introduce a framework that considers the regulatory solvency shortfall probability through the time horizon. In that case the multi-year solvency constraint is based on the Solvency Ratio's chronicle on the whole horizon.

1.1.3 Interpretations of the multi-year solvency

In order to formalize the concepts presented here, we introduce this additional notation. Let SCR_t be the 1-year regulatory capital value at time $t \geq 0$, δ_t be the discount factor at the end of the first $t \geq 1$ periods, R_t be the profit* at time $t \geq 1$, and T be the chosen ORSA time horizon. Additionally, let $\mathscr{F}_t^{RW}$ be the filtration that characterizes the Real-World economic information contained within the t first periods, and $\mathscr{Q}_{\sqcup}$ be a Risk-Neutral measure conditioned by the Real-World financial information known at time t.

Under this notation, the NAV at the end of the t^{th} period ($t \geq 1$) satisfies the formula:

$$NAV_t = \mathbb{E}^{\mathscr{Q}_t}\left[\sum_{u\geq 1}\frac{\delta_u}{\delta_t}R_u\middle|\mathscr{F}_t^{RW}\right].$$

We have (under the same assumption as for SCR_0)

$$SCR_t = NAV_t + K \;/\!:\; K = -VaR_{0.5\%}\left(\frac{\delta_{t+1}}{\delta_t}.NAV_{t+1}\middle|\mathscr{F}_t^{RW}\right) = -q_{0.5\%}\left(\frac{\delta_{t+1}}{\delta_t}.NAV_{t+1}\middle|\mathscr{F}_t^{RW}\right).$$

*. Seen as a NAV variation.

Approaches on economic bankruptcy

The approaches on economic bankruptcy aim at looking at whether the undertaking has enough own funds to carry on its business without capital through a chosen time horizon, under the current economic characteristics. Their objective is to translate the undertaking's overall tolerance into a multi-year constraint on the positivity of the NAV.

Constraint on yearly Net Asset Value distribution

$\forall t \in [\![1;T]\!], \mathbb{P}(NAV_t \geq 0) \geq p$. (SC1)

The aim of such a constraint is to determine the level of own funds at $t = 0$ required to avoid economic bankruptcy at each date of the horizon $[1;T]$ with the same p probability threshold (with p close to 1).

In a simulation framework, such an approach would lead to the separate analyze of the samples of the random variables $(NAV_t)_{t\in[\![1;T]\!]}$, period after period, without considering path-dependence. At each date, every positive outcome is considered solvent (see Figure 1.1).

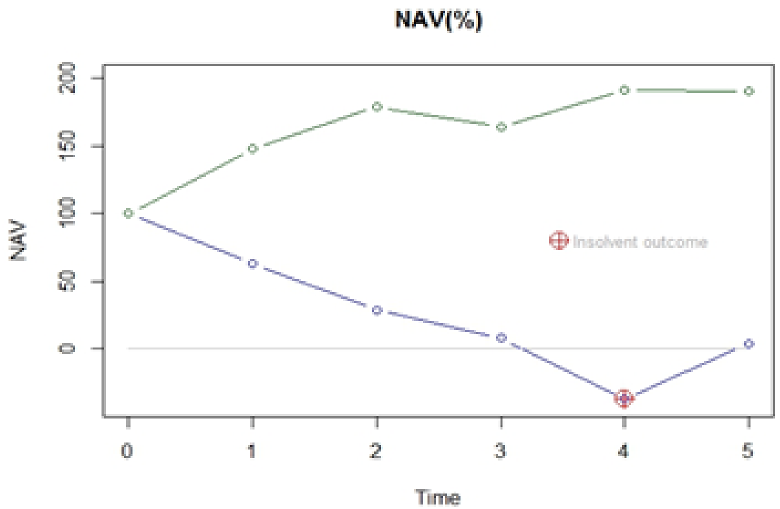

Figure 1.1 – Outcomes of solvent and insolvent NAV through a 5-years horizon (insolvent point targeted in red)

Under the assumption that the additional own funds is invested in risk-free assets, the Overall Solvency Needs can be calculated thanks to the formula:

$\text{RequiredCapital}_{(\text{SC1})} = NAV_0 + K$ /: $K = -min_{0<t\leq T}\left[q_{1-p}\left(\delta_t NAV_t\right)\right]$.

Constraint on the Net Asset Value's paths

$\mathbb{P}\left(\cap_{t=1}^{T}\{NAV_t \geq 0\}\right) \geq p$. (SC2)

The objective of this constraint is to determine the level of own funds required to avoid economic bankruptcy on the whole horizon $[0;T]$ with a p probability threshold.

Considering the intersection $\cap_{t=1}^{T}\{NAV_t \geq 0\}$ enables to model the path-dependence of the random process $(NAV_t)_{t\in[\![1;T]\!]}$. In a simulation framework such an approach would lead to consider sample paths of $(NAV_t)_{t\in[\![1;T]\!]}$. Only paths that lead to positive values at each time are considered as solvent. This is exemplified in Figure 1.2

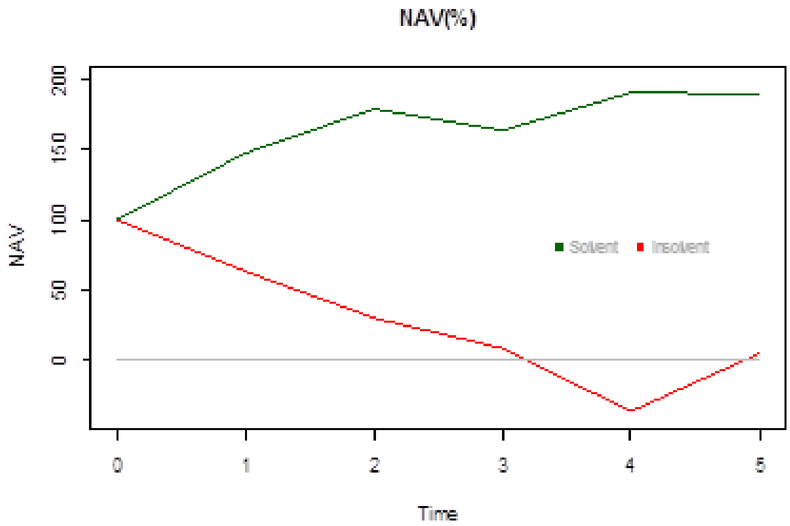

Figure 1.2 – One solvent and one insolvent NAV path for a 5-years horizon

Such a constraint enables to comply with the notion of continuity of business on the whole horizon. Therefore, it seems more realistic from an economic point of view. Under the assumption that the additional own funds are invested in risk-free assets, the Overall Solvency Needs can be calculated thanks to the formula:

$$\text{RequiredCapital}_{(\text{SC2})} = NAV_0 + K \,/ \colon K = Argmin_X \left[\mathbb{P}\left(\cap_{t=1}^{T}\{NAV_t + \frac{X}{\delta_t} \geq 0\}\right) = p\right].$$

Implementation aspects

One possibility of fully simulatory implementation allowing assessment of the Overall Solvency Needs requires the use of multi-year Nested Simulations.

In the Nested Simulations framework, let $Asset_t^n$ be the market value of the firm's asset at time t, for the n^{th} primary simulation, BEL_t^n be the Best Estimate of Liabilities of the firm at time t, for the n^{th} primary simulation, and NAV_t^n be the NAV of the firm at time t, for the n^{th} primary simulation.

The multi-year Nested Simulations procedure is represented in Figure 1.3.

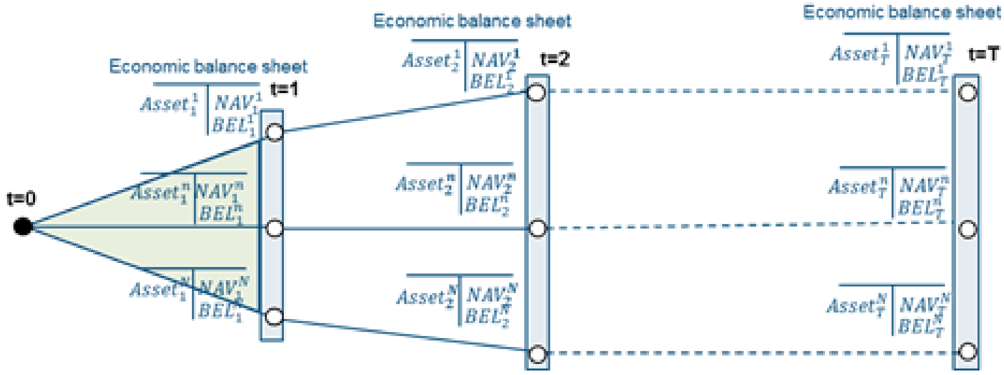

Figure 1.3 – Multi-year Nested Simulations–Generation of sample paths $((NAV_1^n, NAV_2^n, ..., NAV_T^n))_{n \in [\![1;N]\!]}$

The primary simulations leading to each node are obtained by Real-World diffusions of the risks. At each node a Risk-Neutral calculation of the firm's asset, of its Best Estimate of Liabilities (BEL) and of its NAV, is launched.

Such a procedure allows obtaining empirical distributions of NAV paths and percentile values for various thresholds. In practice this approach is hugely time-consuming, especially for life insurance products, for which there is no closed formula to calculate BEL values. However it can be accelerated using efficient proxies of the random variables $(NAV_t)_{t \in [\![1;T]\!]}$. This implementation enables to assess the probability of economic bankruptcy over a chosen time horizon, or the probability of economic bankruptcy at each date of a chosen time horizon, given an own fund's level at $t = 0$. Eventually it permits to calculate an estimate of the Overall Solvency Needs for (SC1) and (SC2).

Generally speaking, the approaches on economic bankruptcy enable to address the strategic problem of continuity of operations. However, this implementation does not bring any information about the compliance with regulatory solvency requirements on the chosen time horizon. The choice of an approach on solvency shortfalls enables one to deal with this particular issue.

Approaches on solvency shortfalls

The approaches on solvency shortfalls translate the undertaking's risk tolerance into a constraint on the value taken by its Solvency Ratio through the time horizon. They aim at assessing whether the undertaking is able to satisfy a regulatory solvency constraint over several years or not.

Constraint on yearly Solvency Ratio distributions

$\forall t \in [\![1;T]\!], \mathbb{P}\left(\frac{NAV_t}{SCR_t} \geq \alpha\right) \geq p.$ (SC3)

The objective of this constraint is to assess the level of own funds at $t = 0$ required to ensure that the Solvency Ratio stays superior to a level α ($\alpha \geq 0$) at each date of the horizon $[1;T]$ with the same p

probability threshold.

In a simulation framework such an approach would lead to analyze separately the outcomes of the random process $\left(\frac{NAV_t}{SCR_t}\right)_{t\in[\![1;T]\!]}$, period after period, without taking path-dependence into account. At each date, every outcome over the limit α is considered as solvent.

The choice of α depends on the risk limits considered in the definition of the firm's risk appetite. If a Solvency Ratio's value is targeted in order to keep a certain credit rating for example, it can be relevant to consider a value $\alpha \geq 100\%$. Such a rating constraint is generally stronger than the regulatory one. However, if the undertaking aims at staying over a lower bound (at least the Minimum Capital Requirement for example), it is possible to consider a value $\alpha \leq 100\%$. Eventually, under the assumption that the additional own funds are invested in risk-free assets, the Overall Solvency Needs can be calculated on the basis of the formula:

$$\text{RequiredCapital}_{(\text{SC3})} = NAV_0 + K \;/\!: \; K = Argmin_X \left[min_{0<t\leq T} \left[q_{1-p}\left(\frac{NAV_t + \frac{X}{\delta_t}}{SCR_t(X)}\right)\right] \geq \alpha \right].$$

Constraint on the Solvency Ratio's paths

$\mathbb{P}\left(\cap_{t=1}^{T}\{\frac{NAV_t}{SCR_t} \geq \alpha\}\right) \geq p$. (SC4)

The objective of this constraint is to assess the level of own funds at $t = 0$ required to ensure the coverage of at least a level α of the regulatory capital on the whole time horizon with a p probability threshold.

In a simulation framework such an approach would lead to focus on each path of the random process $\left(\frac{NAV_t}{SCR_t}\right)_{t\in[\![1;T]\!]}$ as a whole, so as to take path-dependency into account. Only the paths over the α threshold are considered satisfactory.

Under the assumption that the additional own funds are invested in risk-free assets, the Overall Solvency Needs can be calculated on the basis of the formula:

$$\text{RequiredCapital}_{(\text{SC4})} = NAV_0 + K \;/\!: \; K = Argmin_X \left[\mathbb{P}\left(\cap_{t=1}^{T}\{\frac{NAV_t + \frac{X}{\delta_t}}{SCR_t(X)} \geq \alpha\}\right) \geq p \right].$$

Implementation aspects

These constraints can be tested and an estimator of the Overall Solvency Needs can be obtained empirically by using a fully simulatory process (multi-year Nested Simulations). This procedure consists in simulating economic scenarios and calculating a NAV and a 1-year regulatory capital at each node. This approach can be implemented whether the company uses an Internal Model or the Standard Formula approach.

In this framework, let SCR_t^n be the 1-year regulatory capital of the firm at time t, for the n^{th} primary simulation. An exemplification of such implementation is given in Figure 1.4.

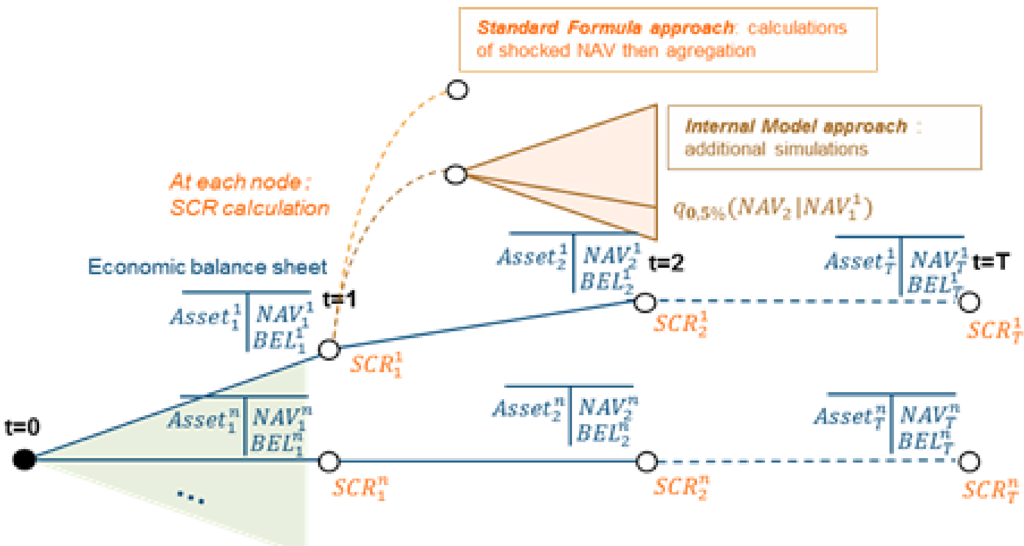

Figure 1.4 – Multi-year Nested Simulations – Generation of joint sample paths $((NAV_1^n, NAV_2^n, \ldots, NAV_T^n))_{n \in [\![1;N]\!]}$ and $((SCR_1^n, SCR_2^n, \ldots, SCR_T^n))_{n \in [\![1;N]\!]}$

In practice a fully simulatory approach is particularly difficult to implement but can be made extremely less time-consuming using proxies and the Standard Formula approach to obtain the SCR outcomes. This implementation enables to assess the probability of meeting a minimum level α of the regulatory capital on a chosen time horizon, or the probability of meeting a minimum level α of the regulatory capital at each date of a chosen time horizon, given an own funds level at $t = 0$. It eventually enables the calculation of an estimate of the Overall Solvency Needs as expressed in (SC3) and (SC4).

Approaches on multi-deterministic scenarios

Both approaches presented above lead to major implementation issues. Therefore, the insurance industry tends towards using of a more pragmatic approach, introducing a looser solvency constraint. The implementation of an approach on multi-deterministic scenarios consists in testing a solvency constraint on a limited number of multi-year stressed scenarios. In practice these scenarios must be chosen depending on their adversity level and their relevance to the underlying risks. Once a set of J stressed scenarios is calibrated, the constraint can be stated as:

$\forall j \in [\![1;J]\!], \forall t \in [\![1;T]\!], \frac{NAV_t^j}{SCR_t^j} \geq \alpha.$ (SC5)

The procedure to test the compliance with this constraint is easier to implement. Indeed, it is possible to assess the outcomes $\left(NAV_t^j, SCR_t^j\right)_{j,t}$ and to test the solvency constraint based on this small number of outcomes.

The major issue of these approaches is the choice of the stressed scenarios. Basically these scenarios are marginally or jointly stressed economic scenarios. The underlying risk factors must be chosen for their significance and the shock intensity must be carefully defined. In particular, for better economic realism, it can be relevant to integrate countercyclicality effects, mean reversion mechanisms and management actions. In practice a small number of scenarios are considered and the Standard Formula approach is chosen for the *SCR* calculations. Therefore few calculations are necessary to test the constraint.

The aim of these approaches is to assess whether the undertaking is able to meet a minimum level α of the regulatory capital through a chosen time horizon and for the selected stressed scenarios.

Under the assumption that the additional own funds are invested in risk-free assets, the Overall Solvency Needs can be calculated on the basis of the formula:

$$\text{RequiredCapital}_{(\text{SC5})} = NAV_0 + K \;/\!:\; K = -min_{0<t\leq T,\, 1\leq j\leq J}\left[\delta_t^j\left(NAV_t^j - \alpha\times SCR_t^j\right)\right].$$

It should be theoretically necessary to take the impact of the initial capital addition on the *SCR*, *K*, into account, for each period. However, these pragmatic approaches usually assume that this impact is negligible.

Eventually, the operational relevance of such an approach is obvious but it leads to difficulties in the choice of the stressed scenarios. Moreover, they provide less information compared to the previous approaches.

In Section 1.2, we develop quantitative proxy methodologies that provide satisfactory alternatives to fully Nested Simulations procedures, for life insurance economic balance sheet projections. These methodologies allow to obtain joint sample paths of the *NAV* and of the *SCR*.

1.2 Quantitative methodologies for the Overall Solvency Needs assessment in a life insurance company framework

Section 1.2 presents in a formal way the fully simulatory implementation and its proxy alternatives. Their common goal is to enable the use of an approach on solvency shortfalls in order to assess the Overall Solvency Needs associated to a life insurance company. The study will be carried out in a context where the *SCR* is calculated through the Standard Formula approach.

1.2.1 Multi-year Nested Simulations

Obtaining of a multi-year Net Asset Value distribution

We aim at obtaining empirical outcomes of the variables $(NAV_t)_{t\in[\![1;T]\!]}$, satisfying the following equation:

$\forall t \in [\![1;T]\!], NAV_t = \mathbb{E}^{\mathscr{Q}_t}\left[\sum_{u\geq 1}\frac{\delta_u}{\delta_t}R_u \middle| \mathscr{F}_t^{RW}\right] = \sum_{u=1}^{t}\frac{\delta_u}{\delta_t}R_u + \mathbb{E}^{\mathscr{Q}_t}\left[\sum_{u>t}\frac{\delta_u}{\delta_t}R_u \middle| \mathscr{F}_t^{RW}\right].$

The Nested Simulations procedure is an empirical methodology based on a Monte-Carlo valuation of $\mathbb{E}^{\mathscr{Q}_t}\left[\sum_{u>t}\frac{\delta_u}{\delta_t}R_u \middle| \mathscr{F}_t^{RW}\right]$. In practice, the implementation of this methodology, in order to obtain N outcomes $(NAV_1^n, ...NAV_T^n)_{n\in[\![1;N]\!]}$, follows this sequence:

- Step 1: Real-World diffusion of N random primary economic scenarios through the period $[0;T]$.
- Step 2: For each primary scenario n and date $t \in [\![1;T]\!]$:
  - Use of a simulation model to calculate the outcomes $\left(\frac{\delta_u^n}{\delta_t^n}R_u^n\right)_{u\in[\![1;t]\!]}$, the profit values at each date through $[0;t]$, capitalized until the t^th period and conditioned by scenario n.
  - Diffusion of P Risk-Neutral secondary economic scenarios through the period $[t;t+H]$, conditioned by the Real-World economic outcomes of scenario n at time t.* These scenarios must be i.i.d. conditionally to $\mathscr{F}_t^{RW}$.
  - Calculation of the related values $\left(\frac{\delta_u^{n,p}}{\delta_t^n}R_u^{n,p}\right)_{u\in[\![t+1;t+H]\!]}$, for each secondary scenario p.
  - Construction of an empirical estimator of the value $NAV_t^n = \sum_{u=1}^{t}\frac{\delta_u^n}{\delta_t^n}R_u^n + \mathbb{E}^{\mathscr{Q}_t}\left[\sum_{u=t+1}^{t+H}\frac{\delta_u}{\delta_t}R_u \middle| \mathscr{F}_t^n\right]$, $\mathscr{F}_t^n$ being the filtration that characterizes the Real-World economic information contained within primary simulation n,

$$\widehat{NAV}_t^n = \sum_{u=1}^{t}\frac{\delta_u^n}{\delta_t^n}R_u^n + \frac{1}{P}\sum_{p=1}^{P}\sum_{u=t+1}^{t+H}\frac{\delta_u^{n,p}}{\delta_t^n}R_u^{n,p}.$$

Eventually, the Nested Simulations procedure requires two kinds of stochastic economic scenarios tables. One table of Nprimary scenarios generated between $t=0$ and $t=T$ under the historical probability. Then, for each primary scenario and each date $t \in [\![1;T]\!]$, one conditional table of P secondary scenarios generated through the period $[t;t+H]$, under the probability measure $\mathscr{Q}_t$. An exemplification of such implementation is given in Figure 1.5.

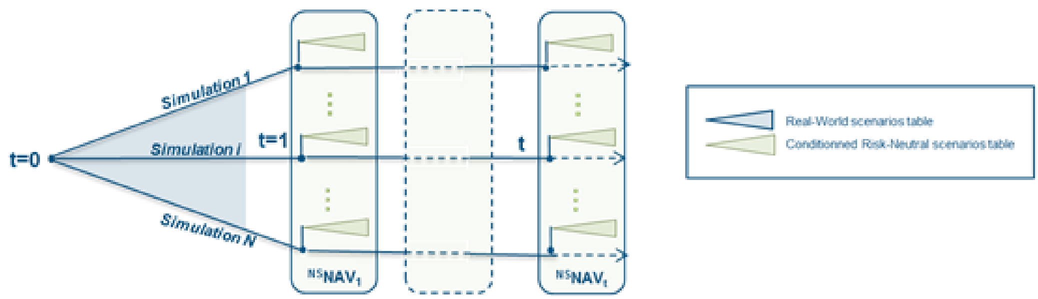

Figure 1.5 – Multi-year Nested Simulations to obtain Net Asset Value distributions

This methodology enables to obtain empirical approximations, assumed exact operationally as soon as P is large enough, of the random vector $((NAV_1^n, NAV_2^n, ..., NAV_T^n))_{n\in[\![1;N]\!]}$. By definition we have, $\forall n \in [\![1;N]\!]$:

*. H is the liability extinction horizon (ranging operationally from 30 to 50 years).

$\widehat{NAV}_t^n \rightarrow_{a.s.}^{P\to+\infty} NAV_t^n$. (strong law of large numbers)

In the rest of the paper we will denote by $\widehat{NAV}_t$ the random variable associated to the approximated *NAV* obtained using a Nested Simulations methodology.

Implementation to obtain a multi-year Solvency Capital Requirement distribution

Taking the Standard Formula approach into account, it is possible to derive the sample paths of *SCR* by duplicating this procedure. Indeed, the application of marginal shocks to the economic conditions at each node, before conditioning the Risk-Neutral tables, enables to calculate marginally shocked *NAV*.

Say we consider I different risks and let $\mathscr{F}_{t^+}^{n,risk\ i}$ be the filtration that characterizes the Real-World economic information contained within primary simulation n until date t and the marginally shocked economic conditions associated to risk number $i \in [\![1;I]\!]$, instantaneously applied after date t. We have, for each primary scenario $n \in [\![1;N]\!]$ and $i \in [\![1;I]\!]$:

$$NAV_t^{n,risk(i)} = \sum_{u=1}^{t} \frac{\delta_u^n}{\delta_t^n} R_u^n + \mathbb{E}^{\mathscr{Q}_t}\left[\sum_{u>t} \frac{\delta_u}{\delta_t} R_u \middle| \mathscr{F}_{t^+}^{n,risk(i)}\right] \approx \sum_{u=1}^{t} \frac{\delta_u^n}{\delta_t^n} R_u + \mathbb{E}^{\mathscr{Q}_t}\left[\sum_{u=t+1}^{t+H} \frac{\delta_u}{\delta_t} R_u \middle| \mathscr{F}_{t^+}^{n,risk(i)}\right]$$

with $NAV_t^{n,risk(i)}$ being the *NAV* at time t associated to the primary scenario n and shocked using the Standard Formula risk i (equity, interest rates, lapse,...). We obtain empirical estimators of the following shape:

$$\widehat{NAV}_t^{n,risk(i)} = \sum_{u=1}^{t} \frac{\delta_u^n}{\delta_t^n} R_u^n + \frac{1}{P} \sum_{p=1}^{P} \sum_{u=t+1}^{t+H} \frac{\left(\delta_u^{n,p}\right)^{risk(i)}}{\delta_t^n} \left(R_u^{n,p}\right)^{risk(i)}.$$

Using the Standard Formula aggregation matrices it is now possible to obtain the joint sample paths $\left(\left(\widehat{NAV}_1^n, ..., \widehat{NAV}_T^n\right)\right)_{n\in[\![1;N]\!]}$ and $\left(\left(\widehat{SCR}_1^n, ..., \widehat{SCR}_T^n\right)\right)_{n\in[\![1;N]\!]}$. We detail below the Standard Formula calculations needed to obtain the sample paths $\left(\left(\widehat{SCR}_1^n, ..., \widehat{SCR}_T^n\right)\right)_{n\in[\![1;N]\!]}$.

Let M be the number of risk modules considered, $Risk_m$ be the set of risks in module $m \in [\![[\![1;M]\!]$. $Risk_m \subset [\![1;I]\!]$, $\hat{C}_t^{n,risk\ i}$ be the stand-alone capital associated to risk i at time t and for the n^{th} scenario, obtained by using Nested Simulations, and $\widehat{SCR}_{m,t}^n$ be the *SCR* associated to module m at time t and for the n^{th} scenario, obtained using Nested Simulations. Moreover, let $\left(\rho_m^{i,j}\right)_{i,j\in Risk_m^2}$ be the Quantitative Impact Studies correlation matrix to aggregate the capitals associated to risk module m, and $\left(\rho^{k,l}\right)_{k,l\in[\![1;M]\!]^2}$ be the QIS correlation matrix to aggregate the *SCR* associated to the various risk modules.

The *SCR* outcomes are calculated on the basis of the following formulas:

$\forall t \in [\![1;T]\!], \forall n \in [\![1;N]\!]$, risk capitals calculation,

$\forall i \in [\![1;I]\!], \hat{C}_t^{n,risk\ i} = max\left[\widehat{NAV}_t^n - \widehat{NAV}_t^{n,risk\ i};0\right].$

Intra-modular aggregation:

$$\forall m \in [\![1;M]\!],\ \widehat{SCR}_{m,t}^n = \sqrt{\sum_{i,j \in Risk_m^2} \rho_m^{i,j}.\hat{C}_t^{n,risk\ i}.\hat{C}_t^{n,risk\ j}}.$$

Inter-modular aggregation *:

$$\widehat{SCR}_t^n = \sqrt{\sum_{k,l \in [\![1;M]\!]^2} \rho^{k,l}.\widehat{SCR}_{k,t}^n.\widehat{SCR}_{l,t}^n}.$$

There is no closed formula to calculate the *NAV* and the *SCR* except when relatively strong model assumptions are made, as in Bonnin et al. (2014). This Monte-Carlo methodology is the most precise approach for deriving joint sample paths of the *NAV* and *SCR*. However it is extremely time-consuming due to the large number of simulations needed to obtain efficient estimators.

The various alternative approaches developed in Subsections 1.2.2 and 1.2.3 aim at approximating the outcomes obtained by implementing such a procedure. The use of these proxies allows for a significant acceleration of this fully simulatory framework.

1.2.2 Multi-year Curve Fitting

The aim of a Curve Fitting approach is to calibrate a polynomial function replicating either the Best Estimate of Liabilities or the *NAV*. Various kinds of regressors can be used but they are easy to calculate and to handle (in this article we will consider polynomial functions of the elementary risk factors of the primary simulations). For more insight about the elementary risk factors definition, the reader may consult Devineau and Chauvigny (2011). This approach is already used in a single-period framework in order to replicate the *NAV* at time $t = 1$ and assess the economic capital. It can be adapted to a larger time horizon with satisfactory results.

Formalization to obtain a multi-year Net Asset Value distribution

Here, we consider a polynomial proxy that replicates the *NAV* in a straightforward fashion. For the sake of simplicity we will only consider two risks: the risk on the stock index and on interest rates. †

The gist of the method is the following. The *NAV* at time t depends on the economic information through the period $[0;t]$, $NAV_t = \mathbb{E}^{\mathcal{Q}_t}\left[\sum_{u=1}^{t+H} \frac{\delta_u}{\delta_t} R_u \middle| \mathcal{F}_t^{RW}\right]$.

When related to the n^{th} outcome obtained by implementing a Nested Simulations methodology, we

*. Here the Solvency Capital Requirement is identified to the Basic Solvency Capital Requirement.

†. We only focus in this article on the level risk and do not consider the volatility risk.

use the approximated NAV at date t introduced above:

$$\widehat{NAV}_t^n = \sum_{u=1}^{t} \frac{\delta_u^n}{\delta_t^n} R_u^n + \frac{1}{P} \sum_{p=1}^{P} \sum_{u=t+1}^{t+H} \frac{\delta_u^{n,p}}{\delta_t^n} R_u^{n,p}.$$

Eventually, every primary economic scenario can be synthesized into sets of yearly elementary risk factors with little or no loss of information, denoted $\left({}^s\varepsilon_1^n, {}^s\varepsilon_2^n, ..., {}^s\varepsilon_t^n\right)$ (resp. $\left({}^{ZC}\varepsilon_1^n, {}^{ZC}\varepsilon_2^n, ..., {}^{ZC}\varepsilon_t^n\right)$) for the stock (resp. interest rate) yearly elementary risk factors. Where ${}^s\varepsilon_u^n$ (resp. ${}^{ZC}\varepsilon_u^n$) stands for the stock (resp. interest rate) elementary risk factor related to the simulation n and the period $[u-1, u]$.

These elementary risk factors can be extracted from the primary scenarios table considering formulas similar to those proposed in Devineau and Chauvigny (2011).

We have $NAV_t^n \approx \{_t \left({}^s\varepsilon_1^n, {}^{ZC}\varepsilon_1^n, {}^s\varepsilon_2^n, {}^{ZC}\varepsilon_2^n, ..., {}^s\varepsilon_t^n, {}^{ZC}\varepsilon_t^n\right)$

for a certain polynomial function $\{_t$.

The notation Z^{tr} stands further for the transpose of a matrix or vector Z.

The implementation of this methodology follows this sequence for each date $t \in [\![1;T]\!]$:

- Step 1: Calculation of a small number N' of outcomes $\left(\widehat{NAV}_t^{n'}\right)_{n' \in [\![1;N']\!]}$.
- Step 2: Calibration of the optimal set of regressors * $X_t = \left(Intercept, {}^1X_t, ..., {}^kX_t\right)$, † with ${}^iX_t = {}^s\varepsilon_t^{x_i}.{}^{ZC}\varepsilon_t^{y_i}$, for each $i \in [\![1;k]\!]$ and certain positive integers x_i and y_i. Then determination of $\hat{\beta}_t = \left({}^I\hat{\beta}_t, {}^1\hat{\beta}_t, ..., {}^k\hat{\beta}_t\right)^{tr}$, the ordinary least squares (OLS) parameters estimator of the multiple regression $\widehat{NAV}_t = X_t.\beta_t + u_t$ where $\widehat{NAV}_t$ is approximated by its conditional expectation given the σ-field generated by the regressors X_t, considered as a linear combination of the regressors. For more insight about multiple regression models the reader may consult Saporta (2006). The underlying assumption of this model can therefore be written:
  $\exists \beta_t, \mathbb{E}\left[\widehat{NAV}_t \middle| X_t\right] = X_t.\beta_t$.
- Step 3: Generation of N independent outcomes $\left(X_t^n = \left(1, {}^1X_t^n, ..., {}^kX_t^n\right)\right)_{n \in [\![1;N]\!]}$ and calculation of the approximated distribution $\left({}^{CF}\right)NAV_t^n = X_t^n.\hat{\beta}_t\Big)_{n \in [\![1;N]\!]}$.

Let us introduce the following notation. Let ${}^{\widehat{NAV}}Y_t = \left(\widehat{NAV}_t^1, ..., \widehat{NAV}_t^{N'}\right)^{tr}$ be the Nested Simulations outcomes used in the calibration step, and $R(X_t) = \left(X_t^1, ..., X_t^{N'}\right)^{tr}$ be the matrix of the N' outcomes of the optimal set of regressors X_t, related to the outcomes $\left(\widehat{NAV}_t^{n'}\right)_{n' \in [\![1;N']\!]}$ used in the calibration step.

With this notation, the parameters estimator is obtained by solving the optimization program

∗. For more insight concerning the calibration of the optimal set of regressors see Subsection 1.2.2

†. Here, k is the number of regressors (not including the intercept) in the calibrated optimal set of regressors. In all generality k depends on time t. This indexation is omitted for the sake of simplicity.

$\hat{\beta}_t = Argmin_{\beta_t} \left[\|({}^{\widehat{NAV}}Y_t - R(X_t).\beta_t\|_1^2 \right].$

Under the assumption that the matrix $R(X_t)^{tr}.R(X_t)$ is invertible (satisfied in practice for k small enough) this program has a unique solution

$\hat{\beta}_t = (R(X_t)^{tr}.R(X_t))^{-1} R(X_t)^{tr}.{}^{\widehat{NAV}}Y_t.$

Implementation issues

This implementation shows two major issues: the choice of the $N^{'}$ primary scenarios used in the calibration step and the choice of the set of optimal regressors.

The scenarios considered so as to calibrate the polynomial function must enable to obtain an estimator $\hat{\beta}_t$ close to the optimal β_t in order to efficiently replicate the overall distribution of NAV_t. Two approaches are usually considered to achieve this. First it can be interesting to consider well-dispersed primary scenarios. It is also possible to consider *extreme* primary scenarios, according to a metric of the scenario's adversity threshold, for example a chosen norm on the underlying risk factor, as developed in Devineau and Loisel (2009*a*).

Both approaches can lead to satisfactory empirical results, depending on the characteristics of the life insurance products, the economic conditions, etc.

Considering the second issue, the general procedure aims at maximizing the R^2 of the considered regression under a significance constraint on the chosen covariates. Various automatic approaches can be used with satisfactory results such as the Stepwise Regression procedures. For more developments about these approaches see Draper and Smith (1981).

Formalization to obtain a multi-year Solvency Capital Requirement distribution

We get a multi-year *SCR* distribution joint with $\left({}^{CF}NAV_1^n, {}^{CF}NAV_2^n, ..., {}^{CF}NAV_T^n\right)_{n\in[\![1,N]\!]}$ easily. It is indeed possible to duplicate this procedure and calibrate proxies *after shock* : $X_t^{risk\ i}.\hat{\beta}_t^{risk\ i}$, $X_t^{risk\ i}$ being the optimal set of regressors of the proxy *after shock i* (equity or interest rate up/down) and $\hat{\beta}_t^{risk\ i}$ the associated parameters estimator.

Basically, it is necessary to duplicate fully the procedure in order to calibrate new proxies *after shock*, to obtain approximated marginally shocked NAV. Indeed, the Standard Formula shocks must be applied after the rebalancing of assets and the asset-mix dependent *central* proxies are not adapted to this new framework. Therefore they cannot be used to approximate the marginally shocked NAV.

The proxies *after shock* are obtained considering a calibration step upon $N^{'}$ shocked outcomes $\left(\widehat{NAV}_t^{n^{'},risk\ i}\right)_{n^{'}\in[\![1;N^{'}]\!]}$, for each considered Standard Formula risk i and date $t \in [\![1;T]\!]$. Then the aggregation of the central and shocked outcomes, $\left(X_t^n.\hat{\beta}_t\right)_{n\in[\![1;N]\!]}$ and $\left(X_t^{n,risk\ i}.\hat{\beta}_t^{risk\ i}\right)_{n\in[\![1;N]\!]}$, enables to obtain joint sample paths of the *NAV* and *SCR*. One can notice that, in practice, the optimal sets $X_t^{risk\ i}$, for all i (equity or interest rate up/down), are very close to X_t^n.* An exemplification of such implementation is given in Figure 1.6.

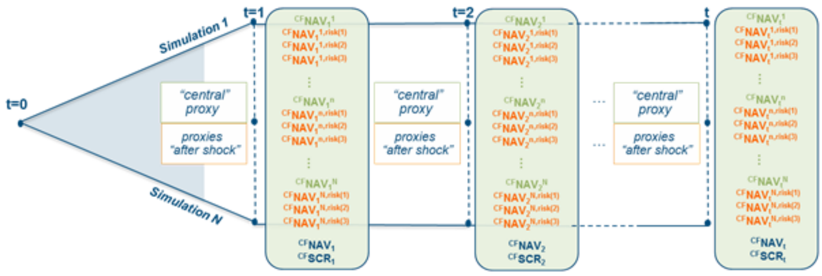

Figure 1.6 – Curve Fitting approach - Simulation of approximated joint sample paths of *NAV* and *SCR* (3 shocks)

N' being usually quite small, this alternative implementation greatly accelerates the Nested Simulations procedure with satisfactory results.

1.2.3 Multi-year Least Squares Monte-Carlo

It is generally considered that the so-called Least Squares Monte-Carlo (*LSMC*) methodology has first been introduced in Longstaff and Schwartz (2001) as a methodology for approximating the value of American options. However, the implementation of the Least Squares Monte-Carlo in the insurance framework is quite different of the financial valuation algorithm developed by Longstaff and Schwartz.

American options pricing is particularly tricky and their valuation leads to a step by step maximization problem. Various methodologies have been proposed to work around this valuation issue. Longstaff and Schwarz consider the framework of an American option that can only be exercised at discrete times $t_1, t_2, ..., t_N$. They propose a backward inducted methodology that uses polynomial functions of the underlying asset to approximate the conditionally expected value of the Net Present Value of future cash-flows at each exercise date. At each date t_k the conditional expected value is compared to the immediate exercise value in order to identify the exercise decision. Then the process is duplicated at time t_{k-1} and recursively until the initial valuation time.

In the insurance framework, the *LSMC* is a forward-looking approach used to approximate the *NAV* or the Best Estimate of Liabilities values by using polynomial functions. Basically this methodology is very close to Curve Fitting. The major difference between both proxy approaches lies in the calibration step.

*. In practice, it seems even possible not to calibrate specific sets of regressors for the proxies *after shock* and just to use the optimal set of regressors calibrated for the *central* proxies. This process is used in Section 1.4. It accelerates the methodology and provides satisfactory results.

Formalization to obtain a multi-year distribution of Net Asset Value

This alternative approach introduces the notion of NPV_t, a Net Present Value of margins at each date $t \in [\![1;T]\!]$, adding with the capitalized profits through $[0;t]$, defined as:

$$NPV_t = \sum_{u \geq 1} \frac{\delta_u}{\delta_t} R_u = \sum_{u=1}^{t+H} \frac{\delta_u}{\delta_t} R_u.$$

One can decompose the $\widehat{NAV}_t$ outcomes as a sum of NPV_t outcomes:

$$\widehat{NAV}_t^n = \sum_{u=1}^{t} \frac{\delta_u^n}{\delta_t^n} R_u^n + \frac{1}{P} \sum_{p=1}^{P} \sum_{u=t+1}^{t+H} \frac{\delta_u^{n,p}}{\delta_t^n} R_u^{n,p} = \frac{1}{P} \sum_{p=1}^{P} NPV_t^{n,p}.$$

With introduction of $NPV_t^{n,p}$, the Net Present Value of margins at time t associated to the n^{th} primary scenario and the p^{th} secondary scenario, adding with the capitalized profits through $[0;t]$.

The basement of the methodology, adapted to our insurance framework, comes from the idea that the use of $\widehat{NAV}_t$ outcomes for the calibration of the polynomial proxies introduces redundancy of information that can be avoided by considering NPV_t outcomes which calculation relies on a single simulated path (see the illustration in Figure 1.7).

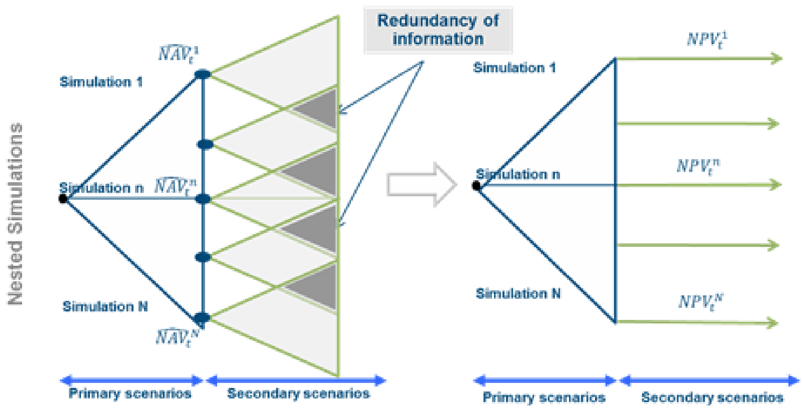

Figure 1.7 – Illustration of the redundancy of information introduced by Nested Simulations

The implementation of this methodology follows this sequence for each date $t \in [\![1;T]\!]$:

— Step 1: Calculation of a large number N' of outcomes $\left(NPV_t^{n',1} = NPV_t^{n'}\right)_{n' \in [\![1;N']\!]}$ obtained considering independent secondary simulations. Basically these outcomes can be considered as a sample of $\widehat{NAV}_t$ obtained by considering one secondary scenario ($P = 1$) per primary scenario. All the secondary scenarios must be generated independently.

— Step 2: Calibration of the optimal regressors set $X_t = \left(Intercept, {}^1X_t, ..., {}^kX_t\right)$, with ${}^iX_t = {}^s\varepsilon_t^{x_i}.{}^{ZC}\varepsilon_t^{y_i}$, for each $i \in [\![1;k]\!]$ and positive integers x_i and y_i. Then assessment of the OLS parameters estimator of the multiple regression $NPV_t = X_t.\beta_t + \nu_t$ where NPV_t is approximated by its conditional expectation given the σ-field generated by the regressors X_t, considered as a

linear combination of the regressors, that is $\hat{\beta}_t = \left({}^I\hat{\beta}_t, {}^1\hat{\beta}_t, ..., {}^k\hat{\beta}_t\right)^{tr}$. The underlying assumption of this model can therefore be written *:
$\exists \beta_t, \mathbb{E}\left[NPV_t X_t | X_t\right] = X_t.\beta_t$.

— Step 3: Generation of N independent outcomes of X_t, $\left(X_t^n = \left(1, {}^1X_t^n, ..., {}^kX_t^n\right)\right)_{n \in [\![1;N]\!]}$, and calculation of an approximated distribution of NAV_t, $\left({}^{LSMC}NAV_t^n = X_t^n.\hat{\beta}_t\right)_{n \in [\![1;N]\!]}$. Indeed, as proven in Subsection 1.3.1, the *LSMC* and the Curve Fitting approaches both converge asymptotically towards the same NAV_t polynomial proxy.

A calculation of NPV_t is much quicker than a calculation of $\widehat{NAV}_t$ as considered in the Curve Fitting methodology (with $P \gg 1$). However it brings little information on the value of the related NAV_t. This justifies the need to consider a large $N^{'}$.

Let us now introduce the following notation. Let ${}^{NPV}Y_t = \left(NPV_t^1, ..., NPV_t^{N^{'}}\right)^{tr}$ be the outcomes used in the calibration step, and $R(X_t) = \left(X_t^1, ..., X_t^{N^{'}}\right)^{tr}$ be the matrix of the $N^{'}$ outcomes of the optimal set of regressors X_t related to the sample $\left(NPV_t^{n^{'}}\right)_{n^{'} \in [\![1;N^{'}]\!]}$ used in the calibration step.

Under this notation, the parameters estimator is obtained by solving the optimization program

$\hat{\beta}_t = Argmin_{\beta_t}\left[\| {}^{NPV}Y_t - R(X_).\beta_t \|_1^2\right]$.

Under the assumption that the matrix $R(X_t)^{tr}.R(X_t)$ is invertible (satisfied in practice for k small enough) this program has a unique solution

$\hat{\beta}_t = \left(R(X_t)^{tr}.R(X_t)\right)^{-1} R(X_t)^{tr}.{}^{NPV}Y_t$.

Implementation issues and formalization to obtain a multi-year distribution of Solvency Capital Requirement

The implementation issues raised by this procedure are similar to those already considered in the Curve Fitting methodology. Similarly one can duplicate the procedure in order to obtain shocked proxies, thus enabling to calculate approximated joint sample paths of shocked *NAV* through the chosen time horizon. These *LSMC* outcomes can be aggregated using the Standard Formula matrices to obtain joint outcomes of the *NAV* and of the *SCR*.

*. This assumption can be loosened by considering $\mathbb{E}\left[X_t.u_t\right] = 0$. This leads to the same results.

1.3 Theoretical formalization and comparison of the polynomial approaches

The main objectives of Section 4 are to prove the asymptotic convergence of both Curve Fitting and *LSMC* estimators and to establish a result enabling the comparison of both Curve Fitting and *LSMC* efficiency. A framework leading to formulas for the estimator error factor in both Curve Fitting and Least Squares Monte-Carlo approaches has been investigated in Kalberer (2012).

In order to theoretically formalize both methodologies we introduce the following notation. Let $\mathscr{P}$ be the Real-World probability measure, and $(\mathscr{P}\otimes\mathscr{Q})_t$ be the probability measure under which our variables indexed by time t are simulated. The indexation by $(\mathscr{P}\otimes\mathscr{Q})_t$ is omitted for the sake of simplicity.

Recall the definition of the following random variables:

$$NPV_t = \sum_{u\geq 1} \frac{\delta_u}{\delta_t} R_u$$

and

$$\widehat{NAV}_t = \frac{1}{P}\sum_{p=1}^{P} NPV_t^p.$$

Considering P i.i.d. random variables conditionally to $\mathscr{F}_t^{RW}$, that follow the same distribution as NPV_t:

$$NPV_t^1, NPV_t^2, ..., NPV_t^P.$$

We have, by definition of NAV_t:

$$NAV_t = \mathbb{E}\left[NPV_t \middle| \mathscr{F}_t^{RW}\right] = \mathbb{E}\left[\widehat{NAV}_t \middle| \mathscr{F}_t^{RW}\right].$$

1.3.1 Preliminary results

Curve Fitting formalization

The model considered here is the following:

$$\widehat{NAV}_t = X_t.^1\beta_t + u_t \tag{1.1}$$

,

where $\widehat{NAV}$ is approximated by its conditional expectation given the *sigma*-field generated by the regressors X_t, considered as a linear combination of the regressors.

The underlying assumption of this model can therefore be written

$\exists^1 \beta_t, \mathbb{E}\left[\widehat{NAV}_t \middle| X_t\right] = X_t.^1\beta_t.$

We introduce the following notation:

$\mathbb{V}[u_t] = {}^u\sigma_t^2, \mathbb{E}\left[\mathbb{V}\left[NPV_t | \mathscr{F}_t^{RW}\right]\right] = {}^{NPV}\sigma_t^2, \mathbb{V}[NAV_t] = {}^{NAV}\sigma_t^2.$

Under this notation, we have

$$ {}^{NAV}\sigma_t^2 + \frac{{}^{NPV}\sigma_t^2}{P} = \mathbb{V}\left[X_t.^1\beta_t\right] + {}^u\sigma_t^2 \tag{1.2} $$

Indeed,

$\mathbb{V}\left[\widehat{NAV}_t\right] = \mathbb{V}\left[\mathbb{E}\left[\widehat{NAV}_t \middle| \mathscr{F}_t^{RW}\right]\right] + \mathbb{E}\left[\mathbb{V}\left[\widehat{NAV}_t \middle| \mathscr{F}_t^{RW}\right]\right] = \mathbb{V}[NAV_t] + \mathbb{E}\left[\mathbb{V}\left[\widehat{NAV}_t \middle| \mathscr{F}_t^{RW}\right]\right],$

$\mathbb{V}\left[\widehat{NAV}_t\right] = {}^{NAV}\sigma_t^2 + \frac{1}{P}\mathbb{E}\left[\mathbb{V}\left[NPV_t | \mathscr{F}_t^{RW}\right]\right] = {}^{NAV}\sigma_t^2 + \frac{{}^{NPV}\sigma_t^2}{P}.$

Moreover,

$\mathbb{E}\left[\widehat{NAV}_t \middle| X_t\right] = X_t.^1\beta_t$

implies that

$\mathbb{E}[X_t.u_t] = \mathbb{E}\left[\mathbb{E}\left[X_t.\left(\widehat{NAV}_t - X_t.^1\beta_t\right) \middle| \mathscr{F}_t^{RW}\right]\right] = (0, ..., 0).$

We have therefore

$\mathbb{V}\left[\widehat{NAV}_t\right] = \mathbb{V}\left[X_t.^1\beta_t + u_t\right] = \mathbb{V}\left[X_t.^1\beta_t\right] + \mathbb{V}[u_t],$

$\mathbb{V}\left[\widehat{NAV}_t\right] = \mathbb{V}\left[X_t.^1\beta_t\right] + {}^u\sigma_t^2.$

Least Squares Monte-Carlo formalization

The model considered here is the following:

$$ NPV_t = X_t.^2\beta_t + v_t. \tag{1.3} $$

In this model, NPV_t is approximated by its conditional expectation given the σ-field generated by the regressors X_t, considered as a linear combination of the regressors.

The underlying assumption of this model can therefore be written as

$\exists^2\beta_t, \mathbb{E}[NPV_t|X_t] = X_t.^2\beta_t.$

We introduce the following notation:

$\mathbb{V}[v_t] = {}^v\sigma_t^2.$

Under this notation, we have

$$^{NAV}\sigma_t^2 + {}^{NPV}\sigma_t^2 = \mathbb{V}\left[X_t^{n'}.^2\beta_t\right] + {}^v\sigma_t^2 \tag{1.4}$$

Indeed,

$\mathbb{V}[NPV_t] = \mathbb{V}\left[\mathbb{E}\left[NPV_t|\mathscr{F}_t^{RW,n'}\right]\right] + \mathbb{E}\left[\mathbb{V}\left[NPV_t|\mathscr{F}_t^{RW,n'}\right]\right] = \mathbb{V}[NAV_t] + \mathbb{E}\left[\mathbb{V}\left[NPV_t|\mathscr{F}_t^{RW,n'}\right]\right],$

$\mathbb{V}[NPV_t] = {}^{NAV}\sigma_t^2 + {}^{NPV}\sigma_t^2.$

Moreover

$\mathbb{E}[NPV_t|X_t] = X_t.^2\beta_t$

implies that

$\mathbb{E}[X_t.v_t] = \mathbb{E}\left[\mathbb{E}\left[X_t.\left(NPV_t - X_t.^2\beta_t\right)|\mathscr{F}_t^{RW,n'}\right]\right] = (0,...,0).$

We have therefore

$\mathbb{V}[NPV_t] = \mathbb{V}\left[X_t.^2\beta_t + v_t\right] = \mathbb{V}\left[X_t.^2\beta_t\right] + \mathbb{V}[v_t],$

$\mathbb{V}[NPV_t] = \mathbb{V}\left[X_t.^2\beta_t\right] + {}^v\sigma_t^2.$

Equivalence of the optimal parameters

Let us now consider the two linear models associated to regressions 1.1 and 1.3, and the additional linear model on the unobserved variable NAV_t associated to regression 1.5, defined as follows:

$$NAV_t = X_t.^3\beta_t + w_t \tag{1.5}$$

In this model, the variable NAV_t is approximated by its conditional expectation given the σ-field generated by the regressors X_t, considered as a linear combination of the regressors.

The underlying assumption of this model can therefore be written

$\exists {}^3\beta_t, \mathbb{E}[NAV_t|X_t] = X_t.{}^3\beta_t.$

We shall make the following assumption (satisfied in practice):

$\overline{\mathscr{H}}$: $\mathbb{E}[X_t^{tr}.X_t]$ exists and is invertible.

Considering the three linear models and assuming $\overline{\mathscr{H}}$, we have the following result, that will be considered in Subsection 1.3.2

$$^1\beta_t = {}^2\beta_t = {}^3\beta_t. \tag{1.6}$$

Indeed,

$\mathbb{E}\left[\widehat{NAV}_t \middle| X_t\right] = X_t.{}^1\beta_t$

implies that

$\mathbb{E}[X_t.u_t] = (0,...,0) \Leftrightarrow \mathbb{E}\left[X_t^{tr}.\left(\widehat{NAV}_t - X_t.{}^1\beta_t\right)\right] = (0,...,0)^{tr}$

$\Leftrightarrow \mathbb{E}\left[X_t^{tr}.\widehat{NAV}_t\right] = \mathbb{E}[X_t^{tr}.X_t]\,{}^1\beta_t.$

Because X_t is $\mathscr{F}_t^{RW}$-measurable,

$\mathbb{E}\left[\widehat{NAV}_t \middle| X_t\right] = X_t.{}^1\beta_t \Rightarrow {}^1\beta_t = \mathbb{E}[X_t^{tr}.X_t]^{-1}.\mathbb{E}\left[X_t^{tr}.\widehat{NAV}_t\right] = \mathbb{E}[X_t^{tr}.X_t]^{-1}.\mathbb{E}\left[X_t^{tr}.\mathbb{E}\left[\widehat{NAV}_t \middle| \mathscr{F}_t^{RW,n'}\right]\right],$

$^1\beta_t = \mathbb{E}[X_t^{tr}.X_t]^{-1}.\mathbb{E}[X_t^{tr}.NAV_t] = {}^3\beta_t.$

We obtain result 1.6 by considering the same demonstration, adapted to 1.3.

1.3.2 Comparison of the Curve Fitting and Least Squares Monte-Carlo efficiency

Comparison Curve Fitting vs. *LSMC*

As for 1.2 and 1.4, we can find a similar formula for the model associated to regression 1.5, considering an adapted notation

$$^{NAV}\sigma_t^2 = \mathbb{V}\left[X_t.{}^3\beta_t\right] + {}^w\sigma_t^2. \tag{1.7}$$

With ${}^{w}\sigma_t^2$, the residuals variance. Eventually, we can derive

$$
{}^{NAV}\sigma_t^2 + \frac{{}^{NPV}\sigma_t^2}{P} = \mathbb{V}\left[X_t.{}^3\beta_t\right] + {}^{u}\sigma_t^2
$$
$$
{}^{NAV}\sigma_t^2 + {}^{NPV}\sigma_t^2 = \mathbb{V}\left[X_t.{}^3\beta_t\right] + {}^{v}\sigma_t^2
$$

and

$$
{}^{v}\sigma_t^2 = {}^{u}\sigma_t^2 + \frac{P-1}{P}{}^{NPV}\sigma_t^2. \tag{1.8}
$$

Under the standard OLS assumption, both Curve Fitting and *LSMC* estimators converge towards ${}^3\beta_t$ with respective asymptotic speed of convergence ${}^{u}\sigma_t\sqrt{\frac{\mathbb{E}[X_t^{tr}.X_t]^{-1}}{N'}}$ and ${}^{v}\sigma_t\sqrt{\frac{\mathbb{E}[X_t^{tr}.X_t]^{-1}}{N'}}$. For a more detailed presentation of the underlying assumptions, see Saporta (2006).

Let ${}^{CF}N$ be the number of scenarios considered to obtain the Curve Fitting estimator and ${}^{LSMC}N$ be the number of scenarios considered to obtain the Least Squares Monte-Carlo estimator. The implementation complexity of both Curve Fitting and *LSMC* can be approximated by the number of *NPV* necessary to implement these methodologies, that is to say exactly ${}^{LSMC}N$ for *LSMC* and ${}^{CF}N \times P$ for Curve Fitting.

We now evaluate the level of ${}^{LSMC}N$ allowing to obtain an equal asymptotic speed of convergence for each methodology, given a fixed value for ${}^{CF}N \times P$, that is to say

$$
{}^{u}\sigma_t\sqrt{\frac{\mathbb{E}\left[(X_t^{tr}.X_t)^{-1}\right]}{{}^{CF}N}} = {}^{v}\sigma_t\sqrt{\frac{\mathbb{E}\left[(X_t^{tr}.X_t)^{-1}\right]}{{}^{LSMC}N}}
$$

which is equivalent to

$$
\frac{{}^{u}\sigma_t^2}{{}^{CF}N} = \frac{{}^{v}\sigma_t^2}{{}^{LSMC}N}.
$$

From equations 1.7 and 1.8 we have

$$
{}^{LSMC}N = {}^{CF}N \times \left(1 + \frac{P-1}{P}\frac{{}^{NPV}\sigma_t^2}{{}^{u}\sigma_t^2}\right) = {}^{CF}N \times \left(1 + \frac{P-1}{P}\frac{{}^{NPV}\sigma_t^2}{{}^{NAV}\sigma_t^2 + \frac{{}^{NPV}\sigma_t^2}{P} - \mathbb{V}[X_t.{}^3\beta_t]}\right)
$$

and therefore

$$
{}^{LSMC}N = {}^{CF}N \times P \times \left(\frac{1 + \frac{{}^{w}\sigma_t^2}{{}^{NPV}\sigma_t^2}}{1 + P \times \frac{{}^{w}\sigma_t^2}{{}^{NPV}\sigma_t^2}}\right) \leq {}^{CF}N \times P. \tag{1.9}
$$

In particular, we obtain the equality ${}^{LSMC}N = {}^{CF}N$ when $P = 1$, which is obvious by definition of $\widehat{NAV}$. According to 1.9 the Least Squares Monte-Carlo approach seems more efficient than the Curve

Fitting methodology. The speed of convergence of its estimator is indeed better than or equal to the Curve Fitting estimator's speed of convergence, for an equivalent algorithmic complexity. This formula enables to compare the efficiency of the *LSMC* and the Curve Fitting approach where the comparative efficiency coefficient $\eta = \sqrt{\frac{1+\frac{{}^{w}\sigma_t^2}{{}^{NPV}\sigma_t^2}}{1+P\times\frac{{}^{w}\sigma_t^2}{{}^{NPV}\sigma_t^2}}}$ appears.

Considering result 1.7, we know that ${}^{w}\sigma_t^2 \in \left[0; {}^{NAV}\sigma_t^2\right]$.Eventually, both formulas obtained for the endpoints of this interval are actually intuitive.

Comparison of extreme cases

Extreme case ${}^{w}\sigma_t^2 = 0$

This case happens when the hidden variable NAV_t can be rewritten exactly as a polynomial function of the underlying risk factors which is the assumption that basically justifies the use of polynomial proxies.

If ${}^{w}\sigma_t^2 = 0$ then ${}^{LSMC}N = {}^{CF}N \times P$. This means that, if regression 1.5 is fully efficient, both procedures *LSMC* and Curve Fitting have the same asymptotic speed of convergence for an equal algorithmic complexity.

As excellent results are generally obtained by using polynomial proxies, we can reasonably assume that ${}^{w}\sigma_t^2$ is close to 0, which leads to ${}^{w}\sigma_t^2 \ll {}^{NPV}\sigma_t^2$ and η is close to 1. Then both approaches Curve Fitting and *LSMC* asymptotically converge with a similar speed.

Extreme case ${}^{w}\sigma_t^2 = {}^{NAV}\sigma_t^2$.

In this scenario the comparative efficiency coefficient takes its lowest possible value. This characterizes the total inefficiency of the replication of variable NAV_t by the regression $X_t.{}^{3}\beta_t$. In this framework we have ${}^{3}\beta_t = \left(\mathbb{E}\left[NAV_t\right], 0, ..., 0\right)^{tr}$. Therefore, the difference between both asymptotic speeds of convergence is entirely explicated by the difference of the asymptotic speed of convergence of the estimators of $\mathbb{E}\left[NAV_t\right]$ obtained from the samples $\left(\widehat{NAV}_t^{n'}\right)_{n' \in [\![1;{}^{CF}N]\!]}$ and $\left(NPV_t^{n'}\right)_{n' \in [\![1;{}^{LSMC}N]\!]}$ used in the calibration steps.

Considering the Central Limit Theorem, the estimators $\frac{1}{{}^{CF}N} \sum_{n'=1}^{{}^{CF}N} NAV_t^{n'}$ and $\frac{1}{{}^{LSMC}N} \sum_{n'=1}^{{}^{LSMC}N} NPV_t^{n'}$ asymptotically converges towards Gaussian distributions that have a same mean $\mathbb{E}\left[NAV_t\right]$ and respective volatilities $\sqrt{\frac{{}^{NAV}\sigma_t^2 + \frac{{}^{NPV}\sigma_t^2}{P}}{{}^{CF}N}}$ and $\sqrt{\frac{{}^{NAV}\sigma_t^2 + {}^{NPV}\sigma_t^2}{{}^{LSMC}N}}$.

Equalizing both asymptotic speeds of convergence we obtain

$$^{LSMC}N = {}^{CF}N \times P\left(\frac{1+\frac{^{NAV}\sigma_t^2}{^{NPV}\sigma_t^2}}{1+P\times\frac{^{NAV}\sigma_t^2}{^{NPV}\sigma_t^2}}\right).$$

Which is the same formula as 1.9, with $^{w}\sigma_t^2 = {}^{NAV}\sigma_t^2$. This observation enables us to validate formula 1.9 *a posteriori*, in this specific case.

1.3.3 Conclusion of this comparison section

Under our assumptions the *LSMC* approach seems quite more efficient asymptotically than the Curve Fitting, for a fixed algorithmic complexity. However we must put this result in perspective. Indeed, for example, if we consider that $\mathscr{F}_t^{RW} = \sigma(X_t)$, which can be the case if $\left({}^{s}\varepsilon_1, {}^{ZC}\varepsilon_1, ..., {}^{s}\varepsilon_t, {}^{ZC}\varepsilon_t\right) \subset \sigma(X_t)$, our assumptions lead us to the result

$$NAV_t = \mathbb{E}\left[NAV_t | \mathscr{F}_t^{RW}\right] = \mathbb{E}\left[NAV_t | X_t\right] = X_t.{}^{3}\beta_t.$$

This situation corresponds to the first extreme case $^{w}\sigma_t^2 = 0$ for which both approaches *LSMC* and Curve Fitting have equal efficiency for an equal algorithmic complexity.

In practice the case $\mathscr{F}_t^{RW} = \sigma(X_t)$ only happens for small values of t (X_t contains a limited number of covariates, when t is too high we have $\sigma(X_t) \subset \mathscr{F}_t^{RW}$ strictly). Considering approximations of $^{w}\sigma_t^2$ and $^{NPV}\sigma_t^2$ calculated with traditional Asset-Liability Management models under current economic conditions, we have obtained orders of magnitude for η, varying decreasingly from around 1 at $t = 1$ to 0.5 at $t = 5$. In this last case, the *LSMC* asymptotical speed of convergence is 2 times greater than the Curve Fitting.

As a conclusion on 1.9, note that one operational complexity element is omitted in the formula. The algorithmic complexity is limited to the number of *NPV* calculated to implement both approaches (^{LSMC}N for the *LSMC* and $^{CF}N \times P$ for the Curve Fitting). However, another complexity factor may be relevant, the implementation time of the methodologies. Indeed, the *LSMC* proxies are longer to calibrate, it requires a large number of outcomes in its calibration step compared to Curve Fitting (in practice ^{LSMC}N is close to $Ptimes^{CF}N$). The impact of this complexity element is difficult to grasp because it highly depends on the calibration software and on the complexity of the underlying insurance products. Thus we have decided to get around this issue and consider a similar implementation time for both approaches.

1.4 Illustration

1.4.1 Implementation framework

The multi-year Curve Fitting and Least Squares Monte-Carlo methodologies are tested on a standard French saving portfolio with low average guaranteed minimum rates (0.81% on average but about

30% of the contracts are 2.5% or more guaranteed rates). We use a projection tool that takes profit sharing mechanisms, target crediting rate and dynamic lapses behaviors of policy holders into account. The credited rate paid to policy holders is a function of the risk-free rate, of the performance of the Eurostoxx index, and of a specific profit sharing rule. The dynamic lapses depend on the credited rate and on a market based target crediting rate. Basically, a low credited rate will lead to a high number of redeemed contracts. The asset reallocation rule is such that the initial asset allocation in maintained over the projections.

Table 1.1 – Initial asset allocation (Market value)

Asset	Allocation
Stock	14%
Real Estate	3%
Bonds	78%
Cash	5%

The tested product is supposed to be subjected to a risk on the stock index and on interest rates. * This leads us to consider one stock risk factor and one interest rate risk factor. The time horizon selected is $T = 5$ years. The economic scenarios tables are calibrated using a specific calibration process is considered for the interest rates model, as developed in 1.4.2. The other economic assumptions have been calibrated as at the 12/31/2009, and not as at the 12/31/2010 or 12/31/2011 in order to avoid simulatory aberrations linked to very low interest rates and very high market implied volatilities. Basically, the calibration date has little impact on the implementation and this issue has only a secondary interest for our study.

The parameters of the polynomial proxies depend on the chosen initial assumptions. The change of one hypothesis (asset allocation, economic information, ALM rules...) implies a new calibration of the proxies.

1.4.2 Real-World interest rates modeling issue

The 1-year horizon considered when calculating the Economic Capital (Pillar I) allows us to consider a standard Real-World model like the 1-factor Heath-Jarrow-Morton. This model integrates historical risk premiums and volatilities per maturity in the Zero-Coupon bonds generation process. For a more developed insight about the Heath-Jarrow-Morton model, the reader may consult Brigo and Mercurio (2007).

Considering the same historical parameters throughout a 5-years horizon can lead to a significant issue when the initial interest rates are low. In particular, the introduction of countercyclicality in the interest rate model is legitimate in the context of low interest rates as at the 12/31/2009. Basically, this is not a major issue for our implementations. We have gotten around it by calibrating the risk premiums so as to obtain on average a specific long-term curve in $t = 5$, the Zero-Coupon bond curve as at the 12/31/2005.

*. We only focus in this article on the level risk and do not consider the volatility risk.

We have been able to verify that this choice has little impact on the efficiency of the methodologies implemented hereafter. Other approaches based on time series theory or Principal Component Analysis, such as the methodology presented in Diebold and Li (2006), may lead to interesting solutions for this issue.

1.4.3 Multi-year Curve Fitting implementation

We have carried out Nested Simulations projections based on 5'000 real world simulations from $t = 1$ to $t = 5$. For each primary simulation and date t, the *NAV* (central and shocked) are estimated with adjusted risk-neutral simulations based on a secondary table consisting of 500 scenarios. For more insight about the adjustment of risk-neutral simulations in the Nested Simulations framework, see Devineau and Chauvigny (2011). This implementation has enabled us to obtain a reference set of joint sample paths $\left(\widehat{NAV}_1^n, ..., \widehat{NAV}_5^n\right)_{n \in [\![1;5'000]\!]}$ and $\left(\widehat{SCR}_1^n, ..., \widehat{SCR}_5^n\right)_{n \in [\![1;5'000]\!]}$. The study of the goodness of fit between the distributions obtained thanks to our proxies (Curve Fitting and Least Squares Monte-Carlo) and the Nested Simulations distributions enables to validate the calculations at each step of the process.

The multi-year Curve Fitting has been calibrated on 100 outcomes of Nested Simulations *NAV* outcomes. This low number of outcomes enables us to implement the Nested Simulations procedure 50 times faster.

Two different approaches of the Curve Fitting methodology have been tested, implementing a calibration on well-dispersed primary scenarios, relying on a multidimensional risk factors' grid (obtained using multidimensional Sobol sequences), and a calibration on extreme primary scenarios. In this latter case, the calibration scenarios have been selected according to the high values of a norm on the risk factors:

$$\forall t \in [\![1;5]\!], \|({}^s\varepsilon_1, {}^{ZC}\varepsilon_1, ..., {}^s\varepsilon_t, {}^{ZC}\varepsilon_t)\| = \sqrt{\sum_{u=1}^{t} {}^s\varepsilon_u^2 + {}^{ZC}\varepsilon_u^2}.$$

Considering this norm, we can select scenarios whose risk factors vectors are associated to low probability thresholds. This type of scenarios' selection is already used to accelerate the calculation of the Economic Capital as in Devineau and Loisel (2009*a*). Both approaches have led to satisfactory results. The results presented in Subsection 1.4.3 have been obtained by using the second approach. [*]

QQ plots

During former tests, we have been able to check that in most cases the final form of the optimal central and shocked proxies were very close if not the same. Therefore, in both the Curve Fitting and *LSMC* implementation, we have chosen not to integrally duplicate the procedure described for central

∗. For more visibility, a little number of points of the top right corners of the QQ plots associated to the Solvency Ratio have been removed from the graphs. These points correspond to the more positive cases and are of little interest here.

NAV when calibrating the shocked proxies. Instead, we have considered the same set of regressors for the shocked proxies as those already calibrated for the central proxies. This has enabled us to accelerate the proxy methodologies implementation by skipping one part of the calibration process for the shocked proxies. The following graphs depict the quantile-quantile plots that compare the quantiles of the reference sets of NAV_t with the quantiles of the sets obtained using the Curve Fitting methodology (see Figure 1.8).

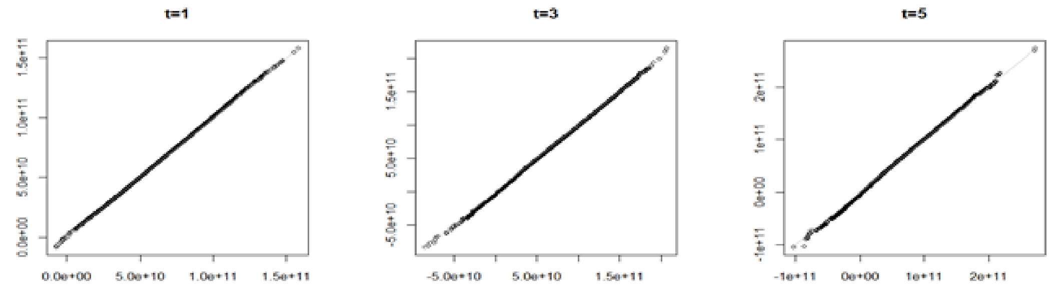

Figure 1.8 – QQ plots $\widehat{NAV}_t$ vs. ${}^{CF}NAV_t$

Denoting $\widehat{SR}_t$ the solvency ratio obtained by Nested Simulations and ${}^{CF}SR_t$ the solvency ratio obtained by the multi-year Curve Fitting approach, we obtain the QQ plots of Figure 1.9.

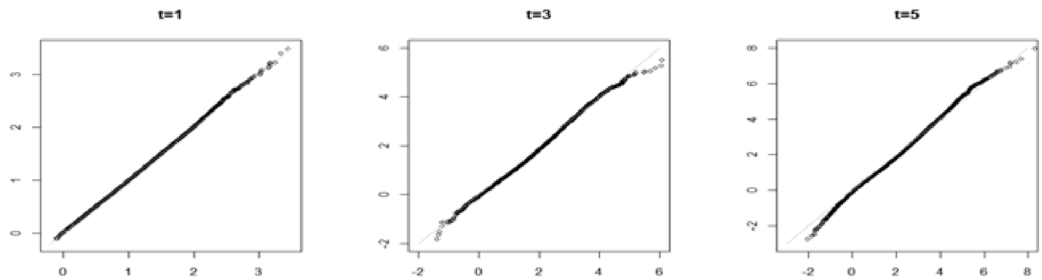

Figure 1.9 – QQ plots $\widehat{SR}_t$ vs. ${}^{CF}SR_t$

The Solvency Ratio is very complex mathematical element and one can observe that it is not as well approximated as the *NAV*. However the QQ plots obtained show satisfactory results for both approximated variables even for $t = 5$.

Relative differences between Nested Simulations and Curve Fitting

The relative differences observed between the Nested Simulations outcomes and the Curve Fitting approximations for chosen quantiles are shown below.

Very low relative differences can be observed as far as the *NAV* approximation is concerned. The relative differences in respect of the Solvency Ratio are much higher, especially for the lowest quantiles, when the outcomes are closer to 0. This was somehow predictable because the Solvency Ratio is a much more complex element than the *NAV*. As one can see further, the *LSMC* implementation led to better relative differences.

Table 1.2 – Initial asset allocation (Market value)

	NAV			*SR*		
Quantile	$t=1$	$t=3$	$t=5$	$t=1$	$t=3$	$t=5$
25%	0.63%	4.11%	1.72%	1.62%	22.05%	15.25%
50%	0.08%	2.65%	-0.59%	1.08%	13.77%	8.71%
75%	-0.01%	2.09%	-0.45%	0.33%	6.18%	1.20%

Final shape of the calibrated proxies

The calibrated "central" proxies present the following shapes:

$^{CF}NAV_1 = I + \alpha_1{}^s\varepsilon_1 + \alpha_2{}^{ZC}\varepsilon_1 + \alpha_3{}^s\varepsilon_1^2 + \alpha_4{}^{ZC}\varepsilon_1^2 + \alpha_5{}^s\varepsilon_1.{}^{ZC}\varepsilon_1 + \alpha_6{}^s\varepsilon_1^3 + \alpha_7{}^s\varepsilon_1^2.{}^{ZC}\varepsilon_1 + \alpha_8{}^s\varepsilon_1.{}^{ZC}\varepsilon_1^2,$

$^{CF}NAV_3 = I + \alpha_1{}^{CF}NAV_2 + \alpha_2{}^s\varepsilon_3 + \alpha_3{}^{ZC}\varepsilon_3 + \alpha_4{}^{ZC}\varepsilon_3^2 + \alpha_5{}^s\varepsilon_3.{}^{ZC}\varepsilon_3 + \alpha_6{}^s\varepsilon_3.{}^s\varepsilon_2 + \alpha_7{}^s\varepsilon_3.{}^s\varepsilon_1$
$+\alpha_8{}^{ZC}\varepsilon_3.{}^{ZC}\varepsilon_2 + \alpha_9{}^{ZC}\varepsilon_3.{}^{ZC}\varepsilon_1 + \alpha_10^{ZC}\varepsilon_2 + \alpha_11^{ZC}\varepsilon_1,$

$^{CF}NAV_5 = I + \alpha_1{}^{CF}NAV_4 + \alpha_2{}^s\varepsilon_5 + \alpha_3{}^{ZC}\varepsilon_3 + \alpha_4{}^s\varepsilon_5^2 + \alpha_5{}^{ZC}\varepsilon_5^2 + \alpha_6{}^s\varepsilon_5.{}^s\varepsilon_3 + \alpha_7{}^{ZC}\varepsilon_5.{}^{ZC}\varepsilon_4 +$
$\alpha_8{}^{ZC}\varepsilon_5.{}^{ZC}\varepsilon_3 + \alpha_9{}^{ZC}\varepsilon_5.{}^{ZC}\varepsilon_2 + \alpha_10^{ZC}\varepsilon_5.{}^{ZC}\varepsilon_1 + \alpha_11^{ZC}\varepsilon_1.$

One can interpret the proxies shapes the following way. The first order regressors reflects the linear sensibility of the *NAV* to the risk. The second order terms permit to introduce the convexity of the *NAV* and the higher order terms enables convexity adjustments. It is noticeable that the term $^{CF}NAV_{t-1}$ is always greatly significant in the regression associated to $^{CF}NAV_t$ ($t \geq 2$).

1.4.4 Multi-year *LSMC* implementation

The multi-year *LSMC* has been calibrated on $50^{'}000$ outcomes of Net Present Value of margins. As for the multi-year Curve Fitting, two different approaches of the *LSMC* methodology have been tested, considering a calibration on well-dispersed primary scenarios or a calibration on extreme primary scenarios. Both methods have led to very satisfactory results. The results presented in this section have been obtained using the approach based on extreme scenarios. *

1.4.5 QQ plots

The graphs in Figure 1.10 depict the QQ plots that compare the quantiles of the reference sets of NAV_t with the quantiles of the sets obtained using the *LSMC* methodology.

Denoting $^{LSMC}SR_t$ the solvency ratio obtained by the multi-year *LSMC* approach, we obtain the QQ

∗. For more visibility, few points of the top right corner of the QQ plots associated to the Solvency Ratio have been removed from the graphs. These points correspond to the best cases and are of little interest here.

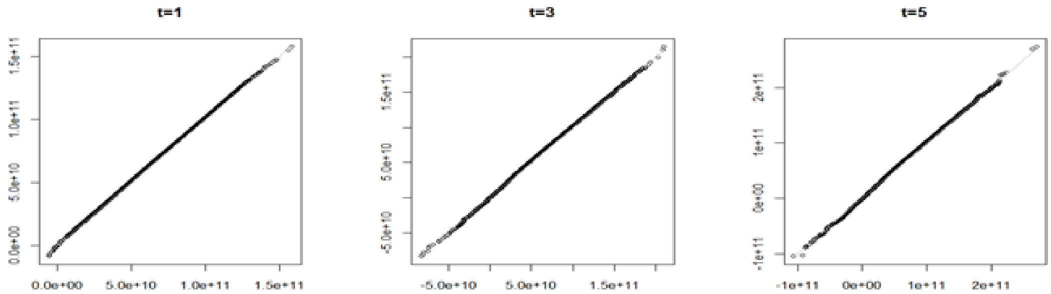

Figure 1.10 – QQ plots $\widehat{NAV}_t$ vs. ${}^{LSMC}NAV_t$

plots depicted in Figure 1.11.

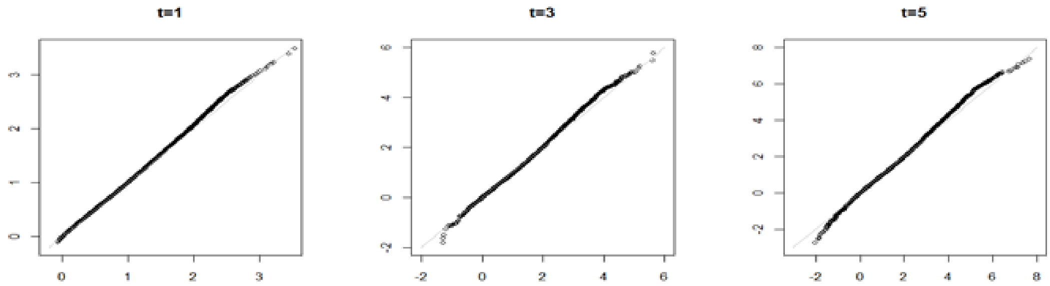

Figure 1.11 – QQ plots $\widehat{SR}_t$ vs. ${}^{LSMC}SR_t$

Similarly to the Curve Fitting results, the Solvency Ratio is not as well approximated as the *NAV*. However the QQ plots obtained show satisfactory results for both approximated variables event for $t = 5$.

Relative differences between Nested Simulations and *LSMC*

We summarize below the relative differences observed between the Nested Simulations outcomes and the *LSMC* approximations for chosen quantiles.

Table 1.3 – Initial asset allocation (Market value)

	NAV			*SR*		
Quantile	$t = 1$	$t = 3$	$t = 5$	$t = 1$	$t = 3$	$t = 5$
25%	-1.92%	-3.58%	-5.51%	0.76%	8.66%	4.80%
50%	-1.79%	-2.63%	-2.47%	0.03%	3.15%	2.60%
75%	-1.39%	-1.28%	-0.79%	-1.42%	-1.95%	-4.75%

Very low relative differences can be observed for both the *NAV* and the Solvency Ratio, even for the lowest quantiles.

1.4.6 Final shape of the calibrated polynomial proxies

The calibrated proxies present the following shapes:

$$^{LSMC}NAV_1 = I + \alpha_1{}^s\varepsilon_1 + \alpha_2{}^{ZC}\varepsilon_1 + \alpha_3{}^s\varepsilon_1^2 + \alpha_4{}^{ZC}\varepsilon_1^2 + \alpha_5{}^s\varepsilon_1.^{ZC}\varepsilon_1 + \alpha_6{}^s\varepsilon_1^3 + \alpha_7{}^s\varepsilon_1^2.^{ZC}\varepsilon_1,$$

$$^{LSMC}NAV_3 = I + \alpha_1{}^{LSMC}NAV_2 + \alpha_2{}^{LSMC}NAV_2.^{ZC}\varepsilon_3 + \alpha_3{}^s\varepsilon_3 + \alpha_4{}^{ZC}\varepsilon_3 + \alpha_5{}^s\varepsilon_3^2 + \alpha_6{}^{ZC}\varepsilon_3^2$$
$$+\alpha_7{}^s\varepsilon_3.^{ZC}\varepsilon_3 + \alpha_8{}^s\varepsilon_3.^s\varepsilon_2 + \alpha_9{}^s\varepsilon_3.^s\varepsilon_1 + \alpha_10^{ZC}\varepsilon_3.^{ZC}\varepsilon_2 + \alpha_11^{ZC}\varepsilon_3.^{ZC}\varepsilon_1 + \alpha_12^{ZC}\varepsilon_2,$$

$$^{LSMC}NAV_5 = I + \alpha_1{}^{LSMC}NAV_4 + \alpha_2{}^{LSMC}NAV_4.^s\varepsilon_5 + \alpha_3{}^s\varepsilon_5 + \alpha_4{}^{ZC}\varepsilon_3 + \alpha_5{}^s\varepsilon_5^2 + \alpha_6{}^{ZC}\varepsilon_5^2$$
$$+\alpha_7{}^{ZC}\varepsilon_5.^{ZC}\varepsilon_4 + \alpha_8{}^{ZC}\varepsilon_5.^{ZC}\varepsilon_3 + \alpha_9{}^{ZC}\varepsilon_5.^{ZC}\varepsilon_2 + \alpha_10^{ZC}\varepsilon_5.^{ZC}\varepsilon_1 + \alpha_11^{ZC}\varepsilon_1.$$

One can observe that both Curve Fitting and *LSMC* proxies have very similar shapes. Basically, the interpretation of the proxies' shapes is similar to the one developed in 1.4.3. Once again the term $^{CF}NAV_{t-1}$ is always highly significant in the regression associated to $^{CF}NAV_t$ $(t \geq 2)$.

1.4.7 Calculation of the capital need associated to a constraint on solvency shortfalls

The solvency constraints have been chosen for their realism, knowing that the initial Solvency Ratio for the considered product, calculated using the Standard Formula, is close to 90%. We have considered two different solvency constraints, one on yearly Solvency Ratio distributions and one on Solvency Ratio's paths.

Multi-year solvency constraint	LSMC	Curve Fitting
(SC3): $\forall t \in [\![1;T]\!], \mathbb{P}\left(\frac{NAV_t}{SCR_t} \geq 110\%\right) \geq 85\%$	7.9%	6.7%
(SC4): $\mathbb{P}\left(\cap_{t=1}^{T} \frac{NAV_t}{SCR_t} \geq 110\%\right) \geq 85\%$	9.3%	11.9%

One can observe that the relative difference between the capital need obtained by using both proxy methodologies and the one obtained by considering Nested Simulations projections, do not exceed 12%. This final result tends to legitimate the use of these methodologies in order to quickly approximate the Overall Solvency Needs related to these highly complex constraints. One can notice that the second constraint is framed such as it leads to a lower risk tolerance than the second constraint, which does not take path-dependence into account.

Conclusion

The Own Risk and Solvency Assessment process implementation leads to numeral strategic issues for insurance undertakings. In this paper we have introduced the issues raised by the Overall Solvency Needs assessment and by the multi-year solvency concept. We have formalized various multi-year solvency metrics that can be used to provide a framework for the Overall Solvency Needs calculation

Several modeling issues have been identified for the implementation of the most complex constraints. In order to work around these points we have developed multi-year proxy methodologies which have provided very interesting results, especially for the Least Squares Monte-Carlo methodology.

The authors notice that the modeling choices considered in this paper do not claim to model the risks in the most appropriate way, especially for long term risk management. However, they are generally considered by practitioners and therefore are relevant in our operational framework.

Concerning the future axes to investigate, it is relevant to address the issue of the proxies recalibration frequency needed to monitor the Overall Solvency Needs in an infra-annual fashion. Indeed, the proxies calibrations greatly depend on the economic assumptions and on the asset-mix at the implementation date.

Eventually, the framework considered in the illustration does not enable us to challenge empirically the efficiency comparison formula obtained in Section 1.3. This direction of work will be investigated further in order to analyze the robustness of our assumptions and to compare efficiently the respective speed of convergence of the proxy methodologies.

Acknowledgement

The authors would like to thank Fabien Ramaharobandro who has actively contributed to this article and Stephane Loisel for his relevant comments throughout the writing of the paper.

The authors would like to extend their thanks to all the employees of Milliman Paris, and in particular the members of the R&D team. Eventually, the authors would also like to express their gratitude to Milliman for having funded this project.

Bibliography

Algorithmics (2011), *Curve Fitting for Calculating Solvency Capital Requirements under Solvency II: Practical insights and best practices from leading European Insurers*.

Asmussen, S. and Albrecher, H. (2010), *Ruin probabilities*, World scientific.

Barrie & Hibbert (2011), *A Least Squares Monte Carlo approach to liability proxy modelling and capital calculation*.

Bonnin, F., Planchet, F. and Juillard, M. (2014), 'Best estimate calculations of savings contracts by closed formulas: application to the orsa', *European Actuarial Journal* **4**(1), 181–196.

Brigo, D. and Mercurio, F. (2007), *Interest rate models-theory and practice: with smile, inflation and credit*, Springer.

Decupère, S. (2011), 'Agrégation des risques et allocation de capital sous Solvabilité II'.

Devineau, L. and Chauvigny, M. (2011), 'Replicating portfolios: calibration techniques for the calculation of the Solvency II economic capital', *Bulletin Français d'Actuariat* **11**(21), 59–97.

Devineau, L. and Loisel, S. (2009*a*), 'Construction d'un algorithme d'accélération de la méthode des "simulations dans les simulations" pour le calcul du capital économique Solvabilité II', *Bulletin Français d'Actuariat* **10**(17), 188–221.

Devineau, L. and Loisel, S. (2009*b*), 'Risk aggregation in Solvency II: How to converge the approaches of the internal models and those of the standard formula?', *Bulletin Français d'Actuariat* **9**(18), 107–145.

Diebold, F. X. and Li, C. (2006), 'Forecasting the term structure of government bond yields', *Journal of Econometrics* **130**(2), 337–364.

Kalberer, T. (2012), 'Stochastic determination of the value at risk for a portfolio of assets and liabilities', *Der Aktuar* **1**, 7–14.

Longstaff, F. A. and Schwartz, E. S. (2001), 'Valuing american options by simulation: A simple least-squares approach', *Review of Financial studies* **14**(1), 113–147.

Nteukam, O. and Planchet, F. (2012), 'Stochastic evaluation of life insurance contracts: Model point on asset trajectories and measurement of the error related to aggregation', *Insurance: Mathematics and Economics* **51**(3), 624–631.

Rulliere, D. and Loisel, S. (2004), 'Another look at the Picard-Lefèvre formula for finite-time ruin probabilities', *Insurance: Mathematics and Economics* **35**(2), 187–203.

Saporta, G. (2006), *Probabilités, analyses des données et statistiques*, editions Technip.

Chapitre 2

Continuous compliance: a proxy-based monitoring framework

Abstract

Within the Own Risk and Solvency Assessment framework, the Solvency II directive introduces the need for insurance undertakings to have efficient tools enabling the companies to assess the continuous compliance with regulatory solvency requirements. Because of the great operational complexity resulting from each complete evaluation of the Solvency Ratio, this monitoring is often complicated to implement in practice. This issue is particularly important for life insurance companies due to the high complexity to project life insurance liabilities. It appears relevant in such a context to use parametric tools, such as Curve Fitting and Least Squares Monte Carlo in order to estimate, on a regular basis, the impact on the economic own funds and on the regulatory capital of the company of any change over time of its underlying risk factors.

In this article, we first outline the principles of the continuous compliance requirement then we propose and implement a possible monitoring tool enabling to approximate the eligible elements and the regulatory capital over time. In a final section we compare the use of the Curve Fitting and the Least Squares Monte Carlo methodologies in a standard empirical finite sample framework, and stress adapted advices for future proxies users.

Keywords : *Solvency II, ORSA, continuous compliance, parametric proxy, Least Squares Monte Carlo, Curve Fitting*.

Introduction

The Solvency II directive (European Directive 2009/138/EC), through the Own Risk and Solvency Assessment process, introduces the necessity for an insurance undertaking to be capable of assessing its regulatory solvency on a continuous yearly basis. This continuous compliance requirement is a crucial issue for insurers especially for life insurance companies. Indeed, due to the various asset-liability interactions and to the granularity of the insured profiles (see *e.g.* Tosetti et al. (2003) and Petauton (2002)), the highly-stochastic projections of life insurance liabilities constitute a tricky framework for the implementation of this requirement.

In the banking industry the notion of continuous solvency has already been investigated through credit risk management and credit risk derivatives valuation, considering an underlying credit model (see e.g. Jarrow et al. (1997) and Longstaff et al. (2005)). The notions of ruin and solvency are different in the insurance industry, due in particular to structural differences and to the specific Solvency II definitions. In a continuous time scheme these have been studied in a non-life ruin theory framework, based on the extensions of the Cramér-Lundberg model Lundberg (1903), see *e.g.* Pentikäinen (1982), Pentikäinen et al. (1989) and Loisel and Gerber (2012). In a life insurance framework, considering more empirical schemes, closed formulas can be found under strong model assumptions. This field has for example been investigated in Bonnin et al. (2014) or Vedani and Virepinte (2011). However, all these approaches are based on relatively strong model assumptions. Moreover, on a continuous basis the use of such approaches generally faces the problem of parameters monitoring and needs adaptations to be extended to the continuous compliance framework.

Monitoring a life insurance liabilities is very complex and will have to introduce several stability assumptions in order to develop a practical solution. The great time and algorithmic complexity to assess the exact value of the Solvency Ratio of an insurance undertaking is another great issue. In practice, an only complete solvency assessment is required by the directive: the insurance undertakings have to implement a complete calculation of their Solvency Capital Requirement and of their eligible own funds at the end of the accounting year. We have identified two possibilities to investigate in order to implement a continuous compliance tool, either to propose a proxy of the Solvency Ratio, easy enough to monitor, or directly to address the solvency state (and not the solvency level). This last possibility leading to little information in terms of risk measurement we have chosen to consider the first one, based on the actual knowledge on the polynomial proxies applied to life insurance Net Asset Value (see *e.g.* Devineau and Chauvigny (2011)) and Solvency Ratios (Vedani and Devineau (2013)), that is to say Least Squares Monte Carlo and Curve Fitting.

Throughout Section 2.1 we lay the foundations of the continuous compliance requirement adapted to life insurance. We underline and discuss the article designing the continuous compliance framework and present the major difficulties to address when implementing a monitoring tool. In Section 2.2 we propose a continuous compliance assessment scheme based on a general polynomial proxy methodology. This tool is implemented in Section 2.3, using a Least Squares Monte Carlo approach, on a standard life insurance product.

2.1 Continuous compliance

The requirement for continuous compliance is introduced in Article 45(1)(b) of the Solvency II Directive (European Commission (2009)): "As part of its risk-management system every insurance undertaking and reinsurance undertaking shall conduct its own risk and solvency assessment. That assessment shall include at least the following: (...) the compliance, on a continuous basis, with the capital requirements, as laid down in Chapter VI, Sections 4 and 5" *.

In this section, we will first remind briefly what these capital requirements are and what they imply in terms of modelling and calculation. We will then discuss continuous compliance, what it entails and what issues it brings up for (re)insurance companies. Finally we will highlight some key elements to the setting of a continuous compliance framework in this business area.

2.1.1 Capital requirements

Regulatory framework

The capital requirements laid down in Chapter VI, Sections 4 and 5 are related to the Solvency Capital Requirement, or *SCR* (Section 4), and the Minimum Capital Requirement, or *MCR* (Section 5).

The *SCR* corresponds to the Value-at-Risk of the basic own funds of the company subject to a confidence level of 99.5% over a one-year period. It has to be calculated and communicated to the supervisory authority. Additionally, companies falling within the scope of the Financial Stability Reporting will have to perform a quarterly calculation (limited to a *best effort basis*) and to report its results. Companies will have to hold eligible own funds higher or equal to the *SCR*. Failing to do so will trigger a supervisory process aiming at recovering a situation where the eligible own funds are in excess of the *SCR*. The *SCR* can be calculated using the Standard Formula - a set of methodological rules set out in the regulatory texts - or an internal model (see below for further details).

The *MCR* is a lower requirement than the *SCR* , calculated and reported quarterly. It can be seen as an emergency floor. A breach of the *MCR* will trigger a supervisory process that will be more severe than in the case of a breach of the *SCR* and could lead to the withdrawal of authorization. The *MCR* is calculated through a factor-based formula. The factors apply to the technical provisions and the written premiums in non-life and to the technical provisions and the capital at risk for life business. It is subject to an absolute floor and a floor based on the *SCR*. It is capped at 45% of the *SCR*.

This paper focuses on the estimation of the eligible own funds and the *SCR*. Basically, the *MCR* will not be used as much as the *SCR* when it comes to risk management, and compliance with the *SCR* will imply compliance with the *MCR*.

*. Article 45(1)(b) also introduces continuous compliance "with the requirements regarding technical provisions, as laid down in Chapter VI, Section 2". This means that the companies should at all times hold technical provisions valued on the Solvency II basis. This implies that they have to be able to monitor the evolution of their technical provisions between two full calculations. The scope of this article is limited to continuous compliance with capital requirements.

Implementation for a life company

The estimation of the eligible own funds and the *SCR* requires to carry out calculations that can be quite heavy. Their complexity depends on the complexity of the company's portfolio and the modeling choices that are made, in particular between the Standard Formula and an internal model. In this section, we present the key issues to be dealt with by a life insurer.

Implementation scheme.

To assess the *SCR* it is necessary to project economic balance sheets and calculate best estimates.

For many companies, the bulk of the balance sheet valuation lies in the estimation of these best estimates. This can imply quite a long and heavy process, since the assessment is carried out through simulations and is subject, amongst other things, to the following constraints:
- — updating the assets and liabilities model points;
- — constructing a set of economic scenarios under the risk-neutral probability and checking its market-consistency;
- — calibrating and validating the stochastic model through a series of tests (*e.g.*: leakage test);
- — running simulations.

The valuation of the financial assets may also be quite time-consuming if a significant part of the portfolio has to be marked to model.

***SCR* calculation through the Standard Formula.**

The calculation of the *SCR* through the Standard Formula is based on the following steps:
- — calculation of the various standalone *SCR*;
- — aggregation;
- — adjustment for the risk absorbing effect of technical provisions and deferred taxes;
- — calculating and adding up the capital charge for operational risk.

Each standalone *SCR* corresponds to a risk factor and is defined as the difference between the current value of the eligible own funds and their value after a pre-defined shock on the risk factor. As a consequence, for the calculation of each standalone *SCR* a balance sheet valuation needs to be carried out, which means that a set of simulations has to be run and that the assets must be valued in the economic conditions after shock.

***SCR* calculation with a stochastic internal model.**

An internal model is a model designed by the company to reflect its risk profile more accurately than the Standard Formula. Companies deciding not to use the Standard Formula have the choice between a full internal model and a partial internal model. The latter is a model where the capital charge for some of the risks is calculated through the Standard Formula while the charge for the other risks is calculated with an entity-specific model. There are two main categories of internal models *:

- models based on approaches similar to that of the Standard Formula, whereby capital charges are calculated on the basis of shocks; the methodology followed in this case is the same as the one described in Subsection 2.1.1;
- fully stochastic models: the purpose of this type of model is to exhibit a probability distribution of the own funds at the end of a 1-year period, in order to subsequently derive the *SCR* , by calculating the difference between the 99.5% quantile and the initial value.

In the latter case, the calculations are based on a methodology called Nested Simulations. It is based on a twofold process of simulations:

- real-world simulations of the risk factors' evolution over 1 year are carried out;
- for each real-world simulation, the balance sheet must be valued at the end of the 1-year period. As per the Solvency II requirements, this valuation has to be market-consistent. It is carried out through simulations under the risk-neutral probability.

More details on Nested Simulations can be found in Broadie et al. (2011) or Devineau and Loisel (2009).

2.1.2 An approach to continuous compliance

In the rest of this article we focus the scope of our study to life insurance.

Defining an approach

As mentioned above, the Solvency II Directive requires companies to permanently cover their *SCR* and *MCR*. This is what we refer to as *continuous compliance* in this paper. The regulatory texts do not impose any specific methodology. Moreover the assessment of continuous compliance is introduced as an element of the Own Risk and Solvency Assessment (*ORSA*), which suggests that the approach is for each company to define.

Different approaches can be envisaged. Here below we present some assessment methodologies that companies can rely on and may combine in a continuous compliance framework.

- **Full calculations**: *i.e.* the same calculations as those carried out for annual reporting to the supervisory authority: this type of calculations can be performed several times during the year. However the process can be heavy and time-consuming, as can be seen from the description made in Subsection 2.1.1. As a consequence, it seems operationally difficult to carry out such calculations more than quarterly (actually most companies are likely to run full calculations only once or twice a year).
- ***Simplified* full calculations**: companies may decide to run calculations similar to those described in the previous item but to freeze some elements. For example they could decide not to update the liabilities model points if the portfolio is stable and if the time elapsed since the last update is short; they could also decide to freeze some modules or sub-modules that are not expected to vary significantly over a short period of time.

*. These approaches can be mixed within one model.

- **Proxies**: companies may develop methods to calculate approximate values of their Solvency Ratio [*] (*SR*). Possible approaches include, among others, abacuses and parametric proxies.
- **Indicators monitoring**: as part of their risk management, companies will monitor risk indicators and set limits to them. These limits may be set so that respecting them ensures that some *SCR* modules stay within a given range.

Overview of the proposed approach

The approach presented in this paper relies on the calibration of proxies allowing to estimate the *SR* quickly and taking as input a limited number of easy-to-produce indicators. It has been developed for life companies using the Standard Formula.

Proxies: generic principles.

Simplifying the calculations requires limiting the number of risks factors that will be monitored and taken into account in the assessment to the most significant. For most life insurance companies, these risk factors will be financial (*e.g.*: stock level, yield curve).

In the framework described in the following sections, the proxies are supposed to be potentially used to calculate the *SR* at any point in time. For operational practicality, the inputs have to be easily available. In particular, for each risk factor, an indicator will be selected for monitoring purpose and to be used as input for the proxy (see Section 2.2 for more insight about proxies). The selected indicators will have to be easily obtainable and reflect the company's risk profile.

As explained in Section 2.2, our approach relies on the development and the calibration of proxies in order to calculate in a quick and simple way the company's Net Asset Value (*NAV*) and the most significant *SCR* sub-modules. The overall *SCR* is then calculated through an aggregation process based on the Standard Formula's structure and using the tools the company uses for its regulatory calculations. As a consequence, a selection has to be made regarding the sub-modules that will be calculated by proxy. The others are frozen or updated proportionally to a volume measure (*e.g.* mortality *SCR* set proportional to the technical provisions).

Continuous compliance framework.

Under Solvency II, companies will set a frequency (at least annual) for the full calculation of the *SCR* [†]. Additionally, they will set a list of pre-defined events and circumstances that will trigger a full calculation whenever they happen. The proxies will be used to estimate the *SR* between two full calculations and should be calibrated every time a full calculation is performed. This process is summarized in Figure 2.1 below.

∗. Solvency Ratio = Eligible Own Funds / *SCR*.

†. We are referring here to full calculations in the broad sense: the infra-annual calculations may be *simplified* full calculations.

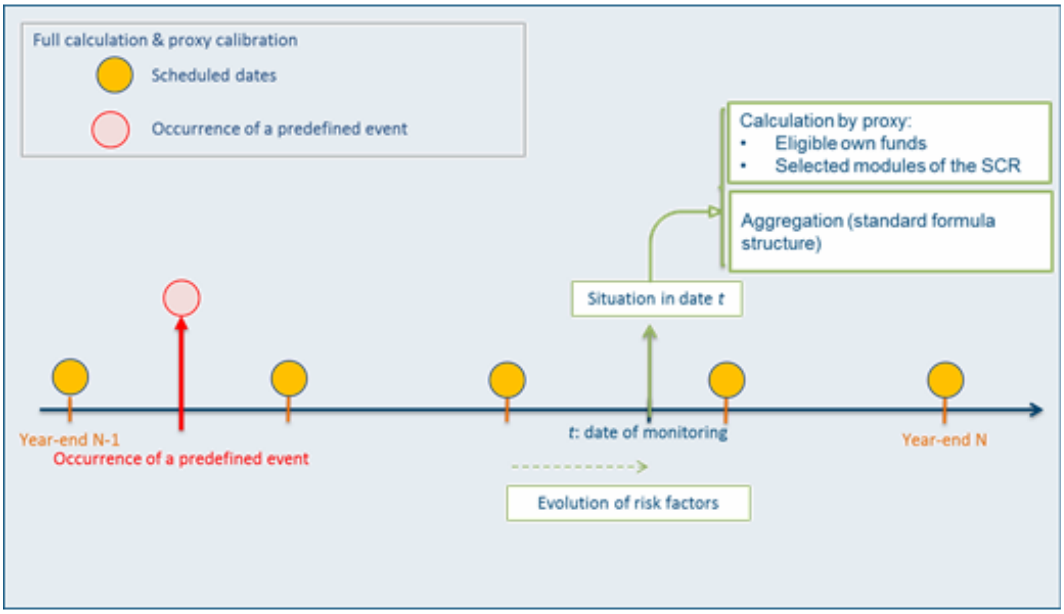

Figure 2.1 – Continuous compliance framework.

Here below are a few examples of pre-defined events and circumstance,
- — external events (*e.g.*: financial events, pandemics),
- — internal decisions (*e.g.*: change in asset mix),
- — risk factors outside the proxies' zone of validity.

2.2 Quantitative approach to assess the continuous compliance

Note first that the study presented in this paper was carried out in a context where the adjustment for the loss-absorbing capacity of technical provisions was lesser than the Future Discretionary Benefits ("FDB") (see Level 2 Implementation Measures - European Commission (2011)). As a consequence, the Value of In-Force and the *NAV* were always calculated net of the loss-absorbing effect of future profit participation. In cases where the loss-absorbing capacity of technical provisions breaches the FDB, further developments (and additional assumptions), not presented in this paper, will be necessary.

In Section 2.2 we present a proxy implementation that enables one to assess the continuous compliance, and the underlying assumptions.

2.2.1 Assumptions underlying the continuous compliance assessment framework

As explained in Subsection 2.1.2, several simplifications will be necessary in order to operationalize the continuous compliance assessment using our methodology.

Selection of the monitored risks

First, we need to assume that the company can be considered subject to a limited number of significant and easily measurable risks with little loss of information.

In most cases this assumption is quite strong. Indeed, there are numerous underlying risks for a life insurance undertaking and these are not always *easily measurable*. For example, the mortality and longevity risks, to cite only those, are greatly difficult to monitor on an infra-year time step, simply because of the lack of data. Moreover the *significant* aspect will have to be justified. For instance, this significance can be defined considering the known impact of the risk on the *SCR* or on the company's balance sheet, or considering its volatility.

In the case of a life insurance business it seems particularly relevant to select the financial risks, easily measurable and monitorable. As a consequence, the selected risk will for example be the stock, interest rates (corporate, sovereign), implicit volatilities (stock / interest rates), illiquidity premium.

In order to enable a frequent monitoring of the selected risks and of their impact, it is necessary to add the assumption that their evolution over time can be satisfyingly replicated by the evolution of composite indexes defined continuously through the monitoring period.

This assumption is a more tangible translation of the measurable aspect of the risks. The objective here is to enable the risks' monitoring through reference indexes.

For example, an undertaking which is mainly exposed to European stocks can consider the EUROSTOXX50 level in order to efficiently synthesize its stock level risk. Another possibility may be to consider weighted European stock indexes to obtain an aggregated indicator more accurate and representative of the entity-specific risk. For example, for the sovereign spread risk, it seems relevant for a given entity to monitor an index set up as a weighted average of the spread extracted from the various bonds in its asset portfolio.

Eventually, the undertaking must aim at developing a indexes table, similar to the following one.

Table 2.1 – Example of indexes table — Significant risks and their associated indicators.

Significant risks	**Associated composite indicators**
Stock (level)	70% CAC40 / 30% EUROSTOXX50
Risk-free rate (level)	Euro swap curve (averaged level evolution)
Spread (sovereign)	Weighted average of the spread by issuing country. Weights : % market value in the asset portfolio
Spread (corporate)	iTraxx Europe Generic 10Y Corporate
Volatility (stock)	VCAC Index
Illiquidity premium	Illiquidity premium (see QIS5 formula - EIOPA (2010))

Generally speaking, all the assumptions presented here are almost induced by the operational constraints linked to the definition of the continuous compliance framework (full calculation frequency /

number of monitored risks). Indeed, it is impossible in practice to monitor each underlying risk day by day. We therefore need to restrict the framework by selecting the most influential risks and indicators enabling their practical monitoring.

In addition, it is irrelevant to consider too stable risks or risks that cannot be monitored infra-annually. In this case, they can simply be assumed frozen, or updated proportionally to a volume measure, through the monitoring period, with little loss of information.

In this simplified framework, a change of the economic conditions over time will be summarize in the realized indexes' level transition. It is then possible to build a proxy enabling one to approximate quickly the *SR* at each monitoring date, knowing the current level of the composite indicators.

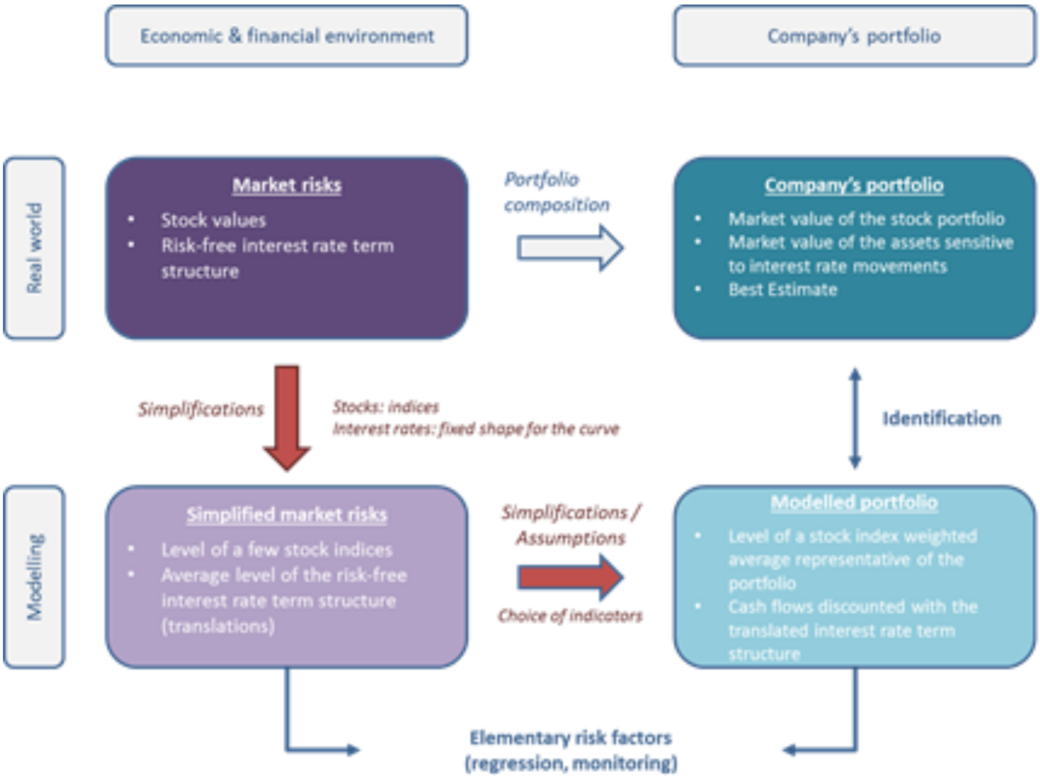

Figure 2.2 – Simplified monitoring framework: Illustration

Figure 2.2 illustrates the process to follow and the underlying assumptions made in a simplified framework. Let us develop a case where the company's asset portfolio can be divided into one stock and one bond pools. Two underlying risks have been identified, the stock level risk and the interest rate level risk (average level change of the rates curve *). Our assumptions lead to consider that, once the risks associated to composite indexes, it is possible to approximate the asset portfolio by a mix between,

- a stock basket with the same returns, composed with the composite stock index only (*e.g.* 70% CAC 40 / 30% EUROSTOXX50),
- a bond basket replicating the cash-flows of the bonds discounted using a rate curve induced

∗. Note that other kinds of interest rates risks can be selected in order to address the term structure risk more precisely, such as the slope and curvature risks. For more insight on this subject see *e.g.* Diebold and Li (2006).

from the initial curve translated of the average variation of the reference rate curve (the "composite" curve, *e.g.* the Euro swap curve).

Eventually we can decompose the process presented in Figure 2.2 between,

— a vertical axe where one simplifies the risks themselves,

— and an horizontal axe where one transforms the risk into composite indexes.

To conclude, note that the assumptions made here will lead to the creation of a basis risk. Indeed, even if the considered indexes are very efficient, one part of the insurance portfolio sensitivity will be omitted due to the approximations. In particular the risks and indexes must be chosen very precisely, entity-specifically. A small mistake can have great repercussions on the approximate *SR*. In order to minimize the basis risk, the undertaking will have to back-test the choices made and the underlying assumptions.

Selection of the monitored marginal *SCR*

The continuous compliance framework and tool presented in this paper applies to companies that use a Standard Formula approach to assess the *SCR* value (but can provide relevant information to companies that use an internal model).

In practice it will not be necessary to monitor every marginal *SCR* of a company. Indeed, some risk modules will be little or not impacted by any infra-annual evolution of the selected risks. Moreover, a certain number of sub-modules have a small weight in the calculation of the Basic Solvency Capital Requirement (*BSCR*). These too small and/or stable marginal *SCR* will be frozen or updated proportionally to a volume measure throughout the monitoring period.

Eventually, the number of risk modules that will have to be updated precisely (the most meaningful marginal *SCR*) should be reduced to less than ten. Note that, among the marginal *SCR* to recalculate, some can correspond to modeled risks factors but others will not correspond to the selected risk factors while being very impacted by those (*e.g.* the massive lapse *SCR*).

This selection of the relevant *SCR* sub-modules will introduce a new assumption and a new basis risk, necessary for our methodology's efficiency. The basis risk associated to this assumption, linked to the fact that some marginal *SCR* will not be updated at each monitoring date, can be reduced by considering a larger number of sub-modules. One will have to apprehend this problem pragmatically, to take a minimal number of risk modules into account in order to limit the number of future calculations, while keeping the error made on the overall *SCR* under control, the best possible way.

2.2.2 Use of parametric proxies to assess the continuous compliance

In the previous section we have defined a reference framework in which we will develop our monitoring tool. The proposed methodology aims at calibrating proxies that replicate the central and shocked *NAV* as functions of the levels taken by the chosen indexes.

Assumption of stability of the asset and liability portfolios

We now work with closed asset and liabilities portfolios, with no trading, claim or premium cash-flow, in order to consider a stable asset-mix and volume of assets and liabilities. Eventually, all the balance sheets movements are now induced by the financial factors.

This new assumption may seem strong at first sight. However, it seems justified on a short term period. In the general case the evolution of these portfolios is slow for mature life insurance companies. This evolution is therefore assumed to have little significance for the monitoring period of our continuous compliance monitoring tool. Eventually, if a significant evolution happens in practice (*e.g.* a portfolio purchase / sale) this will lead to a full recalibration of the tool (see Subsection 2.3.2 for more insight on the monitoring tool governance).

Economic transitions

Let us recall the various assumptions considered until now.

— *H1*: The undertaking's underlying risks can be summarized into a small pool of significant and easily quantifiable risks with little loss of information.

— *H2*: The evolution of these risks can be perfectly replicated by monitoring composite indicators, well defined at each date of the monitoring period.

— *H3*: The number of marginal *SCR* that need to be precisely updated at each monitoring date can be reduced to the most impacting risk modules with little loss of information.

— *H4*: The asset and liability portfolio are assumed frozen between two calibration dates of the monitoring tool.

Under the assumptions *H1*, *H2*, *H3* and *H4* it is possible to summarize the impact of a time evolution of the economic conditions on the considered portfolio into an instant level shock of the selected composite indicators. This instant choc will be denoted "economic transition" and we will see below that it can be identified to a set of elementary risk factors similar to those presented in Devineau and Chauvigny (2011).

Figure 2.3 – Economic transition "$0 \to 0^+$".

Let us consider a two shocks framework: the stock level risk, associated to an index denoted by $S(t)$ at date $t \geq 0$ ($t = 0$ being the tool's calibration date) and an interest rate level risk, associated to zero-coupon prices, denoting by $P(t,m)$ the zero-coupon of maturity m and date $t \geq 0$. Now, let us consider an observed evolution between 0 and a monitoring date $t > 0$Finally, to calculate the *NAV* at date t, under our assumptions, it is only necessary to know the new levels $S(t), P(t,m)$.

The real evolution, from $\left(S(0),(P(0,m))_{m\in[\![1;M]\!]}\right)$ to $\left(S(t),(P(t,m))_{m\in[\![1;M]\!]}\right)$ can eventually be seen as a risk factors couple,

$$\varepsilon = \left({}^{stock}\varepsilon = \ln\left(\frac{S(t)}{S(0)}\right), {}^{ZC}\varepsilon = -\frac{1}{M}\sum_{m=0}^{M} \ln\left(\frac{1}{m}\frac{P(t,m)}{P(0,m)}\right)\right),$$

denoting by ${}^{stock}\varepsilon$ (respectively ${}^{ZC}\varepsilon$) the stock (resp. zero-coupon) risk factor.

This evolution of the economic conditions, translated into a risk factors tuple, is called economic transition in the following sections of this paper and can easily be extended to a greater number of risks. The risk factor will be used in our algorithm to replicate the instant shocks "$0 \to 0^+$" equivalent to the real transitions "$0 \to t$". Moreover, the notion of economic transition will be used to designate either an instant shock or a real evolution of the economic situation between 0 and $t > 0$. In this latter case we will talk about *real* or *realized* economic transition.

Probable space of economic transitions for a given $\alpha\%$ threshold

Let us consider, for example, a 3-months monitoring period (with a full calibrations of the monitoring tool at the start and at the end of the period). It is possible to *a priori* assess a probable space of the probable quarterly economic transitions, under the historic probability $\mathbb{P}$ and for a given threshold $\alpha\%$. One simply has to study a deep enough historical data summary of the quarterly evolutions of the indexes and to assess the interval between the $\frac{1-\alpha\%}{2}$ and the $\frac{1+\alpha\%}{2}$ historical quantiles of the risk factors extracted from the historical data set.

For example, for the stock risk factor ${}^{s}tock\varepsilon$, knowing the historical summary $\left(S_{\frac{i}{4}}\right)_{i\in[\![0,4T+1]\!]}$ one can extract the risk factor's historical data set $\left({}^{stock}\varepsilon_{\frac{i}{4}} = \ln\left(\frac{S_{\frac{i+1}{4}}}{S_{\frac{i}{4}}}\right)\right)_{i\in[\![0,4T]\!]}$ and obtain the probable space of economic transitions for a given $\alpha\%$ threshold,

$$\left[q_{\frac{1-\alpha\%}{2}}\left(\left({}^{stock}\varepsilon_{\frac{i}{4}}\right)_{i\in[\![0;4T]\!]}\right); q_{\frac{1+\alpha\%}{2}}\left(\left({}^{stock}\varepsilon_{\frac{i}{4}}\right)_{i\in[\![0;4T]\!]}\right)\right].$$

In a more general framework, consider economic transitions represented by J-tuples of risk factors $\varepsilon = \left({}^{1}\varepsilon, ..., {}^{J}\varepsilon\right)$ of which one can get an historical summary $\left({}^{1}\varepsilon_{\frac{i}{4}}, ..., {}^{J}\varepsilon_{\frac{i}{4}}\right)_{i\in[\![0;T]\!]}$. The following probable interval of the economic transitions with a $\alpha\%$ threshold can be used,

$$\mathscr{E}^{\alpha} = \left\{\left({}^{1}\varepsilon, ..., {}^{J}\varepsilon\right) \in \prod_{j=1}^{J}\left[q_{\frac{1-\alpha\%}{2}}\left(\left({}^{j}\varepsilon_{\frac{i}{4}}\right)_{i\in[\![0;T]\!]}\right); q_{\frac{1+\alpha\%}{2}}\left(\left({}^{j}\varepsilon_{\frac{i}{4}}\right)_{i\in[\![0;T]\!]}\right)\right]\right\}.$$

Note that such a space does not take correlations into account. Indeed each risk factor's interval is defined independently from the others. In particular, such a space is *prudent*: contains more than $\alpha\%$ of the historically probable economic evolutions.

Implementation — Replication of the central *NAV*

We will now assume that J different risks have already been selected.

The implementation we will now describe aims at calibrating a polynomial proxy that replicates $NAV_{0^+}(\varepsilon)$, the central NAV at the date $t = 0^+$, associated to an economic transition $\varepsilon = \left({}^1\varepsilon, ..., {}^J\varepsilon\right)$. The proxy will allow, at each monitoring date t, after evaluating the observed economic transition ε_t (realized between 0 and t), to obtain a corresponding approximate central NAV value, $NAV_{0^+}^{proxy}(\varepsilon_t)$.

Notation and preliminary definitions.

To build the $NAV_{0^+}^{proxy}(\varepsilon)$ function, our approach is inspired from the Curve Fitting (CF) and Least Squares Monte Carlo ($LSMC$) polynomial proxies approaches proposed in Vedani and Devineau (2013). It is possible to present a generalized implementation plan for these kinds of approaches. They both aim at approximating the NAV using a polynomial function whose monomials are simple and crossed powers of the elements in $\varepsilon = \left({}^1\varepsilon, ..., {}^J\varepsilon\right)$.

Let us introduce the following notation. Let $\mathbf{Q}$ be a risk-neutral measure conditioned by the real-world financial information known at date 0^+, $\mathscr{F}_{0^+}$ the filtration that characterizes the real-world economic information contained within an economic transition between dates 0 and 0^+. Let R_u be the profit realized between $u-1$ and $u \geq 1$, and δ_u the discount factor at date $u \geq 1$. Let H be the liability run-off horizon.

The gist of the method is described here below.

The $NAV_{0^+}(\varepsilon)$ depends on the economic information through the period $[0; 0^+]$:

$$NAV_{0^+}(\varepsilon) = \mathbb{E}^{\mathbf{Q}}\left[\sum_{t=1}^{H} \delta_t R_t | \mathscr{F}_{0^+}\right].$$

For a given transition ε it is possible to estimate $NAV_{0^+}(\varepsilon)$ implementing a standard Asset Liability Management model calculation at date $t = 0^+$. In order to do so one must use an economic scenarios table of P simulations generated under the probability measure $\mathbf{Q}$ between $t = 0^+$ and $t = H$ initialized by the levels (and volatilities if the risk is chosen) of the various economic drivers as induced by transition ε.

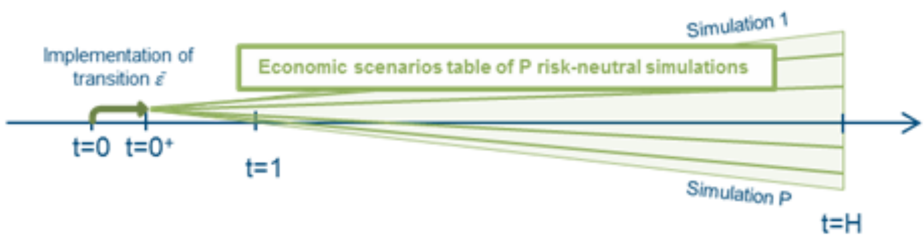

Figure 2.4 – Calculation of an estimator of $NAV_{0^+}(\varepsilon)$ using a Monte Carlo method.

For each simulation $p \in [\![1;P]\!]$ and date $t \in [\![1;H]\!]$, one has to calculate the profit outcome R_t^p using an Asset-Liability Management (ALM) model and, knowing the corresponding discount factor δ_t^p, to assess the Monte Carlo estimator,

$$\widehat{NAV}_{0^+}(\varepsilon) = \frac{1}{P}\sum_{p=1}^{P}\sum_{t=1}^{H}\delta_t^p R_t^p.$$

When $P = 1$ we obtain an inefficient estimator of $NAV_{0^+}(\varepsilon)$ which we will denote by $NPV_{0^+}(\varepsilon)$ (Net Present Value of margins), according to the notation of Vedani and Devineau (2013). Note that for a given transition ε, $NPV_{0^+}(\varepsilon)$ is generally very volatile and it is necessary to have P high to get an efficient estimator of $NAV_{0^+}(\varepsilon)$.

Methodology.

Let us consider a set of N transitions obtained randomly from a probable space of economic transitions and denoted by $\left(\varepsilon^n = \left({}^1\varepsilon^n, \ldots, {}^J\varepsilon^n\right)\right)_{n\in[\![1;N]\!]}$. We now have to aggregate all the N associated risk-neutral scenarios tables, each one initialized by the drivers' levels (and volatilities if needed) corresponding to one of the economic transitions in the set, in a unique table (see Figure 2.5).

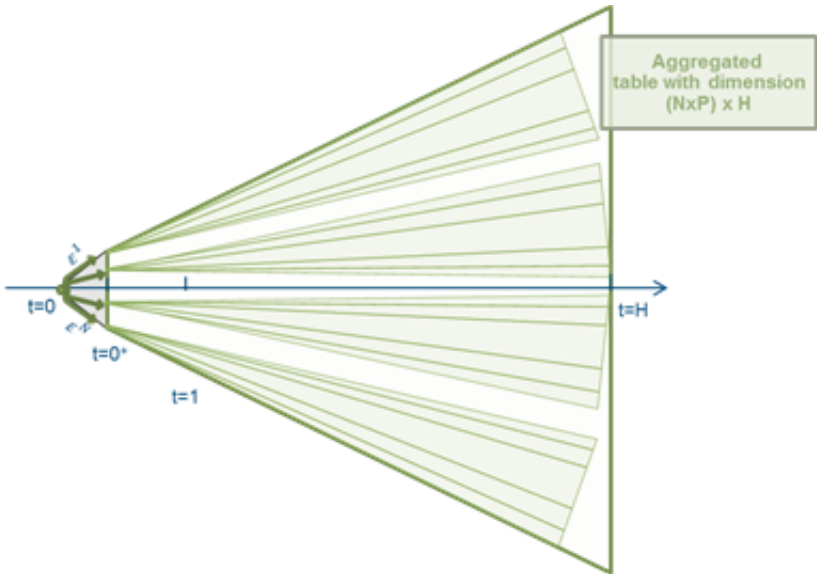

Figure 2.5 – Aggregate table.

The ALM calculations launched on such a table enables one to get $N \times P$ outcomes

$$\left(NPV_{0^+}^p(\varepsilon^n)\right)_{n\in[\![1;N]\!],p\in[\![1;P]\!]},$$

and subsequently a N sample

$$\left(NAV_{0^+}(\varepsilon^n) = \frac{1}{P}\sum_{p=1}^{P} NPV_{0^+}^p(\varepsilon^n)\right)_{n\in[\![1;N]\!]}.$$

Then, the outcomes $\left(\widehat{NAV}_{0^+}(\varepsilon^n)\right)_{n\in[\![1;N]\!]}$ are regressed on simple and crossed monomials of the risk factors in $\varepsilon = \left({}^1\varepsilon,...,{}^J\varepsilon\right)$. The regression is made by Ordinary Least Squares (OLS) and the optimal regressors $x = \left(\text{Intercept}, {}^1x,...,{}^Kx\right)$ (with, for all $k \in [\![1;K]\!]$,${}^kx = \prod_{j=1}^J \left({}^j\varepsilon\right)^{{}^k\alpha_j}$, for a certain J-tuple $\left({}^k\alpha_1,...,{}^k\alpha_J\right)$ in $\mathbb{N}^J$) are selected using a stepwise methodology. For more developments about these approaches see Draper et al. (1966) or Hastie et al. (2009).

Let $\beta = \left({}^{Int}\beta, {}^1\beta,...,{}^K\beta\right)$ be the optimal multilinear regression parameters.

The considered multilinear regression can therefore be written under a matricial form $Y = X\beta + U$, denoting by

$$Y = \begin{pmatrix} NAV_{0^+}\left(\varepsilon^1\right) \\ \vdots \\ NAV_{0^+}\left(\varepsilon^N\right) \end{pmatrix},$$

$$X = \begin{pmatrix} x^1 \\ \vdots \\ x^N \end{pmatrix}$$

with, for all $n \in [\![1;N]\!]$, $x^n = \left(1, {}^1x^n,...,{}^Kx^n\right)$, for all $k \in (|1;K|)$, ${}^kx^n = \prod_{j=1}^J \left({}^j\varepsilon^n\right)^{{}^k\alpha_j}$ and $U = Y - X\beta$.

In this regression, the conditional expectation of $NAV_{0^+}(\varepsilon^n)$ given the σ-field generated by the regressors matrix X is simply seen as a linear combination of the regressors. For more insight about multiple regression models the reader may consult Saporta (2006).

The underlying assumption of this model can also be written $\exists\beta, \mathbb{E}[Y|X] = X\beta$.

Under the assumption that $X'X$ is invertible (with Z' the transposition of a given vector or matrix Z), the estimated vector of the parameters is,

$$\hat{\beta} = (X'X)^{-1}X'Y.$$

Moreover, for a given economic transition $\bar{\varepsilon}$ and its associated set of optimal regressors $\bar{x}$, $\bar{x}.\hat{\beta}$ is an unbiased and consistent estimator of $\mathbb{E}\left[\widehat{NAV}_{0^+}(\bar{\varepsilon})|\bar{x}\right] = \mathbb{E}\left[NAV_{0^+}(\bar{\varepsilon})|\bar{x}\right]$. When $\sigma(x) = \mathscr{F}_{0^+}$, which is generally the case in practice, $\bar{x}\hat{\beta}$ is an efficient estimator of $NAV_{0^+}(\varepsilon)$ and we get an efficient polynomial proxy of the central NAV for every economic transition.

Eventually, it is necessary to test the goodness of fit. The idea is now to calculate several approximate outcomes of central NAV, associated to an *out of sample* * set of economic transition, using a Monte Carlo method on a great number of secondary scenarios, and to compare these outcomes to those obtained using the proxy.

*. Scenarios that are not included in the set used during the calibration steps.

Implementation — Replication of the shocked *NAV*

At each monitoring date, we aim at knowing each pertinent marginal *SCR* value, for each chosen risk modules. With the proxy calibrated in the previous section one can calculate an approximate value of the central *NAV*. We now have to duplicate the methodology presented in Subsection 2.2.2, adapted for each marginally shocked *NAV* (considering the Standard Formula shocks). *

The implementation is fully similar except the fact that the shocked proxies are calibrated on N outcomes of marginally shocked NAV_{0^+}. Indeed each marginal *SCR* is a difference between the central *NAV* and a *NAV* after the application of the marginal shock. We therefore need the *NAV* after shock that takes the conditions associated to an economic transition into account.

This enables one to obtain, for each shock *nb i*, a set $\left(NAV_{0^+}^{shock\ nb\ i}(\varepsilon^n)\right)_{n\in[\![1;N]\!]}$, a new optimal regressors set $\left(\text{Intercept}, {}^1x^{shock\ nb\ i}, \ldots, {}^Kx^{shock\ nb\ i}\right)$ and new optimal parameters estimators $\hat{\beta}^{shock\ nb\ i}$.

Practical monitoring

Once the methodology has been implemented, the obtained polynomial proxies enable one, at each date within the monitoring period, to evaluate the central and shocked *NAV* values knowing the realized economic transition.

At each monitoring date t, the process is the following.

- Assessment of the realized transition between 0 and t, $\hat{\varepsilon}$.
- Derivation of the values of the optimal regressors set for each proxy:
  - $\bar{x}$ the realized regressors set for the central proxy,
  - $\bar{x}^{shock\ nb\ 1}, \ldots, \bar{x}^{shock\ nb\ J}$ the regressors set for the J shocked proxy.
- Calculation of the approximate central and shocked *NAV* levels at date t:
  - $\bar{x}\hat{\beta}$, the approximate central *NAV*,
  - $\bar{x}^{shock\ nb\ 1}\hat{\beta}^{shock\ nb\ 1}, \ldots, \bar{x}^{shock\ nb\ J}\hat{\beta}^{shock\ nb\ J}$ the J approximate shocked *NAV*.
- Calculation of the approximate marginal *SCR* and, considering frozen values, or values that are updated proportionally to a volume measure, for the other marginal *SCR*, Standard Formula aggregation to evaluate the approximate overall *SCR* and *SR* †.

2.2.3 Least-Squares Monte Carlo vs. Curve Fitting — The large dimensioning issue

The implementation developed in Subsection 2.2.2 is an adapted application, generalized to the $N \times P$ framework, of the polynomial approaches such as *LSMC* and *CF*, already used in previous studies to project *NAV* values at t years ($t \geq 1$). For more insight about these approaches, see for example

∗. Note that it is necessary to calibrate new "after shock" proxies because it is impossible to assimilate a Standard Formula shock to a transition shock...

†. For more insight concerning the Standard Formula aggregation, especially about the evaluation of the differed taxes, see Subsection 2.3.1.

Vedani and Devineau (2013), Algorithmics (2011) or Barrie & Hibbert (2011).

When $P = 1$ and N is very large (basically the proxies are calibrated on Net Present Values of margins / *NPV*), we are in the case of a *LSMC* approach. On the contrary, when N is rather small and P large, we are in the case of a *CF* approach.

Both approaches generally deliver similar results. However the *LSMC* is often seen as more stable than a *CF* when a large number of regressors are embedded in the proxy. This clearly matches the continuous compliance case where the user generally considers a larger number of risk factors compared to the usual *LSMC* methodologies, used to accelerate Nested Simulations for example. In our case, this large dimensioning issue makes a lot of sense.

In Section 2.3 we will apply the methodology on four distinct risk factors, the stock level risk, the interest rates level risk, the widening of corporates spread and of sovereign spread risks. We have chosen to implement this application using a *LSMC* method. In Section 5 we eventually try to challenge the commonly agreed idea that this methodology is more robust than *CF* in a large dimension context.

2.3 *LSMC* approach adapted to the continuous compliance issue

In Section 2.3 we will implement the presented methodology, in a standard savings product framework. The ALM model used for the projections takes profit sharing mechanisms, target crediting rate and dynamic lapses behaviors of policy holders into account. Its characteristics are similar to those of the model used in Section 5 Vedani and Devineau (2013). The economic assumptions are those of 31/12/2012.

2.3.1 Implementation of the monitoring tool — Initialization step and proxies calibration

Firstly it is necessary to shape the exact framework of the study. We have to select the significant risks to be monitored, to choose representative indexes and then to identify the risk modules that will be updated. Note that the other risk modules will be considered frozen through the monitoring period.

The monitoring period must be chosen short enough to ensure a good validity of our stability assumptions for the risk modules that are not updated and for the balance sheet composition. However, it also defines the time during two complete proxy calibrations and, as a consequence, it must be chosen long enough not to force too frequent calibrations, which are highly time-consuming. In this study we have therefore chosen to consider a quarterly monitoring period.

Initialization step — Implementation of a complete regulatory solvency calculation

In order to quantify the relative relevance of the various marginal *SCR* of the Standard Formula, it is recommended to implement, as a preliminary step, a complete regulatory solvency calculation before a calibration of the monitoring tool. Moreover, seen as an *out of sample* scenario, this central calculation can be used as a validation point for the calibrated proxies. *

It is also possible to select the marginal *SCR* based on expert statements or on the undertaking's expertise, knowing the products sensitivities to the various shocks and economic cycles at the calibration date (and the previous *SCR* calculations).

Initialization step — Risk factor and monitored indexes selection

We have selected four major risks and built the following indexes table.

Table 2.2 – Selected risks and associated indicators.

Selected risks	Composite indicators
Stock (level)	100% EUROSTOXX50
Risk-free rate (level)	Euro swap curve (averaged level evolution)
Spread (sovereign)	Average spread French bonds rate vs. Euro swap rate
Spread (corporate)	iTraxx Europe Generic 10Y Corporate

These four risks generally have a great impact on the *NAV* and *SCR* in the case of savings products, even on a short monitoring period. Moreover, they are highly volatile at the calibration date (31/12/12). In particular, the division of the spread risk in two categories (sovereign and corporate) is absolutely necessary within the European sovereign debt context.

A wide range of risks have been set aside of this study that is just intended to be a simple example. In practice both the stock and interest rates implicit volatility risks are also relevant risks that can be added in the methodology's implementation with no major issue. For the stock implicit volatility risk it is possible to monitor market volatility indexes such as the VIX. Note that the interest rates implicit volatility risk raises several questions related to the application of the risk in the instant economic transitions, in the calibration scenarios. These issues can be set aside considering recalibration/regeneration approaches (see Devineau (2011)) and will not be discussed in this paper.

*. The implementation of two to four complete regulatory solvency calculations may be a strong constraint for most insurance undertakings however, due to the several assumptions made to implement the monitoring tool, we recommend to consider monitoring period no longer than six months.

Initialization step — Choice of the monitored marginal *SCR*

Considering the updated risk modules to update, we have chosen the most significant in the Standard Formula aggregation process (see Table 2.3). These are also the less stable trough time:
- the stock *SCR*,
- the interest rates *SCR*,
- the spread *SCR*,
- the liquidity *SCR*.

The lapse risk *SCR*, generally highly significant, has not been considered here. Indeed with the very low rates, as at 31/12/2012, the lapse risk *SCR* is close to zero. Some other significant *SCR* sub-modules such as the real estate *SCR* have been omitted because of their low infra-year volatility.

Table 2.3 – Market marginal *SCR* as at 31/12/2012.

Market *SCR*	**Value as at 31/12/2012**
IR *SCR*	968
Stock *SCR*	3930
Real Estate *SCR*	943
Spread *SCR*	2658
Liquidity *SCR*	3928
Concentration *SCR*	661
Currency *SCR*	127
...	...

Proxies calibration and validation

The calibration of the various proxies is made through the same process as developed in Vedani and Devineau (2013). The proxy is obtained by implementing a standard *OLS* methodology and the optimal regressors are selected through a stepwise approach. This enables the process to be completely automated. The validation of each proxy is made by considering ten *out of the sample* scenarios. These are scenarios that have not be used to calibrate the proxies but on which we have calculated shocked and central outcomes of $\widehat{NAV}_{0^+}$. These "true" outcomes are then compared to the approximate outcomes obtained from our proxies.

To select the *out of the sample* scenarios we have chosen to define them as the 10 scenarios that go step by step from the "initial" position to the "worst case" situation (the calibrated *worst case* limit of the monitored risks).

For each risk factor ε^j:
- the "initial" position is $\varepsilon^j_{init} = 0$,
- the "worst case" situation * is $\varepsilon^j_{w.c.} = q_{\frac{1-\alpha\%}{2}}\left(\left(\varepsilon_{\frac{i}{4}}\right)_{i\in[\![0;T]\!]}\right)$ or $q_{\frac{1+\alpha\%}{2}}\left(\left(\varepsilon_{\frac{i}{4}}\right)_{i\in[\![0;T]\!]}\right)$, depending on the direction of the worst case for each risk,

∗. = the 10^{th} *out of sample* scenario

— the k^{th} ($k \in [\![1;9]\!]$) *out of sample* scenario is $\varepsilon^{j}_{nb.\ k} = \frac{k}{10}\varepsilon^{j}_{w.c.} + \frac{10-k}{10}\varepsilon^{j}_{init}$.
Below (in Table 2.4) are shown the relative deviations, between the proxies outcomes and the corresponding *out of sample* fully-calculated scenarios, obtained on the first five validation scenarios.

As one can see, the relative deviations are always close to 0 apart from the illiquidity shocked *NAV* proxy. In practice this proxy is the most complex to calibrate due to the high volatility of the illiquidity shocked *NAV*. To avoid this issue, the user can add more calibration scenarios or select more potential regressors when implementing the stepwise methodology. In our study we have chosen to validate our proxy, staying critical on the underlying approximate marginal *SCR* illiquidity.

We do not discuss in this paper the optimal way to select the validation scenarios. More generally we acknowledge the need for a reflection about the calculation error due to the use of proxies in Life insurance projections. This reflection is underway and will be the subject of future articles. In the framework presented in this article, the analysis of the proxies' accuracy could be threefold:

- validation of the polynomials on the basis of out-of-sample scenarios (as mentioned above, the scenario selection process will be investigated further in the future);
- at different dates during the monitoring period, update the proxies and run simplified full calculations on the basis of the risk factors' observed values. By doing this, we are adding new out-of-sample scenarios, selected ex-post for their relevance. Note that in the simplified full calculations, the only inputs updated are the risk factors selected for the proxies;
- comparison at different dates (prior to and after the calibration dates) between the results obtained with the proxies and those obtained by full calculations, with all the inputs of the full calculations being updated. This would be a way to confirm that the risk factor selection is satisfactory. For example, if the proxies are calibrated at dates t_{-1}, t_0 and t_1, the results obtained with the proxies calibrated in t_0 could be compared with the full calculations at dates t_{-1}, t_0 and t_1.

Note however that the number of tests performed will have to be limited so that the implementation of proxies remains operationally beneficial.

Table 2.4 – Relative deviations proxies vs. full-calculation *NAV* (check on the five first validation steps).

Validation scenarios	**1**	**2**	**3**	**4**	**5**
Central *NAV*	-0.07%	1.65%	1.56%	1.05%	0.29%
IR shocked *NAV*	-0.18%	1.67%	1.14%	0.44%	-0.83%
"Global" Stock shocked *NAV*	0.24%	1.93%	1.56%	1.15%	0.28%
"Other" Stock shocked *NAV*	0.19%	1.95%	1.78%	1.31%	0.27%
Spread shocked *NAV*	0.01%	2.29%	2.15%	1.06%	0.18%
Illiquidity shocked *NAV*	-5.35%	-3.27%	-2.43%	-3.03%	-2.39%

All the proxies being eventually calibrated and validated, it is now necessary to rebuild the Standard Formula aggregation process in order to assess the approximate overall *SCR* value.

Proxies aggregation through the Standard Formula process

In practice the overall *SCR* is calculated as an aggregation of three quantities, the *BSCR*, the operational *SCR* (*SCRop*) and the tax adjustments (Adj).

As far as the *BSCR* is concerned, no particular issue is raised by its calculation. At each monitoring date, the selected marginal *SCR* are approximated using the proxies and the other *SCR* are assumed frozen. The *BSCR* is simply obtained through a Standard Formula aggregation (see for example Devineau and Loisel (2009)).

To derive the operational *SCR*, we consider that this capital is also stable through time, which is in practice an acceptable assumption for a half-yearly or quarterly monitoring period (and consistent with the asset and liability portfolios stability assumption).

The Tax adjustments approximation leads to the greatest issue. Indeed we need to know the approximate Value of In-Force (VIF) at the monitoring date. We obtain the approximate VIF as the approximate central *NAV* $\left(\widehat{NAV}^{central proxy}\right)$ minus a fixed amount calculated as the sum of the tier-one own funds ($tier\_one\_OF$) and of the subordinated debt (*SD*) minus the financial management fees (FMF), as at the calibration date. Let t be the monitoring date and 0 be the proxies' calibration date ($t > 0$),

$$\widehat{VIF}_t \approx NAV_t^{central\ proxy} - (tier\_one\_OF_0 + SD_0 - FMF_0).$$

Assuming a frozen corporation tax rate of 34.43% (French corporation tax rate), the approximated level of deferred tax liability $\widehat{DTL}$ is obtained as,

$$\widehat{DTL}_t = 34.43\% \times \widehat{VIF}_t.$$

Eventually, the income tax recovery associated to new business $\left(ITR^{NB}\right)$ is assumed frozen through the monitoring period and the approximate tax adjustments at the monitoring date is obtained as,

$$\widehat{Adj}_t = ITR_0^{NB} + \widehat{DTL}_t.$$

Knowing the approximate values $\widehat{BSCR}_t$ and $\widehat{Adj}_t$, and the initial value $SCRop_0$, one can obtain the approximate overall *SCR* (simply denoted by $\widehat{SCR}$) at the monitoring date as,

$$\widehat{SCR}_t = \widehat{BSCR}_t + SCRop_0 - \widehat{Adj}_t.$$

Eventually, in order to obtain the *SR* approximation we obtain the approximate eligible own funds $\widehat{OF}$ as,

$$\widehat{OF}_t = (tier\_one\_OF_0 + SD_0 - FMF_0) + \widehat{VIF}_t \times (1 - 34.43\%).$$

Eventually, the approximate *SR* at the monitoring date is,

$$\widehat{SR}_t = \frac{\widehat{OF}_t}{\widehat{SCR}_t}.$$

2.3.2 Practical use of the monitoring tool

In subsection 2.3.2 we will first see the issues raised by the practical continuous compliance's monitoring through our tool, and the tool's governance. In a second part we will develop the other possible uses of the monitoring tool, especially in the area of the risk management and for the development of preventive measures.

Monitoring the continuous compliance

At each monitoring date the process to assess the regulatory compliance is the same as presented in Subsection 2.2.2.

- Assessment of the realized transition between 0 and t, $\hat{\varepsilon}$.
- Derivation of the values of the optimal regressors set for each proxy.
- Calculation of the approximate central and shocked *NAV* levels at date t.
- Calculation of the levels of each approximate marginal *SCR* at date t (the other marginal *SCR* are assumed frozen through the monitoring period).

This, with other stability assumptions such as stability of the tax rate and of the tier-one own funds, enables one to reconstruct the Basic *SCR*, the operational *SCR* and the Tax adjustments and, eventually, to approximate the overall *SCR* and the *SR* at the monitoring date.

Note that this process can be automated to provide a monitoring diagram such as the one depicted below and a set of outputs such as the eligible own funds, the overall *SCR*, the *SR*, but also the various marginal *SCR* (see Figure 2.6).

Monitoring the daily evolution of the *SR*

In practice the ability to monitor the *SR* day by day is very interesting and provides a good idea of the empirical volatility of such a ratio (see Figure 2.7).

In particular, in an ORSA framework it could be relevant to consider an artificially smoothed *SR*, for example using a 2-week moving average, in order to depict a more consistent solvency indicator. Considering the same data as presented in the previous figure we would obtain the following two graphs (see Figure 2.8).

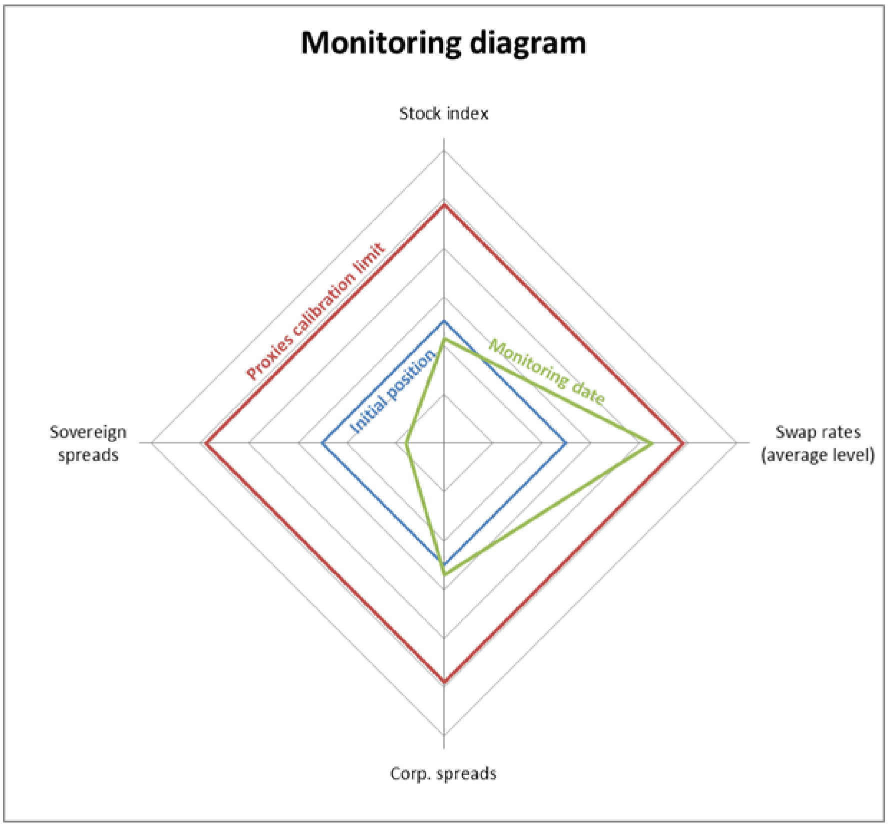

Figure 2.6 – Diagram used to monitor the evolution of the risk factors

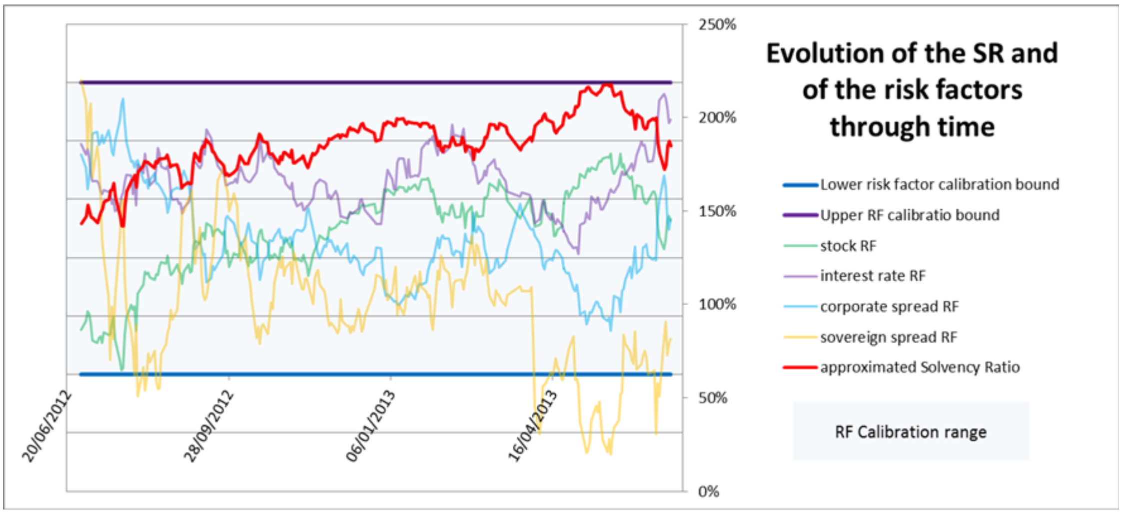

Figure 2.7 – Monitoring of the approximate *SR* and of the four underlying risk factors, from 30/06/12 to 30/06/13.

Monitoring tool governance

Several assumptions are made to provide the approximate *SR* but we can observe in practice a good replication of the risk impacts and of the *SR* variations. However the use of this monitoring tool only

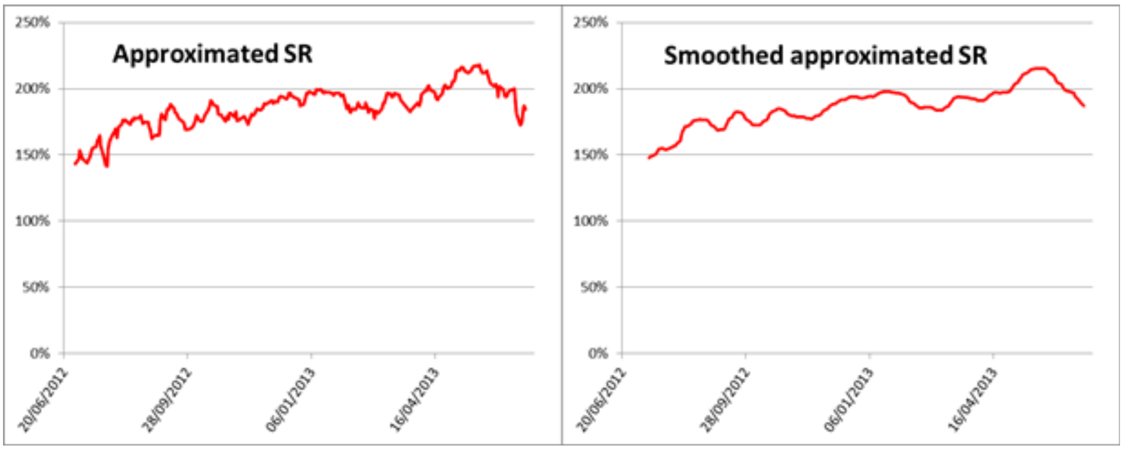

Figure 2.8 – Comparison of the standard approximate *SR* and of a smoothed approximate *SR* - Monitoring from 30/06/12 to 30/06/13.

provides a proxy and therefore the results must be used with caution and its governance must be managed very carefully.

The governance of the tool can be divided into three parts.

- Firstly it is necessary to *a priori* define the recalibration frequency. The monitoring period associated to each total calibration of the tool should not be too long. The authors believe it should not exceed half a year.
- Secondly it is important to identify clearly the data to update for each recalibration. These data especially cover the asset and liability data.
- Finally the user must define the conditions leading to a total (unplanned) recalibration of the tool. In particular, these conditions must include updates following management decisions (financial strategy changes inside the mode, asset mix changes,...) and updates triggered by the evolution of the economic situation.

Alternative uses of the tool

This monitoring tool enables the risk managers to run a certain number of studies, even at the beginning of the monitoring period, in order ton anticipate the impact of future risk deviations for example.

Sensitivity study and stress testing.

The parametric proxy that replicates the central *NAV* can also be used to stress the marginal and joint sensitivities of the *NAV* to the various risks embedded in our proxies. Even more interesting for the risk managers, it is possible to assess a complete sensitivity study directly on the *SR* of the company, which is very difficult to compute without using an approximation tool (see Figures 2.9 and 2.10).

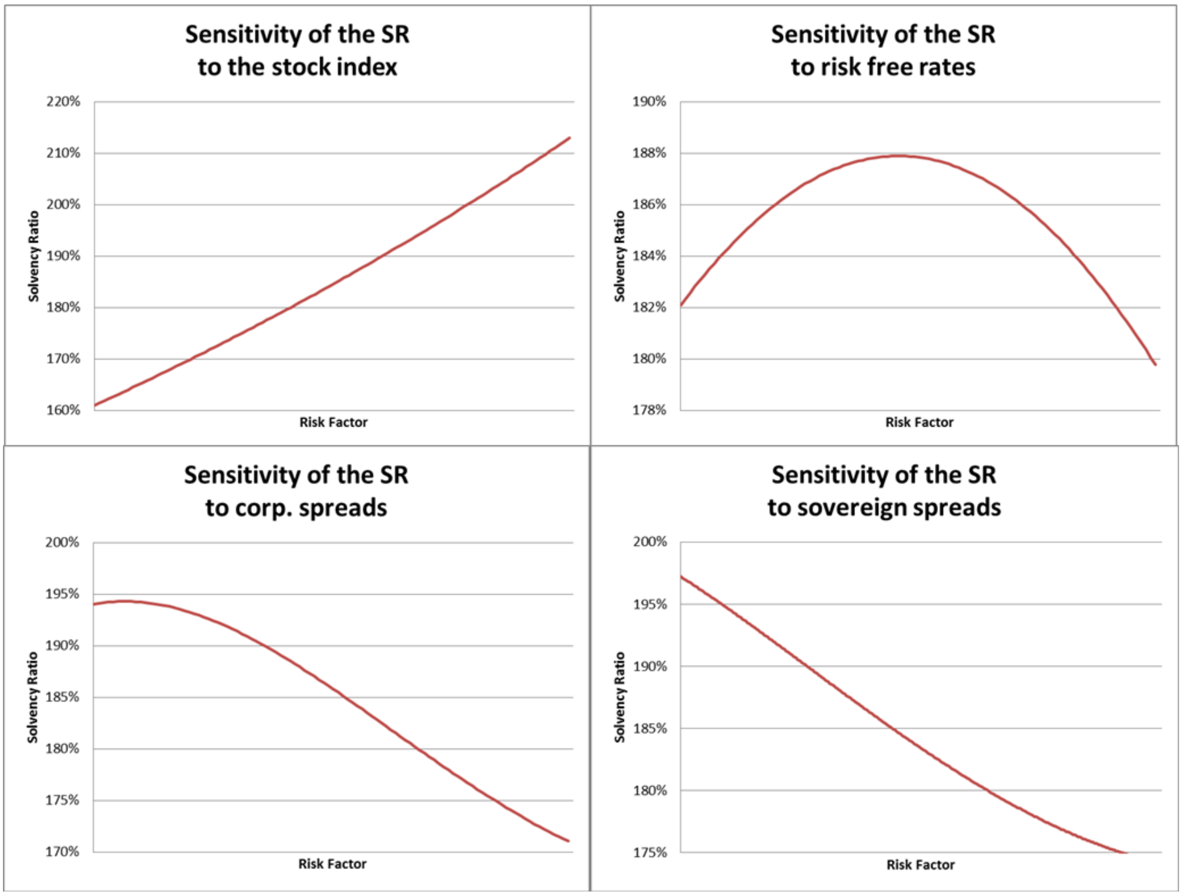

Figure 2.9 – 1D solvency ratio sensitivities.

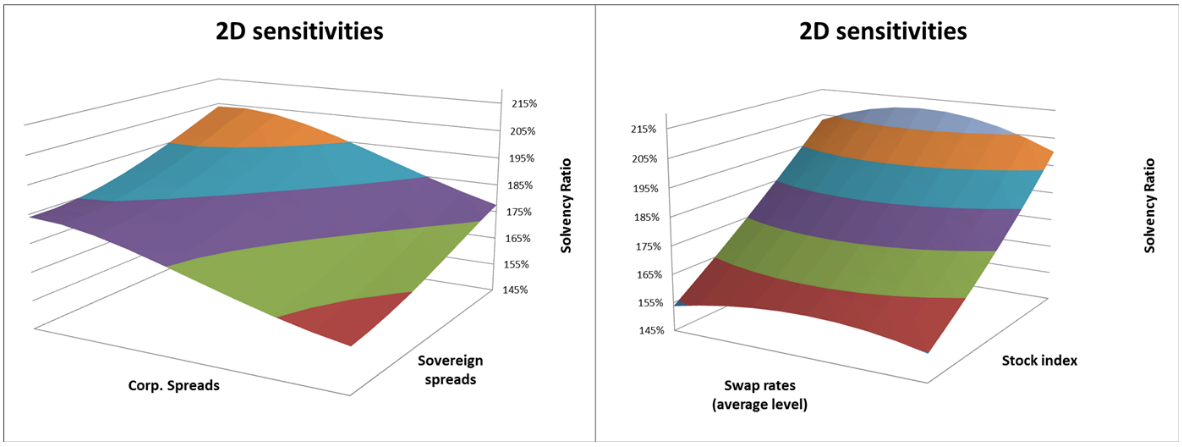

Figure 2.10 – 2D solvency ratio sensitivities.

This sensitivity analyzes needs no additional calculations to the proxies' assessment and enables the risk managers to compute as many "approximate" stress tests as needed. In practice such a use of the tool enables to gain better insight about the impact of each risk, taken either individually or jointly, on the *SR*.

Monitoring the marginal impacts of the risks and market anticipations.

Using our monitoring tool it is possible to trace the evolution of the *SR* risk after risk (only for the monitored risks). Figures 2.11 and 2.12 correspond to a fictitious evolution of the risks implemented between the calibration date and a "virtual" monitoring date).

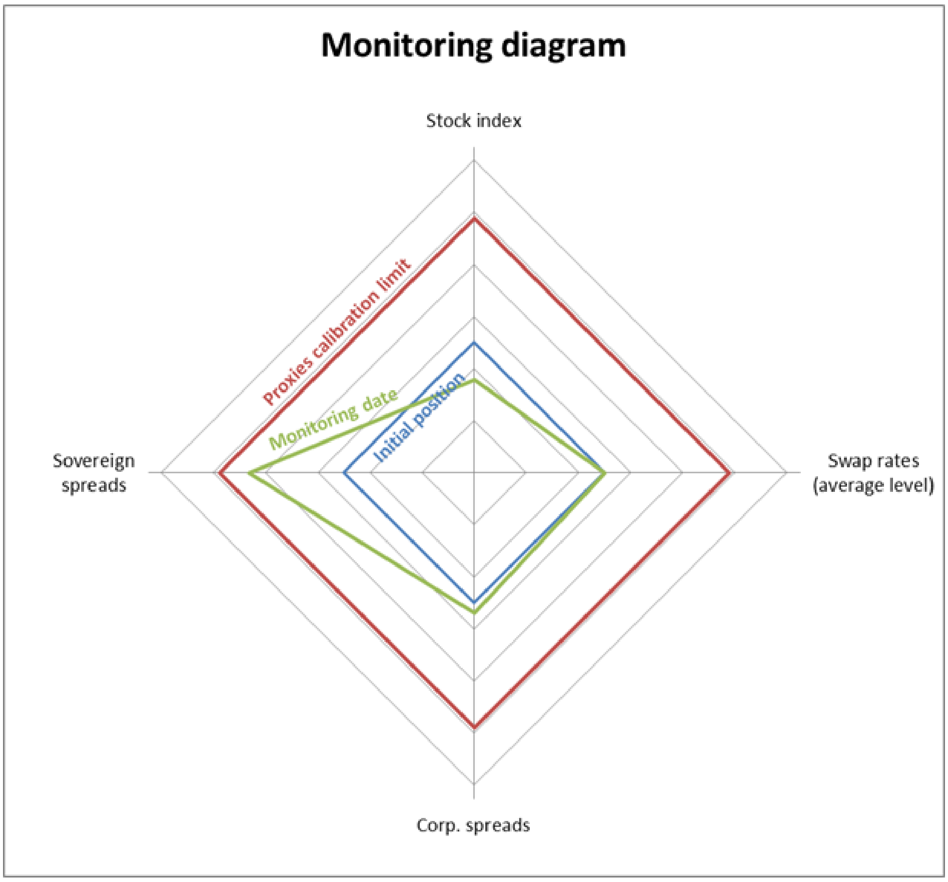

Figure 2.11 – Monitoring diagram after the fictitious evolution of the monitored risks.

Such a study can be run at each monitoring date, or on fictitious scenarios (*e.g.* market anticipations), in order to provide better insight about the *SR* movements through time.

Concerning market anticipations, if a risk manager anticipates a rise or a fall of the stocks / interest rates / spread, he can directly, through our tool, evaluate the corresponding impact on the undertaking's *SR*. In particular, such a study can be relevant to propose quantitative preventive measures.

In practice it also seems possible for the user to add asset-mix weights in the monitored risk factors set. This would enable the user to *a priori* test asset-mix re-balancing possibilities in order to select specific preventive measures and prevent the worse market anticipation. This implementation has not been done yet but will be part of the major future developments of the monitoring tool.

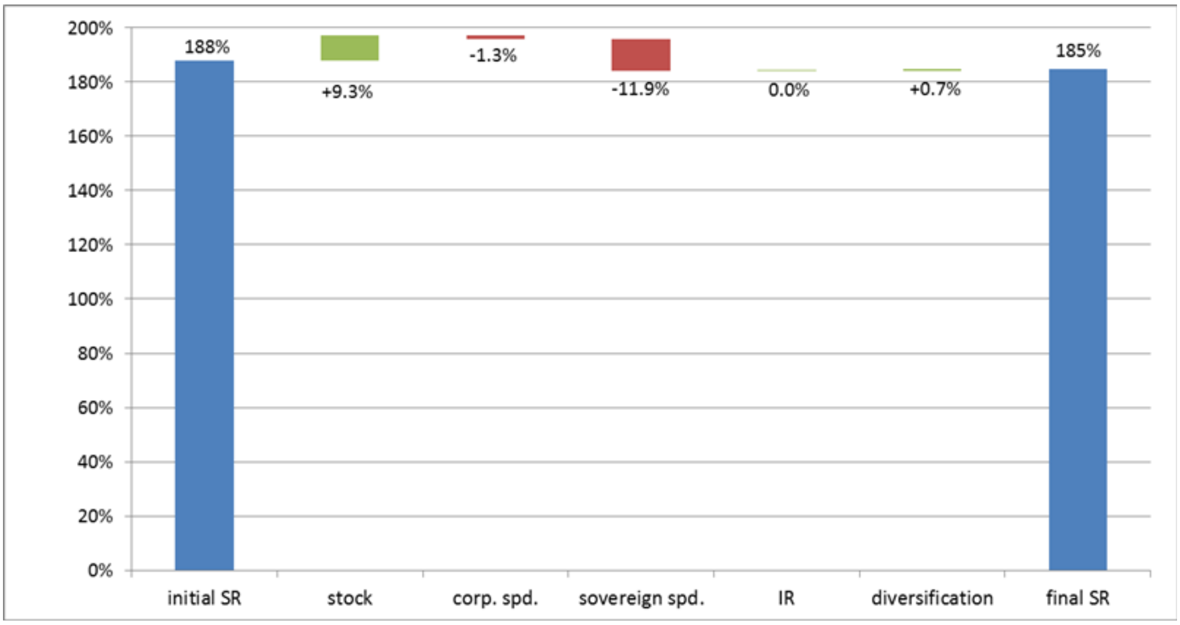

Figure 2.12 – Marginal impact of the risks on the *SR* between the calibration date and a "virtual" monitoring date.

Conclusion

The continuous compliance requirement is a key strategic issue for European insurers. In this article, we have presented the various characteristics of this problem from a practical point of view and provided a monitoring tool to answer it. Our monitoring scheme is based on the implementation of parametric proxies, already used among the insurance players to project the Net Asset Value over time, adapted to fit the ORSA continuous compliance requirements. This pragmatic tool has been implemented on a realistic life insurance portfolio to present the main features of both the development and the use of the monitoring scheme. In particular, several other alternative risk management uses for the tool are proposed.

Note that the monitoring tool only provides approximate values and is based on assumptions that can be discussed. The authors notice that the modeling choices can lead to errors. In particular we can only advice the future users of our tool to update the proxies frequently in order to make sure that the underlying stability assumptions are reasonable. One of the future axes to investigate is clearly to aim at a better control of the error and to address in depth the issue of the proxies recalibration frequency.

In addition the possibility to add asset-mix weights in the monitored risk factors set should be tested. This would greatly help asset managers to select optimal asset-mixes, consistently with the risk strategy of the undertaking.

Eventually we intend to investigate the issue of the calculation error due to the use of proxies in Life insurance projections and the various possibilities provided by the financial and econometric theories to optimize the proxies calibration process, in order to increase the estimators convergence speed and decrease the heteroskedasticity of our models.

Acknowledgement

The authors would like to address special thanks to Fabien Conneau, Laurent Devineau and Christophe Vallet for their help all along the redaction of this paper.

Moreover, we would like to extend our thanks to all the employees of Milliman Paris, and in particular the members of the R&D team.

Bibliography

Algorithmics (2011), *Curve Fitting for Calculating Solvency Capital Requirements under Solvency II: Practical insights and best practices from leading European Insurers*.

Barrie & Hibbert (2011), *A Least Squares Monte Carlo approach to liability proxy modelling and capital calculation*.

Bonnin, F., Planchet, F. and Juillard, M. (2014), 'Best estimate calculations of savings contracts by closed formulas: application to the orsa', *European Actuarial Journal* **4**(1), 181–196.

Broadie, M., Du, Y. and Moallemi, C. C. (2011), 'Efficient risk estimation via nested sequential simulation', *Management Science* **57**(6), 1172–1194.

Devineau, L. (2011), *La méthode des simulations dans les simulations. Mise en œuvre d'un modèle actif / passif en assurance-vie : quelles techniques ?, Part 2.*, Milliman. slides.

Devineau, L. and Chauvigny, M. (2011), 'Replicating portfolios: calibration techniques for the calculation of the Solvency II economic capital', *Bulletin Français d'Actuariat* **11**(21), 59–97.

Devineau, L. and Loisel, S. (2009), 'Risk aggregation in Solvency II: How to converge the approaches of the internal models and those of the standard formula?', *Bulletin Français d'Actuariat* **9**(18), 107–145.

Diebold, F. X. and Li, C. (2006), 'Forecasting the term structure of government bond yields', *Journal of Econometrics* **130**(2), 337–364.

Draper, N. R., Smith, H. and Pownell, E. (1966), *Applied regression analysis*, Wiley New York.

EIOPA (2010), 'QIS5 technical specifications'.

European Commission (2009), *Solvency II - EC directive 2009/138*.

European Commission (2011), 'Draft Implementing measures Solvency II'.

Hastie, T., Tibshirani, R. and Friedman, J. (2009), 'The elements of statistical learning'.

Jarrow, R. A., Lando, D. and Turnbull, S. M. (1997), 'A Markov model for the term structure of credit risk spreads', *Review of Financial Studies* **10**(2), 481–523.

Loisel, S. and Gerber, H.-U. (2012), Why ruin theory should be of interest for insurance practitioners and risk managers nowadays, *in* 'Proceedings of the AFMATH Conference 2012', pp. 17–21.

Longstaff, F. A., Mithal, S. and Neis, E. (2005), 'Corporate yield spreads: Default risk or liquidity? new evidence from the credit default swap market', *The Journal of Finance* **60**(5), 2213–2253.

Lundberg, F. (1903), *I. Approximerad framställning af sannolikhetsfunktionen. II. Återförsäkring af kollektivrisker.*, Almqvist och Wiksells boktryckeri.

Pentikäinen, T. (1982), *Solvency of insurers and equalization reserves. Volume I: General aspects*, Helsinki: Insurance Publishing Company Ltd.

Pentikäinen, T., Bonsdorff, H., Pesonen, M., Rantala, J. and Ruohonen, M. (1989), *Insurance solvency and financial strength*, Finnish Insurance Training and Publishing Company Helsinki.

Petauton, P. (2002), *Théorie et pratique des opérations de l'assurance vie*, Dunod, Paris.

Saporta, G. (2006), *Probabilités, analyses des données et statistiques*, editions Technip.

Tosetti, A., Paris, B., Paslky, P. and Le Vallois, F. (2003), *Gestion Actif Passif en Assurance vie: réglementation, outils, méthodes*, Économica.

Vedani, J. and Devineau, L. (2013), 'Solvency assessment within the ORSA framework: issues and quantitative methodologies', *Bulletin Français d'Actuariat* **13**(25), 35–71.

Vedani, J. and Virepinte, P.-A. (2011), 'Modèle structurel de crédit dans une optique assurantielle'.

Chapitre 3

Economic balance sheet proxies improvement: variance reduction and metamodel approach

Abstract

The use of parametric proxies (Replicating Portfolios, Curve Fitting, Least Squares Monte Carlo), for the economic values' distributions projection through time, is becoming more and more standard on the European life insurance market. These methodologies can either be used for 1-year net asset values projections in order to assess the Solvency Capital Requirement or, considering either a shorter or longer time horizon, for Own Risk and Solvency Assessment purposes. Through the calibration, specification assumptions, and convergence speed of their estimators, these methodologies embed several sources of error that can lead to Solvency Capital Requirement misestimations. To tackle this issue, *a posteriori* tests, on out of sample scenarios, are available but these validation processes display important weaknesses. They can only allow a partial validation of the proxies.

In this paper we provide and implement tools and methodologies, first to get an optimized proxy, through variance reduction techniques (leading to a reduction of the estimator volatility of more than 70%), and second to help practitioners managing their validation issue through a conservative approach, using a metamodel adjustment. This methodology enables a reduction of the error made on the assessed extreme quantile (from 18% to less than 3%) and a global improvement of the risk profile given by the proxies.

Finally, we address the proxy calibration issue. We present and justify the use of uniform meshes as a prudent best practice for any proxy calibration.

Keywords : *Solvency II, parametric proxy, Least Squares Monte Carlo, Curve Fitting, Replicating Portfolio, metamodel.*

Introduction

In this paper we consider different tools to improve the proxy methodologies used by numerous European insurance undertakings in order to fasten the projection of economic own funds distributions through time.

The projection of life insurance net asset values (*NAV*) is one of the major and trickiest operational requirement induced by the Solvency II Directive implementation. For instance, the calculation of the Solvency Capital Requirement (*SCR*) in an Internal Model approach requires *NAV* projections at $t = 1$ year ; some methodologies implemented within the ORSA process (Own Risk and Solvency Assessment, see article 45 of the Solvency II - 2009/138/EC directive) can also require multi-year (see Vedani and Devineau (2013)) or instantly shocked (between $t = 0$ and $t = 0^+$) *NAV* projections.

These projections lead to complex operational issues due to their highly stochastic nature [*]. Indeed, the NAV_t (*NAV* at time $t \geq 0^+$), being based on numerical approaches, is very difficult to efficiently assess. Some closed formulas have been proposed in life insurance (see Bonnin et al. (2014) or Nteukam and Planchet (2012)) but they rely on strong model assumptions and do not reflect the general implementation framework considered by the European life insurance market. Actuarial practitioners can, however use a Monte Carlo approximation methodology, the Nested Simulations. This approach leads to convergent estimators of NAV_t but is greatly complex to implement and very time-consuming.

To accelerate this process, various other methodologies have been proposed, based on a direct acceleration of Nested Simulations (see Devineau and Loisel (2009*a*), Broadie et al. (2011)) or on statistical learning methods (see *e.g.* Martial and Garnier (2013)). But another possibility offered to life insurance, and currently more and more used by actuarial practitioners, is to consider proxy methodologies. These proxy methodologies quickly provide parametric estimators of NAV_t that can bring efficient information in terms of levels (and adversity), sensitivities and finally be used to simulate numerous approximated NAV_t outcomes. They are generally very similar (in their theory and in their implementation) and based on Ordinary Least Squares (OLS) estimated regressions. We, therefore, talk about parametric proxies.

The most crucial issue with the use of such proxies is their validation. The OLS estimations can be improved through different methods, including the variance reduction techniques presented, and mathematically justified, in this paper. However, even the best of proxies relies on a strong specification assumption and the validation and management of the proxies error are often achieved through expert judgements, a pragmatic balance between efficiency and time-operational consumption. In this paper, we propose a first tool to manage this strategic issue through the use of a metamodel approach. Complementary to proxy calibration (and improvement) this approach tries to model the modelling error.

In Section 3.1 we lay the foundations of our study and choose a standard proxy model, a well-known Least Squares Monte Carlo approach. In the following Section we stress the main directions for improved proxy estimation. We introduce the theoretical and operational basics of methodologies to

∗. These projections should, in particular, embed a large number of financial and technical risks (see Planchet et al. (2009)), complex Asset-Liability Management (ALM) rules (see Tosetti et al. (2003) or Fromenteau and Petauton (2012)).

successively improve the standard LSMC approach, based on two usual variance reduction techniques. These methodologies aim at increasing the convergence speed of the proxy's estimator. However these techniques do not correct the error linked with the specification assumption, the assumption on the proxy shape, and there are other sources of approximation. This issue is addressed in Section 3.3 where we develop a metamodeling methodology to estimate the residual modeling error, in the specific case of the Solvency Capital Requirement estimation. In Section 3.4 we implement a case study to test the techniques introduced in Sections 3.2 and 3.3. This Section is the opportunity to guide actuarial practitioners towards better proxy implementations and error management. In Section 3.5 we study a transversal but also a strategic issue. We consider the two main proxy calibration methodologies proposed on the life insurance market: uniform non-random meshes or random samples that follow true distributions. Is it justified to use non-random samples? Do these two approaches lead to similar asymptotic results? These questions are crucial as far as the practical use of any proxy is concerned, but they have never been addressed from the insurance operational practice point of view.

3.1 Life insurance Net Asset Value projections methodologies

The most accurate way to project NAV_t values is to consider a Nested Simulations methodology. This methodology provides Monte Carlo estimators which convergence results from the Law of Large Numbers, without any additional model specification assumption. The main issues with this approach are its implementation and time complexity. The use of proxies leads to quicker results but requires additional model assumptions.

3.1.1 Nested Simulations

Consider a typical Nested Simulation (NS) framework (see *e.g.* Vedani and Devineau (2013) or Devineau and Loisel (2009*a*)), using N economic Real World scenarios through horizon $[0,t]$ (with $t \geq 0^+$), then P random economic Risk Neutral scenarios through a given horizon $[t, t+H]$, to assess the Monte Carlo NAV_t estimators (see Figure 3.1).

Below, we denote by $R_u^{n,p}$ the profit cash-flow (resp. R_u), $\delta_u^{n,p}$ the discount factor (resp. δ_u), associated to primary scenario n and secondary scenario p (resp. the corresponding random variables). We denote by $\mathscr{Q}_t$ the risk-neutral measure under which the secondary scenarios are projected and by $\mathscr{F}_t^{RW}$ the filtration induced by the economic information projected between 0 and t. We have in particular

$$NAV_t = \mathbb{E}^{\mathscr{Q}_t}\left[\sum_{u\geq 1} \frac{\delta_u}{\delta_t} R_u | \mathscr{F}_t^{RW}\right] = \sum_{1\leq u\leq t} \frac{\delta_u}{\delta_t} R_u + \mathbb{E}^{\mathscr{Q}_t}\left[\sum_{u>t} \frac{\delta_u}{\delta_t} R_u | \mathscr{F}_t^{RW}\right].$$

The Nested Simulations procedure enables one to get Monte Carlo approximations of $\mathbb{E}^{\mathscr{Q}_t}\left[\sum_{u>t} \frac{\delta_u}{\delta_t} R_u | \mathscr{F}_t^{RW}\right]$,

$$\widehat{NAV}_t^n = \sum_{1\leq u\leq t} \frac{\delta_u^n}{\delta_t^n} R_u^n + \frac{1}{P} \sum_{p=1}^{P} \sum_{u=t+1}^{t+H} \frac{\delta_u^{n,p}}{\delta_t^n} R_u^{n,p}.$$

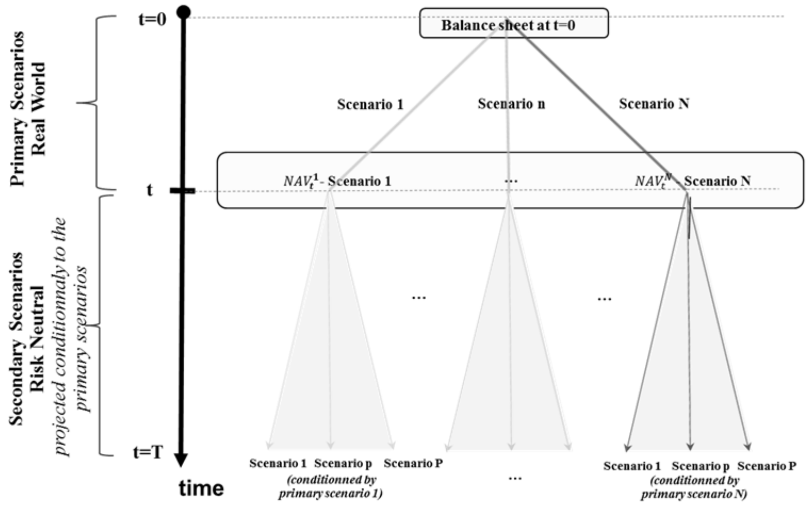

Figure 3.1 – Nested Simulations - Illustration (source: Milliman)

By definition we have, $\forall n \in [|1,N|]$, $\widehat{NAV}_t^n \underset{P\to+\infty}{\overset{a.s.}{\to}} NAV_t^n$ (strong law of large numbers).

The author note that the economic valuation scheme is not totally satisfactory from a theoretecal point of view (see *e.g.* El Karoui et al. (2015)) but we have chosen to address an empirical field and processes currently implemented, though improvable.

In the rest of the paper we denote by $\widehat{NAV}_t$ the random variable associated to the approximated *NAV* obtained using a Nested Simulations methodology. In addition we introduce the notion of Net Present Value of profit at time t, defined as $NPV_t = \sum_{u\geq 1} \frac{\delta_u}{\delta_t} R_u$, and their outcomes, associated to each primary scenario n and secondary scenario p, $NPV_t^{n,p} = \sum_{u=1}^{t} \frac{\delta_u^n}{\delta_t^n} R_u^n + \sum_{u=t+1}^{t+H} \frac{\delta_u^{n,p}}{\delta_t^n} R_u^{n,p}$.* In practice it is also useful to introduce NPV_t^n, a Net Present Value outcome associated to the n^{th} primary scenario and calculated considering a random secondary scenario.

In practice, H is chosen such as, at date $t+H$ the liabilities are extincted. We have, $NAV_t = \mathbb{E}^{\mathscr{Q}_t}\left[\sum_{u\geq 1} \frac{\delta_u}{\delta_t} R_u | \mathscr{F}_t^{RW}\right] = \mathbb{E}^{\mathscr{Q}_t}\left[\sum_{u=1}^{t+H} \frac{\delta_u}{\delta_t} R_u | \mathscr{F}_t^{RW}\right]$.

3.1.2 Parametric proxies – General mathematical formalization

We consider now multi-linear proxies that replicate the *NAV* in a straightforward fashion. Remind the *NAV* at time t depends on the economic information projected through $[0,t]$, $NAV_t = \mathbb{E}^{\mathscr{Q}_t}\left[\sum_{u=1}^{t+H} \frac{\delta_u}{\delta_t} R_u | \mathscr{F}_t^{RW}\right]$.

*. We have, in particular, $\widehat{NAV}_t^n = \frac{1}{P} \sum_{p=1}^{P} NPV_t^{n,p}$.

The idea of these proxy methodologies is to define a set of relevant variables that summarizes efficiently the economic information through $[0,t]$. Denote this variables set by $x_t = \left(1, {}^1x_t, {}^2x_t, ..., {}^Kx_t\right)$ for a given $K \in \mathbb{N}^*$. The objective is to have $\sigma(x_t) \approx \mathscr{F}_t^{RW}$. In this paper we assume $\sigma(x_t) = \mathscr{F}_t^{RW}$. *
This assumption legitimates the use, below, of notations $NAV_t(x_t)$ (resp. $\widehat{NAV}_t(x_t)$ and $NPV_t(x_t)$) for the value of variable NAV_t (resp. variables $\widehat{NAV}_t$, and NPV_t) associated to primary scenario x_t.

As far as parametric proxies used for life insurance *NAV* projections are concerned, we include three methodologies, the Curve Fitting (CF, see *e.g.* Algorithmics (2011)), the Least Squares Monte Carlo (LSMC, see Longstaff and Schwartz (2001) or Vedani and Devineau (2013)) and the Replicating Portfolio methodology (RP, see Devineau and Chauvigny (2011) or Kalberer (2012)). The general models associated to such methodologies are very similar and can be seen as a general regression, †

$y_t = x_t.\beta + u$ under the assumption $\mathbb{E}^{\mathscr{Q}_t}[y_t|x_t] = x_t.\beta$.

The main difference between these methodologies is the choice of the 1-dimension variable y_t and of the K-dimension vector x_t (though we still must verify $\sigma(x_t) = \mathscr{F}_t^{RW}$), the idea being to keep the equality $\mathbb{E}^{\mathscr{Q}_t}[y_t|x_t] = NAV_t(x_t)$.

Under the CF or LSMC methodologies x_t is composed of risk factors monomials (under the RP methodology, x_t is composed of market assets prices ‡

For the CF methodology, variable y_t is the Monte Carlo approximation of NAV_t obtained by a NS-like methodology, $\widehat{NAV}_t$. For the LSMC methodology, y_t is the Net Present Value of profits NPV_t. For the RP methodology y_t is either $\widehat{NAV}_t$ or NPV_t, depending on the used proxy methodology.

We can propose the following proxy methodologies description table:

∗. Which is often verified in practice.

†. When a vector of parameters is used in these proxies we always use the notation $\beta = \begin{pmatrix} {}^1\beta \\ {}^2\beta \\ \vdots \\ {}^K\beta \end{pmatrix}$

‡. Note that some methodologies tend to consider market assets cash-flows at times over the valuation date t (see *e.g.* Revelen (2009) or Schrager (2008)) as regressors, so that $\sigma(x_t)$ is very different of $\mathscr{F}_t$. As the reader can see below, we do not include these approaches due to specific theoretical and practical difficulties.

Proxy method	Multiple regression models (see *e.g.* Saporta (2006))	Model specification assumption
Curve Fitting	$\widehat{NAV}_t(x_t) = x_t.{}^{CF}\beta + {}^{CF}u$ where the ${}^i x_t$ ($i \in [\|1,K\|]$) are risk factor monomials.	$\mathbb{E}^{\mathcal{Q}_t}\left[\widehat{NAV}_t(x_t)\|x_t\right] = x_t.{}^{CF}\beta$
LSMC	$NPV_t(x_t) = x_t.{}^{LSMC}\beta + {}^{LSMC}u$ where the ${}^i x_t$ ($i \in [\|1,K\|]$) are risk factor monomials.	$\mathbb{E}^{\mathcal{Q}_t}\left[NPV_t(x_t)\|x_t\right] = x_t.{}^{LSMC}\beta$
Replicating Portfolio (v1)	$\widehat{NAV}_t(x_t) = x_t.{}^{RP1}\beta + {}^{RP1}u$ where the ${}^i x_t$ ($i \in [\|1,K\|]$) are simple assets prices.	$\mathbb{E}^{\mathcal{Q}_t}\left[\widehat{NAV}_t(x_t)\|x_t\right] = x_t.{}^{RP1}\beta$
Replicating Portfolio (v2)	$NPV_t(x_t) = x_t.{}^{RP2}\beta + {}^{RP2}u$ where the ${}^i x_t$ ($i \in [\|1,K\|]$) are simple assets prices or cash-flows, depending on the chosen approach.	$\mathbb{E}^{\mathcal{Q}_t}\left[NPV_t(x_t)\|x_t\right] = x_t.{}^{RP2}\beta$

Note that these models assume the strong exogenity of the covariates *. This assumption is generally admitted in practice. We discuss it in Section 3.5. Remark that if all the model specification assumptions are true we have clearly ${}^{CF}\beta = {}^{LSMC}\beta$ (see Vedani and Devineau (2013) and Kalberer (2012)) if the risk factor monomials considered in the CF and in the LSMC methods are the same, and ${}^{RP1}\beta = {}^{RP2}\beta$ if the assets considered in the RP (v1) and in the RP (v2) methods are the same (see for example Kalberer (2012) and Vedani and Devineau (2013)). In addition, note that under our assumption $\sigma(x_t) = \mathcal{F}_t^{RW}$ we have $\mathbb{E}^{\mathcal{Q}_t}\left[\widehat{NAV}_t(x_t)|x_t\right] = \mathbb{E}^{\mathcal{Q}_t}\left[NPV_t(x_t)|x_t\right] = NAV_t(x_t)$.

This mathematical formalization was important in order to underline the two error sources embedded within these proxy methods and their use. First, these proxies are calibrated on a finite number of outcomes, which leads to an error when approximating the vector of parameters. This is the parameters' error. It can lead to large deviations when trying to assess the proxy $NAV_t(x_t)$ value, $x_t.\beta$. The second error source lies in the model specification assumption, which can be quite strong in reality. We address both error sources in this paper.

As one can see, the mathematical formalization of all proxies methodologies is very similar, though they all have their own particularities, pros and cons, from a theoretical and practical point of view. In this paper, for the sake of simplicity, we have chosen to deal only with an LSMC proxy, a commonly used methodology. However the methods developed below can be easily extended and tested on other proxy methodologies. We consider the following proxy model:

$$\begin{cases} NPV_t(x_t) = x_t.\beta + u, \text{ where the } {}^i x_t \; (i \in [|1,K|]) \text{ are risk factors monomials,} \\ \text{under the model specification assumption } \mathbb{E}^{\mathcal{Q}_t}\left[NPV_t(x_t)|x_t\right] = x_t.\beta \end{cases} \quad . \textit{ model 1}$$

In order to simplify the notation we now omit the $\mathcal{Q}_t$ indexation.

$*$. The assumption that the projection of the explained variable on the explanatory filtration is linear in the explanatory variables (*strong exogeneity*). Instead of this assumption, it is weaker to assume that the explanatory variables and the residuals are decorrelated (*weak exogeneity*), but this assumption is rarely made in life insurance practices

3.1.3 Standard Ordinary Least Squares model

In practice, the model depicted above is estimated through the use of a great number of $(NPV_t(x_t), x_t)$ i.i.d. outcomes, say N, denoted by $\left(\left(NPV_t(x_t^n), x_t^n = \left(1, {}^1x_t^n, {}^2x_t^n, ..., {}^Kx_t^n\right)\right)\right)_{(n \in [|1,N|])}$.

We can re-write the model in its matricial, more econometric, form,

$$\begin{cases} Y = X.\beta + U, \\ \text{under the model specification assumption } \mathbb{E}[Y|X] = X.\beta \end{cases} \qquad \textit{model 1},$$

with $Y = \begin{pmatrix} y^1 = NPV_t(x_t^1) \\ \vdots \\ y^N = NPV_t(x_t^N) \end{pmatrix}, X = \begin{pmatrix} x_t^1 \\ \vdots \\ x_t^N \end{pmatrix}$

and

$$U = \begin{pmatrix} u^1 = y^1 - x_t^1.\beta \\ \vdots \\ u^N = y^N - x_t^N.\beta \end{pmatrix}.$$

Under the assumption (always true in practice) that $X'X$ is invertible (denoting by V' the transposition of a vector or matrix V), the Ordinary Least Squares (OLS) estimator of β is $\hat{\beta} = \left(X'X\right)^{-1} X'Y$. This estimator is unbiased and asymptotically convergent.

Adding the homoscedasticity assumptions ($\mathbb{V}[U|X] = \sigma_U^2.I_N$, denoting by I_N the rank N identity matrix), the OLS estimator is the Best Linear Unbiased Estimator (BLUE) of β, that is to say there exist no other linear estimator of β with a variance (conditionally on X) lower than the conditional variance of $\hat{\beta}$. This is what demonstrates the Gauss-Markov theorem (see Plackett (1950)). However, the homoscedasticity assumption is rarely observed in practice.

Note that we have, under our specification assumption and with I_{K+1} the identity matrix of dimension $K+1$,*

$$\mathbb{V}\left[\hat{\beta}|X\right]^{-\frac{1}{2}} \left(\beta - \hat{\beta}\right) \longrightarrow^d \mathcal{N}(0, I_{K+1}) \text{ where } \mathbb{V}\left[\hat{\beta}|X\right] = \left(X'X\right)^{-1} X'\mathbb{V}[U|X]X\left(X'X\right)^{-1}.$$

3.2 Increasing the estimator convergence speed

In this section, we try to optimize the proxy calibration by increasing the estimator's convergence speed at best, always keeping the same simulation budget. The proposed methodologies are useful as

*. Denoting by $M^{\frac{1}{2}}$ the square root of any symetric positive definite matrix M (such as $M^{\frac{1}{2}} * M^{\frac{1}{2}'} = M$)

far as any sort of proxy (RP, CF or LSMC) is used.

Note that these methodologies have already been proposed for operational LSMC implementation purpose (see *e.g.* Barrie & Hibbert (2007) or Barrie & Hibbert (2011)), with no theoretical justification. We aim here to theoretically analyze their efficiency.

3.2.1 Antithetic variables

The antithetic variables (AV) methodology is a well-known variance reduction approach, often used in finance and simulation in general (see *e.g.* Paskov and Traub (1995) or Glasserman (2003)). The idea here is to speed up the Monte Carlo estimation of a variable mean by considering an alternative variable with same mean but lower variance.

Application to $\widehat{NAV}_t$ estimations

We have, $\forall n \in [|1,N|]$, $y^n = NPV_t(x_t^n)$. The studied variable is, therefore, NPV_t. This random variable is obtained based on one primary scenario, which information can be summarized in variable x_t (leading to the x_t^n outcomes), but also thanks to a unique secondary scenario. Let ε be the set of (secondary) risk factors embedded in this scenario. ε is assumed to be fully known. We have $NPV_t(x_t) \in \sigma(x_t, \varepsilon)$, but neither $NPV_t(x_t) \in \sigma(x_t)$ nor $NPV_t(x_t) \in \sigma(\varepsilon)$.

Note that, in the economic scenarios generators used to simulate the secondary scenarios, it is rather easy to extract ε, which we can easily assume to be a Gaussian vector (composed with the Wiener processes increments used to project the economic drivers - stock indexes, Zero-Coupon prices,... - through the projection horizon $[t, t+H]$) without loss of generality. We can write, considering a given x_t outcome / scenario, $NPV_t(x_t, \varepsilon)$ to denote the NPV_t value associated to primary scenario x_t and secondary scenario ε.

We can now propose a first adaptation of the AV methodology in our framework. Indeed, consider a new studied variable equal to the mean between $NPV_t(x_t, \varepsilon)$ and $NPV_t(x_t, -\varepsilon)$ *

$$z_t = \frac{(NPV_t(x_t,\varepsilon) + NPV_t(x_t,-\varepsilon))}{2}.$$

In practice we often verify that this new variable has a lower variance than an average of two NPV_t calculated conditionally on the same x_t, but considering two random ε outcomes, say ${}_1\varepsilon$ and ${}_2\varepsilon$. The idea here is that, once we assume $CoVar[NPV_t(x_t, \varepsilon), NPV_t(x_t, -\varepsilon)|x_t] < 0$, we have

$$\mathbb{V}[z_t|x_t] = \frac{(\mathbb{V}[NPV_t(x_t,\varepsilon)|x_t] + \mathbb{V}[NPV_t(x_t,-\varepsilon)|x_t])}{4} + \frac{1}{2} CoVar[NPV_t(x_t,\varepsilon), NPV_t(x_t,-\varepsilon)|x_t]$$

$$\mathbb{V}[z_t|x_t] < \frac{(\mathbb{V}[NPV_t(x_t,\varepsilon)|x_t] + \mathbb{V}[NPV_t(x_t,-\varepsilon)|x_t])}{4} = \frac{\mathbb{V}[NPV_t(x_t,\varepsilon)|x_t]}{2}$$

*. It is noticeable that ε and $-\varepsilon$ are assumed to follow the same distribution, which is true when ε only embeds the simulated financial drivers' Brownian motion increments (as in most practical cases).

with still

$$\mathbb{E}[z_t|x_t] = \frac{1}{2}(\mathbb{E}[NPV_t(x_t,\varepsilon)|x_t] + \mathbb{E}[NPV_t(x_t,-\varepsilon)|x_t]) = \mathbb{E}[NPV_t(x_t)|x_t] = NAV_t(x_t).$$

So, if we want to estimate $NAV_t(x_t)$ it is more efficient to consider the Monte Carlo estimator calculated as the average of J random z_t outcomes than the Monte Carlo estimator calculated on $2J$ random $NPV_t(x_t)$ outcomes. The first estimator being less volatile than the second, it converges faster.

Use of AV to speed up the proxies' estimators convergence

Now consider two LSMC models, the one already presented:

$$\begin{cases} NPV_t(x_t) = x_t.{}_1\beta + u, \text{ where the } {}_ix_t \ (i \in [|1,K|]) \text{ are risk factors monomials,} \\ \text{under the model specification assumption } \mathbb{E}[NPV_t(x_t)|x_t] = x_t.{}_1\beta \end{cases} \quad \textit{model OLS std}$$

and an AV-adapted one:

$$\begin{cases} z_t = x_t.{}_2\beta + v, \text{ where the } {}_ix_t \ (i \in [|1,K|]) \text{ are risk factors monomials,} \\ \text{under the model specification assumption } \mathbb{E}[z_t|x_t] = x_t.{}_2\beta \end{cases} \quad \textit{model AV-adapted.}$$

Note first that under our model specification assumptions, the true parameters' sets are equal, ${}_1\beta = {}_2\beta = \beta$. The demonstration of this result is almost immediate (see Kalberer (2012) or Vedani and Devineau (2013)), due to the fact that $\mathbb{E}[NPV_t|x_t] = \mathbb{E}[z_t|x_t] = NAV_t(x_t)$. *model AV-adapted* is a simple adaptation of the standard OLS framework, sometimes proposed in operational literature with lower theoretical justification.

Assume now the use of N (with N even) $((NPV_t(x_t^n), x_t^n))_{n\in[|1,N|]}$ i.i.d. outcomes to calibrate *model OLS std*, and N $((z^n, x_t^n))_{n\in[|1,N|]}$ i.i.d. outcomes, with the same $(x_t^n)_{n\in[|1,N|]}$, to calibrate *model AV-adapted*). In terms of ALM simulation budget, the *model AV-adapted* calibration is twice more consuming than the *model OLS std* calibration. But it is questionable whether the convergence of the estimator is twice faster (or more).

The matricial models are

$$\begin{cases} Y = X.\beta + U, \\ \text{under the model specification assumption } \mathbb{E}[Y|X] = X.\beta \end{cases} \quad \textit{model OLS std}$$

with $Y = \begin{pmatrix} y^1 = NPV_t(x_t^1) \\ \vdots \\ y^N = NPV_t(x_t^N) \end{pmatrix}$, $X = \begin{pmatrix} x_t^1 \\ \vdots \\ x_t^N \end{pmatrix}$ and

$$U = \begin{pmatrix} u^1 = y^1 - x_t^1.\beta \\ \vdots \\ u^N = y^N - x_t^N.\beta \end{pmatrix}$$

and

$$\begin{cases} Z = X.\beta + V, \\ \text{under the model specification assumption } \mathbb{E}[Z|X] = X.\beta \end{cases} \quad \textit{model AV-adapted}$$

with

$$Z = \begin{pmatrix} z^1 \\ \vdots \\ z^N \end{pmatrix}, \text{ and } V = \begin{pmatrix} v^1 = z^1 - x_t^1.\beta \\ \vdots \\ v^N = z^N - x_t^N.\beta \end{pmatrix}.$$

Let us now compare the conditional variances of the two OLS estimators of β obtained respectively from model OLS std and *model AV-adapted*, ${}_1\hat{\beta} = \left(X'X\right)^{-1} X'Y$ and ${}_2\hat{\beta} = \left(X'X\right)^{-1} X'Z$

with

$$\begin{cases} \mathbb{V}\left[{}_1\hat{\beta}|X\right] = \left(X'X\right)^{-1} X'\mathbb{V}[U|X]X'\left(X'X\right)^{-1} = \left(X'X\right)^{-1} X'\mathbb{V}[Y|X]X'\left(X'X\right)^{-1} \text{ and} \\ \mathbb{V}\left[{}_2\hat{\beta}|X\right] = \left(X'X\right)^{-1} X'\mathbb{V}[V|X]X'\left(X'X\right)^{-1} = \left(X'X\right)^{-1} X'\mathbb{V}[Z|X]X'\left(X'X\right)^{-1}. \end{cases}$$

Remind that we have $\mathbb{V}[z_t|x_t] < \frac{\mathbb{V}[NPV_t(x_t,{}_1\varepsilon)|x_t]}{2}$, so that $\mathbb{V}[Z|X] < \frac{V[Y|X]}{2}$ (considering the partial order on the Hilbertian matrices as presented *e.g.* in Horn and Johnson (1990)).

This theoretical computation proves that doubling the simulation budget and using the AV technique leads to a division of the OLS estimators variance by more than two* $\mathbb{V}\left[{}_2\hat{\beta}|X\right] < \frac{1}{2}\mathbb{V}\left[{}_2\hat{\beta}|X\right]$.

To get the same simulation budget as when implementing model OLS std, we now divide the number of ALM simulations used to calibrate *model AV-adapted* by two. Under the assumption that $\mathbb{V}[z_t|x_t] < \frac{\mathbb{V}[NPV_t(x_t,{}_1\varepsilon)|x_t]}{2}$, we should still increase the convergence of our estimator when considering *model AV-adapted*. In practice, the AV adaptation generally enables proxy users to significantly increase the convergence speed of their estimators or, similarly, to decrease the simulation budget while keeping the same calibration efficiency.

*. This result also requires the (empirically verified) assumption that $\left(X'X\right)^{-1} X'$ is full ranked (see Horn and Johnson (1990)).

3.2.2 Control variate

Control variate (CV) is another variance reduction methodology, very well-known in financial engineering (see *e.g.* Hull and White (1988) or Glasserman (2003)). It can be easily adapted to our proxy methodologies.

Application to $\widehat{NAV}_t(x_t)$ estimations

We still want to estimate the NAV_t value associated to a given x_t. In order to achieve this estimation we use J outcomes of variable $NPV_t(x_t)$, $\left(\left(NPV_t^j(x_t)\right)\right)_{j\in[|1,J|]}$. The standard Monte Carlo estimator is $\frac{1}{J}\sum_j NPV_t^j(x_t)$. The idea of the CV methodology is to decrease the volatility of this estimator by using an alternative underlying variable with same mean but lower variance than $NPV_t(x_t)$.

Assume we know how to jointly project $NPV_t(x_t)$ and a new variable ζ_t outcomes, so that we get the joint outcomes $\left(\left(\zeta_t^j, NPV_t^j(x_t)\right)\right)_{j\in[|1,J|]}$. ζ_t should be a variable such that $CoVar(\zeta_t, NPV_t(x_t)|x_t) \neq 0$ and such that we know easily the value of $\mathbb{E}[\zeta_t|x_t]$. The variable used in the CV methodology is not $NPV_t(x_t)$ but $NPV_t(x_t) + c(\zeta_t - \mathbb{E}[\zeta_t|x_t])$, for a well-chosen c parameter.

The conditional mean and variance of this variable are

$$\begin{cases}\mathbb{E}[NPV_t(x_t) + c(\zeta_t - \mathbb{E}[\zeta_t|x_t])|x_t] = \mathbb{E}[NPV_t(x_t)|x_t] = NAV_t(x_t) \text{ and} \\ \mathbb{V}[NPV_t(x_t) + c(\zeta_t - \mathbb{E}[\zeta_t|x_t])|x_t] = \mathbb{V}[NPV_t(x_t)|x_t] + c^2\mathbb{V}[\zeta_t|x_t] + 2c.CoVar[NPV_t(x_t), \zeta_t|x_t].\end{cases}$$

In particular, this variance is minimal for $c^*(x_t) = -\frac{CoVar[NPV_t(x_t),\zeta_t|x_t]}{\mathbb{V}[\zeta_t|x_t]}$ and lower than $\mathbb{V}[NPV_t(x_t)|x_t]$ as soon as $CoVar[NPV_t(x_t), \zeta_t|x_t] \neq 0$.

Assume now that we know the real value of $c^*(x_t)$. The $\left(NPV_t^j(x_t) + c^*(x_t)\left(\zeta_t^j - \mathbb{E}[\zeta_t|x_t]\right)\right)_{j\in[|1,J|]}$ i.i.d. outcomes enable us to build the estimator $\frac{1}{J}\sum_j NPV_t^j(x_t) + c^*(x_t)\left(\zeta_t^j - \mathbb{E}[\zeta_t|x_t]\right)$, we obtain a more efficient estimator of $NAV_t(x_t)$ than the standard Monte-Carlo one, due to its lower variance,

$$\mathbb{V}\left[\frac{1}{J}\sum_j\left(NPV_t^j(x_t) + c^*(x_t)\left(\zeta_t^j - \mathbb{E}[\zeta_t|x_t]\right)\right)|x_t\right] = \mathbb{V}\left[\frac{1}{J}\sum_j NPV_t^j(x_t)|x_t\right] - \frac{CoVar[NPV_t,\zeta_t|x_t]^2}{J.\mathbb{V}[\zeta_t|x_t]},$$

$\mathbb{V}\left[\frac{1}{J}\sum_j NPV_t^j(x_t)|x_t\right]$ being the variance of the standard Monte Carlo estimator of $NAV_t(x_t)$.

This subsection states the CV theory through a practical example fully embedded in our life insurance scheme. The great difficulties of this process are clearly first to choose an efficient variable ζ_t, both easily calculated and highly (positively or negatively) correlated with variable $NPV_t(x_t)$, and second to assess $c^*(x_t)$ (especially when one considers different potential x_t outcomes) and $\mathbb{E}[\zeta_t|x_t]$, which is sometimes very complex. In particula, a wrong estimation of $c^*(x_t)$ can lead to less efficient estimators.

Let us now adapt the methodology for our proxy framework.

Use to speed up our proxies' estimators convergence

Similarly to Subsection 3.2.1 for the antithetic variables method, we now consider two model, *model 1* (or also *model OLS std*),

$$\begin{cases} NPV_t(x_t) = x_t.{}_1\beta + u, \text{ where the } x_t^i \, (i \in [|1,K|]) \text{ are risk factors monomials,} \\ \text{under the model specification assumption } \mathbb{E}\left[NPV_t(x_t)|x_t\right] = x_t.{}_1\beta \end{cases} \quad \textit{model OLS std},$$

and its CV adaptation,

$$\begin{cases} \widetilde{NPV}_t(x_t) = x_t.{}_3\beta + w, \text{ where the } x_t^i \, (i \in [|1,K|]) \text{ are risk factors monomials,} \\ \text{under the model specification assumption } \mathbb{E}\left[\widetilde{NPV}_t(x_t)|x_t\right] = x_t.{}_3\beta \end{cases} \quad \textit{model CV-adapted},$$

with $\widetilde{NPV}_t(x_t) = NPV_t(x_t) + c^*\left(\zeta_t - \mathbb{E}\left[\zeta_t|x_t\right]\right)$ (we still assume variable ζ_t to be already chosen, highly correlated with NPV_t and we assume functional $c^*(x_t)$, or at least a good estimator, to be known). Note that under our two specification assumptions, we have ${}_1\beta = {}_3\beta = \beta$.

These two models being estimated thanks to N NPV_t / $\widetilde{NPV}_t$ i.i.d. outcomes, respectively $\left(NPV_t(x_t^n)\right)_{n\in[|1,N|]}$ and $\left(\widetilde{NPV}_t(x_t^n)\right)_{n\in[|1,N|]}$, this enables to consider their matricial adaptations,

$$\begin{cases} Y = X.\beta + U, \\ \text{under the model specification assumption } \mathbb{E}\left[Y|X\right] = X.\beta \end{cases} \quad \textit{model 1 / model OLS std}$$

and

$$\begin{cases} \tilde{Y} = X.\beta + W, \\ \text{under the model specification assumption } \mathbb{E}\left[\tilde{Y}|X\right] = X.\beta \end{cases} \quad \textit{model CV-adapted},$$

with $\tilde{Y} = \begin{pmatrix} \tilde{y}^1 = \widetilde{NPV}_t(x_t^1) \\ \vdots \\ \tilde{y}^N = \widetilde{NPV}_t(x_t^N) \end{pmatrix}$

and

$$W = \begin{pmatrix} w^1 = \tilde{y}^1 - x_t^1.\beta \\ \vdots \\ w^N = \tilde{y}^N - x_t^N.\beta \end{pmatrix}.$$

Let us now compare the conditional variances of the two OLS estimators of *model OLS std* and *model CV-adapted*, respectively ${}_1\hat{\beta}\ \left(= \left(X'X\right)^{-1} X'Y\right)$ and ${}^3\hat{\beta}\ \left(= \left(X'X\right)^{-1} X'\tilde{Y}\right)$ where

$$\begin{cases} \mathbb{V}\left[{}_1\hat{\beta}|X\right] = \left(X'X\right)^{-1} X'\mathbb{V}[U|X]X'\left(X'X\right)^{-1} = \left(X'X\right)^{-1} X'\mathbb{V}[Y|X]X'\left(X'X\right)^{-1} \text{ and} \\ \mathbb{V}\left[{}_3\hat{\beta}|X\right] = \left(X'X\right)^{-1} X'\mathbb{V}[W|X]X'\left(X'X\right)^{-1} = \left(X'X\right)^{-1} X'\mathbb{V}\left[\tilde{Y}|X\right]X'\left(X'X\right)^{-1} \end{cases}$$

In particular, remind from Subsection 3.2.2 that $\mathbb{V}[\tilde{y}_t|x_t] < \mathbb{V}[y_t|x_t]$ and $\mathbb{V}\left[\tilde{Y}|X\right] < \mathbb{V}[Y|X]$ (still according to the partial order on the Hilbertian matrices). Finally, assuming $\left(X'X\right)^{-1} X'$ is full ranked, we have $\mathbb{V}\left[{}_3\hat{\beta}|X\right] < \mathbb{V}\left[{}_1\hat{\beta}|X\right]$ and the use of *model CV-adapted* leads to a more efficient estimator than *model OLS std*.

The two variance reduction methodologies presented in Subsections 3.2.1 and 3.2.2 can separately, under small additional assumptions (generally verified in practice, in a life insurance framework), provide OLS estimators that converge quicker towards the true parameters than the standard OLS approach. These two methodologies can also be easily jointly implemented to try and produce an improved methodology. However, even using both approaches, the final proxy still suffers from estimation errors due to both the fact that the true β stays unknown and that the model may not be totally well-specified. In Section 3.3, we address this proxy error issue by proposing a metamodel approach to *model the modeling error*. This methodology is relevant as far as any determination of NAV_t quantiles is concerned (which is almost always the case, be it for the Solvency Capital Requirement calculation or for ORSA uses).

3.3 Metamodeling

One often assimilates metamodeling to stochastic kriging implementation techniques (see *e.g.* Ankenman et al. (2010), Liu and Staum (2010) or Ribereau and Rullière (2011)). In a more general scheme, metamodels aim at providing a response function in a limited number of reference points (design points) in order to fit it using a stochastic model, enabling *in fine* to provide an improved prediction.

Nevertheless, the stochastic models' structures developed through the literature is too elementary to display efficiently the complexity of a NAV response function. Indeed, these models generally require the implementation of a maximum likelihood calibration and the construction of predictors minimizing a mean square error, which is very useful but quite simple in our framework.

This Section is freely inspired by the work of Kalberer (2012).

3.3.1 Conceptualization

We denote below $Res(x_t) = NAV_t(x_t) - B(x_t)$, the modeling error associated to a primary scenario x_t (recall that we assume, from Section 3.1 on, that $\sigma(x_t) = \mathscr{F}_t$), seen as the difference between the real NAV_t value associated to scenario x_t, and $B(x_t)$ the corresponding value estimated with the calibrated improved proxy (using *e.g.* an AV then CV-adapted model).

The aim of our proxy is to obtain a result based on an almost sure equality

$NAV(x_t) =^{a.s.} B(x_t)$.

The issue is that the proxy leads to a model error. The equality is actually

$NAV_t(x_t) = B(x_t) + Res(x_t)$.

But the estimation of the $Res(x_t)$ distribution is hardly feasible without very strong assumptions. The idea of our metamodel is to control this issue and estimate the law of $Res(x_t)$ knowing $B(x_t)$, a variable denoted below by $Res(B(x_t))$. This enables us, not to build an almost sure equality, but an equality *in distribution*,

$NAV_t(x_t) =^{d} B(x_t) + Res(B(x_t))$.

Once the distribution of $Res(B(x_t))$ is estimated, it is easy to assess the α-quantile of $B(x_t) + Res(B(x_t))$, equal to the α-quantile of $NAV_t(x_t)$, we aimed at estimating precisely. We must note two important points concerning this methodology.

First, it directly addresses the model specification assumption issue, regarding the proxy definition and calibration. This metamodel is mainly specification assumption-free, which is maybe the major advantage of this approach. However this approach has its own limits: this model specification assumption is transformed, through the use of *metamodel-corrected proxy method*, into a distribution assumption for $Res(B(x_t))$. This new assumption is weaker, can be tested, and may, as presented in Section 3.4, not be that necessary for the method to provide efficient results.

Second, by using this approach we switch from an *almost sure* to an *in distribution* characterization. In particular, the final *in distribution* result keeps the quantile values but we lose the link between a primary scenario (or an x_t outcome) and the associated NAV_t values. This aspect can be relevant as far as risk management is concerned and our methodology alone, without additional developments, is of little interest for such practices.

3.3.2 Estimation of variable $Res\left(x_t\right)$ knowing $B\left(x_t\right)$ distribution

As abovementionned the metamodeling implementation requires a distribution assumption for variable $Res\left(x_t\right)$, given $B\left(x_t\right)$ (variable $Res\left(B\left(x_t\right)\right)$). If the assumed distribution is highly parameterized the distribution assumption is weaker but the estimation of these parameters gets quite complicated. For the sake of simplicity, we assume that variable $Res\left(x_t\right)$ given $B\left(x_t\right)$ follows a Gaussian distribution. The evident advantages of this distribution are the fact that it requires only two parameters (mean and variance) and that it is defined on $\mathbb{R}$.* Besides, we propose below a justification for this assumption choice.

We now have to estimate functions $\mu\left(B\left(x_t\right)\right)$ (mean of $Res\left(B\left(x_t\right)\right)$) and $\sigma^2\left(B\left(x_t\right)\right)$ (variance of $Res\left(B\left(x_t\right)\right)$).

Estimation of $\mu\left(B\left(x_t\right)\right)$

It is in practice quite complicated to estimate $\mu\left(B\left(x_t\right)\right)$ and $\sigma^2\left(B\left(x_t\right)\right)$. We propose an estimation process that introduces an embedded model specification assumption. Remind that $\mu\left(B\left(x_t\right)\right) = \mathbb{E}\left[\widehat{NAV}_t\left(x_t\right) - B\left(x_t\right)|B\left(x_t\right)\right]$ where $\widehat{NAV}_t\left(x_t\right)$ is the random variable associated to a Monte Carlo estimation of NAV_t based on P secondary scenarios. Denoting by $Res^{'}$ the quantity $\widehat{NAV}_t\left(x_t\right) - B\left(x_t\right)$, we then consider $\mathbb{E}\left[Res^{'}\left(x_t\right)|B\left(x_t\right)\right]$.

To approximate this quantity, it is possible to calculate several $Res^{'}\left(x_t\right)$ outcomes, associated to random x_t outcomes, then to estimate the following model:

$$\begin{cases} Res^{'}\left(x_t\right) = {}^0\alpha + {}^1\alpha.B\left(x_t\right) + {}^2\alpha.B\left(x_t\right)^2 + ... + {}^k\alpha.B\left(x_t\right)^k + \nu, \\ \text{under the model specification assumption } \mathbb{E}\left[Res^{'}\left(x_t\right)|B\left(x_t\right)\right] = {}^0\alpha + {}^1\alpha.B\left(x_t\right) + {}^2\alpha.B\left(x_t\right)^2 + ... + {}^k\alpha.B\left(x_t\right)^k \end{cases}$$

This estimation enables to approximate function $\mu\left(B\left(x_t\right)\right)$. In practice k can be optimized through a simple stepwise regression. Other model specification assumptions can be considered the idea being to use a regression-type smoothing technique on $\mathbb{E}\left[Res^{'}\left(x_t\right)|B\left(x_t\right)\right]$. Finally, the user can add other simple functions of $B(x_t)$ in the regression, such as $\frac{1}{B(x_t)}$,... The idea is here to select the most significant regressors while keeping the $\sigma\left(B(x_t)\right)$ conditioning.

To calibrate this model, the user can simply choose to consider the outcomes already calculated and used to calibrate the proxy ($Res^{'}\left(x_t\right) = \tilde{z}_t\left(x_t\right) - B\left(x_t\right)$ for example). This implementation does not require any additional simulation budget.

*. Note that this is also the case for the Student distribution (only two parameters and defined on $\mathbb{R}$) for instance, which can also be a good candidate for the $Res\left(x_t\right)$ knowing $B\left(x_t\right)$ distribution. The Gaussian distribution is often well-adapted to our settings; however, further tests will be required to compare the efficiency of such assumptions

Another way to estimate function $\mu(B(x_t)) = \mathbb{E}\left[Res'(x_t)|B(x_t)\right] = \mathbb{E}[\tilde{z}_t(x_t) - B(x_t)|B(x_t)]$ is to calculate the $B(x_t)$ values associated to the $(x_t^n)_{n\in[|1,N|]}$ outcomes used to estimate the improved proxy, then after sorting the $((\tilde{z}_t^n, B(x_t^n)))_{n\in[|1,N|]}$ outcomes by order of $B(x_t^n)$, to estimate $\mathbb{E}[\tilde{z}_t(x_t) - B(x_t)|B(x_t)]$ by sliding window, based on an estimation on closest neighbors (and using a smoothing method - Friedman's supersmoother, multilinear regressions,... - if the result is too erratic). This kind of implementation can provide very efficient results.

Estimation of $\sigma^2(B(x_t))$

First it is important to simplify our analysis due to the great complexity of the $\sigma^2(B(x_t))$ direct estimation. We, therefore, base our work on the following result:

$\sigma^2(B(x_t)) = \mathbb{V}\left[Res'(x_t)|B(x_t)\right] - \mathbb{E}\left[\frac{\mathbb{V}[NPV_t|x_t]}{P}|B(x_t)\right]$, denoting by Res' the quantity $\widehat{NAV}_t(x_t) - B(x_t)$.

To estimate $\mathbb{V}\left[Res'(x_t)|B(x_t)\right]$, it is possible to get a $Res'(x_t)^2$ sample, associated to random x_t outcomes, then to estimate the following model:

$$\begin{cases} \left(Res'\right)^2(x_t) = {}^0\beta + {}^1\beta.B(x_t) + {}^2\beta.B(x_t)^2 + \eta, \\ \text{under the model specification assumption } \mathbb{E}\left[\left(Res'\right)^2(x_t)|B(x_t)\right] = {}^0\beta + {}^1\beta.B(x_t) + {}^2\beta.B(x_t)^2 \end{cases}$$

This estimation, plus the estimation of $\mu(B(x_t))$ enables to evaluate function $\mathbb{V}\left[Res'(x_t)|B(x_t)\right]$ (we obtain $\hat{\mathbb{V}}\left[Res'(x_t)|B(x_t)\right] = {}^0\hat{\beta} + {}^1\hat{\beta}.B(x_t) + {}^2\hat{\beta}.B(x_t)^2 - \hat{\mu}^2(B(x_t))$). Other model specification assumptions can be used, the idea being to consider a regression-type smoothing technique on $\mathbb{E}\left[\left(Res'\right)^2(x_t)|B(x_t)\right]$.

To calibrate this model, the user can simply choose $P = 1$ or $P = 2$ and consider the outcomes already calculated and used to calibrate its proxy ($\left(Res'\right)^2(x_t) = (NPV_t(x_t) - B(x_t))^2$ for example). This implementation does not require any additional simulation budget.

Concerning the estimation of $\mathbb{E}[\mathbb{V}[\tilde{z}_t|x_t]|B(x_t)]$. It is possible to first estimate $\mathbb{V}[\tilde{z}_t|x_t] = \mathbb{V}[\tilde{u}_t|x_t]$ through a regression of x_t on $\hat{u}_t^2$. It is also possible to calibrate an exponential model: regression of x_t on $\omega_t = ln\left(\hat{u}_t^2\right)$ then use of the estimator $\hat{\mathbb{V}}[\tilde{z}_t|x_t] = exp(\hat{\omega}_t)$. This model leads to a convergent but biased estimator which has the good property of being always positive. Then the regression of $\hat{\mathbb{V}}[\tilde{z}_t|x_t]$ on powers of $B(x_t)$ leads to an estimator of $\mathbb{E}[\mathbb{V}[\tilde{z}_t|x_t]|B(x_t)]$.

We obtain an estimated function $\widehat{\sigma^2}(B(x_t))$.

Similarly to function $\mu(B(x_t))$ estimation, several alternative possibilities can be considered to estimate the intermediaries $\mathbb{V}\left[Res^{'}(x_t)\,|B(x_t)\right]$ and $\mathbb{E}\left[\mathbb{V}\left[NPV_t|x_t\right]|B(x_t)\right]$. In particular, the estimation of $\mathbb{V}\left[Res^{'}(x_t)\,|B(x_t)\right]$ is quite delicate in practice and different methodologies can lead to very different results. The *a posteriori* test of the estimators is required to check the goodness of our estimators.

An alternative methodology that generally provides good results is to use the efficient $\widehat{NAV}_t$ estimated on the validation (*out of sample*, OOS) scenarios considered to validate the proxy * (cf. the validation step in Vedani and Devineau (2013)). The $\widehat{NAV}_t$ outcomes obtained are based on a large P value, it is, therefore, possible to assume that these Monte-Carlo estimators have converged and to use the corresponding $Res(x_t) \approx \widehat{NAV}_t(x_t) - B(x_t)$ to estimate the regression

$$\begin{cases} \left(log(\nu(x_t)^2)\right)(x_t) = {}^0\gamma + {}^1\gamma.B(x_t) + {}^2\gamma.B(x_t)^2\,\eta, \\ \text{under the model specification assumption,} \\ \mathbb{E}\left[log\left(\nu(x_t)^2\right)|B(x_t)\right] = {}^0\gamma + {}^1\gamma.B(x_t) + {}^2\gamma.B(x_t)^2, \\ \text{and where,} \\ \nu = Res(x_t) - \hat{\mu}(x_t). \end{cases}$$

The final estimator of $\sigma^2(B(x_t))$ is $\widehat{\sigma^2}(B(x_t)) = e^{{}^0\hat{\gamma}+{}^1\hat{\gamma}.B(x_t)+{}^2\hat{\gamma}.B(x_t)^2}$. It is biased but convergent and it generally provides more stable approximations of $\sigma^2(B(x_t))$ than those obtained using NPV_t outcomes, this variable being much too volatile.

3.3.3 Metamodeling validation

Remind the structural relation,

$NAV_t(x_t) = B(x_t) + Res(x_t)$.

The idea of our metamodeling validation step is to test our assumptions based on several efficient $\widehat{NAV}_t(x_t)$ outcomes estimated on a given set of OOS scenarios. In order to limit the simulation budget increase, this set should, in particular, include the set of validation scenarios used to test the efficiency of the proxy. The obtained $\widehat{NAV}_t(x_t)$ outcomes should be calculated thanks to a great number of secondary scenarios in order to be very close to the real $NAV_t(x_t)$ values.

Once the out of sample $\widehat{NAV}_t(x_t)$ outcomes calculated, the approximated standardized $Res(x_t)$ can be assessed, (denoted by $\widetilde{Res}(x_t)$) these outcomes being obtained after estimating $\mu(B(x_t))$ and $\sigma^2(B(x_t))$ where

$\widetilde{Res}(x_t) = \frac{\left(\widehat{NAV}_t(x_t) - B(x_t)\right) - \hat{\mu}(B(x_t))}{\sqrt{\widehat{\sigma^2}(B(x_t))}}$.

*. 10 to 100 in practice.

If our assumptions are efficient, the outcomes of variable $\widetilde{Res}(x_t)$ should be independent standard Gaussian outcomes, so that we can test the normality of our approximated standardized residuals, and both the validity of our assumptions and of our estimators $\hat{\mu}(B(x_t))$ and $\widehat{\sigma^2}(B(x_t))$ thanks to visual (QQ-plots) and theoretical (Kolmogorov-Smirnov test for example) normality adequation tests.

Note that thanks to a QQ-plot comparing the ordered $\widetilde{Res}(x_t)$ outcomes to the exact standard Gaussian quantiles, it is possible to detect the OOS scenarios that deviate the most from the theoretical quantiles (see Figure 3.2). Considering their associated $B(x_t)$ values, it is possible to simulate new OOS scenarios x_t, where the $B(x_t)$ values are close, and to use the ALM calculations (NPV_t / $\widetilde{NAV}_t$ outcomes) to better estimate functions μ and σ^2 for these $B(x_t)$ values.

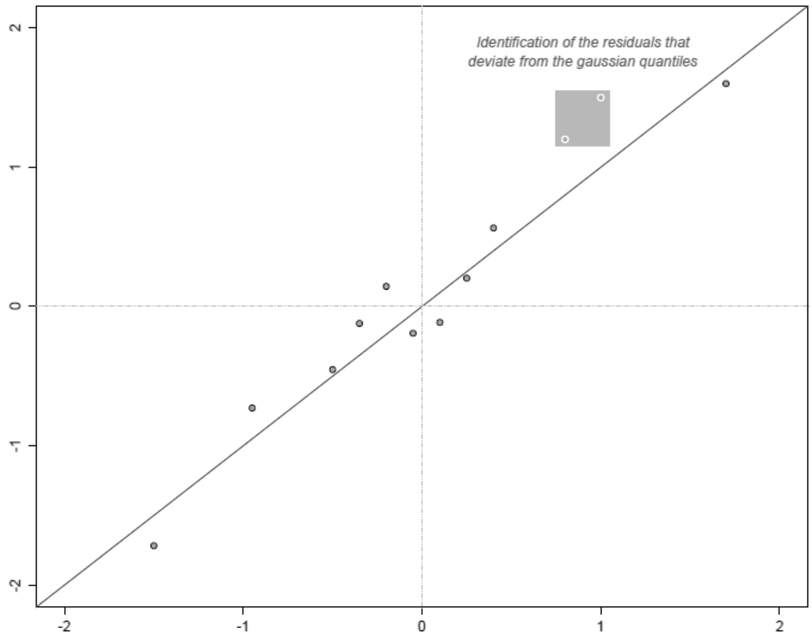

Figure 3.2 – Metamodel validation - QQ-plot true Gaussian quantiles vs. standardized residuals

Under our Gaussian distribution assumption, the estimators should converge. Therefore, the new QQ-plots should be more efficient.

This loop between *addition of OOS scenarios* and *normality test* may not converge (the normal distribution assumption being potentially too strong). A visual validation based on the empirical density of the $\widetilde{Res}(x_t)$ outcomes can often provide a good and pragmatic indication on the overall acceptability of the assumption and be a pragmatic alternative criterion to stop the loop.

3.3.4 Quantile calculation

The metamodel calibration procedure can lead to an infinite loop while never validating the normal distribution assumption. In practice, in most empirical implementation made by the authors, a visual validation test helps to conclude the procedure and leads to very efficient results when assessing the target quantile of NAV_t.

The idea, once the metamodel is calibrated*, is to simulate a large sample of $B(x_t) + Res(B(x_t))$ where

$$Res(B(x_t)) \sim \mathcal{N}\left(\hat{\mu}(B(x_t)), \widehat{\sigma^2}(B(x_t))\right)$$

and estimate the empirical target quantile based on this distribution. Indeed, recall the following relation:

$$NAV_t(x_t) =^d B(x_t) + Res(B(x_t)).$$

Consider now the user aims at estimating the Solvency Capital Requirement, based on the formula $SCR = NAV_0 - P(0,1)q_{0.5\%}(NAV_1)$ (true under good assumptions, see for example Devineau and Loisel (2009*b*)), denoting by $q_{0.5\%}(NAV_1)$ the 0.5% quantile of NAV_1.

The user has already implemented an LSMC methodology (plus the eventual optimization approaches presented in Sections 3.2.1 and 3.2.2) and obtained a proxy $B(x_1)$ that replicates $NAV_1(x_1)$. A metamodel is also implemented, enabling to estimate the distribution of $Res(B(x_1))$.The final SCR assessment algorithm is finally,

- Step 1 - Simulation of a great number of $B(x_1)$ outcomes and assessment the LSMC 0.5% empirical quantile, $\hat{q}_{0.5\%}(B(x_1))$.
- Step 2 - Derivation of the $B(x_1) + \sqrt{\widehat{\sigma^2}(B(x_1))}\varepsilon + \hat{\mu}(x_1) \left(=^d NAV_1(x_1)\right)$ outcomes, where $\varepsilon \sim \mathcal{N}(0,1)$, and assessment of the $LSMC + Metamodel$ 0.5% quantile $\hat{q}_{0.5\%}\left(B(x_1) + \sqrt{\widehat{\sigma^2}(B(x_1))}\varepsilon + \hat{\mu}(B(\right.$
- Step 3 - Calculation of
  $$SCR = max\left(NAV_0 - P(0,1)\hat{q}_{0.5\%}(B(x_1)), NAV_0 - P(0,1)\hat{q}_{0.5\%}\left(B(x_1) + \sqrt{\widehat{\sigma^2}(B(x_1))}\varepsilon + \hat{\mu}(B(x_1))\right)\right)$$

The idea of Step 3 is that the metamodel implementation being generally efficient but sometimes inexact, the SCR assessment should be cautious and protective. Instead of using simply the proxy-estimated SCR, our approach provides more robustness to the final SCR thanks to the $max(.,.)$ function.

Note that the idea of our metamodel is not to legitimate the use of whichever proxy. If the proxy is efficient enough, it passes without any compromise the validation step and there is no need for the calibration of a metamodel. In addition, the more efficient the proxy, the closer to 0 variable Res. Therefore, if the proxy provides good results, the metamodel should have little impact on the final quantile estimation.

*. As explained above, if the validation loop does not stop before, the user may artificially stop it after 4 or 5 loops.

3.3.5 Concerning the normal distribution assumption

The trickiest aspect of the proposed metamodeling technique is clearly the normal distribution assumption. Nevertheless, it is possible to legitimate this assumption under some simpler framework assumptions that do not seem too exaggerated.

First note that we can often assume the set of, say M, primary risk factors ($\left({}^1\varepsilon, {}^2\varepsilon, ..., {}^M\varepsilon\right)$, see *e.g.* Devineau and Chauvigny (2011) for more insight on the elementary risk factors notion) to be a Gaussian vector. Indeed, in most cases these primary risk factors are simulated in the Economic Scenarios Generators as Gaussian vector outcomes.

Assume now that $\left({}^1\varepsilon, {}^2\varepsilon, ..., {}^M\varepsilon\right)$ knowing $B(x_t)$ is still a Gaussian vector *, and that the real $NAV_t(x_t)$ value can be seen as the sum of $B(x_t)$ and of a linear combination, with parameters depending on $B(x_t)$, of the underlying primary risk factors,

$NAV_t(x_t) = B(x_t) + \sum_m {}^m\alpha\left(B(x_t)\right).{}^m\varepsilon.$

Then, under these simplification assumptions, the distribution of $NAV_t(x_t) - B(x_t)$ knowing $B(x_t)$ is Gaussian.

Note that the last assumption (the real $NAV_t(x_t)$ value can be seen as the sum of $B(x_t)$ and a linear combination, with parameters depending on $B(x_t)$, of the underlying primary risk factors) is weak when the calibrated improved proxy is good enough.

3.3.6 General metamodel-corrected proxy implementation process

The final Proxy+Metamodel improved implementation can be depicted as follow:

- Step 1: Standard OLS proxy calculation, based on an AV/CV joint variance reduction implementation (see Section 3.2).
  - Step 1.1: Implementation of a hybrid economic scenarios table where the economic drivers are simulated based on the Antithetic Variables approach.
  - Step 1.2: Selection of a Control Variate variable (*e.g.* simple RP). Calculation of the $\widetilde{NPV}_t$ outcomes.
  - Step 1.3: Estimation of the standard OLS model replicating the $\widetilde{NPV}_t$ outcomes.
- Step 2: Proxy validation based on OOS scenarios.
  - If the validation is only partially successful $\longmapsto$ Step 3
  - If the validation is fully successful $\longmapsto$ Step 4
- Step 3: Metamodel implementation (Section 3.3).
  - Step 3.1: Based on the improved proxy selected through steps 1 and 2, estimation of functions $\mu\left(B(x_t)\right)$ and $\sigma^2\left(B(x_t)\right)$.
  - Step 3.2: Validation of the normal distribution assumption based on the OOS scenarios

*. True if the proxy is a linear combination of the Gaussian primary risk factors.

  - — If the validation fails, addition of OOS scenarios and re-estimation of $\mu\left(B\left(x_t\right)\right)$ and $\sigma^2\left(B\left(x_t\right)\right)$. Then $\longmapsto$ Step 3.2.
  - — If the validation is fully successful $\longmapsto$ Step 4.
- — Step 4: Assessment of the target NAV_t quantile ($q_{0.5\%}\left(NAV_1\right)$ in the case of an SCR calculation).

This implementation process provides an efficient empirical scheme for proxies' uses. In particular, this framework, coming as an additional stone to the standard scheme, is much more robust and cautious than the standard OLS proxies calibration. In addition, note that this process can be implemented to estimate either an LSMC, a CF or an RP proxy, with little or no additional simulation budget.

3.4 Metamodel-corrected proxy Implementation

In this section, we have considered a simplified life insurance product to implement the methodologies presented in Sections 3.2 and 3.3. This product is assumed to be subject to three risk factors: ${}^{S}\varepsilon$ a stock risk factor, ${}^{r1}\varepsilon$ a short-term interest rates and ${}^{r2}\varepsilon$ a middle-term interest rates risk factors.

After having caibrated an LSMC proxy of variable NAV_1 in a standard fashion, we try to improve its efficiency considering the AV, then the CV adaptation. The product used being simple enough to enable a quick estimation of the true 0.5% percentile of NAV_1, we compare the true quantile to the one estimated using our proxy. In a final step, we implement an additional metamodel to try to reduce the error due to the optimized proxy. *

3.4.1 Improved LSMC implementation

Standard LSMC Implementation

We use the standard OLS calibration process presented in Vedani and Devineau (2013) and estimate first parameter β. The calibration process is implemented based on 10'000 NPV_1 outcomes.

The final parametric form is the following:

$$_{LSMC}NAV_1 = {}^{0}\beta + {}^{1}\beta_{S}\varepsilon + {}^{2}\beta_{r1}\varepsilon + {}^{3}\beta_{r2}\varepsilon + {}^{4}\beta_{S}\varepsilon^2 + {}^{5}\beta_{r1}\varepsilon^2 + {}^{6}\beta_{r2}\varepsilon^2 + {}^{7}\beta_{S}\varepsilon._{r1}\varepsilon + {}^{8}\beta_{S}\varepsilon._{r2}\varepsilon + {}^{9}\beta_{r1}\varepsilon._{r2}\varepsilon$$

with OLS parameters

∗. The time index is omitted in Section 3.4 and 3.5 for the sake of simplification

Table 3.1 – Standard LSMC OLS parameters

${}^{0}\hat{\beta}$	${}^{1}\hat{\beta}$	${}^{2}\hat{\beta}$	${}^{3}\hat{\beta}$	${}^{4}\hat{\beta}$	${}^{5}\hat{\beta}$	${}^{6}\hat{\beta}$	${}^{7}\hat{\beta}$	${}^{8}\hat{\beta}$	${}^{9}\hat{\beta}$
15314	-1800	1738	-967	-652	216	-19	71	-20	-173

Use of the AV-adaptation

Instead of 10'000 NPV_1 outcomes we now consider 5'000 z outcomes (z being a mean of two antithetic NPV_1 outcomes, see Subsection 3.2.1) to calibrate the LSMC parameters.

We obtain the following parameters:

Table 3.2 – AV-adapted LSMC parameters

${}^{0}_{AV}\hat{\beta}$	${}^{1}_{AV}\hat{\beta}$	${}^{2}_{AV}\hat{\beta}$	${}^{3}_{AV}\hat{\beta}$	${}^{4}_{AV}\hat{\beta}$	${}^{5}_{AV}\hat{\beta}$	${}^{6}_{AV}\hat{\beta}$	${}^{7}_{AV}\hat{\beta}$	${}^{8}_{AV}\hat{\beta}$	${}^{9}_{AV}\hat{\beta}$
15371	-1785	1788	-1003	-656	144	79	153	-66	-268

These new parameters are not very different from the ones obtained after the standard calibration implementation. Now, it is possible to estimate the matricial variances of the estimators (conditionally on the regressors outcomes) *e.g.* using the White's estimators (see White (1980)). Then, it is possible to compare these two estimated matrices by studying the eigenvalues of their difference. Indeed, if one variance is lower than the other (that is to say the estimation is more accurate), then the difference between the second and the first matrices is positive definite and its eigenvalues are all positive.

We estimate the variance-covariance matrix $\Sigma = \mathbb{V}\left[\hat{\beta}|X\right]$ and ${}_{AV}\Sigma = \mathbb{V}\left[{}_{AV}\hat{\beta}|{}_{AV}X\right]$, denoting by X (resp. ${}_{AV}X$) the matricial set of regressors outcomes used to estimate the standard LSMC (resp. AV-adapted) model. Then, we study the eigenvalues of ${}_{1}V = \Sigma - {}_{AV}\Sigma$. We have obtained one small negative eigenvalue. We have calculated the estimated variances of the proxy's $NAV_1(x)$ estimators, $\hat{\mathbb{V}}\left[x.\hat{\beta}|X\right]$ and $\hat{\mathbb{V}}\left[x.{}_{AV}\hat{\beta}|{}_{AV}X\right]$. We obtain that, in more than 99% of the cases, the estimators obtained using ${}_{AV}\hat{\beta}$ have a lower volatility and are more accurate than those obtained using the estimator $\hat{\beta}$, estimated considering a standard LSMC implementation.

Use of the CV adaptation

The AV adaptation of previous subsection has proven to provide efficient results. Now we add the CV adaptation considering, as a control variate, a simplified asset portfolio only composed with a Stock index and Zero-Coupon bonds. This portfolio has been previously optimized so that it is well correlated with variable z. In this implementation we have approximated $c^*(x_t)$ by $\mathbb{E}[c^*(x_t)]$.

We obtain the following parameters:

We also estimate the variance-covariance matrix ${}_{AV/CV}\Sigma = \mathbb{V}\left[{}_{AV/CV}\hat{\beta}|{}_{AV/CV}X\right]$, denoting by ${}_{AV/CV}X$

Table 3.3 – AV/CV-adapted LSMC parameters

$^{0}_{AV/CV}\hat{\beta}$	$^{1}_{AV/CV}\hat{\beta}$	$^{2}_{AV/CV}\hat{\beta}$	$^{3}_{AV/CV}\hat{\beta}$	$^{4}_{AV/CV}\hat{\beta}$
15366	-1776	1797	-1010	-651
$^{5}_{AV/CV}\hat{\beta}$	$^{6}_{AV/CV}\hat{\beta}$	$^{7}_{AV/CV}\hat{\beta}$	$^{8}_{AV/CV}\hat{\beta}$	$^{9}_{AV/CV}\hat{\beta}$
163	79	131	-68	-267

$(= {}_{AV/CV}X)$ the matricial set of regressors' outcomes used to estimate the AV/CV-adapted model. Then we study the eigenvalues of ${}_2V = {}_{AV}\Sigma - {}_{AV/CV}\Sigma$. These are all positive and it is directly evident that the estimator ${}_{AV/CV}\hat{\beta}$ is more efficient than ${}_{AV}\hat{\beta}$

With our two variance reduction approaches, we have greatly decreased the volatility of our standard LSMC estimator (of more than 70% according to our White's estimators). We have obtained very efficient results through this proxy optimization Subsection but one should remind this is true for our specific simplified life insurance product.

Improved proxy results

We now assess the 0.5% quantile on a 100.000 random sample of approximated NAV obtained using the *improved* AV/CV LSMC proxy. We have chosen a very simple product so that we know the exact value of the 0.5% NAV_1 quantile. The results and the associated relative deviance between both values are the following:

Table 3.4 – improved proxy results and comparison with the true NAV_1 quantile

Proxy's NAV_1 0.5%-quantile	**True NAV_1 0.5%-quantile**	Relative deviance
6504	5533	17.6%

The quantile vs. quantile plot (QQ-plot) obtained comparing the 100'000 proxy's and the true NAV_1 outcomes is given in Figure 3.3.

As one can see, the error made on the 0.5%-quantile, using the proxy, is not negligible (most of all, this deviation leads to a *non-prudent* quantile) and the QQ-plot shows a bad estimation of a generous portion of the distribution. We implement a metamodel methodology in the following Subsection.

3.4.2 Metamodel implementation

Thanks to our simplified product we can easily calculate a large number of both variables $\widehat{NAV}(x)$ and $NAV_1(x)$. Throughout the previous subsection, we have already estimated an optimized LSMC

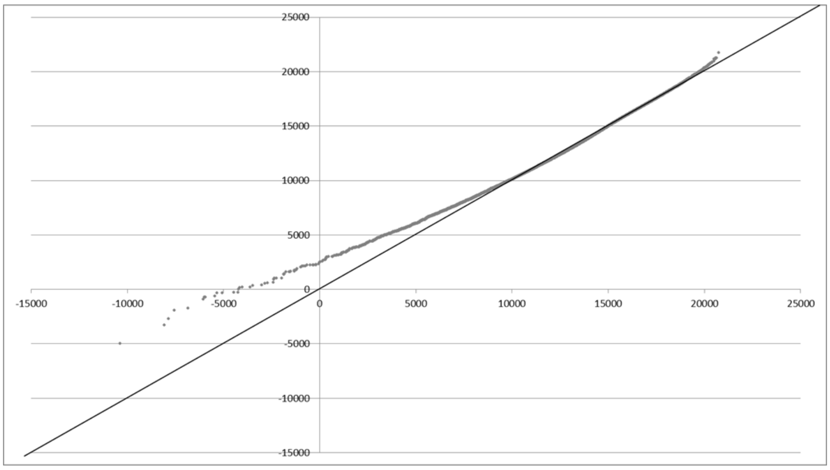

Figure 3.3 – QQ-plot improved proxy vs. true distribution (100'000 outcomes)

proxy $B(x)$. This proxy is not good enough as far as the estimation of the 0.5%-quantile of $NAV_1(x)$ is concerned.

We have all the required information to estimate the metamodel associated to $B(x)$ to try and improve both the quantile estimation and the QQ-plot of $B(x)$ vs. $NAV_1(x)$.

Estimation of functions $\mu(B(x))$ and $\sigma^2(B(x))$

Knowing both outcomes of $B(x)$ and $NAV_1(x)$ we can have a sample of $Res(x)$ outcomes (though in real operational conditions we would have to base our analyze on $Res^{'}(x)$ outcomes). Considering these outcomes we can estimate directly the functional $\mu(B(x))$ and $\sigma^2(B(x))$ and test the normality of $\frac{Res(x)-\mu(B(x))}{\sqrt{\sigma^2(B(x))}}$. This *theoretical* implementation enables us to test the potential efficiency of our Gaussian assumption.

We first get a random sample of 100.000 $Res'(x) = NPV_t(x) - B(x)$ outcomes (in practice, these could be the ones simulated to get the improved LSMC proxy) to estimate the following regression:

$$\begin{cases} Res'(x) = {}^0\alpha + {}^1\alpha.B(x) + {}^2\alpha.B(x)^2 + {}^3\alpha.B(x)^3 + {}^4\alpha.B(x)^4 + \nu, \\ \text{under the model specification assumption} \\ \mathbb{E}[Res(x)|B(x)] = {}^0\alpha + {}^1\alpha.B(x) + {}^2\alpha.B(x)^2 + {}^3\alpha.B(x)^3 + {}^4\alpha.log\left(B(x)^2\right). \end{cases}$$

Then we calculate a random sample of N $Res(x)$ ($N = 100$ & $N = 25$) outcomes * on which we estimate the parameters of the following regression:

∗. In practice, such data would already be estimated through the out of sample scenarios validation of the parametric proxy. Indeed, we have $Res'(x)$ outcomes but we assume here that the convergence of the $\widehat{NAV}$ estimated on the validation scenarios towards the true NAV value is reached (*e.g.* with a number of 5'000 secondary scenarios).

$$\begin{cases} log\left(\nu^2\right) = {}^0\gamma + {}^1\gamma.B\left(x\right) + {}^2\gamma.B\left(x\right)^2 + \eta, \\ \text{under the model specification assumption} \\ \mathbb{E}\left[log\left(\nu^2\right)|B\left(x\right)\right] = {}^0\gamma + {}^1\gamma.B\left(x\right) + {}^2\gamma.B\left(x\right)^2. \end{cases}$$

And we finally build the estimators,

$$\begin{cases} \hat{\mu}\left(B(x)\right) = {}^0\alpha + {}^1\alpha.B\left(x\right) + {}^2\alpha.B\left(x\right)^2 + {}^3\alpha.B\left(x\right)^3 + {}^4\alpha.B\left(x\right)^4, \\ \widehat{\sigma^2}\left(B(x)\right) = e^{{}^0\hat{\gamma} + {}^1\hat{\gamma}.B(x) + {}^2\hat{\gamma}.B(x)^2}. \end{cases}$$

The results on the 0.5%-quantile estimation and on the QQ-plot (vs. 100 000 $NAV_1(x)$ outcomes) are given in Table 3.5, Figures 3.4 and 3.5.

Table 3.5 – Metamodel-corrected proxy 0.5%-quantile comparison with the true NAV_1 quantile

	Proxy alone	**Metamodel** $N = 100$	**Metamodel** $N = 25$	NAV_1
0.5%-quantile value	6504	5697	5375	5533
Relative deviance	17.6%	2.9%	-2.8%	/

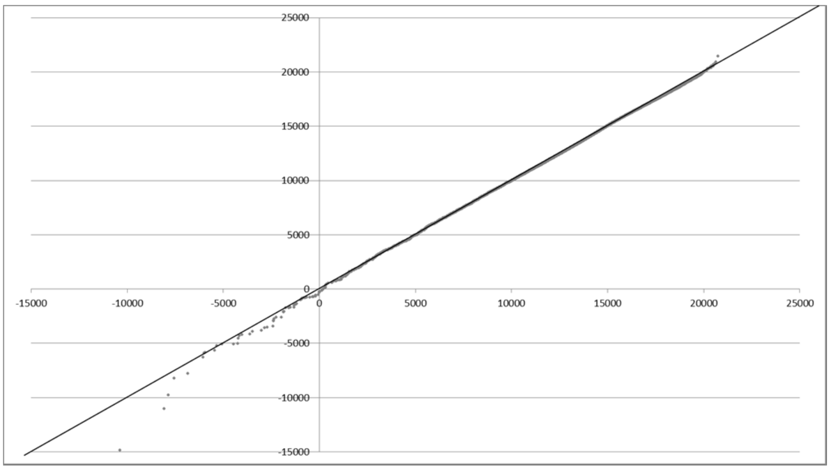

Figure 3.4 – QQ-plot proxy + metamodel (N=100) vs. true $NAV_1(x)$ distribution

In both cases, with a simple implementation of the metamodel not even checking for Gaussianity, we obtain both a better estimation of the quantile and a better QQ-plot. This last point must, however, be balanced as far as the QQ-plot obtained for the proxy+metamodel when $N = 25$ shows a strong deviance for the most extreme percentiles, that is lower than 0.1% (left tail). But, in most cases, the metamodel helps to linearize the very majority of the QQ-plot from Figure 3.3.

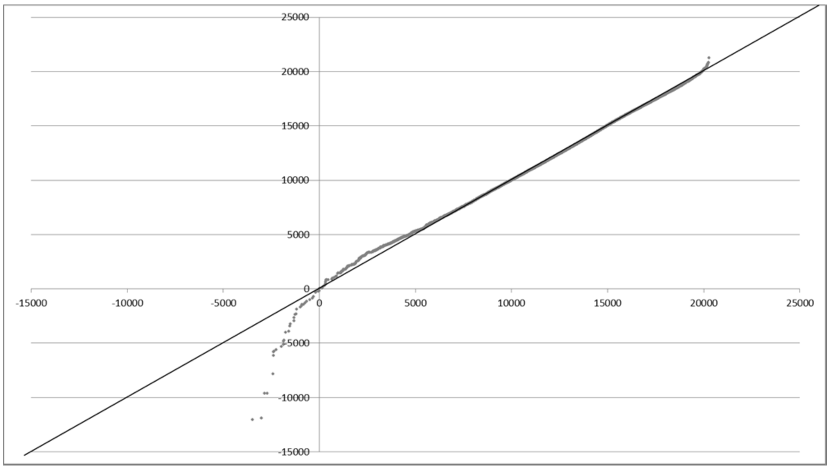

Figure 3.5 – QQ-plot proxy + metamodel (N=25) vs. true $NAV_1(x)$ distribution

Normality test

Remind that the major assumption of our metamodel implementation is the normality of the proxy's error, conditionally on $B(x)$. Our simplistic implementation has provided efficient results. However, we have chosen to test the metamodel without checking the Gaussianity of the standardized outcomes $\frac{Res(x)-\hat{\mu}(B(x))}{\sqrt{\widehat{\sigma^2}(B(x))}}$.

To test the asymptotic efficiency of our assumption, we base our Normality test on the true $\hat{\mu}(B(x))$ and $\widehat{\sigma^2}(B(x))$ functionals. This enables us to check if the algorithm converges or not.

One can easily see that we do not get standard Gaussian residuals in the major part of the $B(x_t)$ distribution. However we observe more efficient results when we focus in the left tail of this distribution, the one of interest for our calculation. We have tested it in a small neighborhood of the 0.5% percentile of $B(x)$ with more efficient results (see Figure 3.6 below).

Finally, the methodology proposed (and on this specific example) is highly relevant to correct the proxies' results and provides very efficient corrections. As can be observed, a local efficiency of the Normality assumption seems to be enough for the metamodel to achieve its goal. However, we still observe a significant improvement of the whole proxy distribution and this implementation finally makes us believe that the global Normal assumption may not be necessary for the metamodel approach to improve the proxy methodology.

3.5 Proxy calibration set issue

In this section, we consider one practical issue linked with the LSMC calibration process. Most operational papers on the LSMC methodology, such asBarrie & Hibbert (2011), tend to estimate their polynomial proxy parameters ($\hat{\beta}$) considering calibration primary scenarios which risk factors sets are

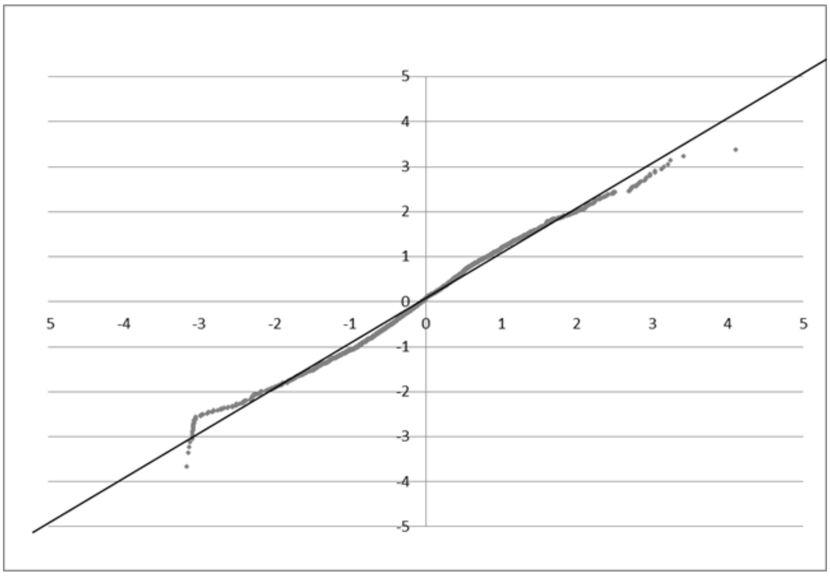

Figure 3.6 – Local validation of the Normality assumption - Proxy + metamodel standardized $Res(x)$ outcomes vs. true standard Gaussian quantiles (considering the true [theoretical] μ and σ^2 functionals)

well distributed, so as to cover efficiently the distribution space of potential primary scenarios, using *e.g.* a low-discrepancy quasi-random sequence and following a process similar to the one below.

Consider a set $\varepsilon = \left({}^1\varepsilon, {}^2\varepsilon, ..., {}^K\varepsilon\right)$ of risk factors, following a probability distribution $\mathscr{L}$. We assume $\mathscr{L}$ support is equal to $\mathbb{R}^K$. This is the case when ε is a non-degenerated Gaussian vector, which is often the case in practice. The procedure used to simulate *well distributed* primary scenarios consists first in assessing boundaries $\left({}^1a, ..., {}^Ka\right)$ and $\left({}^1b, ..., {}^Kb\right)$ such as $\mathbb{P}\left[\varepsilon \in \left[{}^1a, {}^1b\right] \times \left[{}^2a, {}^2b\right] \times ... \times \left[{}^Ka, {}^Kb\right]\right]$ is close to 1. Then the idea is to uniformly discretize $\left[{}^1a, {}^1b\right] \times \left[{}^2a, {}^2b\right] \times ... \times \left[{}^Ka, {}^Kb\right]$, either by considering the outcomes of a low discrepancy sequence (Sobol, Van der Corput,...) or by considering any deterministic uniformly distributed mesh of the hypercube.

The idea under such a process is easily understandable. Such an implementation should provide an estimator almost efficient everywhere on $\left[{}^1a, {}^1b\right] \times \left[{}^2a, {}^2b\right] \times ... \times \left[{}^Ka, {}^Kb\right]$. In addition, considering such a procedure should enable to overweigh the extreme percentiles of distribution $\mathscr{L}$ and to be more efficient when estimating extreme values of NAV(e) (the NAV associated to a primary scenario identified to the set e). Indeed, in most cases, these extreme values are obtained for extreme percentiles of the distribution (see Devineau and Loisel (2009*a*)).

The papers introducing these process and intuition never theoretically justify it.

We study here this issue from a theoretical point of view. In particular, we wonder if such a process leads to estimators that converge toward the same parameters as in the case of a standard calibration (based on risk factors sets distributed according to the real distribution $\mathscr{L}$). We also discuss the comparative efficiency of the uniform and standard calibration processes under usual LSMC assumptions.

3.5.1 Notation and first assumptions

We consider a standard Nested Simulation framework, as presented in Section 3.1. Below, we denote by $\mathscr{F}$ the filtration generated by the primary scenarios financial information.

Let x be the random vector of the LSMC regressors, when the underlying risk factors follow the standard probability distribution $\mathscr{L}$, and $\dot{x}$ the same LSMC regressors, but when the underlying risk factors follow the uniform probability distribution. We assume that $\mathscr{F} = \sigma(x)$, which is the case when all the risk factors are embedded in x (generally verified in practice), that is to say the function $f : \varepsilon \longrightarrow x$ is a bijection. Let $NPV(x;\varepsilon)$ be the Net Present Value of future margins random variable associated to the vector of primary variables x, and secondary risk factors ε (ε is the risk factor's set associated to the secondary risk-neutral scenario used to assess $NPV(x;\varepsilon)$). Let $u(x;\varepsilon)$ be the residual of the *true/standard* LSMC regression (regression of x on $NPV(x;\varepsilon)$), and $\dot{u}(\dot{x};\varepsilon)$ be the residual of the *uniform* regression (regression of $\dot{x}$ on $NPV(\dot{x};\varepsilon)$). Let β (resp. $\dot{\beta}$) be the vector of the true parameters of the *true/standard* (resp. *uniform*) model.

Let Ω (resp. $\dot{\Omega}$) be the support of x (resp. $\dot{x}$), such that $\dot{\Omega} \subset \Omega$, and such that function $\mathbb{E}\left[NPV(x;\varepsilon)|x\right]$ is equal to function $\mathbb{E}\left[NPV(\dot{x};\varepsilon)|\dot{x}\right]$ on the whole $\dot{\Omega}$. In addition, we assume that $\dot{\Omega}$ embeds an infinity of potential scenarios (and so does Ω).

These assumptions are verified in practice (as far as $\mathscr{F} = \sigma(x)$).

The two models considered are

$$\begin{cases} NPV\,(x,\varepsilon) = x.\beta + u(x,\varepsilon), \\ \text{under a chosen heterogeneity (strong or weak) assumption} \end{cases} \qquad \textit{model m0}$$

and

$$\begin{cases} NPV\,(\dot{x},\varepsilon) = \dot{x}.\dot{\beta} + \dot{u}(\dot{x},\varepsilon), \\ \text{under a chosen heterogeneity (strong or weak) assumption} \end{cases} \qquad \textit{model m1}.$$

3.5.2 Theoretical analysis

Results under the strong exogeneity assumption

A generalized but strong assumption Until now we have chosen the strong exogeneity assumption. This assumption is often *a priori* chosen by practitioners but can be released by considering a weaker exogeneity assumption (non-correlation between the covariates and the residuals) so that the OLS estimator still converges towards the true parameter.

This assumption is often made by practitioners. It leads, in our case, to consider the following equalities:

$$\begin{cases}\mathbb{E}\left[NPV\left(x,\varepsilon\right)\right] = NAV(x) = x.\beta \\ \mathbb{E}\left[NPV\left(\dot{x},\varepsilon\right)\right] = NAV(\dot{x}) = \dot{x}.\dot{\beta}\end{cases}$$

This strong assumption can become weaker if the covariates vector x embeds increasing material, that is increasing risk factors monomial.

The weak exogeneity assumption (see below) can be tested, through a Hausman test, for example (see Hausman (1978)), if an instrument variable is available. If the Hausman test leads to refuse the weak exogeneity assumption, then the strong exogeneity assumption is not true either.

However, the strong exogeneity assumption cannot be tested in a direct fashion. We assume, in this paper, and like most practitioners, that the covariates vector embeds enough risk factors monomials, so that the strong exogeneity assumptions are acceptable in our LSMC regressions.

First result: $\beta = \hat{\beta}$

This is obtained easily from the equality between functions $\mathbb{E}\left[NPV(x;\varepsilon)|x\right]$ and $\mathbb{E}\left[NPV(\dot{x};\varepsilon)|\dot{x}\right]$ on the whole $\dot{\Omega}$. Indeed, as $\dot{\Omega} \subset \Omega$ embeds an infinity of potential scenarios $\dot{x}$ such that $\dot{x}\dot{\beta} = \dot{x}\beta$, we have necessarily, $\beta = \dot{\beta}$.

It is interesting to notice that the strong exogeneity assumption of *model m1* can be relaxed if the strong exogeneity assumption of *model m0* stands. If a weak exogeneity hypothesis is assumed the equality $\beta = \dot{\beta}$ still stands. Indeed, in such a case, the strong exogeneity on all Ω is easily transmitted to $\dot{\Omega}$, leading to the equalities $\beta = \dot{\beta}$ and $u = \dot{u}$.

Conclusion on the selection of a best LSMC calibration framework under tehe strong exogeneity assumption

The results obtained above are easily checked in the simple case where the covariates vector is scalar, either in a homoscedastic ($\forall x \in \Omega, \mathbb{E}\left[u(x,\varepsilon)^2|x\right] = \sigma^2$) or in a heteroscedastic ($\forall x \in \Omega, \mathbb{E}\left[u(x,\varepsilon)^2|x\right] = \sigma^2(x)$) scheme (see Figures 3.7 and 3.8 below, for an arbitrarily chosen space Ω, segment of $\mathbb{R}$.

As it can be seen, the trend estimated through an OLS method, considering our assumptions, is always the same and does not depend on the probability distribution followed by x, the scalar covariate. It corresponds to $x\beta$.

Our conclusions are the following:

— Under the additional homoscedasticity assumption *, there exists no best calibration approach.

$*$. An assumption that can be easily tested considering, for example, a White's test.

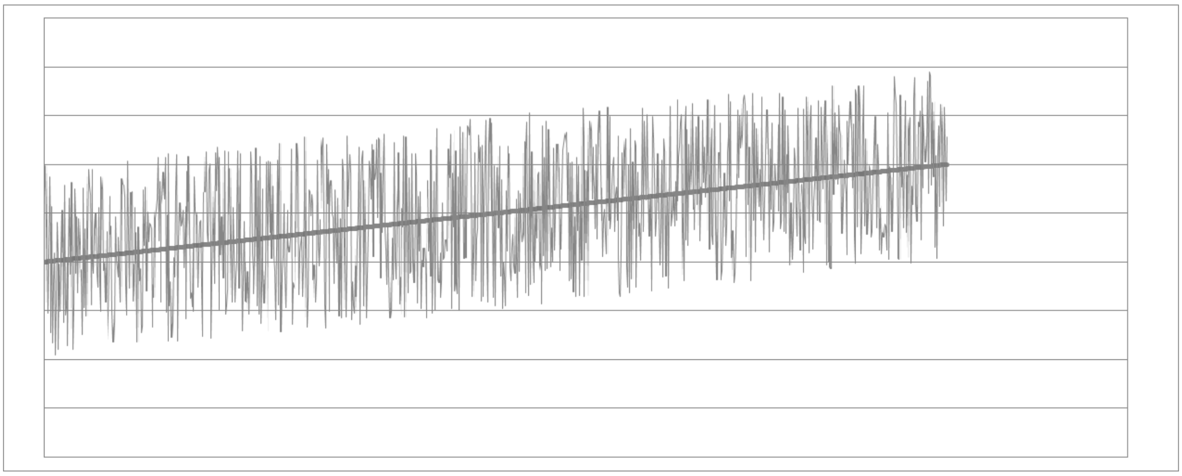

Figure 3.7 – Simplified scalar example - homoscedastic scheme (Own Funds in thick grey / NPV outcomes in thin grey)

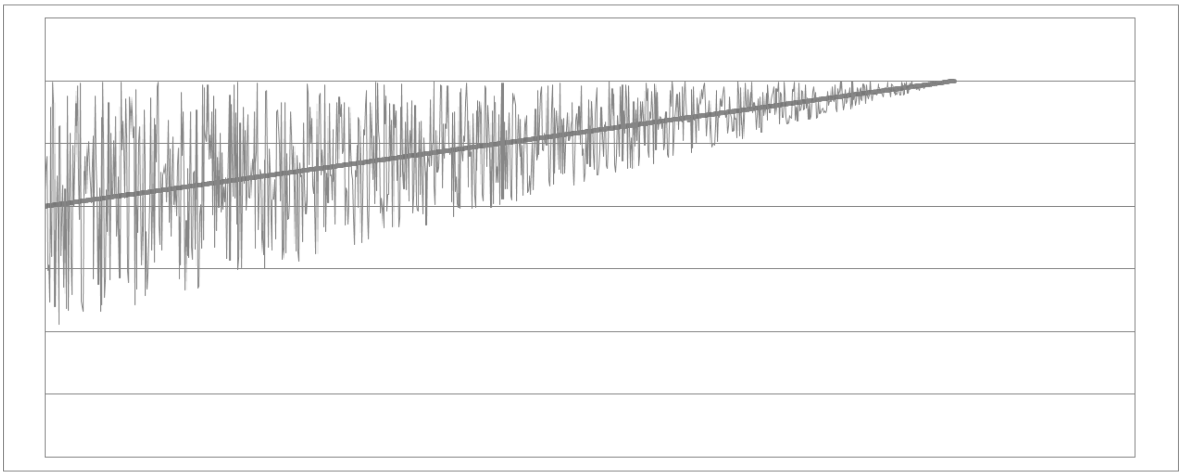

Figure 3.8 – Simplified scalar example - heteroscedastic scheme (Own Funds in thick grey / NPV outcomes in thin grey)

The only way to minimize the OLS estimator's variance is to consider the highest possible number of calibration scenarios. In particular, there is no particular advantage to estimate the LSMC in a *uniform* fashion, compared to a more *standard* fashion. It is evident in the case when the regressor's set is scalar and easily extendable to a higher dimension.

— Under the more general heteroscedasticity case, the best calibration approach leans in considering calibration scenarios that lead to the minimal values of function $\sigma^2(x)$. However, it is in practice very rare to have an a priori information concerning function $\sigma^2(x)$. One possibility could be to use a reduced number of calibration scenarios in order to estimate this function (*e.g.* using one of the methods presented in Subsection 3.2.2), then to locate the scenarios leading to lower values of $\sigma^2(x)$, and to calibrate the proxy based on these scenarios. The efficiency of such a process is however quite unpredictable. Indeed, there is always an error embedded in the estimation of $\sigma^2(x)$ and the process may be more time- and scenarios-consuming than a simpler approach. Finally, without any information on $\sigma^2(x)$, it is relevant to consider a *uniform* approach (as presented above), considering a large space $\dot{\Omega}$, to calibrate the proxy, in order to equally weigh each of the parts of Ω. In particular, if a small space $\dot{\Omega}$ is chosen (if $\forall i \in [1;K]; a_i \approx b_i$), this would arbitrarily overweight a small part of Ω, the risk being that this small part is associated to high values of function $\sigma^2(x)$.

In any case, it is relevant to apply variance reduction techniques to the regression implemented by the practitioner.

Assuming the weak exogeneity assumption for (m0) and (m1)

The strong exogeneity assumption, though it is often assumed by actuarial practitioners, is complex to test and may be too strong. It is interesting to see what happens when only the weak exogeneity hypotheses is assumed. Under such assumptions the regressions become,

$$\begin{cases} NPV\,(x,\varepsilon) = x.\beta + u(x,\varepsilon), \\ \text{under the assumption } \mathbb{E}\left[u(x,\varepsilon).x\right] = 0_{K+1} \end{cases} \qquad \textit{model } \tilde{m}0$$

and

$$\begin{cases} NPV\,(\dot{x},\varepsilon) = \dot{x}.\dot{\beta} + \dot{u}(\dot{x},\varepsilon), \\ \text{under the assumption } \mathbb{E}\left[\dot{u}(\dot{x},\varepsilon).\dot{x}\right] = 0_{K+1} \end{cases} \qquad \textit{model } \tilde{m}1$$

The result $\beta = \dot{b}$ does not stand anymore; indeed,

$$\dot{\beta} = \mathbb{E}\left[\dot{x}'.\dot{x}\right]^{-1}\mathbb{E}\left[\dot{x}'NPV\,(\dot{x},\varepsilon)\right] = \mathbb{E}\left[\dot{x}'.\dot{x}\right]^{-1}\mathbb{E}\left[\dot{x}'\left(\dot{x}\beta + u\,(\dot{x},\varepsilon)\right)\right] = \beta + \mathbb{E}\left[\dot{x}'.\dot{x}\right]^{-1}\mathbb{E}\left[\dot{x}'u\,(\dot{x},\varepsilon)\right]$$

where $\mathbb{E}\left[\dot{x}'.\dot{x}\right]^{-1}\mathbb{E}\left[\dot{x}'.u\,(\dot{x},\varepsilon)\right] \neq 0$ in general.

It is, in particular, very difficult to conclude regarding the improved LSMC calibration method. If the practitioner wants to estimate a global trend on the whole Ω, and has no information on the conditional volatility of the regression's residuals, it also seems relevant to consider a uniform (random or deterministic mesh) approach, with a large space $\dot{\Omega}$, to calibrate the proxy, in order to equally weigh each of the parts of Ω.

As an example of such scheme, consider a 1-dimension NAV, continuous and piecewise linear, but with non-derivable singularities, such as the one presented in Figure 3.9.

In such a scheme, which can be true in practice, the NAV is piecewise polynomial (linear) and the strong exogeneity assumption does not stand though the weak exogeneity can be achieved. It is typically a case where the *uniform* calibration set can provide a better global estimation than a standard

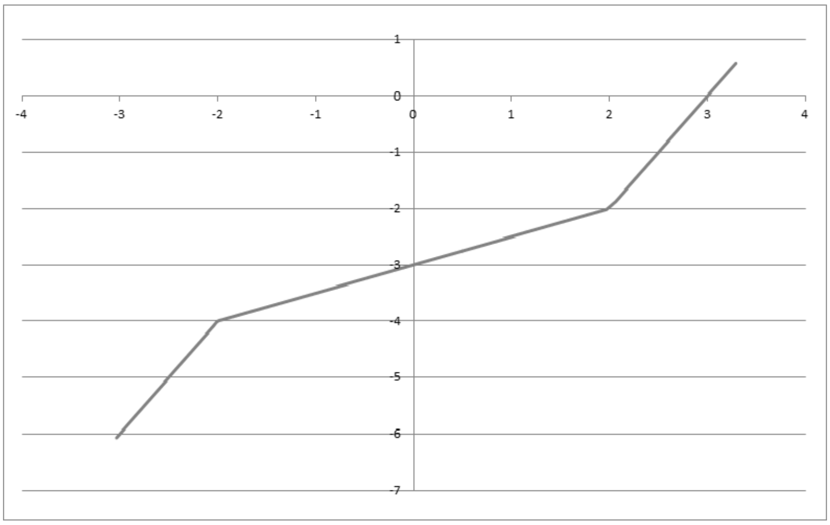

Figure 3.9 – Simplified scalar example without strong exogeneity (Own Funds function in thick grey)

Gaussian calibration set. We have applied both implementations and obtained the results presented in Figure 3.10.

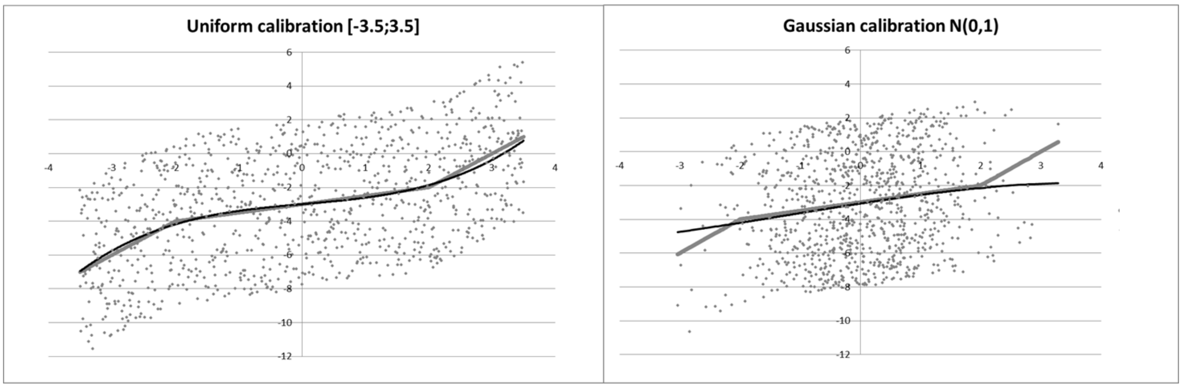

Figure 3.10 – Simplified scalar example with homoscedasticity but without strong exogeneity - comparison between gaussian and uniform calibrations (Own Funds in thick grey / NPV outcomes in thin grey / estimated proxy in dark)

As one can see the choice of a large range in the uniform calibration (*e.g.* $[3.5;3.5]$ for $\mathscr{L} = \mathscr{N}(0;1)$) leads to a more efficient estimation of the extreme NAV quantiles, compared to a Gaussian calibration, which overweights the central quantiles.

The weak exogeneity sounds, in practice, more *assumable*. In particular, the NAV may be a continuous function of the regressors but it seems strong to assume that it follows the same polynomial function, be it for central or extreme values. In all generality (even under a false specification), it seems more prudent to use a uniform mesh approach, with a large range, to calibrate any LSMC.

Note that this first analysis did not aim at comparing the efficiency of the different uniform (random or deterministic mesh) approaches (random uniform outcomes, uniform grid or low-discepancy sequence), just at estimating the legitimacy of its use.

Conclusion

As the European insurance market investigates deeper the complex requirements induced by Directive Solvency II, proxy methodologies tend to be more and more used by life insurance undertakings. Be it proxy optimization or error management, these two subjects are very strategical and have to be managed by practitioners.

We have considered and theoretically analyzed different methods to improve the standard proxy implementations and work on the error embedded in its approximations. These approaches have been implemented in a last Section with efficient results, on a synthetic life insurance product. The authors acknowledge both the fact that the results presented here have been obtained on an only product and that the Normal assumption required by the metamodel proposed in the paper has been proved inexact.

Considering these two limitations, note that the proposed approaches have also been empirically tested on other, more realistic, products and have always proved to improve the initial proxy quantile. Note also that our paper's first objective was to introduce the problem of the proxy's error management. The metamodel methodology proposed may be incomplete but it is a pragmatic tool, rather easy to implement and can provide a first solution to this issue.

Further developments are currently tested, especially to improve our metamodel approach, such as testing it on new products, proposing other distribution assumptions for the metamodel and more accurate methodologies to estimate the $\mu(B(x))$ and $\sigma^2(B(x))$ functionals.

Three major points must be kept in mind by proxy/metamodel users.

First, the choice to use a maximum when assessing the global SCR ensures the conservative feature of the metamodel approach proposed in this paper.

Second, this method has to be used in pair with the proxy optimization approaches. They need no additional simulation budget while decreasing the proxy error and leading to a less impacting metamodel.

Third, the strong exogeneity assumption is too quickly assumed by practitioners. To assume a weak exogeneity requires accepting fully the defaults of proxies approaches. The NAV can probably not be exactly replicated. These pragmatic tools are **simple approximations** but they can provide **great help** to actuarial users. In addition, without any *a priori* on the volatility of the NPV, conditioned by the risk factors knowledge, it is prudent to use a uniform calibration to assess the proxies parameters.

Bibliography

Algorithmics (2011), *Curve Fitting for Calculating Solvency Capital Requirements under Solvency II: Practical insights and best practices from leading European Insurers*.

Ankenman, B., Nelson, B. L. and Staum, J. (2010), 'Stochastic kriging for simulation metamodeling', *Operations research* **58**(2), 371–382.

Barrie & Hibbert (2007), *Smart nested simulations: learning from option traders.*

Barrie & Hibbert (2011), *A Least Squares Monte Carlo approach to liability proxy modelling and capital calculation*.

Bonnin, F., Planchet, F. and Juillard, M. (2014), 'Best estimate calculations of savings contracts by closed formulas: application to the orsa', *European Actuarial Journal* **4**(1), 181–196.

Broadie, M., Du, Y. and Moallemi, C. C. (2011), 'Efficient risk estimation via nested sequential simulation', *Management Science* **57**(6), 1172–1194.

Devineau, L. and Chauvigny, M. (2011), 'Replicating portfolios: calibration techniques for the calculation of the Solvency II economic capital', *Bulletin Français d'Actuariat* **11**(21), 59–97.

Devineau, L. and Loisel, S. (2009*a*), 'Construction d'un algorithme d'accélération de la méthode des "simulations dans les simulations" pour le calcul du capital économique Solvabilité II', *Bulletin Français d'Actuariat* **10**(17), 188–221.

Devineau, L. and Loisel, S. (2009*b*), 'Risk aggregation in Solvency II: How to converge the approaches of the internal models and those of the standard formula?', *Bulletin Français d'Actuariat* **9**(18), 107–145.

El Karoui, N., Loisel, S., Prigent, J.-L. and Vedani, J. (2015), 'Market inconsistencies of the market-consistent European life insurance economic valuations: pitfalls and practical solutions'.

Fromenteau, M. and Petauton, P. (2012), *Théorie et pratique de l'assurance vie. : Cours et exercices corrigés*, Dunod.

Glasserman, P. (2003), *Monte Carlo methods in financial engineering*, Springer.

Hausman, J. A. (1978), 'Specification tests in econometrics', *Econometrica* **46**(6), 1251–1271.

Horn, R. A. and Johnson, C. R. (1990), *Matrix analysis.*, Cambridge University Press.

Hull, J. and White, A. (1988), 'The use of the control variate technique in option pricing', *Journal of Financial and Quantitative analysis* **23**(3), 237–251.

Kalberer, T. (2012), 'Stochastic determination of the value at risk for a portfolio of assets and liabilities', *Der Aktuar* **1**, 7–14.

Liu, M. and Staum, J. (2010), 'Stochastic kriging for efficient nested simulation of expected shortfall', *Journal of Risk* **12**(3), 3–27.

Longstaff, F. A. and Schwartz, E. S. (2001), 'Valuing american options by simulation: A simple least-squares approach', *Review of Financial studies* **14**(1), 113–147.

Martial, A.-C. and Garnier, J.-B. (2013), 'Optimisation du calcul du capital économique d'une compagnie d'assurance vie par la méthode des simulations dans les simulations'.

Nteukam, O. and Planchet, F. (2012), 'Stochastic evaluation of life insurance contracts: Model point on asset trajectories and measurement of the error related to aggregation', *Insurance: Mathematics and Economics* **51**(3), 624–631.

Paskov, S. H. and Traub, J. F. (1995), 'Faster valuation of financial derivatives', *The Journal of Portfolio Management* **22**(1), 113–123.

Plackett, R. L. (1950), 'Some theorems in least squares', *Biometrika* **37**(1), 149–157.

Planchet, F., Thérond, P.-E. and Kamega, A. (2009), *Scénarios économiques en assurance-Modélisation et simulation*, Économica.

Revelen, J. (2009), 'Replicating Portoflio et capital économique en assurance vie'.

Ribereau, P. and Rullière, D. (2011), Agrégation d'informations et alternative au krigeage en environnement aléatoire.

Saporta, G. (2006), *Probabilités, analyses des données et statistiques*, editions Technip.

Schrager, D. (2008), 'Replicating portfolios for insurance liabilities'.

Tosetti, A., Paris, B., Paslky, P. and Le Vallois, F. (2003), *Gestion Actif Passif en Assurance vie: réglementation, outils, méthodes*, Économica.

Vedani, J. and Devineau, L. (2013), 'Solvency assessment within the ORSA framework: issues and quantitative methodologies', *Bulletin Français d'Actuariat* **13**(25), 35–71.

White, H. (1980), 'A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity', *Econometrica* **48**(4), 817–838.

Chapitre 4

Propriétés structurelles de la *VAN forward* - Application à l'optimisation de l'approche Simulations dans les Simulations en assurance vie

Abstract

Avec l'implémentation du cadre défini par la directive Solvabilité II, les assureurs ont la possibilité de développer un modèle interne, afin de mieux refléter les spécificités de leurs risques propres, plutôt qu'utiliser une approche plus simplifiée et moins spécifique, la formule standard. En assurance vie, la méthode des Simulations dans les Simulations (SdS) est une première approche à la disposition des opérationnels pour évaluer le capital de solvabilité requis. Cette approche très simulatoire permet de disposer d'une distribution complète des fonds propres économiques à un an, sans nécessiter d'hypothèse restrictive. Toutefois, sa complexité algorithmique la rend opérationnellement lourde à implémenter.

L'accélérateur SdS proposé par Devineau et Loisel (2009) permet de réduire le nombre de simulations, à l'aide d'une procédure de sélection *a priori* des scénarios les plus adverses. Toutefois, la sélection est basée sur un critère donné, qui, bien que relativement efficace, peut se révéler simpliste et apparait optimisable. La Valeur Actuelle Nette de marges *forward* est une Valeur Actuelle Nette de marges calculée sur un scénario central, dans lequel toutes les variables financières suivent leur trajectoire espérée. Elle permet d'apporter une information de type *équivalent certain* (à une prime de risque près) quant à l'espérance des VAN de marge, les fonds propres économiques.

Cet article, présente des trauvaux initiés lors d'un mémoire d'actuariat réalisé par les auteurs. Il s'intéresse aux propriétés de cette Valeur Actuelle Nette de marges, dite *forward*, et aux moyens de l'utiliser dans un cadre opérationnel pour accélérer les approches simulatoires de calcul du capital économique en assurance vie.

Mots clés : *modèle interne, Solvabilité II, Simulations dans les Simulations, accélérateur SdS, VAN forward.*

Introduction

Les spécificités de l'activité d'assurance, et notamment l'inversion de son cycle de production, ont créé le besoin d'une réglementation adaptée, dont les missions principales sont la protection des assurés et la stabilité du système financier. Dans ce contexte, la directive européenne Solvabilité I a défini des critères quantitatifs de solvabilité pour tous les acteurs. Les règles de calcul, souvent trop générales, ont rapidement montré leurs limites. C'est pourquoi une nouvelle directive, Solvabilité II, doit être appliquée dès 2016.

Solvabilité II repose sur trois piliers, définissant à la fois des critères quantitatifs et qualitatifs pour garantir la solvabilité des assureurs. En particulier, le capital de solvabilité requis (*solvency capital requirement*, SCR) est le niveau de fonds propres dont doit disposer l'assureur pour éviter la ruine dans 99,5 % des cas sur un horizon d'un an. Celui-ci peut être calculé de deux façons différentes, dont le choix est laissé à la libre appréciation de l'assureur : par la formule standard, ou par un modèle interne, qu'il soit partiel ou total. La formule standard fournit des paramètres et des formules calibrés sur l'ensemble des données du marché. Elle présente l'inconvénient de ne pas prendre en compte fidèlement le profil de risque spécifique de l'assureur. Dans cet article, l'approche considérée est celle d'un modèle interne (complet ou partiel), c'est-à-dire entièrement développé en adéquation avec les risques auxquels est effectivement exposé le portefeuille.

S'agissant du calcul du SCR, la méthode de référence est celle des simulations imbriquées, appelée également méthode des simulations dans les simulations (SdS, voir Gordy and Juneja (2010), Broadie et al. (2011), Devineau (2011)). Elle consiste en la projection dans l'univers historique de facteurs économiques et financiers reflétant les différents scénarios possibles à l'horizon 1 an. Le bilan de la compagnie est ensuite valorisé, conditionnellement à ces projections primaires, à l'aide de nouvelles simulations, appelées simulations secondaires [∗]. Les Simulations dans les Simulations (SdS) constituent l'approche la plus précise, mais sa complexité algorithmique la rend difficilement utilisable opérationnellement. C'est pourquoi des méthodes alternatives ont été développées .

Il existe deux grands types d'approches alternatives. Premièrement, les méthodes dites *proxy*, qui reposent sur de nombreuses hypothèses de modélisation mais peuvent fournir de bons résultats, bien qu'approchés (voir, par exemple,Devineau and Chauvigny (2011), Bauer et al. (2010), Vedani and Devineau (2013)). Elles ont l'avantage d'être peu coûteuses en temps de calcul et de produire une distribution complète de la variable d'intérêt (fonds propres économiques) à 1 an. Deuxièmement, les méthodes basées sur la réduction du nombre de simulations qui nécessitent moins d'hypothèses, et donnent des résultats plus précis mais soit elles sont relativement lourdes à mettre en place (comme l'approche de Broadie et al. (2011) ou l'approche multi-level Monte Carlo, voir Giles (2008), Lemaire et al. (2014) et Lemaire et al. (2015)) soit elles ne permettent d'obtenir qu'une partie (la plus adverse) de la distribution de la variable d'intérêt). L'accélérateur SdS appartient à cette deuxième famille de méthodes, et s'avère très efficace opérationnellement, bien que ne permettant d'obtenir que les valeurs les plus adverses de la distribution empirique recherchée. Il consiste à sélectionner uniquement les scénarios primaires les plus adverses avant de procéder aux projections secondaires, afin de cibler directement le quantile à 0,5 % des fonds propres économiques. Une norme, définie sur l'espace des facteurs de risque, fournit un indicateur d'adversité des scénarios. Nous introduisons, dans ce qui

∗. Pour plus d'informations sur cette méthodologie, voir Devineau and Loisel (2009*b*).

suit, un nouveau critère permettant de mieux sélectionner les scénarios primaires et d'accélérer la convergence de l'accélérateur SdS : la Valeur Actuelle Nette de marges (*VAN*) *forward*.

Cet article débute par un rappel des concepts liés à l'activité d'assurance et aux contrats d'assurance vie épargne, faisant l'objet de l'étude. Puis, le contexte réglementaire et Solvabilité II sont introduits dans la deuxième partie, présentant ainsi le SCR et les notions de solvabilité économique. Ensuite, une revue de littérature concernant les méthodes de calcul du SCR, et en particulier l'accélérateur SdS, est menée dans la troisième partie. Les quatrième et cinquième parties introduisent la notion de *VAN forward* et étudient ses caractéristiques et sa pertinence, en vue d'un couplage avec l'accélérateur SdS. La sixième partie est dédiée à la comparaison des performances de l'accélérateur SdS avec plusieurs critères, dont la *VAN forward*. Enfin, la dernière partie est consacrée au développement d'une méthode bayésienne alternative à l'accélérateur SdS.

4.1 Approche simulatoire pour le calcul du capital réglementaire en assurance vie

4.1.1 Avant-propos : contexte réglementaire

Contrairement à la directive Solvabilité I, qui s'adapte aux principes comptables domestiques européens, la réglementation Solvabilité II considère le bilan sous l'angle *économique*, ce qui signifie que ses postes sont évalués à leur valeur de marché. Celle-ci correspond à la somme des flux futurs, actualisés sous une probabilité risque neutre $\mathscr{Q}$. En particulier, l'univers *market-consistent* permet de valoriser les fonds propres économiques FP_t, sans que l'aversion au risque ne biaise les résultats. On a :

$$FP_t = \mathbb{E}^{\mathscr{Q}}\left[\mathrm{RA}_t \mid \mathscr{F}_t\right],$$

où RA représente la somme des résultats futurs actualisés, et $\mathscr{F}_t$ l'ensemble de l'information financière disponible à la date t.

Solvabilité II impose aux assureurs de disposer d'une réserve de fonds propres suffisante pour honorer leurs engagements. Le pilier 1 de la directive fournit des critères quantitatifs précis, et définit notamment le capital économique .

Définition 4.1.1. *Le Solvency Capital Requirement (SCR), appelé également capital économique, correspond au montant des fonds propres dont doit disposer l'assureur en* $t = 0$ *pour éviter la ruine économique dans un an avec une probabilité de 99,5 %.*

Le SCR est généralement déterminé à l'aide de la formule suivante (sous certaines hypothèses),

$$SCR = FP_0 - \cdot q_{0,5\%}(D(0,1).FP_1),$$

avec FP_0 les fonds propres économiques initiaux de l'assureur, FP_1 ses fonds propres à l'horizon 1 an, $q_{0,5\%}(FP_1)$ le quantile à 99,5% de la distribution associée et $D(0,1)$ le déflateur permettant d'actualiser les flux entre $t = 0$ et $t = 1$.

Le calcul du capital réglementaire repose donc sur la connaissance à la date $t = 0$ de la distribution des fonds propres à horizon 1 an. Le problème posé à l'assureur est *l'absence de formules fermées* permettant de déterminer cette distribution. En effet, il existe des dépendances complexes entre l'actif et le passif du bilan économique de l'assureur. On peut citer, par exemple, le phénomène des rachats dynamiques.

Des modèles ont été développés dans la littérature pour valoriser le portefeuille par formules fermées, mais ceci au prix d'hypothèses restrictives. Guibert et al. (2010) montrent qu'une approximation du SCR peut être obtenue conditionnellement à la connaissance de caractéristiques liées aux contrats (premiers moments de la distribution des flux de trésorerie, etc.). Bonnin et al. (2014) s'appuient sur ce modèle et obtiennent des formules fermées dans le cas des contrats en euros, pour la valorisation des provisions en *Best Estimate* dans un cadre adapté à l'ORSA.

4.1.2 L'approche Simulations dans les Simulations

Le capital économique est inconnu à la date de valorisation puisqu'il dépend d'un grand nombre d'évènements pouvant intervenir de manière aléatoire dans le futur. Certains de ces évènements sont relatifs aux contrats eux-même, comme les rachats, la mortalité, les trajectoires possibles de nombreux indicateurs financiers, tels que des indices action ou des taux. Ils déterminent les valeurs du bilan économique de l'entreprise et sont donc une source d'incertitude. La méthode la plus précise pour estimer le SCR est basée sur la simulation de ces évènements multiples et le recalcul complet de l'actif et du passif en vision économique, dans un an.

La première étape de cette méthode nécessite la projection de scénarios économiques (les facteurs de risque) sur un horizon d'un an. Ces projections, appelées "simulations primaires", permettent de développer une grande variété de contextes économiques potentiels à 1 an. Ils sont ensuite utilisés pour conditionner la projection des flux futurs relatifs aux contrats et l'évolution de leur provision mathématique. Chacune de ces "simulations secondaires" permet de déterminer une VAN en $t = 1$ (VAN_1), correspondant à la valeur actualisée des cash-flows de résultats prospectifs à engranger par l'assureur. Les projections secondaires sont risque-neutres et ont un objectif de valorisation, distinct de l'objectif de réalisme des projections primaires. Leur horizon correspond à celui nécessaire à l'extinction du passif considéré, soit entre 30 et 50 ans en pratique.

Les fonds propres économiques en date $t = 1$ (FP_1) sont obtenus par moyenne des VAN_1 pour chaque simulation primaire p : $FP_1 = \mathbb{E}^{\mathscr{Q}}[VAN_1 \mid \mathscr{F}_1]$ où $\mathscr{F}_1$ est la filtration induite par l'information économique disponible en date $t = 1$, et $\mathscr{Q}$ est la probabilité de valorisation risque-neutre utilisée.

Ce calcul permet d'obtenir l'ensemble de la distribution de FP_1, mais requiert un grand nombre de simulations, ce qui est souvent très consommateur en temps de calcul. Cette méthode combinant simulations primaires puis secondaires dans une approche de type Monte-Carlo imbriqué est appelée Simulations dans les Simulations. Elle est très largement développée dans la littérature opérationnelle (voir Devineau and Loisel (2009*b*), Bergmann (2011)).

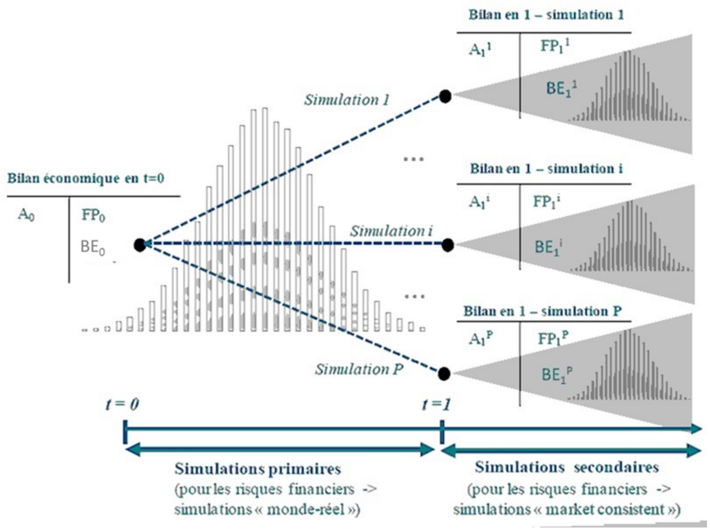

FIGURE 4.1 – Illustration de l'approche Simulations dans les Simulations (source : Milliman).

4.2 L'accélérateur SdS

En pratique, le SdS est fortement consommateur en ressources matérielles, humaines ainsi qu'en temps de calcul. De ce fait, plusieurs méthodes alternatives ont été récemment développées pour réduire le temps de calcul induit. Cette étude s'intéresse plus particulièrement à une des approches les plus efficaces, l'accélérateur SdS.

L'accélérateur SdS a été introduit pour la première fois dans Devineau and Loisel (2009*a*).

L'idée principale est la suivante : le calcul du SCR ne requiert pas la connaissance de l'intégralité de la distribution de la variable FP_1, mais uniquement le quantile à 0.5%. L'accélérateur va donc s'intéresser à la localisation des scénarios primaires les plus adverses, menant aux valeurs de FP_1 les plus faibles. Au lieu de générer l'intégralité des simulations secondaires, seules celles correspondant aux scénarios primaires sélectionnés vont être lancées.

L'adversité d'un scénario est mesurée par une norme sur les facteurs de risque primaires, c'est à dire, dans notre étude, un indice action et le taux d'intêret. D'autres normes peuvent être utilisées, comme la norme sensibilité qui pondère chaque facteur de risque par son influence sur la variable FP_1. C'est une version améliorée de la norme standard, permettant d'introduire un impact asymétrique des facteurs de risque.

4.2.1 Les facteurs de risque

Les simulations primaires consistent en la projection d'indicateurs économiques et financiers suivant des modèles de diffusion choisis : prix de l'action, valeur des taux, etc. Dans la suite, et ceci dans un objectif de simplification, les indicateurs considérés se limitent à l'indice action et à une courbe de taux zéro-coupons, mais ils sont souvent bien plus nombreux dans la réalité.

De ces scénarios sont extraits des *facteurs de risque*, qui synthétisent les informations liées aux simulations et reflètent leur adversité marginale en chaque sous-risque (action / taux). Ces facteurs de risque peuvent être différents des indicateurs projetés, mais leur connaissance permet de reconstituer les indicateurs initiaux. En d'autres termes, ils contiennent la totalité de l'information liée à un scénario donné.

Le choix des facteurs dépend de la méthode utilisée pour la projection. En pratique, plusieurs cas peuvent se présenter.

— Cas 1 : si l'on a accès à l'intégralité du modèle de projection primaire, il est possible de récupérer directement les facteurs de risque. Il s'agit du cas le plus simple, mais pas le plus courant.
— Cas 2 : le plus souvent, l'assureur dispose d'une table de scénarios économiques contenant uniquement les valeurs des indices action et taux pour la date $t = 1$. Il doit alors en déduire les facteurs de risque en inversant les équations des modèles. On appelle ce procédé une *obtention a posteriori*.

L'extraction des facteurs de risque à partir des scénarios primaires peut être réalisée dans l'algorithme de génération des simulations en univers monde-réel (voir L. Devineau et S. Loisel (2009)) toutefois cela peut aussi être réalisé *a posteriori*, connaissant la table des P scénarios historiques utilisée.

Il est ainsi possible de retrouver les facteurs de risque, pour le scénario p, par les formules suivantes, en notant $S(t)^p$ l'indice action associé à la simulation primaire p, en date $t \in \{0;1\}$ et $ZC(t,m)^p$ le prix zéro-coupon de maturité $m \in [\![1;m_{max}]\!]$, associé à la simulation primaire p, en date $t \in \{0;1\}$;

$$\varepsilon_1^{a,p} = \frac{1}{\hat{\sigma}^a} \left(\ln\left(\frac{S^p(1)}{S(0)}\right) - \hat{\mu}^a \right), \tag{4.1}$$

pour le facteur action et

$$\varepsilon_1^{\mathrm{ZC},p} = \frac{1}{m_{\max}-1} \sum_{m=1}^{m_{\max}} \frac{1}{\hat{\sigma}_m^{\mathrm{ZC}}} \left(\ln\left(\frac{ZC^p(1,1+m)}{ZC(0,1+m)}\right) - \hat{\mu}_m^{\mathrm{ZC}} \right) \tag{4.2}$$

pour le facteur taux.

On note dans ces formules

—

$$(\sigma^a)^2 = \mathbb{E}\left[\left(\ln\left(\frac{S(1)}{S(0)}\right) - \mathbb{E}\left[\ln\left(\frac{S(1)}{S(0)}\right) \mid \mathscr{F}_1\right] \mid \mathscr{F}_1\right)^2\right],$$

$$\Rightarrow \quad \hat{\sigma}^a = \sqrt{\frac{1}{P-1}\sum_{p=1}^{P}\left(\ln\left(\frac{S^p(1)}{S(0)}\right) - \frac{1}{P}\sum_{i=1}^{P}\ln\left(\frac{S^i(1)}{S(0)}\right)\right)^2}, \tag{4.3}$$

l'estimateur de la volatilité des log-rendements action entre $t = 0$ et $t = 1$,

—

$$\mu^a = \mathbb{E}\left[\ln\left(\frac{S(1)}{S(0)}\right) \mid \mathscr{F}_1\right]$$
$$\Rightarrow \quad \hat{\mu}^a = \frac{1}{P}\sum_{p=1}^{P}\ln\left(\frac{S^p(1)}{S(0)}\right), \tag{4.4}$$

l'estimateur du log-rendement action moyen entre $t = 0$ et $t = 1$,

—

$$\left(\sigma_m^{ZC}\right)^2 = \mathbb{E}\left[\left(\ln\left(\frac{ZC(1,m+1)}{ZC(1,1)ZC(0,m)}\right) - \mathbb{E}\left[\ln\left(\frac{ZC(1,m+1)}{ZC(1,1)ZC(0,m)}\right) \mid \mathscr{F}_1\right] \mid \mathscr{F}_1\right)^2\right]$$
$$\Rightarrow \quad \hat{\sigma}_m^{ZC} = \sqrt{\frac{1}{P-1}\sum_{p=1}^{P}\left(\ln\left(\frac{ZC^p(1,m+1)}{ZC^p(1,1)ZC(0,m)}\right) - \frac{1}{P}\sum_{i=1}^{P}\ln\left(\frac{ZC^i(1,m+1)}{ZC^p(1,1)ZC(0,m)}\right)\right)^2}, \tag{4.5}$$

l'estimateur de la volatilité des log-rendements zéro-coupon entre $t = 0$ et $t = 1$, pour la maturité $m \in [\![1; m_{max} - 1]\!]$,

—

$$\mu_m^{ZC} = \mathbb{E}\left[\ln\left(\frac{ZC(1,m+1)}{ZC(1,1)ZC(0,m}\right) \mid \mathscr{F}_1\right] \tag{4.6}$$
$$\Rightarrow \quad \hat{\mu}_m^{ZC} = \frac{1}{P}\sum_{p=1}^{P}\ln\left(\frac{ZC^p(1,m+1)}{ZC(1,1)ZC(0,m)}\right), \tag{4.7}$$

l'estimateur du log-rendement zéro-coupon moyen entre $t = 0$ et $t = 1$, pour la maturité $m \in [\![1; m_{max} - 1]\!]$.

Remarque 4.2.1. *La méthode de l'accélérateur SdS, à travers ses modèles de diffusion, repose sur l'hypothèse fondamentale suivante : les facteurs de risque forment un* vecteur gaussien dont les marginales sont centrées réduites.

Les facteurs de risque utilisés pour quantifier l'adversité des scénarios forment donc un vecteur gaussien centré réduit $\varepsilon = (\varepsilon_1^a, \varepsilon_1^{ZC})$. Plus précisément, il existe un vecteur pour chaque simulation primaire, et les facteurs de risque forment une matrice de dimension $P \times 2$: $(\varepsilon^1, \ldots, \varepsilon^P)$.

On remarque qu'une *hausse* du facteur de risque ε^{ZC} correspond à une hausse du prix du zéro-coupon, et donc à une *baisse des taux.*

Sélection des scénarios extrêmes sur la base d'une norme standard

La procédure de sélection des scénarios primaires repose sur la définition de l'adversité pour le portefeuille considéré. Il s'agit de déterminer si celui-ci est exposé à la hausse ou à la baisse des taux, à la baisse de l'indice action, etc.

Intuitivement, il est clair que les scénarios les plus adverses seront ceux pour lesquels les composantes de $\varepsilon^p = (\varepsilon^{a,p}, \varepsilon^{ZC,p})$ prennent des valeurs extrêmes, dont l'orientation est à déterminer. Les scénarios dits "centraux", c'est-à-dire, dont les aléas sont relativement proches de 0 entre les dates $t = 0$ et $t = 1$, ne conduiront pas à des valeurs extrêmes de fonds propres.

L'objectif est donc de déterminer *l'intensité du risque* ε^p associé à chaque scénario primaire p, afin de procéder à la sélection. Le facteur de risque choisi pour discriminer les scénarios contient l'ensemble de l'information permettant de refléter leur adversité et de prédire la position approximative des fonds propres correspondants dans la distribution finale.

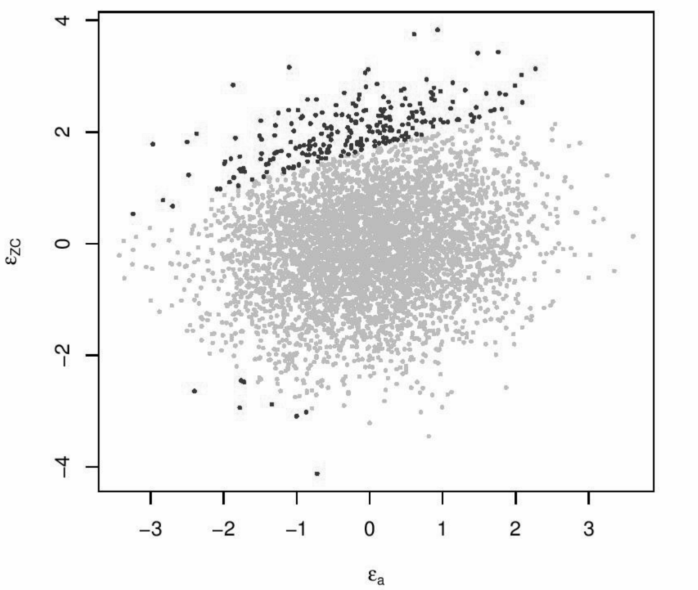

FIGURE 4.2 – Facteurs de risque primaires $(\varepsilon^a, \varepsilon^{ZC})$.

La Figure 4.2 est basée sur des données réelles obtenues à date du 31/12/2011. Elle représente l'ensemble du nuage $\varepsilon = (\varepsilon^a, \varepsilon^{ZC})$ des P aléas de première période. En noir sont représentés les points correspondant aux fonds propres de la queue de distribution au-delà du quantile à 5 %. Le produit considéré est un produit d'épargne réel avec taux minimum garanti prenant en compte des rachats dynamiques dépendant des taux de rendement d'actifs. L'actif associé au produit est composé d'une poche obligation, d'une poche action, d'une poche immobilier et de cash.

On remarque que ceux des scénarios les plus adverses sont situés en périphérie du nuage de points. Plus précisément, l'assureur est très exposé à la baisse des taux (hausse du prix zéro-coupon) et peu sensible à un mouvement de l'action (bien que plus exposé à la baisse). En règle générale, un produit d'épargne est plutôt exposé à une hausse des taux (associée à des rachats), et sensible à l'action. Ces observations inhabituelles s'expliquent par le contexte économique au moment de la simulation (historique utilisé). La Figure 4.2 reprend les données à fin 2011. Or, à cette date, les taux directeurs étaient extrêmement bas. Par conséquent, une baisse supplémentaire des taux est plus à craindre qu'une hausse (ce qui n'est pas le cas en général). Le niveau des taux explique aussi la faible sensibilité au risque action, dans la mesure où une baisse supplémentaire des taux, déjà très bas, a plus de conséquences qu'une baisse de l'action. Les sensibilités des fonds propres aux différents risques sont donc très déséquilibrées dans ce cas.

Remarque 4.2.2. *Dans toute la suite, et sauf indication contraire, les études utilisent les données issues d'un modèle de gestion Actif-Passif standard à la date du 31/12/2011. Elles sont caractérisées par un nombre de scénarios primaires $P = 5\,000$, et un nombre de scénarios secondaires $S = 500$*. C'est pourquoi, on parlera désormais des 25 pires fonds propres, correspondant à la queue de distribution au-delà du quantile à 0,5 % :* $\alpha \cdot P = 5\,000 \cdot 0,5\ \% = 25$.

De façon à mettre en évidence la sensibilité de l'adversité à la situation économique, ces données peuvent être comparées à celles de fin 2010 (avec le même générateur). Le nouveau nuage de points est présenté dans la Figure 4.3.

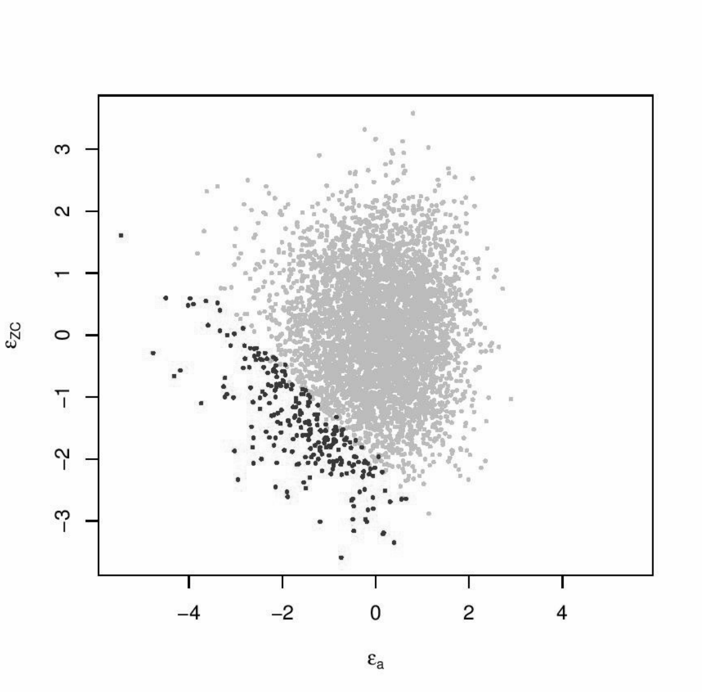

FIGURE 4.3 – Facteurs de risque primaires $(\varepsilon^a, \varepsilon^{ZC})$.

Avec ces données, les taux de base initiaux sont plus élevés (pour exemple, le taux directeur de la BCE était de 1 % en décembre 2011, contre 1,25 % fin 2010). Ainsi, le contexte économique est relativement plus usuel et l'exposition présente les caractéristiques plus standard pour les assurances vie : exposition à la hausse des taux et à la baisse de l'action, avec des sensibilités équivalentes.

Ces graphiques démontrent que, selon la situation dans laquelle se trouve l'économie au moment de la projection, les adversités seront définies différemment. Elles dépendent de plusieurs paramètres,

1. les sensibilités respectives aux taux et à l'action ;

*. Ces dimensions simulatoires sont relativement faibles mais fournissent un premier cadre d'expérimentation rapide et pertinent pour notre analyse

2. l'exposition aux taux : elle peut être à la hausse, à la baisse, ou aux deux, dépendant de la sensibilité du produit au risque de rachat (risque de hausse des taux) comparé au risque lié au versement du taux minimum garanti (risque de baisse des taux) ;
3. l'exposition à l'action est, quant à elle, toujours à la baisse. Cependant, elle peut être faible si la sensibilité au taux devient très importante (comme c'est le cas sur les données de 2011).

Malgré le fait que l'adversité des scénarios dépende de la conjoncture économique, il est clair que les scénarios extrêmes en termes de fonds propres sont systématiquement situés en périphérie du nuage de points. Cela justifie le choix d'une norme comme indicateur d'adversité, car elle évalue la distance d'un scénario donné au scénario "moyen" (situé au centre du nuage). Il s'agit donc d'une mesure de l'extrémité du scénario considéré.

Pour rappel, d'après le chapitre 4.2.1, le vecteur ε est gaussien, et, par conséquent, la densité de celui-ci s'écrit :

$$\begin{aligned} f_\varepsilon(\varepsilon^a, \varepsilon^{ZC}) &= \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(\varepsilon^a)^2 + (\varepsilon^{ZC})^2 - 2\rho\,\varepsilon^a\,\varepsilon^{ZC}}{2(1-\rho^2)}\right) \\ &= K_1 \exp\left(K_2((\varepsilon^a)^2 + (\varepsilon^{ZC})^2 - 2\rho\,\varepsilon^a\,\varepsilon^{ZC})\right), \end{aligned}$$

où K_1 et K_2 sont deux constantes.

La densité des scénarios adverses est faible, car ceux-ci sont définis comme ayant des aléas extrêmes. Cela permet de définir la norme suivante comme mesure d'adversité :

$$\left\|\varepsilon^a, \varepsilon^{ZC}\right\| = \sqrt{(\varepsilon^a)^2 + (\varepsilon^{ZC})^2 - 2\rho\,\varepsilon^a\,\varepsilon^{ZC}}. \tag{4.8}$$

On reconnaît au passage l'équation d'une ellipse. Plus cette norme est grande, plus le scénario est adverse. On définit alors une zone d'exécution F_h de niveau h, regroupant les scénarios primaires dont la norme est supérieure au seuil h,

$$F_h = \{(\varepsilon^a, \varepsilon^{ZC}) : \left\|\varepsilon^a, \varepsilon^{ZC}\right\| \geq h\}, \tag{4.9}$$

où h est un seuil qui dépend du nombre de scénarios que l'on souhaite prendre en compte dans F_h.

La Figure 4.4 représente les zones d'exécution successives associées à un nombre de points $M = 100$. Elles correspondent aux surfaces entre ellipses, allant de la périphérie vers le centre. À chaque étape, h est calculé de telle sorte que,

$$\mathrm{Card}(F_{h+1}) - \mathrm{Card}(F_h) = 100$$

.

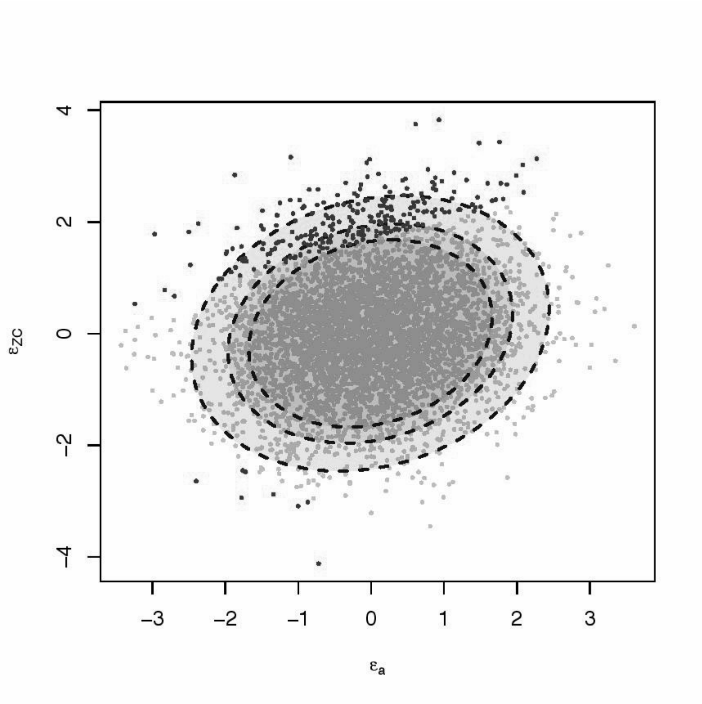

FIGURE 4.4 – Ellipses de sélection des facteurs de risque primaires $(\varepsilon^a, \varepsilon^{ZC})$ (source : données Milliman).

Sources d'amélioration

L'utilisation de la norme définie dans la partie précédente ne permet pas d'écarter des scénarios extrêmes mais pouvant être bénéfiques pour l'assureur. Une très forte hausse de l'indice action, par exemple, sera considérée comme un scénario adverse, alors qu'il ne l'est pas. En effet, l'assureur est un investisseur institutionnel et ne spécule donc pas à la baisse sur le marché boursier.

Par conséquent, une amélioration proposée par les auteurs est la définition de régions de non-exécution. Celles-ci définissent un ensemble de scénarios extrêmes ne pouvant conduire *de facto* à des fonds propres faibles, et accélèrent davantage la convergence de l'algorithme. Grâce à cette méthode, le temps de calcul est réduit en moyenne d'un facteur 25.

Une autre amélioration importante concerne le cas des facteurs multiples. En effet, il est possible que les facteurs de risque soient nombreux, mais que l'adversité des scénarios ne soit due qu'à certains d'entre eux. Dans ce cas, les autres facteurs seront des bruits et n'apporteront pas d'information significative, ralentissant ainsi l'algorithme. Pour pallier ce problème, les auteurs ont introduit la notion de *sensibilité à chaque facteur*. Elle permet de les pondérer et de donner moins de poids à ceux qui impactent peu la valeur des fonds propres.

On définit d'abord le risque pur associé à un facteur de risque. Notons pour cela $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ le vecteur gaussien des n aléas de première période (facteurs de risque).

Soit $\Omega = VV'$ la matrice de variance-covariance du vecteur ε et sa décomposition de Cholesky. Ainsi,

$\zeta = V^{-1}\,\varepsilon$ est un nouveau vecteur gaussien dont les marginales sont indépendantes, et correspond aux risques purs associés à chacun des facteurs de risque de ε.

La norme ajustée à la sensibilité de chaque facteur est alors :

$$\|\varepsilon\|^{\text{sensi}} = \sqrt{s_1\,\zeta_1^2 + \cdots + s_n\,\zeta_n^2}, \tag{4.10}$$

où le vecteur $(s_1, \ldots, s_n)$ représente les sensibilités des fonds propres aux différents facteurs. Cela permet de sous-pondérer les facteurs peu influents, de façon à ce qu'ils n'aient que peu d'impact sur la norme.

Remarque 4.2.3. *En pratique, les s_i sont obtenus après avoir appliqué des chocs de même amplitude aux différents facteurs de risque, puis en mesurant l'impact relatif de chacun d'entre eux sur les fonds propres.*

Une norme alternative peut être introduite afin d'affiner les sensibilités précédentes : la norme sensibilité asymétrique. Elle permet de pondérer différemment les facteurs à la hausse et à la baisse. Dans cette étude, les sensibilités sont toujours symétriques pour l'action, mais différentes pour les hausses et les baisses de taux. La nouvelle norme s'écrit :

$$\|\varepsilon\|^{\text{sensi asymétrique}} = \sqrt{s_1\,\zeta_1^2 + s_2\,\zeta_2^2\,\infty_{(\zeta_2 \geq 0)} + s_3\,\zeta_2^2\,\infty_{(\zeta_2 < 0)}},$$

avec s_1, s_2 et s_3 les sensibilités à l'action, la baisse et la hausse des taux respectivement.

4.2.2 Algorithme

L'algorithme suivi est le suivant,

1. Selection des M scénarios les plus adverses au sens de la norme choisie. Dans l'article original de L. Devineau et S. Loisel, $M = 4N$, avec $N = 0.5\ \% \times$ nombre de scénarios primaires.
2. Simulation des scénarios secondaires associés aux scénarios primaires sélectionnés.
3. Sélection des 0.5 % pires valeurs de fonds propres, parmi les M précédents.
4. Sélection des M scénarios les plus adverses suivants, et itération des étapes précédentes jusqu'à ce que le vecteur des 0.5 % pires valeurs se stabilise.

Remarque 4.2.4. *Le choix de M résulte d'un compromis entre efficacité et précision. Si M est trop élevé, la vitesse de convergence est réduite du fait des nombreuses valeurs de fonds propres à recalculer à chaque étape de l'algorithme. Si M est trop faible, la probabilité de se tromper et de manquer une des vraies valeurs adverses va augmenter et l'algorithme peut converger vers un mauvais vecteur des pires valeurs.*

4.3 La *VAN* *forward*

L'objectif est ici d'étudier un nouvel indicateur d'adversité pour les scénarios primaires, la *VAN forward*. Intuitivement, une *VAN* intègre plus d'informations que la norme sensibilité puisque cette dernière est une mesure marginale qui n'intègre pas les effets croisés entre les risques. De plus une *VAN* donne une information sur la manière dont le modèle ALM réagit conjointement aux différents facteurs de risque. Les effets croisés ne doivent pas être négligés car ils peuvent causer de graves pertes dans des scénarios de crise (par exemple, lorsqu'une baisse de l'indice action exacerbe l'effet d'un rachat massif). Toutefois, une *VAN* peut aussi se révéler extrêmement volatile car elle intègre une grosse partie du risque lié au scénario secondaire considéré.

La *VAN forward* est une *VAN* particulière, obtenue à l'aide d'un scénario secondaire *déterministe* pour lequel les facteurs de risque (indice action et taux d'intérêt dans notre exemple) suivent respectivement leurs trajectoires espérées ("forward"). Ces trajectoires sont obtenues en capitalisant simplement les indices aux taux *forwards*,

$$\forall t \in [\![2, H]\!], \qquad {}^{f}S(t) = {}^{f}S(t-1)\,\frac{1}{{}^{f}ZC(t-1,t)} \tag{4.11}$$

et

$$\forall t \in [\![2, H]\!], \forall m \in [\![1, m_{\max}]\!], \qquad {}^{f}ZC(t, t+m) = {}^{f}ZC(t-1, t+m)\,\frac{1}{{}^{f}ZC(t-1,t) = \frac{{}^{f}ZC(1,t+m)}{{}^{f}ZC(1,t)}}, \tag{4.12}$$

en notant les valeurs initiales ${}^{f}S(1) = S(1)$ et ${}^{f}ZC(1, 1+m) = ZC(1, 1+m)$.

Dans le cas des simulations primaires, les facteurs de risque sont représentés par les aléas associés aux simulations en univers monde-réel $\varepsilon = (\varepsilon_1^a, \varepsilon_1^{ZC})$. Des facteurs de risque peuvent également être définis pour les simulations secondaires. Soit η ces jeux de facteurs secondaires.

Remarque 4.3.1. *En pratique il existe $H-1$ facteurs de risque secondaires action et $(H-1)\cdot m_{max}$ facteurs de risque secondaires taux.*

En réalité la *VAN* est une fonctionnelle de jeux de facteurs de risque primaire et secondaire,

$VAN = f(\varepsilon, \eta)$,

et la VAN *forward* est associée à des facteurs de risque secondaires particuliers,

${}^{f}VAN = f(\varepsilon, {}^{f}\eta)$.

Le concept de la *VAN forward* repose sur l'utilisation des diffusions *forward* pour les facteurs de risque lors des simulations secondaires. L'aléa de seconde période et la simulation sont déterministes vue de l'instant $t = 1$.

Ainsi, au lieu de simuler S trajectoires secondaires permettant d'obtenir S valeurs possibles de VAN_1, *une seule simulation déterministe* est effectuée pour calculer la *VAN forward* en $t = 1$, ${}^{f}VAN_1$, à partir des trajectoires $(S(t))_{t\in[\![2,H]\!]}$ et $(ZC(t, t+m))_{(t,m)\in[\![2,H]\!]\times[\![1,m_{\max}]\!]}$.

Le scénario *forward* est un scénario *central* donnant une information non bruitée (déterministe), bien qu'imparfaite, sur la valeur probable des fonds propres correspondants. En pratique, on voit que le graphe fonds propres vs. *VAN forward* montre une importante corrélation entre ces deux variables (voir Figure 4.5), ce qui suggère que la *VAN forward* est un indicateur de l'adversité des scénarios primaires relativement efficace. D'autre part, notons que le calcul des *P VAN forward* (associées à nos *P* scénarios primaires), ne nécessitant qu'un unique scénario secondaire pour chaque scénario primaire, est très rapide.

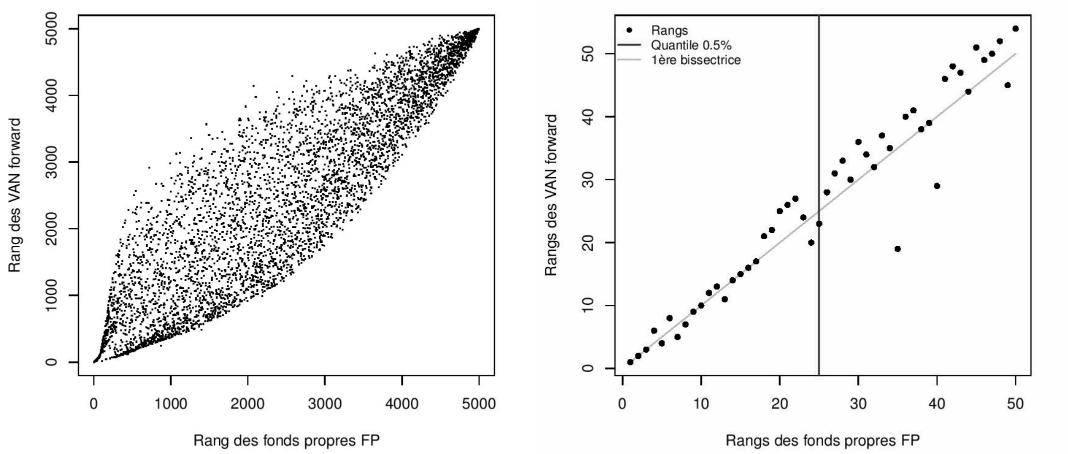

FIGURE 4.5 – Rangs des *VAN forward* en fonction des rangs des fonds propres (à gauche) et zoom sur la queue basse (à droite).

La première étape de notre étude est de prouver la supériorité de la *VAN forward* comparé à un tirage (aléatoire) de *VAN* ordinaire, et de quantifier cette différence. Pour cela, proposons une nouvelle manière de visualiser la *VAN*,

$$VAN_1 = FP_1 + \varepsilon,$$

où ε est une variable centrée réduite. Cela vient du fait que les fonds propres à 1 an se calculent comme une moyenne de VAN_1.

Utilisant cette vision plus statistique afin de comparer *VAN* et *VAN forward*, il est possible d'étudier les propriétés structurelles de la *VAN forward*. Nous intuitons en particulier qu'elle représente un scénario relativement central. Pour confirmer empiriquement cette hypothèse nous avons implémenté un SdS complet sur un produit d'épargne synthétique, afin de déterminer les distributions conditionnelles (par rapport au réalisé primaire) des *VAN*. Nous obtenons finalement les résultats suivants,

— la forme de la volatilité des *VAN* pour les réalisations les plus adverses des fonds propres (La queue de distribution sur les 30 premiers pourcents) ;
— le quantile de la distribution des *VAN* auquel correspond la *VAN forward*, pour tous les scénarios primaires considérés.

La Figure 4.6 illustre la position quantile de la *VAN forward* ainsi que la volatilité de la variable de bruit (ε), en fonction de la valeur des fonds propres associée. Elle montre que la *VAN forward* est toujours comprise entre les quantiles à 50 % et à 75 % de la distribution des *VAN*, quelle que soit la valeur des fonds propres économiques associée. Ce résultat tend à confirmer notre première intuition : la *VAN forward* est bien une *VAN* "centrale". De plus, la volatilité décroit *grossièrement* linéairement avec les fonds propres.

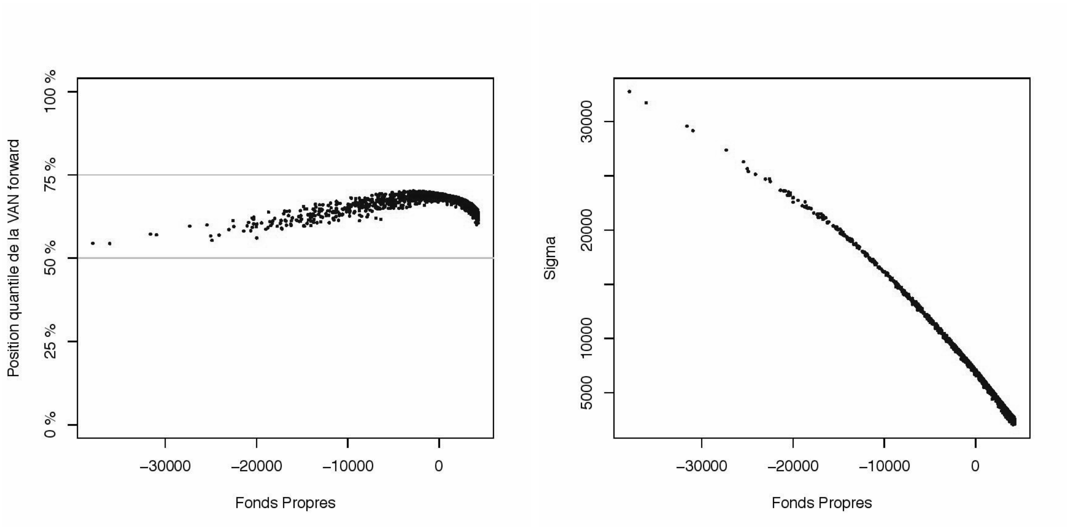

FIGURE 4.6 – Position quantile de la *VAN forward* (gauche) et volatilité empirique de la *VAN*, conditionnellement à la valeur des fonds propres associée (droite).

Ces résultats, ainsi que la définition ci-dessus, sont ensuite utilisés dans le cadre d'une études statistique. La *VAN* est supposée suivre une loi Gaussienne centrée sur les fonds propres économiques, et avec pour volatilité $\sigma(FP_1)$, une fonction linéaire décroissante de FP_1. La *VAN forward* est simulée en considérant un bruit modifié ε', tiré aléatoirement entre les quantiles à 50 % et à 70 % de $\varepsilon \sim \mathcal{N}(0, \sigma(FP_1)^2)$. Notons que la volatilité est définie relativement à la valeur des fonds propres associés. Elle correspond à la volatilité de la *VAN*, conditionnellement à la réalisation des facteurs de risque primaires ayant menés à la valeur de FP_1. Le choix de ce cadre hétéroscédastique est plus général que de considérer une volatilité stable, hypothèse peu réaliste en pratique.

4.3.1 Probabilité de croisement

Sous l'hypothèse de linéarité pour $\sigma(FP_1)$, il est possible d'évaluer la probabilité pour deux valeurs successives de *VAN*, que les valeurs associées de FP_1 aient un ordre inversé. Moins ce type de "croisement" a de risque d'apparaître, plus l'information fournie par la *VAN* est précise.

La procédure d'estimation se base sur des simulations numériques. Considérant le vecteur des 5000 valeurs de fonds propres en $t = 1$ obtenu par SdS complet, nous simulons $B = 1\,000$ vecteurs de 5000 *VAN* aléatoires et de *VAN forward*, en utilisant le modèle proposé plus haut($VAN = FP + \varepsilon$ et ${}^{f}VAN = FP + \varepsilon'$). Pour chaque simulation, la probabilité de croisement est définie comme la proportion des couples de réalisations de *VAN* dont les rangs sont inversés comparé à ceux des fonds propres correspondants.

La Figure 4.7 représente la probabilité de croisement, dans la queue de distribution à 0.5% de FP_1, pour la *VAN* ordinaire (en bleu) et pour la *VAN forward* (en rouge). Cette dernière est très intéressante du fait de sa haute corrélation avec la variable FP_1, mais aussi de par le fait que l'ordre d'adversité est mieux préservé. Dans l'objectif d'appliquer l'accélérateur SdS, la *VAN forward* semble donc être un critère de sélection particulièrement pertinent.

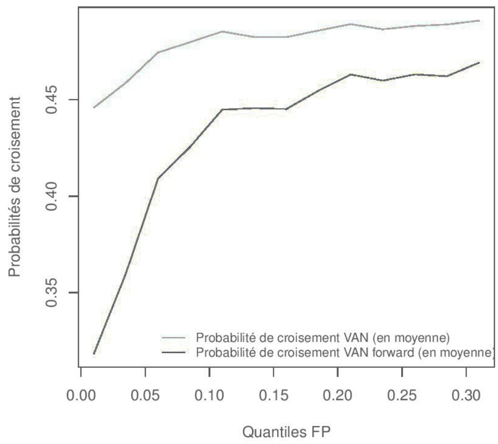

FIGURE 4.7 – Probabilités de croisement dans la queue de distribution, pour des *VAN* ordinaires (en gris) et pour la *VAN forward* (en noir)

4.3.2 Complexité algorithmique

La comparaison entre *VAN* ordinaire et *VAN forward* a été réalisée dans la perspective de combiner leurs propriétés à l'algorithme de l'accélérateur SdS. Une nouvelle fois, nous étayons notre réflexion à l'aide de données empiriques afin d'évaluer la vitesse de convergence de cet outil lorsque la *VAN forward* est utilisée comme critère d'adversité.

Cette mise en œuvre de l'algorithme est similaire à la précédente. À partir de nos B échantillons simulés de *VAN* et de *VAN forward*, nous observons le rang de la *VAN* nécessaire afin que l'algorithme converge (sur les 25 pires valeurs de FP_1). Soit N_{min} ce nombre d'exploration de *VAN* (i.e. de calcul de fonds propres). Grâce aux B simulations, nous obtenons une distribution empirique de N_{min}.

Les étapes de la méthode sont les suivantes, pour une simulation b donnée,

1. Bruitage, de toutes les valeurs FP_1^i par un bruit gaussien de variance définie. On note,

$$\Lambda_b = (VAN_1^{1,b}, \dots, VAN_1^{P,b}),$$

le vecteur des variables bruitées, soit, $VAN_1^i \overset{\mathscr{L}}{=} FP_1^i + U_i$. Dans le cas de cette étude, on considère, par convention, que les FP_1^i sont triés, c'est-à-dire, $FP_1^1 < FP_1^2 < \cdots < FP_1^P$ (les *VAN* ne le sont pas car il peut y avoir des croisements).

2. Observation des rangs de $VAN_1^{1,b}, \ldots, VAN_1^{25,b}$ dans le vecteur Λ_b. Le rang de $VAN_1^{i,b}$ correspond au nombre de réalisations de VAN_1 qui doivent être explorées avant de trouver les $i^{\text{ème}}$ pires fonds propres (dans le cas où l'on utiliserait l'algorithme SdS). On peut donc en déduire le nombre minimal de points à prendre en compte dans l'algorithme avant d'obtenir les 25 pires fonds propres (noté N_{min}),

$$N_{min,b} = \max_{i \in [\![1,25]\!]} \text{rang}(VAN_1^{i,b}).$$

Plus $N_{min,b}$ est faible, plus l'algorithme sera efficace, car il convergera en peu d'étapes (le bruitage entraîne peu de croisements).

3. Ces étapes sont répétées un grand nombre B de fois, de façon à obtenir une distribution pour la variable N_{min},

$$(N_{min,1}, \ldots, N_{min,B}).$$

La Figure 4.8 donne une illustration graphique de ce procédé.

Sur le schéma, les fonds propres triés $FP_1^{(1)}, \ldots, FP_1^{(5000)}$ correspondent à $FP_1^1, \ldots, FP_1^{5000}$ avec les notations ci-dessus. Le bruitage des fonds propres les plus faibles (notés $FP^{(1)}$ sur la figure et FP_1^1 ci-dessus) donne $VAN_1^{(17)}$, correspondant à VAN_1^1 (VAN associée aux fonds propres de rang 1 initialement). Ainsi, l'algorithme SdS devra explorer au moins 17 *VAN* avant de trouver le vecteur $FP_1^{(1)}, \ldots, FP_1^{(25)}$.

La Figure 4.10 donne la distribution empirique de N_{min} pour la *VAN* ordinaire et pour la *VAN forward*, lorsque l'hypothèse sur $\sigma(FP_1)$ est en adéquation avec la réalité observée. On observe, sans surprise, que la *VAN forward* est un critère beaucoup plus efficace que la *VAN* ordinaire. De fait, la distribution de N_{min} pour cette variable est concentrée autour de 27 et n'excède jamais 36. Au contraire, la distribution associée à la *VAN* ordinaire à une queue de distribution épaisse montrant, en particulier, que la probabilité que l'algorithme converge après plus de 2000 calculs de fonds propres n'est pas nulle.

Ces résultats, bien qu'ils reposent sur des hypothèses particulières de modélisation, suggèrent que la *VAN forward* possède des propriétés très intéressantes.

— la distribution de N_{min} montre que la combinaison de cette variable avec l'accélérateur SdS est très efficace, du fait que 30 calculs de FP_1 sont généralement suffisants afin d'obtenir l'intégralité de la queue de distribution à 0.5 % ;
— elle montre aussi que la probabilité de piégeage (l'algorithme converge, mais vers de mauvaises valeurs) est insignifiante. En conséquence, M peut être choisi plus proche de 25, ce qui évite des calculs inutiles au cours de la dernière itération de l'algorithme, qui ne sert qu' à valider la convergence.

4.3.3 Combiner la *VAN forward* avec l'accélérateur SdS

Les premières analyses numériques ont permis de souligner les bonnes propriétés de la *VAN forward* et suggèrent une bonne efficacité pour le couplage à l'accélérateur SdS. Toutefois, les hypothèses simulatoires sont basées sur les résultats d'un modèle de gestion actif-passif simplifié ce qui , par

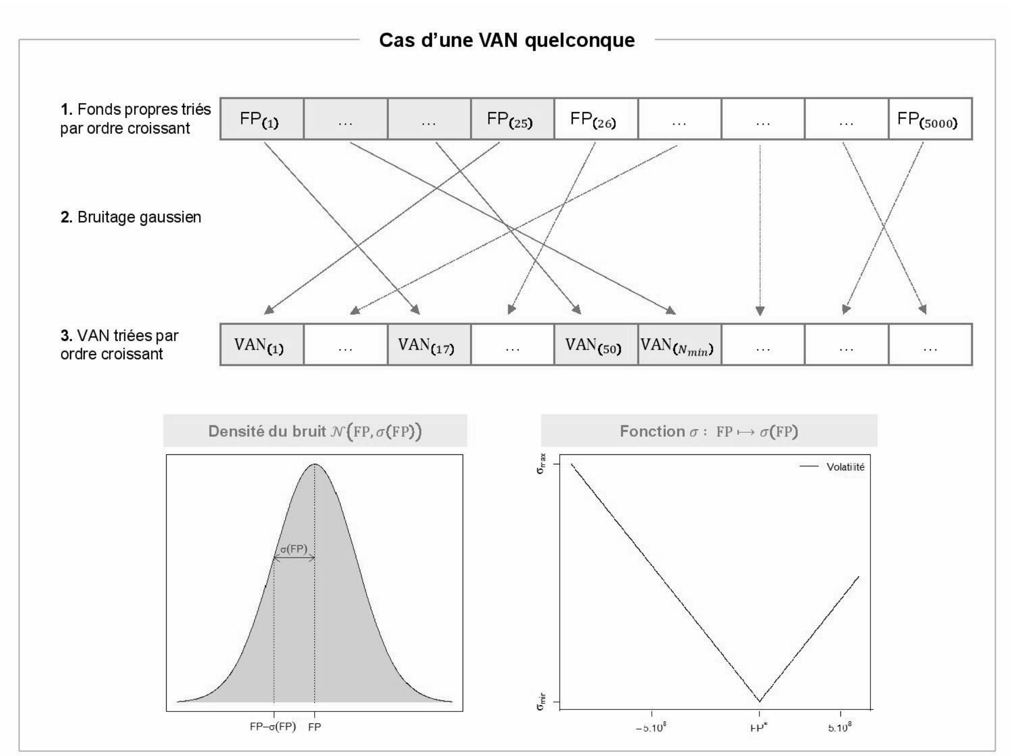

FIGURE 4.8 – Bruitage de la distribution pour trouver N_{min} dans le cas d'une *VAN* quelconque.

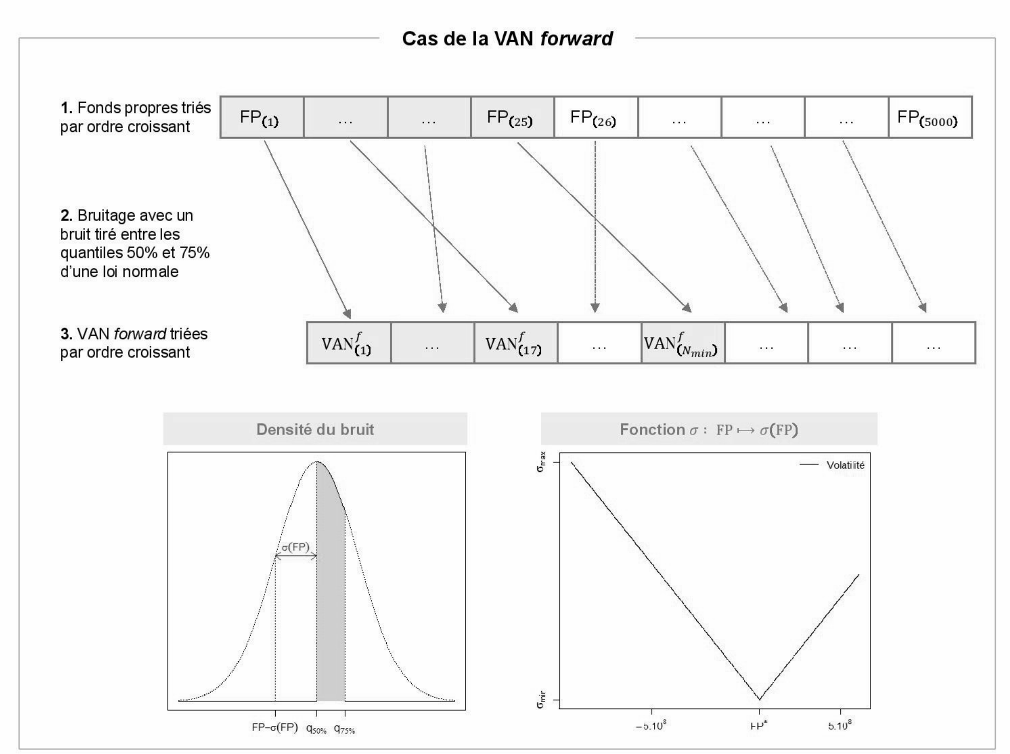

FIGURE 4.9 – Bruitage de la distribution pour trouver N_{min} dans le cas d'une *VAN forward*.

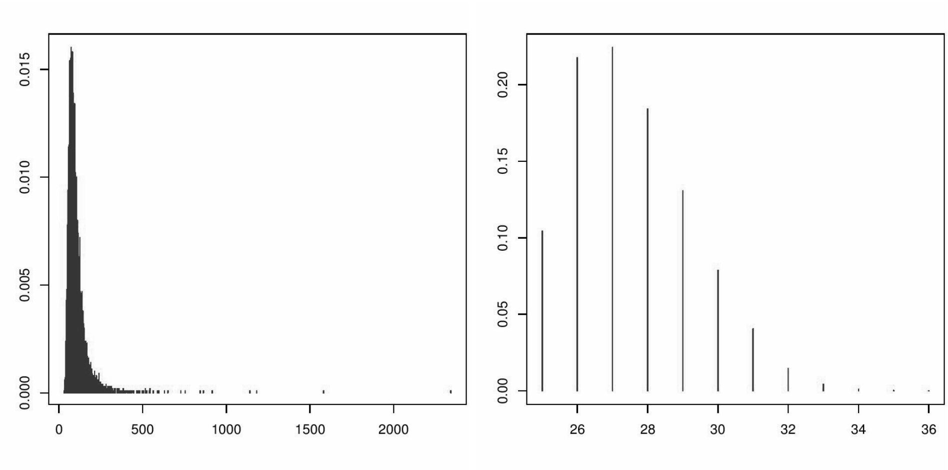

FIGURE 4.10 – Distribution de N_{min} pour une *VAN* ordinaire (gauche) et pour la *VAN forward* (droite).

définition, peut s'avérer éloigné de la réalité. L'objectif de cette sous-section est de confirmer la performance de la *VAN forward* en appliquant l'algorithme à des données obtenues pour un produit d'épargne réel avec taux minimum garanti. Le modèle ALM utilisé permet de modéliser le mécanisme de participation aux bénéfices, ainsi que les comportements de rachats dynamiques des assurés. Il intègre de plus des hypothèses réalistes de rebalancements d'actif. Les données finales consistent en un échantillon de valeurs de FP_1 obtenu par approche SdS, ainsi que l'échantillon des *VAN forward* associées.

L'idée de notre méthode d'accélération est de pallier au manque d'information de la norme sensibilité en utilisant la *VAN forward* comme critère d'adversité dans l'algorithme standard de l'accélérateur SdS. La performance de cette approche est challengée par l'application de l'algorithme en utilisant d'autres critères d'adversité : la norme standard et la norme sensibilité. L'objectif est ainsi de comparer le nombre d'itérations requises pour atteindre la convergence de l'algorithme, lorsque M varie. Les résultats sont présentés dans la Figure 4.11.

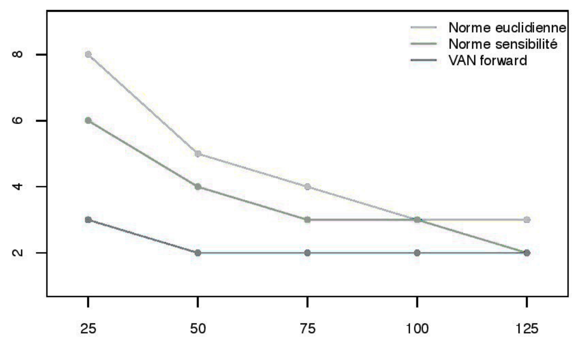

FIGURE 4.11 – Vitesses des convergences (en nombre d'itérations de l'algorithme par valeur de M fixée), pour différents critères d'adversité

La convergence est effectivement largement accélérée avec la *VAN forward*, ce qui confirme notre intuition concernant ses qualités de prédiction de l'adversité des scénarios primaires. À la vue de cette forte efficacité, il apparaît pertinent de choisir M plus faible. Au final, le gain en termes de temps de calcul serait fortement significatif.

Toutefois, il convient de noter que, dans notre cadre, seuls des facteurs de niveau (action et taux) sont pris en compte. Si la VAN *forward* semble particulièrement adaptée pour des risques de niveau, ce ne serait pas le cas pour des risques de volatilité. En effet, les mouvements de volatilité (actions et taux) ne peuvent être retranscrits dans un scénario *forward*

4.4 Approche Bayesienne

La *VAN forward* est un critère puissant pour mesurer l'adversité d'un scénario primaire. Couplé à l'accélérateur SdS, cet outil permet d'obtenir des résultats bien plus efficaces que ceux obtenus par des critères plus standards. Cela peut s'expliquer par la bonne qualité (toutefois toujours imparfaite) de l'information contenue dans cet indicateur. L'accélérateur SdS permet alors de réduire, de manière significative, le nombre de calculs complets de fonds propres économiques à un an. Cependant, il possède aussi certains défauts. Premièrement, il n'exploite pas totalement la forte corrélation entre les fonds propres et la *VAN forward*, dans les queues de distribution. De plus, sa performance dépend du choix de M. Bien que notre étude statistique puisse aider à faire ce choix, initialiser M reste relativement subjectif et peut fortement impacter la vitesse de convergence.

De plus, une fois ce paramètre défini, l'algorithme explore la liste des scénarios de manière déterministe, jusqu'à ce que la convergence soit atteinte. Un tirage aléatoire, réalisé en exploitant les propriétés de la *VAN forward*, pourrait s'avérer un moyen efficace d'optimiser la vitesse de l'algorithme. Dans cette optique, les approches Bayesiennes se révèlent souvent pertinentes. Les statistiques Bayesiennes se basent sur un *a priori* nécessitant d'être défini. Les études précédentes, concernant les caractéristiques de la *VAN forward*, permettent d'intuiter une information précise afin de définir cet *a priori*.

L'algorithme du recuit simulé (voir Davis (1987), Aarts and Korst (1988), Hwang (1988)) semble une alternative particulièrement pertinente à l'accélérateur SdS. Originaire d'une pratique issue de l'industrie de la métallurgie, cette méthodologie permet de minimiser une fonction de coût.

4.4.1 Optimisation discrète

Dans notre cadre, cette fonction de coût est,

$$FP_1 = \psi({}^{f}VAN_1),$$

où ${}^{f}VAN_1$ est la *VAN forward* en $t = 1$.

La Figure 4.12 représente cette fonction dans la queue et sur l'ensemble de la distribution des fonds propres. Les conditions nécessaires à l'application de l'algorithme du recuit simulé sont remplies, car

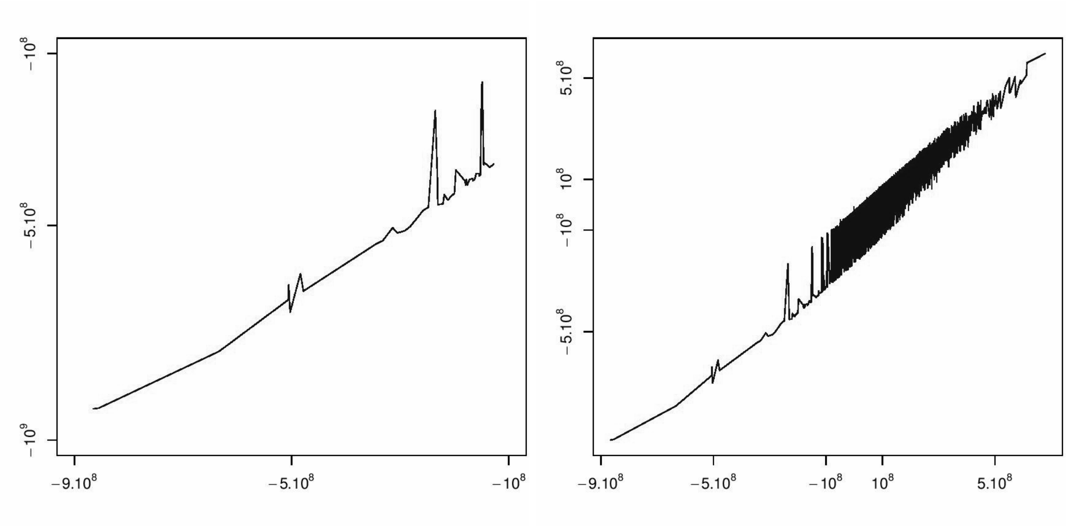

FIGURE 4.12 – Fonds propres en fonction de la *VAN forward* sur les domaines $[\![1, 50]\!]$ (à gauche) et $[\![1, 5000]\!]$ (à droite).

la fonction présente de nombreux minima locaux, dus à la corrélation imparfaite des rangs entre les fonds propres et la *VAN forward*.

Tout d'abord, on remarque qu'ici la fonction à minimiser ne peut être évaluée qu'en un nombre fini de points, correspondant aux P scénarios primaires, ou encore aux P *VAN forward* calculées. Le seul *input* est donc le vecteur des P *VAN forward*, trié par ordre croissant. On le note,

$$V = \left({}^{f}VAN_1^{(1)}, \ldots, {}^{f}VAN_1^{(P)}\right),$$

avec $P = 5000$.

L'utilisation d'un algorithme spécifique pour la minimisation de cette fonction est justifiée par le coût lié à son évaluation. En effet, le but est d'estimer la quantité $\psi({}^{f}VAN_1^{(i)})$ pour un nombre de *VAN forward* aussi faible que possible.

Dans le cas discret, le bruitage de la variable explicative ${}^{f}VAN_1$ ne peut pas prendre la forme d'une loi de probabilité continue. Il s'agira, au contraire, de remplacer un bruitage aléatoire continu par le passage d'une *VAN* à une autre au sein du vecteur V. Pour cela, on procède à un *bruitage sur les rangs* des *VAN forward*, et non directement sur leur valeur.

La loi de bruitage est donc discrète, et son choix dépend de la forme attendue des trajectoires d'exploration. Dans le cas du recuit simulé, on souhaite explorer les *VAN forward* voisines avec la plus grande probabilité, c'est à dire de manière à ce que les sauts les plus probables sont d'amplitude faible. Une loi discrète présentant cette caractéristique est la loi binomiale négative $\mathscr{BN}(r, p)$.

La Figure 4.13 illustre le principe du bruitage sur les 10 points correspondants aux *VAN forward* les plus faibles. Les chiffres en bleu sont les rangs des fonds propres associés. Le bruitage est caractérisé par le passage, dans cet exemple, de la *VAN forward* de rang 3 à celle de rang 6. Il a pour amplitude

$R = 3$ et permet de passer des fonds propres de rang 3 aux fonds propres de rang 4.

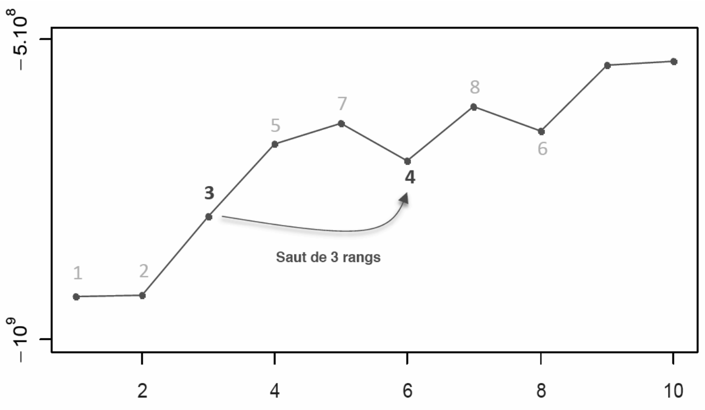

FIGURE 4.13 – Fonds propres en fonction des 10 premiers rangs de la *VAN forward*.

La variable d'amplitude du saut, permettant à l'algorithme de se déplacer entre deux rangs, constitue l'*a priori* choisi pour la mise en œuvre du recuit simulé.

4.4.2 Calcul d'un quantile par recuit simulé

L'application d'un tel algorithme au calcul du SCR n'est pas immédiate, dans la mesure où la valeur recherchée est un quantile et non un simple minimum. Cependant, il est possible d'adapter le problème de recherche du quantile à un problème de minimisation standard compatible avec le recuit simulé.

Le principe de la méthode est le suivant,

1. la minimisation de la fonction ψ par recuit simulé sur le vecteur $V = ({}^fVAN_1^{(i)})_{i \in [\![1,P]\!]}$ donne,

$$FP_1^{l_1} = \underset{x \in V}{\text{argmin}}\, \psi(x) ;$$

2. le vecteur V est mis à jour en supprimant la *VAN forward* correspondant aux fonds propres $FP_1^{l_1}$, c'est-à-dire, aux fonds propres les plus faibles : on suppose ici que l'algorithme a bien convergé vers le minimum global sur V ;
3. si le recuit simulé a été exécuté 25 fois, alors on arrête, sinon, on retourne à l'étape 1 afin de lancer une nouvelle minimisation de la fonction ψ.

Plus simplement, chercher la valeur de rang 25 dans un vecteur de fonds propres revient à appliquer 25 recuits simulés successifs en supprimant, à l'issue de chacun des algorithmes, la valeur la plus faible trouvée. La Figure 4.14 illustre le principe de combinaison de recuits simulés consécutifs.

Le quantile du vecteur de fonds propres est obtenu par application de l'algorithme 1.

Algorithm 1 Algorithme du recuit simulé appliqué au calcul du quantile

Initialiser le rang de départ dans le domaine de définition $r_0 = 1$.
Initialiser la température T_0.
Initialiser le vecteur des *VAN forward* $V = ({}^{f}VAN_1^{(1)}, \dots, {}^{f}VAN_1^{(P)})$
Initialiser le compteur de calcul du nombre de fonds propres $n_{\text{FP}} = 0$.

for $k \in [\![1, 25]\!]$ **do**
  for $i \in [\![1, M]\!]$ **do**
    Modifier aléatoirement r_{i-1} pour obtenir $r_{new} = r_{i-1} + R_i$.
    if les fonds propres $\psi({}^{f}VAN_1^{(r_{new})})$ sont inconnus **then**
      Calculer les fonds propres.
      Mettre à jour le compteur : $n_{\text{FP}} = n_{\text{FP}} + 1$.
    end if

    Calculer $\Delta FP_1 = \psi({}^{f}VAN_1^{(r_{new})}) - \psi({}^{f}VAN_1^{(r_{i-1})})$.
    Mettre à jour la température $T_i = \gamma T_{i-1}$.

    if $\Delta\text{FP} \leq 0$ **then**
      $r_i = r_{new}$
    else
      Tirer une variable aléatoire $u \sim \mathscr{U}(0, 1)$.
      if $u \leq \pi(T_i, \Delta\text{FP})$ **then**
        $r_i = r_{new}$
      else
        $r_i = r_{i-1}$
      end if
    end if

  end for

  Enregistrement de la solution trouvée : FP_{l_k}.
  Suppression dans V de la *VAN forward* correspondant à FP_{l_k}.
end for

Le quantile à 0,5 % des fonds propres correspond à $\text{FP}_{l_{25}}$.

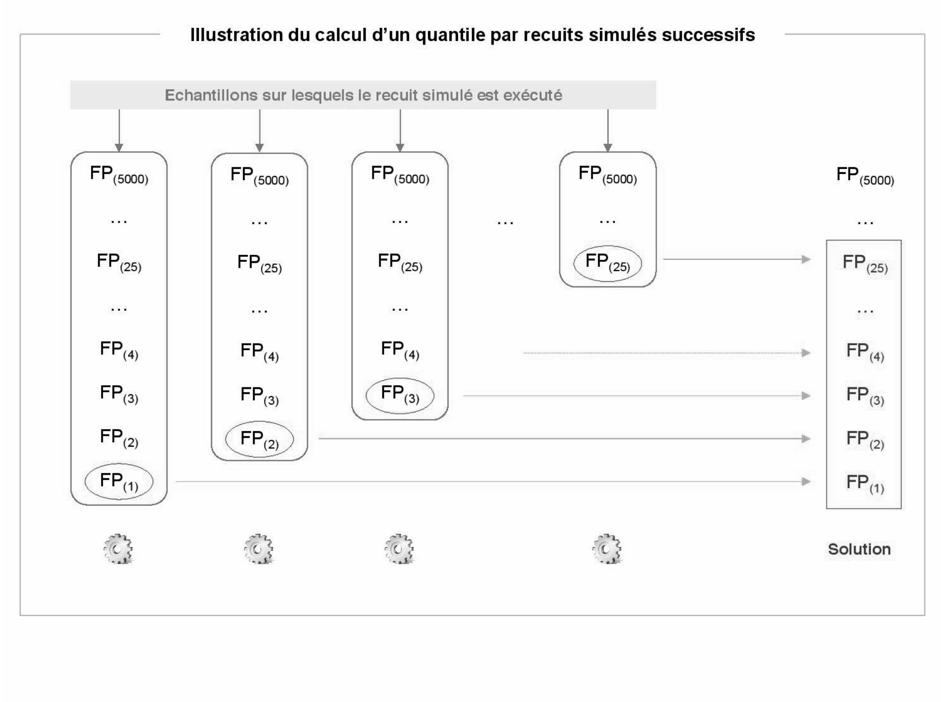

FIGURE 4.14 – Illustration du calcul d'un quantile par recuits simulés successifs.

4.4.3 Résultats

L'algorithme 1 est appliqué au vecteur des *VAN forward* issues des données Milliman, avec les paramètres suivants,

- les paramètres du bruitage de loi binomiale négative sont $r = 1$ et $p = 60\ \%$;
- la température initiale est $T_0 = 2 \cdot 10^8$ et $\gamma = 99\ \%$;
- l'algorithme s'arrête après $M = 2\,000$ itérations.

Il fournit le nombre n_{FP} de fonds propres qui ont dû être calculés pour obtenir les 25 plus faibles valeurs de fonds propres.

La procédure est répétée $N = 1\,000$ fois, de sorte que l'on dispose d'autant de valeurs pour n_{FP}. On trouve que l'algorithme conduit, en moyenne, au calcul de $\overline{n}_{\text{FP}} = 62$ fonds propres. Cela représente un gain moyen de 13 par rapport à l'accélérateur SdS appliqué avec $M = 25$.

De façon à appréhender l'influence du paramètre p sur le coût algorithmique du recuit simulé n_{FP}, on réitère la procédure 100 fois pour différentes valeurs de p,

$$p \in \{70\%,\ 50\%,\ 30\%,\ 10\%\}\,.$$

Le critère d'arrêt reste le même. La Figure 4.15 représente la moyenne sur les 100 simulations du nombre de fonds propres calculé, $\overline{n}_{\text{FP}}$, en fonction de p. Elle confirme l'intuition selon laquelle un p faible entraîne un coût algorithmique conséquent. Il est important de remarquer que l'efficacité de l'algorithme dépend fortement du choix de p.

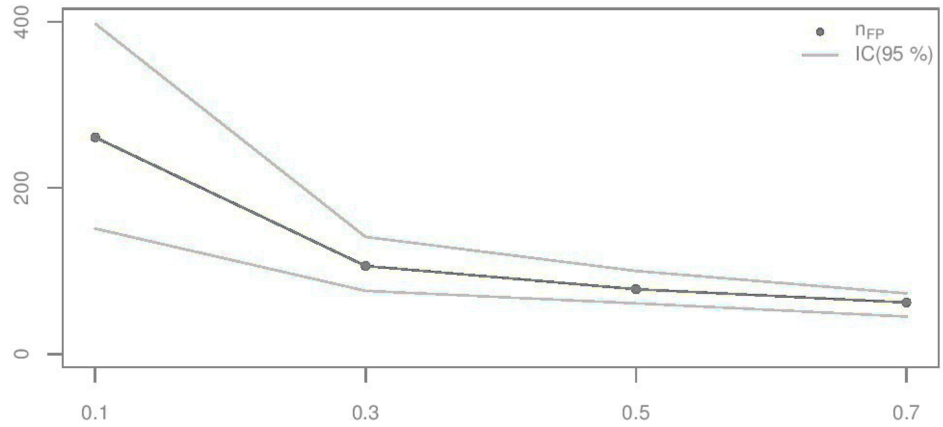

FIGURE 4.15 – Influence du choix de p sur le nombre moyen de fonds propres calculés $\overline{n}_{FP}$.

Remarque 4.4.1. *Pour les valeurs de p testées ici, l'algorithme converge pour la quasi-totalité des simulations (plus de 99 %). En revanche, pour $p > 70$ %, il reste souvent piégé, car l'exploration n'est pas assez profonde. Un compromis doit être trouvé entre la minimisation de n_{FP} et la fiabilité de la convergence. Dans notre cas, la valeur optimale est $p = 70$ %.*

De façon à comparer les performances du recuit simulé pour les différentes normes précédemment étudiées, on applique l'algorithme en remplaçant successivement le vecteur des *VAN forward* par celui des normes standard, sensibilité et sensibilité asymétrique. Pour garantir la comparabilité des simulations, les paramètres de température et le critère d'arrêt sont gardés constants. En revanche, il n'est pas pertinent de prendre un p identique à toutes les normes. À chacune d'entre elles est associée une valeur de p optimale, garantissant à la fois une convergence fiable et un coût raisonnable.

On lance ensuite les trois procédures standard, sensibilité et sensibilité asymétrique, $N = 1\,000$ fois chacune. Les valeurs de p, du nombre moyen $\overline{n}_{\mathrm{FP}}$ de fonds propres calculés et de leur intervalle de confiance à 95 % sont renseignées dans la table 4.1. L'intervalle de confiance est défini à partir des quantiles empiriques à 2,5 % et 97,5 % de l'échantillon des n_{FP}. La table fait également apparaître les taux de réussite, définis comme les proportions de recuits simulés ayant convergé vers les bonnes valeurs de fonds propres.

Norme	p	n_{FP} moyen	$\mathrm{IC}_{95\,\%}$	Taux de réussite
Norme standard	2 %	927	$[600, 1\,459]$	60,3 %
Norme sensibilité	5 %	425	$[283, 660]$	95,8 %
Norme sensibilité asymétrique	10 %	242	$[166, 351]$	99,5 %
VAN *forward*	70 %	62	$[44, 76]$	99,7 %

TABLE 4.1 – Nombre de fonds propres calculés au cours de l'algorithme du recuit simulé, pour différentes normes.

Sans surprise, $\overline{n}_{\mathrm{FP}}$ augmente avec la qualité prédictive de la norme, et diminue avec p. On remarque également qu'à budget d'itérations M fixé, la convergence est moins bonne pour la norme standard (60,3 %), que pour la *VAN forward* (99,7 %). On constate également que la largeur des intervalles de confiance augmente lorsque le pouvoir prédictif de la norme diminue. En pratique, plus l'exploration est profonde (p faible), plus le nombre de simulations doit être grand pour permettre la bonne convergence de l'algorithme.

À titre d'illustration, la Figure 4.16 représente l'évolution de la variable n_{FP} au cours des itérations, et ce pour les quatre normes. Ce résultat est obtenu pour une simulation, et on constate que les n_{FP} sont bien compris dans leurs intervalles de confiance respectifs, présentés dans la table 4.1. Plus le pouvoir prédictif de la norme est grand, plus n_{FP} se stabilise rapidement, et sur une valeur faible.

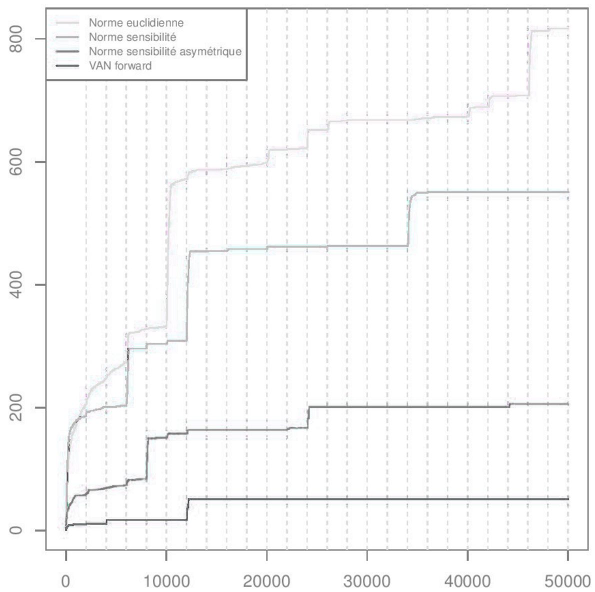

FIGURE 4.16 – Évolution du nombre de fonds propres calculés n_{FP} au cours des itérations pour le recuit simulé en utilisant la *VAN forward*, la norme sensibilité asymétrique, la norme sensibilité ou la norme standard.

4.5 Comparaison à l'accélérateur SdS

Si l'on compare les valeurs de $\overline{n}_{\text{FP}}$ aux résultats obtenus avec l'accélérateur SdS (100 calculs de fonds propres, en prenant $M = 50$), il apparaît que le recuit simulé est plus performant dans le cas de la *VAN forward*, mais pas pour les autres normes.

Les sauts importants que l'on observe pour certaines normes obligent à choisir une valeur de p assez faible pour que l'exploration soit profonde. Pour la norme standard, par exemple, la distribution du bruitage doit être telle qu'un saut de 122 se produise avec une probabilité non négligeable. Le choix d'un p faible est donc nécessaire pour converger vers les bonnes valeurs de fonds propres. Dans notre exemple, $p = 2$ %, et $\mathbb{E}[\![R]\!] = 50$. Par conséquent, le nombre de fonds propres calculés à chaque étape de l'algorithme est important.

La faiblesse du recuit simulé, par rapport à l'accélérateur SdS, vient du fait que les fonds propres doivent être découverts dans un ordre précis (des plus faibles aux plus élevés), ce qui nécessite des sauts de rangs plus conséquents que dans le cas où l'ordre n'a pas d'importance (cas de l'accélérateur SdS). Cela n'a que peu d'impact avec la *VAN forward*, puisque les rangs sont presque parfaitement corrélés. En revanche, il est évident que le recuit simulé sera moins adapté pour des normes peu puissantes telles que la norme standard ou la norme sensibilité. On peut d'ailleurs remarquer que, malgré les bonnes propriétés de la norme sensibilité asymétrique, le nombre moyen de fonds propres calculés est de 242, contre 62 pour la *VAN forward*.

L'objectif est de localiser la $25^{\text{ième}}$ pire valeur de FP_1, et non pas de minimiser la fonction ψ au sens propre. Toutefois, le problème peut être résolu en itérant 25 algorithmes consécutifs. À chaque étape, la plus faible valeur est supprimée de l'échantillon de manière à ce que les 25 pires scénarios soient finalement isolés. Le quantile empirique à 0.5% est alors la $25^{\text{ième}}$ pire valeur, c'est à dire le résultat de la dernière étape simulatoire.

Combiné à la *VAN forward*, cet algorithme est très performant, et même supérieur à l'accélérateur SdS. Toutefois il serait moins efficace utilisé avec d'autres normes car l'algorithme requiert une forte corrélation entre le critère et les fonds propres. L'avantage principal de l'accélérateur SdS est qu'il réunit plus aisément les scénarios proches entre eux. Cependant ce processus simulatoire est particulièrement bien adapté à notre problème, où il est possible de le combiner à la connaissance d'un indicateur d'adversité efficace, la *VAN forward*.

Conclusion

La mise en place d'un modèle interne (complet ou partiel), dans le cadre de Solvabilité II, se heurte à de nombreuses difficultés opérationnelles. En particulier, les simulations imbriquées, théoriquement adaptées pour le calcul du *solvency capital requirement* (SCR), impliquent une complexité algorithmique particulièrement lourde. Des méthodes alternatives ont été développées pour réduire les temps de calcul et améliorer l'efficacité opérationnelle.

L'accélérateur SdS, développé par Devineau et Loisel, repose sur le constat que la détermination du SCR ne requiert la connaissance que du quantile à 0,5 % de la distribution des fonds propres à 1 an. Son fonctionnement est fondé sur l'utilisation d'une norme pour prédire l'adversité des différents scénarios primaires et leur appartenance à ce quantile critique. L'objet de cet article était d'étudier la pertinence de la *VAN forward* et son pouvoir prédictif concernant l'adversité d'un scénario donné.

La *VAN forward* est le résultat d'une projection entièrement déterministe, dans laquelle l'ensemble des variables (taux d'intérêt et indice action) évoluent suivant leur taux *forward*. Ce concept de *forward* est déjà présent dans les problématiques de valorisation de portefeuille, avec la composante "certainty equivalent" dans la *Market Consistant Embedded Value*. Il apparaît comme la meilleure façon de prédire l'évolution d'une variable à partir d'une unique simulation déterministe, et possède un caractère "central" particulièrement intéressant dans le cadre de notre étude. Les études statistiques développées dans cet article ont montré que la *VAN forward* est effectivement centrale, car comprise entre les quantiles 50 % et 75 % de la distribution des *VAN* (cas spécifique de notre portefeuille). Ce résultat traduit une excellente corrélation des rangs entre fonds propres et *VAN forward*, confirmée par des études sur les probabilités de croisement dans les queues, bien plus faibles que pour une *VAN* quelconque. Le choix de cette *VAN* particulière s'avère donc pertinent.

La forte corrélation de la *VAN forward* avec les fonds propres en fait un candidat idéal pour la prévision de l'adversité des scénarios. Son couplage avec l'accélérateur SdS a donc été comparé à l'utilisation d'autres mesures d'adversité (normes standard et sensibilité). La vitesse de convergence de l'algorithme est très élevée, et représente des gains de facteurs 3 et 2 par rapport aux normes standard et sensibilité respectivement. La qualité prédictive de la *VAN forward* provient de la quantité d'informations contenue, puisqu'elle intègre l'impacts des scénarios primaires sur les modèles ALM, ainsi qu'une partie des effets croisés entre risques. Sa grande force est qu'elle représente un indicateur complet, mais aussi beaucoup plus simple à manipuler qu'une norme lorsque la dimension des facteurs de risque devient grande (ce qui est le cas en pratique).

La dernière partie de l'article a été consacrée à l'élaboration d'une méthode alternative à l'accélérateur SdS, mettant en œuvre la *VAN forward*. Les méthodes d'optimisation bayésiennes trouvent leur application dans des domaines très variés, et s'avèrent souvent efficaces dans la résolution de problèmes complexes. L'adaptation du recuit simulé au problème de la recherche de quantile a donné des résultats très satisfaisants, et a conduit à un coût algorithmique inférieur à celui de l'accélérateur SdS. La forte corrélation entre fonds propres et *VAN forward* rend cette méthode très adaptée, car l'exploration des solutions reste limitée dans l'espace. L'algorithme heuristique développé dans cet article a pourtant l'inconvénient d'être sensible au choix des paramètres, ce qui peut poser des problèmes dans la pratique.

La *VAN forward* est donc un indicateur d'adversité d'une grande puissance, car, premièrement, elle combine les informations primaires et secondaires, ainsi qu'une partie des effets croisés, contrairement aux normes usuelles. Deuxièmement, elle est de dimension 1, et est donc simple à appréhender et à manipuler. Troisièmement, elle est fortement corrélée avec les fonds propres, ce qui lui confère un pouvoir prédictif très supérieur à celui des autres mesures d'adversité.

Il convient de noter, toutefois, que si le cadre de notre étude, considérant uniquement des risques de niveau, est fortement adapté à l'utilisation de cet outil, la prise en compte de risques de volatilité pourrait fournir des résultats relativement peu probants du fait que ces risques ne peuvent être pris en compte dans les scénarios *forwards*, déterministes. Malgré ce point la *VAN forward* s'adapte à un large panel d'études ALM complexes pouvant induire de futurs développements. Ces études vont du calcul de capital économique à des mises en œuvre de type ORSA (en adaptant son calcul à un horizon supérieur à un an) où à des études de sensibilité des fonds propres économiques, adaptées aux problématiques de gestion des risques internes.

Bibliographie

Aarts, E. and Korst, J. (1988), *Simulated annealing and Boltzmann machines*, New York, NY ; John Wiley and Sons Inc.

Bauer, D., Bergmann, D. and Reuss, A. (2010), Solvency II and nested simulations–a least-squares Monte Carlo approach, *in* 'Proceedings of the 2010 ICA congress'.

Bergmann, D. (2011), *Nested simulations in life insurance*, IFA.

Bonnin, F., Planchet, F. and Juillard, M. (2014), ‘Best estimate calculations of savings contracts by closed formulas : application to the orsa’, *European Actuarial Journal* **4**(1), 181–196.

Broadie, M., Du, Y. and Moallemi, C. C. (2011), ‘Efficient risk estimation via nested sequential simulation’, *Management Science* **57**(6), 1172–1194.

Davis, L. (1987), *Genetic algorithms and simulated annealing*, Pitman Publishing.

Devineau, L. (2011), *La méthode des simulations dans les simulations. Mise en œuvre d’un modèle actif / passif en assurance-vie : quelles techniques ?, Part 2.*, Milliman. slides.

Devineau, L. and Chauvigny, M. (2011), ‘Replicating portfolios : calibration techniques for the calculation of the Solvency II economic capital’, *Bulletin Français d’Actuariat* **11**(21), 59–97.

Devineau, L. and Loisel, S. (2009*a*), ‘Construction d’un algorithme d’accélération de la méthode des "simulations dans les simulations" pour le calcul du capital économique Solvabilité II’, *Bulletin Français d’Actuariat* **10**(17), 188–221.

Devineau, L. and Loisel, S. (2009*b*), ‘Risk aggregation in Solvency II : How to converge the approaches of the internal models and those of the standard formula ?’, *Bulletin Français d’Actuariat* **9**(18), 107–145.

Giles, M. B. (2008), ‘Multilevel Monte Carlo path simulation’, *Operations Research* **56**(3), 607–617.

Gordy, M. B. and Juneja, S. (2010), ‘Nested simulation in portfolio risk measurement’, *Management Science* **56**(10), 1833–1848.

Guibert, Q., Planchet, F. and Juillard, M. (2010), ‘Un cadre de référence pour un modèle interne partiel en assurance de personnes’, *Bulletin Français d’Actuariat* **10**(20), 5–34.

Hwang, C.-R. (1988), ‘Simulated annealing : theory and applications’, *Acta Applicandae Mathematicae* **12**(1), 108–111.

Lemaire, V., Pages, G., Panloup, F. et al. (2015), Invariant measure of duplicated diffusions and application to Richardson-Romberg extrapolation, *in* ‘Annales de l’Institut Henri Poincaré, Probabilités et Statistiques’, Vol. 51, Institut Henri Poincaré, pp. 1562–1596.

Lemaire, V. et al. (2014), ‘Multilevel richardson-romberg extrapolation’, *Available at SSRN 2539114* .

Vedani, J. and Devineau, L. (2013), ‘Solvency assessment within the ORSA framework : issues and quantitative methodologies’, *Bulletin Français d’Actuariat* **13**(25), 35–71.

Chapitre 5

Market inconsistencies of the *market-consistent* European life insurance economic valuations: pitfalls and practical solutions

Abstract

The Solvency II directive has introduced a specific so-called risk-neutral framework to valuate economic accounting quantities throughout European life insurance companies. The adaptation of this theoretical notion for regulatory purposes requires the addition of a specific criterion, namely *market-consistency*, in order to objectify the choice of the valuation probability measure.

This paper points out and fixes some of the major risk sources embedded in the current regulatory life insurance valuation scheme. We compare actuarial and financial valuation schemes. We then first address operational issues and potential market manipulation sources in life insurance, induced by both theoretical and regulatory pitfalls. For example, we show that the economic own funds of a representative French life insurance company can vary by almost 140%, as already observed by market practitioners, when the interest rate model is calibrated in October or on the 31^{st} of December. We then propose various modifications of the current implementation, including a first product-specific valuation scheme, to limit the impact of these market-inconsistencies.

Keywords : *risk-neutral valuation, economic valuation, market-consistency, European regulation, life insurance*.

Introduction

Since the mid-2000s, the European life insurance industry regulation has undertaken strong evolutions aiming at a better understanding and quantifying of the insurance liabilities portfolios underlying risks. This has led to an increasingly complex valuation framework shaped by the successive Solvency I and Solvency II directives. Nowadays, to valuate there complex portfolios, the market actors tend to consider Monte Carlo type valuations, already well used in financial markets.

These evolutions have induced a focus on market risks. In practice, the life insurance liabilities valuations are therefore implemented based on so-called *economic scenarios*. These scenarios are simulations of the risk factors, which enable a quantification of the valuated liabilities portfolio cash-flows on a long time horizon. The actuaries can then establish a best estimate of their liabilities portfolios by discounting and averaging these cash-flows. The general process is very similar to Monte Carlo financial asset pricing methodologies (see *e.g.* Glasserman (2003) or Kroese et al. (2013)). The probability measure considered to simulate the economic scenarios is currently denoted as *risk-neutral* by practitioners (see Devineau and Loisel (2009), Vedani and Devineau (2013)), with the underlying idea that the final values are obtained under a risk-neutral probability measure. This use of the financial (see Black and Scholes (1973), Merton (1973) etc.) terminology reflects a will to bring the insurance regulation closer to the financial one, but is quite different from its original formulation.

The valuation framework used by actuaries leads to various homogenization issues. Non-uniqueness of the valuation measure requires the introduction of an additional probability measure selection constraint, the *market-consistency* criterion. The final valuation process is pragmatic, supposedly easy to implement and regulate.

While financial valuation focuses on short term horizons, locality in modeling and valuation is justified. The long-term view of the tasks in life insurance however needs stable and reliable inputs and models that have a global perspective in view

In addition to a detailed presentation and comparison of both financial and actuarial risk-neutral valuation schemes, the main contribution of our paper is to point out and exemplify several major theoretical and regulatory pitfalls, and introduce several subjective choices and manipulation opportunities. We also propose first alternatives to the regulatory valuation scheme such as new, product-specific, and less manipulable calibration processes. These refinements aim to provide a safer and more comparable actuarial valuation framework.

In Section 5.1, we develop the European life insurance use of the risk-neutral valuation, beginning by regulation evolutions, and recall the way market finance practitioners deal with it, while comparing the finance and insurance valuation practices. This leads us to introduce the *market-consistency* criterion, considering the definition generally accepted through the operational actuarial literature. In Section 5.2, we address specific operational issues leading to relative market-inconsistencies, easy market manipulations and strongly incomparable final valuations. Through various illustrations of these points, we bring up new, important research questions and propose first alternatives to the current valuation scheme.

5.1 The risk-neutral probability measure in life insurance

This section introduces the main points in the European regulation historical developments related to the use of the risk-neutral probability measure in life insurance. We then provides a wide development of operational actuarial practices using this valuation scheme. As a by-product, it emphasizes the differences between finance versus insurance practices concerning the use of such valuation rule.

5.1.1 The European regulatory framework

Economic valuation and the economic balance sheet

When considering the recent evolutions of the European regulation and the introduction of risk-neutral valuation in actuarial practices, it is important to first focus on the new international accounting norms. The International Financial Reporting Standards (IFRS) introduce the notion of fair valuation for insurance liabilities. European undertakings have to assess such valuations, which can be seen as trading values.

In parallel with the development of the insurance liabilities fair value notion and with the wide spreading of the valuation approaches by projection and discounting of future cash-flows (as in corporate finance), the European regulation has focused towards a scheme that better takes account of optionalities, time value, and other products specificities. The European Commission thus enacts, in 2009, a new solvency regulation, Solvency II, improving the previous one (Solvency I, 2002). This directive bases its implementation on the so-called economic valuation. The underlying idea here is to build a valuation scheme on objective data (external to the insurance market and common to every practitioners: financial market data).

Risk neutral/economic valuation and economic accounting

This short overview about regulation enables us to focus more specifically on the Solvency II-specific notion of *economic* value and on the concept of *economic* balance sheet. This atypical balance sheet is, in its shape, quite close to the accounting asset and liability balance sheet. The liability side is generally separated between own funds and *less liquid* liabilities corresponding to the provision associated with the insurance liabilities towards its policyholders.

In the Solvency II accounting scheme, the main difference with a standard accounting balance sheet is the methodology used to evaluate its various items. This leads to the definition of the economic value of assets (their market value), of the liability portfolio (thought of as the best estimate of liability), and finally, of the own funds.

These values are expectations assessed under an *actuarial* valuation probability measure. The choice of this kind of valuation probability measure is an answer to the framework defined by the Solvency II directive.

Section 2 of the Solvency II directive, article 77.2 states that:

> "The best estimate shall correspond to the probability-weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free IR term structure."
> Then,
> "The calculation of the best estimate shall be based upon up-to-date and credible information and realistic assumptions and be performed using adequate, applicable and relevant actuarial and statistical methods."

Considering the last regulatory information, the process and parameters used to *estimate* the "relevant risk-free interest rates term structure" to be used by life insurance companies for economic valuations are provided and updated on a regular basis by EIOPA (*European Insurance and Occupational Pensions Authority*), so as to homogenize the economic valuation process. It is based on the Euroswap term structure but integrates a specific processing to be adapted to each specific insurance market. Though this homogenization attempt of EIOPA may originally be a good idea, it also leads to new practical and theoretical issues. In particular, new difficulties happen when practitioners try to reconcile market prices with the *regulatory* interest rates term structure.

Regarding the requirement to choose a probability measure to assess the "probability-weighted average of future cash-flows", the directive enforces the economic balance sheet to be fully valuated under a risk-neutral fashion, quite different to the current accounting standards (through IFRS are currently in process). Indeed, the models used to assess the products prices should be parametrized using the easily available data given by financial markets, which aims to provide an efficient answer to impartiality.These markets generally valuate and disclose data under a risk-neutral probability measure, so the risks have to be projected under a risk-neutral probability.

5.1.2 Risk neutral valuation in finance compared to insurance

Risk neutral valuation in finance

The reference to risk-neutral valuation rises from the development of derivative markets in the 70s (Chicago 1973) and the notion of hedging portfolios. The idea to introduce dynamic portfolios in option theory is due to Black and Scholes (1973) and Merton (1973) influenced by the modern portfolio theory proposed by Markowitz (1952). In the derivative markets, where contracts grant future cash-flows on tradable assets, Black, Scholes and Merton's solution is based on hedging requirement providing an inter-temporal day-to day diversification based on trading the underlying asset of the contract. Finally, the everyday work of traders includes, for each derivative, to frequently update the corresponding hedging portfolio: it is a prerequisite for any trade. In practice, it is necessary to associate an hedging portfolio with any contract to make it neutral with respect to its underlying market risk. Then the risk-neutral valuation may be used. Roughly speaking, the value of the contract is equal to the value of its perfect hedging portfolio. As a consequence, option price processes must be consistent with respect to the available information revealed over time and must preclude any arbitrage opportunity. Therefore, as proved by Black, Scholes and Merton, option prices (in a given numeraire) must be computed under a so called risk-neutral probability measure: they are equal to the conditional *risk-neutral expectations* of their terminal cash-flows given the available information.

Obviously, the perfect hedging strategy does not exist since there are too many sources of risk to be hedged (this corresponds to market incompleteness). The original risk-neutral theory can be extended to the incomplete market case, leading to theoretical no-arbitrage bid-ask spreads for option pricing (see Kreps (1981)). The market compromise consists in selecting one risk-neutral probability measure well-adapted to the contract to be hedged. In any practical case, when introducing a new derivative asset, the practitioner has to choose a particular price among a set of no-arbitrage values since the *rational* option price is no longer unique. As seen in the section devoted to economic valuation in practice, the use of the risk-neutral valuation for insurance can also lead to a multiplicity of potential valuation probability measures. This issue is partly solved by the use of a constraint when choosing the valuation measure, namely the *market-consistency*.

Comparative elements - finance versus insurance practices

At this point, it is relevant to compare insurance and finance practices concerning the use of risk-neutral valuation schemes. It is frequent, in Solvency II, to find inspiration sources from banking practices and regulation (see *e.g.* the very explicit paper of Schubert and Grießmann (2004)). But it has to be noted that these two uses of risk-neutrality are different, sometimes in opposition, in their goal, conceptualization and management. This is structural, due to the difference between these two businesses.

First, main objectives are different. Traders use risk-neutral valuation to get fair trading prices, while actuaries estimate economic values for solvency purposes. They assess the economic value on a yearly basis for regulatory reasons, but there is no structural need for it to be a market value: it is not traded.

The second difference lies in the management of risk-neutral valuation schemes. The market provides instantaneous data that change in a continuous fashion to adjust prices towards their true values. These data, implied volatilities in particular, are build to change after each trade. The valuation measures also adjust trade after trade. In comparison, in insurance you only need one valuation per year to respond to quantitative regulatory requirements. This leads to great differences when considering the market data importance. When traders adjust their price continuously and are more interested by the historical movements, day by day, of market data, actuaries valuate their yearly solvency capital need based only on the single last trade of the year.

Finally, the main difference between both approaches probably lies in hedging. Life insurance bears biometric and other technical risks that cannot be hedged since not all risk factors of life insurance products are of financial nature (longevity, lapse, etc.). This is a problem because financial markets do not provide data for such non-hedgeable risks. Therefore, the valuation of optionality only focuses on financial risks. The non-hedgeable risks are managed separately through a cost-of-capital approach (see Floreani (2011)). Unlike bank trading practices, life insurers do not hedge their products partly because there is no solution better than risk mitigation. Some insurers buy simple financial instruments (Caplets, Floorlets, Swaptions, Calls, Puts, etc.) to protect a part of their asset portfolios against financial movements, but this is not true hedging. One has to remember that risk-neutral valuation is legitimated by hedging.

There is little theoretical justification for insurance risk-neutral valuation. Risk-neutral valuation is intimately linked to a specific business and this leads to great difficulties when trying to switch from its original use. It is not well-adapted to the insurance industry and this leads to an unusual mix of practices, making it difficult and subjective to compare valuations. This issue is solved in finance by trades and the continuous comparison of market prices by buyers and sellers. The resulting framework is operationally complex for insurance practitioners. The assessed values are difficult to justify and to be checked by control authorities. They are easy to manipulate by the biggest companies.

5.1.3 Economic valuation in practice

As previously seen, on the whole, in life insurance, the valuation process requires projecting and discounting of future cash-flows under a *good* probability measure. This measure is in practice (this is not fully spelled out in the directive) a martingale measure associated with a *numéraire* that is the exponential integral of a regulatory-driven risk-free rate, a measure that neutralizes other market trends, and therefore a *risk-neutral* measure. In spite of its obvious theoretical gaps, this pragmatic valuation scheme is mostly scalable and adaptable to entity-specific actuarial practices.

Economic scenarios & market-consistency

In practice any economic valuation process is implemented with Monte Carlo simulations to estimate the complex potential behavior of life insurance product options depending on the stochastic evolution of market risks. It is indeed almost impossible to assess closed formula for liability portfolios valuation, without very strong model assumptions (see Bonnin et al. (2014)).

Each Monte Carlo estimation requires actuaries to dispose of a table of valuation scenarios where underlying financial risks are diffused under a good probability measure.

As explained in Subsection 5.1.1, the economic valuation framework requires one to consider stochastic financial risks (stock, risk free interest rates, spread risk, etc.) and deterministic technical risks (mortality and longevity risks, lapse risk, etc.). It therefore involves the simulation of *economic scenarios*, composed with risk-neutral projections of the financial risks. The technical risks evolve through the valuation horizon according to deterministic chronicles (mortality rates, lapse rates by projection year) or as deterministic functions of the financial drivers outcomes (*e.g.* dynamic lapse), calibrated on historical data. A valuation table, called Economic Scenarios Table (EST), is a set of a sufficient number of such economic scenarios.

However there is still an issue with the choice of the risk-neutral probability measure to be used for the economic scenarios generation. Considering real market conditions, the inaccuracy of the market efficiency assumption leads to the existence of an *infinity* of risk-neutral probability measures under the *numeraire* chosen by the actuary. To select their measure choice, the companies have to add a selection criterion, based on so-called *market-consistency*.

This criterion is given by the Directive in Article 76:

> "The calculation of technical provisions shall make use of and be consistent with information provided by the financial markets and generally available data on underwriting risks (market-consistency)."

The notion of market-consistency has a considerable yet recent history in the actuarial literature (see Sheldon and Smith (2004), Wüthrich et al. (2008), Malamud et al. (2008) or Moehr (2011)). Probably the first attempt to legitimate it, when the European life insurance industry is concerned, is the book of Kemp (2009). This criterion is easy to understand under the complete market assumption. Indeed, under this assumption, the market-consistent valuation of an asset is its unique trading value. The main difficulty in applying market-consistency appears when the complete market requirements are not met. In an incomplete market, the idea of market-consistency valuation is to assess a *best estimate* of what its price should be if a complete market existed for its trade. This is exactly what Kemp says:

> "A market-consistent value of an asset or liability is its market value, if it is readily traded on a market at the point in time that the valuation is struck, and, for any other asset or liability, a reasoned best estimate of what its market value would have been had it been readily traded at the relevant valuation point."

According to current actuarial practice, empirical application of the market-consistency requirement is done by adding a measure choice criterion based on the measured efficiency to replicate financial market prices. This is an example of actuarial interpretation that has nowadays a huge impact on the whole European life insurance market.

The goal of practitioners is to select/calibrate the valuation probability measure so that the EST generated under the chosen measure enables them to re-assess, by Monte Carlo valuations, options prices very closely to the prices observed on the financial market at the valuation date (at least for some well-chosen liquid derivatives). This market-consistent calibration is a source of practical difficulties and theoretical concessions, but it provides a straightforward valuation process. actuaries deal with it through a five-step implementation, associating *probability measure selection* with *financial models calibration*.

Step a Choose models to project any needed market variable (stock indices, interest rates, credit spreads, etc.). These financial drivers are selected so as to efficiently represent or summarize the company's liability portfolio underlying (financial) risk (*e.g.* for a French company, the chosen stock driver could be the CAC40 index). These models are *market* models that provide martingale scenarios under the chosen risk-neutral *numeraire*.

Step b Select certain options and get their prices or other market data, *e.g.* implied volatilities, on the market at the valuation date. These options values are used to calibrate the models selected through the first step.

Step c Calibrate the financial models so that their parameters optimally fit the market data.

Step d Using an economic scenarios generator, simulate a valuation EST, given the chosen models and calibrated parameters.

Step e Check the market-consistency of the obtained EST, that is its efficiency to re-estimate the calibration options prices. Also, check the martingale property of the discounted financial drivers, associated with the *risk-neutral property* of the EST.

In practice, the market-consistent (economic) values obtained strongly depend on the financial models and on the financial calibration data set (the options prices or volatilities) selected by the practitioners. In general the chosen calibration instruments are more or less similar among practitioners, even if it is questionable that they are not always related to the set of derivatives instruments purchased by the company. For example, one often chooses a set of at-the-money call and puts to calibrate the model used to project stock indexes (Black & Scholes, Stochastic Volatility Jump Diffusion, Heston model, etc.). To calibrate the zero-coupon rate (or the instantaneous interest rate) model (Hull-White, Black-Karazinsky, LIBOR Market Model [*], etc.), one often uses a set of receiver swaptions implied volatilities or prices (for various maturities, tenors).

This implementation may lead to various homogenization issues as a certain degree of freedom is given to companies to choose the financial models and the calibration data sets. They can greatly differ, in maturities and tenors in particular, from one entity to another. Even the same model can be parametrized very differently when the calibration data set changes, especially because there almost exists one risk-neutral measure per financial asset due to the incompleteness of financial markets [†]. Similarly, a change in model can produce large, unpredictable movements in economic balance sheet items.

Standard valuation process

We have presented the way the valuation probability measure is selected, and how financial models are calibrated. It is now possible to propose a standard three-step implementation process for an economic valuation aiming at assessing a best estimate of liability.

∗. In reality this is an insurance-specific use of LIBOR Market Model. In particular it is calibrated in a standardized market fashion but the parameters of the calibration date are used for 30 to 60 years projections. This model is not adapted to such simulations, its first objective being to project the LIBOR forward yield on very short horizons (a few days) for hedging and not to project zero-coupon curves on very long term horizon. For this reason, we denote this model, in its insurance version, by LMM_{ins} below. The reader may consult the Appendix for further information on this model use and calibration.

†. See *e.g.* Mukerji and Tallon (2001), Martin and Rey (2004)

In Step 1, the company updates its Asset-Liability Management (ALM) hypotheses to reflect the economic reality. For the company, this means updating its asset/liability portfolio, arbitrage law, financial strategy, commission rates, lapse and mortality assumptions, etc. but also getting the new economy parameters at the valuation date (new interest rates levels, implied volatilities, etc.).

In Step 2, the company chooses and calibrates its risk-neutral projection models. Then the practitioner simulates a large number of random economic scenarios at a 30/60 years horizon (the liability portfolio extinction horizon if no new business is added to the portfolio). This allows her to get a risk-neutral EST, which market-consistent efficiency is *a posteriori* checked.

In Step 3, using an ALM model, the company projects the liability cash-flows through the EST so that it obtains one cash-flow for each projection date and for each economic scenario in the EST. The best estimate of liability is obtained as the average of the net present (discounted) value of future liability cash-flows.

The process is very similar when the practitioner aims to valuate economic own funds. The only difference lies in the projected cash-flows, which are future margin cash-flows. The capital obtained as the expected net present value of future margin cash-flows is called the Value of In-Force (VIF). Then the economic own funds correspond to the sum of the VIF and the Revalued Net Asset (RNA).

5.2 Analysis of practical issues

Various theoretical failures have been described in Section 5.1. One of the goals of the Solvency II regulations is to encourage practitioners themselves to improve the process. The market should make its own laws, through healthy emulation and competition, leading to economic optimality. In this section, we consider various practical issues induced by the current implementation of the directive. We analyze these, and put forward some initial ideas to make the actuarial valuation scheme clearer and safer.

5.2.1 Interest rates projections in economic scenario generators

There are wwo major issues with the treatment used to forecast yield curves in the economic valuation framework. First, the extremely complex practical scheme used to calibrate interest rates models involves several flaws that induce a disconnect with fair valuation. Second, the use of risk-neutral market models leads to a greak lack of realism in the projected trajectories, which can have a great and unpredictable impact on the way medium and long-term risks are taken into account in valuation.

Calibration issues

Let us consider a model-free calibration procedure used in practice by most European life insurers. A more in-depth version of Section 5.1.3 is required to better understand the *model calibration sub-step*. Assume that an actuary wants to calibrate their models on December 31st of year N (12/31/N).

- On 12/31/N (the calibration & valuation date), a set of at-the-money receiver swaption implied volatilities (5x5 10x10, 20x20, or limited to certain maturities/tenors) is obtained from a market data provider. In parallel, a reference zero-coupon (ZC) curve is provided by EIOPA (for the date 12/31/N). This curve is based on the Euroswap. To obtain the final *regulatory-specific* curve, a *Credit Risk Adjustment* spread (CRA) is subtracted from the re-compounded rates. The curve is now "risk-free". Then, a Smith-Wilson method is used to make the 1-year ZC forward rates converge towards an *Ultimate Forward Rate* (UFR) between maturities of 20 (the "Last Liquid point") and 60 years. Lastly, a *Volatility Adjustment* (VA) treatment is applied to the ZC curve [∗], which has to be used to assess economic valuation, in spite of the fact that it is no longer risk-free. Thanks to these two inputs (ZC *regulatory-specific* curve and swaptions volatilities), one re-builds swaption rates and prices (using Black formulas).
- In the second step, a model is chosen to project the yield curve (LMM_{ins}, with stochastic volatility, 1-factor Hull-White, etc.). A key decision criterion is the availability of a closed formula for swaption prices, which must only depend on the interest rate curve and on the set of parameters required to simulate interest rates.
- Lastly, an optimization algorithm is run to minimize the distance between the *theoretical* swaption prices and the model swaption prices. Performing this algorithm leads the user to obtain the *market-consistent model's parameters*, as at date 12/31/N.

Interest rate risk is one of the major hazard sources for life insurance, both *under real life conditions*, and *in economic risk management*. However, this risk is unusually dealt with in the economic valuation scheme. It induces several theoretical and practical issues.

It is known and understood in practice that each swaption is valued under a probability measure specific to its maturity and tenor. Knowledge of this measure is absolutely necessary when using the Black pricing formula and the externally-provided implied volatility. **The yield curve used to assess the swaption rates, the numeraire values, and the ultimate theoretical prices, are not correct**, because the *regulatory-specific* yield curve does not enable practitioners to get the true swaption rates. The impact on final market prices can lead to arbitrage sources. Indeed, the true swaption rate surfaces used in the market have a very specific shape that satisfies certain order conditions necessary to retain the no-arbitrage property. Futhermore, this property is absolutely necessary when using or even calibrating a risk-neutral valuation.

In addition, let us consider the way market data providers (Bloomberg, in particular) re-calculate the matrix of swaptions implied volatilities. In practice, not every swaption in the 10x10 matrix is traded each day, and only some sets of maturities and tenors can be associated with trading prices and "true" implied volatilities. For the other sets in the matrix, providers use an algorithm (SABR for Bloomberg) to interpolate and extrapolate the other points in the matrix. Note also that some data

∗. See EIOPA (2015) for developments and justification of such treatments, used to formalize, in practice, the idea of "relevant risk-free interest rates term structure".

used by actuaries are assessed by private firms, and subject to exogenous expert judgment.

The last point is less problematic than the first, due to the large distortion seen in the yield curve induced by applying CRA/UFR/VA. These regulatory issues lead to irrelevance of the final calibrated parameters.

However, another important structural issue comes from to the use of risk-neutral interest rate models when projecting interest rates in the long term.

The importance of forward rates

When market models are used to project yield curves, two major points have to be taken into account. First, the models are only used in financial markets for (very) short term simulations (e.g., 1 day/week). They do not provide a fair valuation of medium and long-term interest rate risk. This is due to the second important fact induced by their use: the simulation of flat curves in the long run.

The market models used by actuaries (1-factor Hull-White, LMM_{ins}, and others) all match the no-arbitrage property. Now, let $F(t,s,\theta)$ be the θ-year forward ZC price at time $s > t$, seen at $t \geq 0$. For simulated ZC curves, we have, under the t-forward measure,

$$\forall \theta \in [\![1;M]\!], \mathbb{E}\left[P(t,\theta)\right] \simeq F(0,t,\theta) = \frac{P(0,t+\theta)}{P(0,t)}.$$

The diffusion leads to a mixture between the forward ZC curve and a stochastic variation. Due to the reduced number of risk factors (between 1 and 3 in practice), the simulated curve either moves close to the forward curve, or diverges. In the long run, the curves either diverge or stay flat (due to the flat forward curve), close to 0, in order to compensate for the numerous LMM_{ins} diverging curves on average. This may have a limited impact on at-the-money or short term options pricing, but can lead to a large mis-valuation of the medium and long-term time value of financial options and guarantees, embedded in insurance products.

Forward rates are important in every kind of market valuation. This is particularly true for economic valuations, where ZC curves can be projected for up to 60 or more years, for life insurance products such as pensions and savings products. It is remarkable that EIOPA discloses regulatory yield curves and adjustment methodologies/parameters (CRA/VA/UFR), which have a great impact on the final ZC curve, forward rates, and valuations. These *market-inconsistent* developments have little theoretical and practical legitimacy, and lead to a strong disconnect between insurance economic valuation and both *true risk-neutral valuation* and *economic reality*.

Impact on economic scenarios

To describe the operational impact of the current economic valuation choices, consider the yield curve explicitly given by EIOPA on 12/31/14.

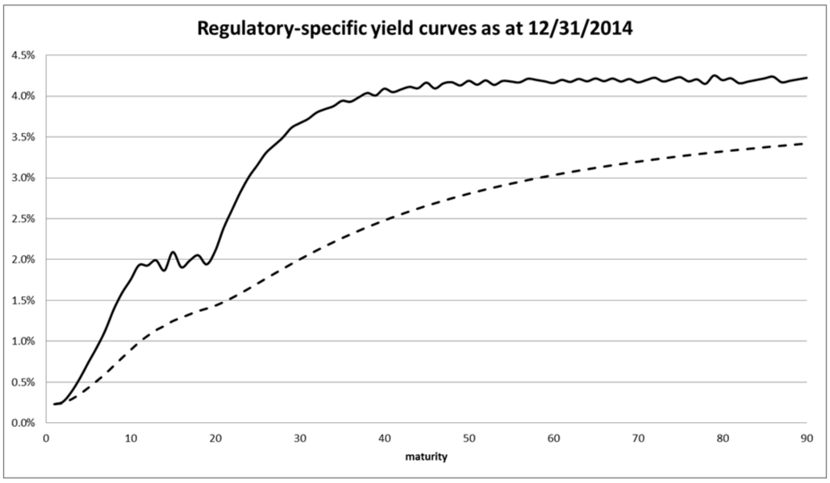

Figure 5.1 – Yield curves at $t = 0$: ZC yield curve (dashed) and 1-year Forward curve.

Figure 5.1 shows the regulatory-specific ZC yield curve by maturity (1 to 90) and the 1-year forward rates at $t = 0$ ($F(0,s,1)$ for $s \in [\![0;90]\!]$). The yield curve is unrealistic, mostly due to the impact of the UFR. In particular, the 1-year forward rates seen at $t = 0$ have an erratic structure after 60 years (around the UFR, 4.2%). This is structurally due to the Smith-Wilson methodology, and can be found on each and every regulatory yield curve. * Another surprising fact is the erratic form of the forward curve between maturities of 10 and 20 (see Figure 5.2 – a zoom of Figure 5.1).

This kind of forward curve is far from what is observed in practice, and essentially due to EIOPA's interpolation methodology.

Considering the swaption implied volatilities matrix at date 12/31/14 extracted from Bloomberg, we have calibrated a LMM_{ins} in a market-consistent fashion †. Then, we simulated 1000 random trajectories of the yield curve on 30 maturities, through a time horizon of 60 years. ‡ As this model tends to produce diverging curves, we have capped the yields at 70% in the Economic Scenarios Generator. This is typical market practice.

∗. In most other practical fields, when using such types of convergence algorithm, the erratic values after 40 years would be smoothed by the user, in order not to introduce any additional disturbance. It is remarkable that this is not done for the EIOPA yield curve.

†. The LMM_{ins} is only used here for illustration purposes. A pure finance practitioner may find many theoretical and practical issues when using this insurance adaptation of the natural *LMM*. This question is not developed further here, but will be addressed in subsequent papers.

‡. We have chosen to consider a LMM_{ins} because it is currently one of the most-used models, but the results would be similar for other market interest rates models.

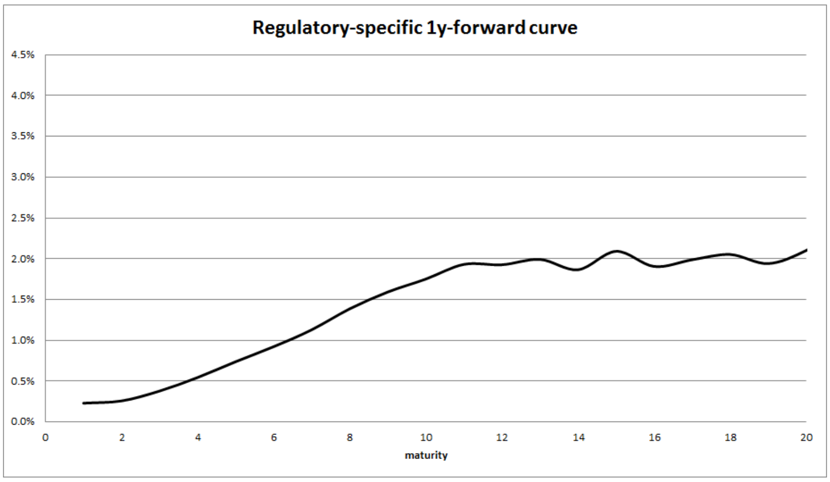

Figure 5.2 – 1y-forward curve - zoom in on maturities 0 to 20

Ten randomly drawn curves, simulated at time $t = 10$ and $t = 30$, are presented in Figures 5.3 and 5.4. In addition, we represent the corresponding forward yield curves as seen at $t = 0$.

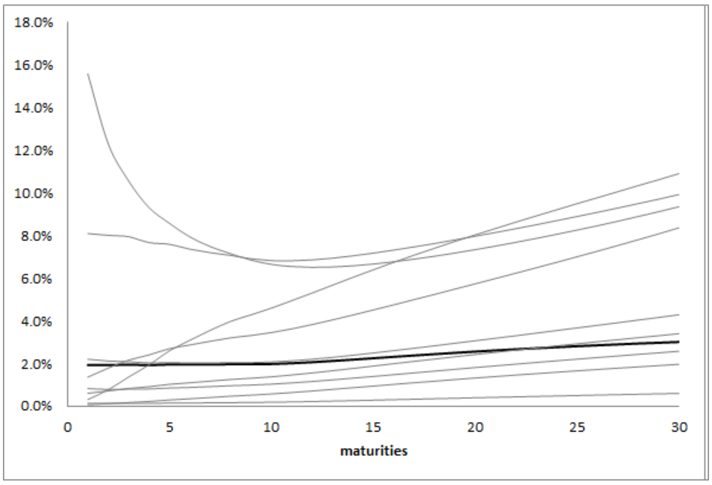

Figure 5.3 – $t = 10y$ – 10 randomly drawn curves (grey lines) + forward curve at $t = 0$ (darker line).

These two graphs show the strong impact of the diverging rates. The cap is indeed reached for 5 out of the 1000 simulations at $t = 10y$, for 75 out of the 1000 simulations at $t = 30y$, and for 243 out of the 1000 simulations at $t = 60y$. In practice, without a cap (here at 70%), every yield either asymptotically goes to infinity or zero *, which is a structural fact for LMM_{ins} and for most risk-neutral financial models (see El Karoui et al. (1997), Dybvig et al. (1996)). This is one of the reasons why they are only used in very short runs by financial practitioners. In particular, we must wonder how policyholders and lapse rates would behave in such unrealistic situations.

*. For the 10 randomly drawn curves shown in Figure 5.4, we already have two curves capped at 70% and two close to zero.

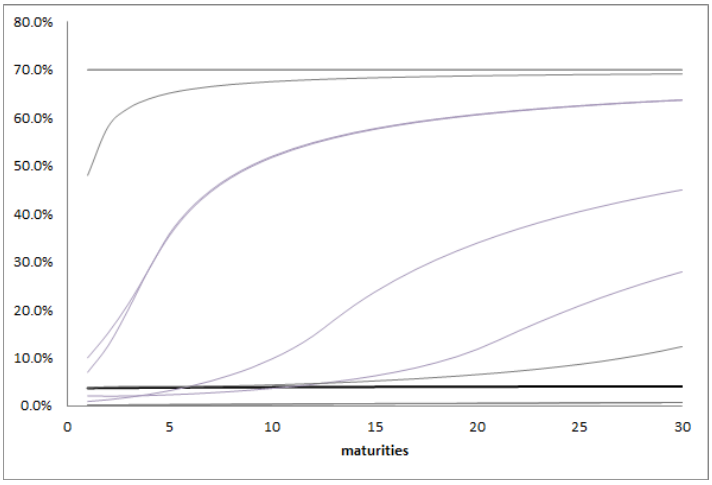

Figure 5.4 – $t = 30y$ – 10 randomly drawn curves (grey lines) + forward curve at $t = 0$ (darker line).

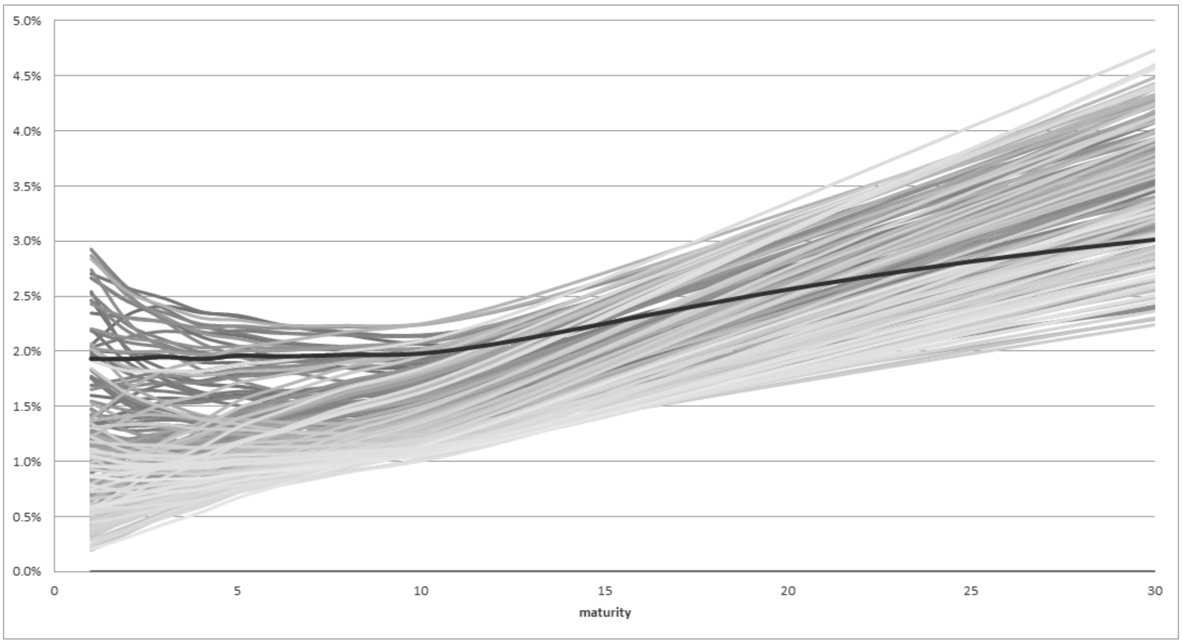

Figure 5.5 – $t = 10y$ - Selected 20% of the simulated curves (grey lines) + forward curve at $t = 0$ (darker line)

In the medium run, most curves are concentrated around the forward curve (see Figure 5.5 for $t = 10y$, where we show 20% of the simulated curves, chosen for their plausibility: no diverging rates, nor flat and close to zero curves). This is structural (not cyclical) for these risk-neutral models, every simulated curve at time $t = \theta$ years is deduced from the θ years-forward ZC curve at $t = 0$, to which diffusion adds time-increasing fluctuations based on a limited number of standardized Gaussian random values.

To conclude, in Section 5.2.1, we have described market inconsistencies resulting from the choice of swaptions implied volatilities data, and from the regulatory-specific yield curve. In addition, we have illustrated the fact that, though they involve high technical complexity in terms of calibration and simulation, risk-neutral models, used by insurance undertakings to project interest rates, have similar and relatively simple properties, based on the replication of forward curves.

This illustrates the impact of try to adapt financial practices to completely different insurance topics. Simpler and more realistic models, like the intuitive Nelson-Siegel model, are often used and studied by central banks (see e.g., the works of Söderlind and Svensson (1997), Christensen et al. (2009) and Coroneo et al. (2011) for the European Central Bank). These works provide realistic curves that could give much fairer valuations of insurance Time-Value of Future Options and Guarantees. The model is not used for risk-neutral valuation in the actuarial sense. However, some finance practitioners have considered this model in an arbitrage-free scheme, or for long term interest rate trading (see Christensen et al. (2011) and Turc and Ungari (2009)).

5.2.2 The local character of market-consistency and high volatility in economic valuation

The actuarial market-consistency criterion leads to quite volatile valuations. It is directly subject to market movements, and depends on the calibration sets chosen by the actuary. We therefore claim that this criterion is local in time and space. In this section, we further develop this idea, in order to underline the necessity for strong sculpting of implementations, taking locality better into account.

Locality in time and space

What actuaries call market-consistent one day will have changed one day later Financial markets move all the time; it is a structural property of environments where risk-neutral valuation is used fairly. This naturally leads to a high cyclicality of economic balance sheets, and of economic own funds in particular.

- When everything is going fine in financial markets, stock prices tend to rise, interest rates have a “profitable” level, assets are liquid, and implied volatilities are low. Economic own funds rise.
- However, when the market is hit by a crisis, and is unstable and illiquid, one observes high implied volatilities, and the market drift switches in an adverse direction. Everything starts to go wrong, and insurance companies undergo a decrease in their economic own funds (and in some cases a consequent rise in their capital requirements).

This pro-cyclicality has led regulators to introduce counter-cyclicality valuation methods, with mixed results. For more insight into these recent developments, see works concerning *liquidity premium* (e.g., Danielsson et al. (2011) and Delcour (2012)), which were first introduced in 2010, and concern the currently used Volatility Adjuster, recently published by EIOPA (2015).

The local aspect of market-consistency also comes from a *spatial* component. Indeed, this criterion largely depends on data chosen to calibrate the models. Given: (i) the non-uniqueness of the risk-neutral probability measure, (ii) the numerous approximations induced by the actuarial calibration of the models, and (iii) the numerous optionality sources embedded in life insurance products, market-consistent valuations can strongly vary if users change the set of market data used to calibrate the market-consistent valuation probability measure.

This locality also exists in the model space used for economic valuations. Indeed, for each economic driver, numerous models are possible. All of them have the same risk-neutrality properties, but provide rather different time-values for insurance products. One can see such impacts in Market-Consistent Embedded Value (MCEV) reports, disclosed by most listed firms. The MCEV is a market-consistent way to assess the embedded value of insurance firms (see CFO Forum (2009)). The valuation process is finally very close to the one used to assess economic own funds. The entities which calculate their MCEV also disclose a report that explains the deviation between their MCEV and its previous year's value. This deviation can have numerous origins, but one is called *model changes* in the report. According to such reports, it often leads to 10%-deviations for prospective estimations, and even more.

Impact of using market data as at the 12/31 for economic valuations

Considering locality in time, practitioners should also beware that the date 12/31 is not ideal for calibrating a model with market data (and proposing an insurance economic valuation). Indeed, at this date, traders observe numerous unpredictable movements linked with the accounting closing date, leading to accounting optimization deals, sales and purchases of large positions. Markets are known to be specifically untrue, and to provide inexact, highly volatile prices and implied data, in December. This well-known fact is supported by numerous analyses (see the many works on the December last trading day effect, e.g., Lakonishok and Smidt (1988), Chen and Singal (2003) and Sum (2010)). It seems therefore quite hazardous to base a valuation framework on such data.

As an example, we present some statistics based on the implied volatility of the swaption of maturity 5y/tenor 5y [*] in Figure 5.6. We have extracted the historical data associated with this volatility, from 10/01/14 to 01/31/15. We show the evolution of the implied volatility and the 1-month moving volatility of this value, on historical data. December 14 is highlighted, along with its erratic profiles.

Year 2014 gives a good idea of the impact of the local aspect of market-consistency and of the choice of calibration at 12/31 on the robustness of final outputs. It is well-known that economic values are

∗. These parameters have been chosen because the receiver swaption of maturity 5y/tenor 5y is the most liquid one.

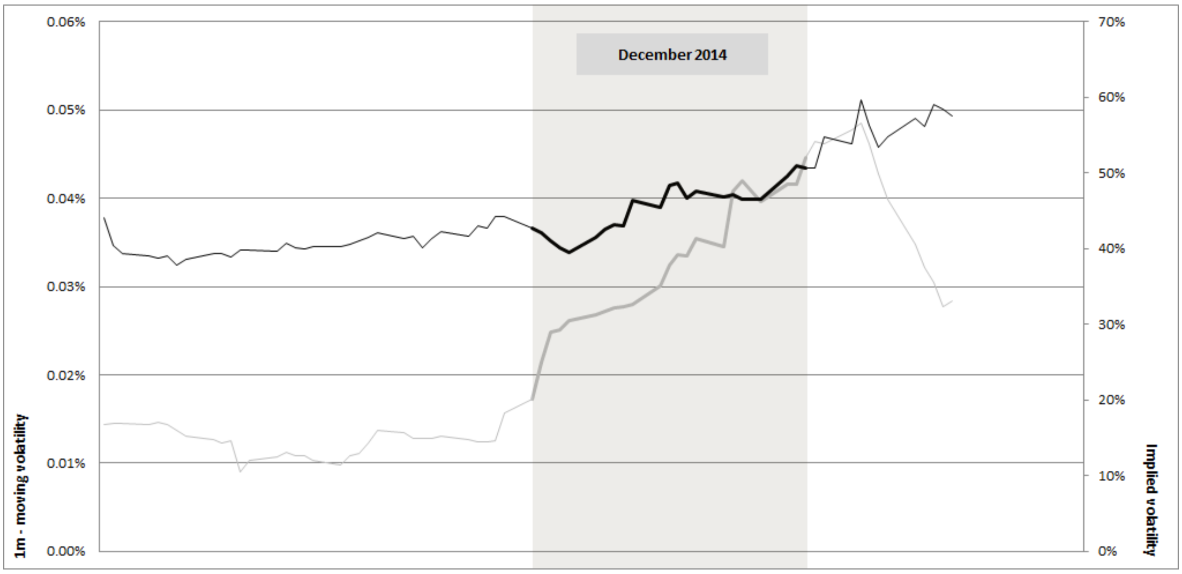

Figure 5.6 – Statistics on the receiver swaption implied volatility - maturity 5y/tenor 5y; darker line: 1m-moving volatility from 10/15/14 to 01/15/15; grey line: implied volatility from 10/15/14 to 01/15/15.

highly driven by market cycles, but when actuaries do a valuation, they also give high importance to unexpected accounting optimization malpractice. In the long run, the choice of a single date to calibrate data could even lead to additional insurance practices to optimize their economic values and required capital, such as buying or selling the appropriate assets.

Note that this is just a first illustration of the "turn of the year" effect. For more insight on the periodicity impact of this day on market data, the reader can find a wide range of papers on the subject (see Lakonishok and Smidt (1988), Hirt and Block (2006) and related literature).

Alternative proposition

To compute a more stable, well-supported, and robust economic valuation in life insurance using a well-adapted market-consistent constraint, an easily implementable approach could be to use averaged calibration data sets. As an example, we propose a calibration of the interest rates model (using a typical LIBOR Market Model) based on a replication of an averaged swaption implied volatilities matrix. Based on a realistic saving product, the valuation of the economic own funds is implemented for 4 types of calibration.

The first calibration is based on the implied volatilities matrix, averaged over the whole month of October 2014. Only the receiver swaptions of maturity 5, tenors 1 to 10, and the receiver swaptions of tenor 5, maturities 1 to 10, are used in the calibration set, to focus on the most liquid options.

The second calibration is based on the implied volatilities matrix averaged on the whole months of October and November 2014. Only the receiver swaptions of maturity 5, tenors 1 to 10, and the receiver swaptions of tenor 5, maturities 1 to 10, are used in the calibration set, to focus on the most liquid options.

Two *different* valuations are made at 12/31/14. *Both of them can be considered as market-consistent*, but are performed using different sets of calibration assets: for the third calibration, we use the receiver swaptions of maturity 10, tenors 1 to 10, and the receiver swaptions of tenor 10, maturities 1 to 10 (v1). For the fourth calibration, we use the receiver swaptions of maturity 5, tenors 1 to 10, and the receiver swaptions of tenor 5, maturities 1 to 10 (v2).

The final results, obtained using a 30-year EST of 1 000 random simulations * initialized with the same seed, are presented in Table 5.1.

Table 5.1 – Comparison between obtained economic own funds values

	October 14'	Oct. & Nov. 14'	12/31/14 v1	12/31/14 v2
Economic own funds	16'898	15'614	7'046	10'000
95% confidence intervals	[14'493,19'305]	[13'156,18'073]	[4'227,9'865]	[7'292,12'708]

Several interesting points can be raised. Locality in time is well illustrated by the large difference between 12/31 and October/November values (approx. 100% relative difference). Locality in space is highlighted by the strong difference between v1 and v2 valuations (more than 40% relative difference). The stronger decrease associated with the calibration v1 can have various sources. We have seen in Section 5.2.2 that the implied volatilities have on averaged increased between 10/01/14 and 12/31/14. In addition, the product being considered may have various specific characteristics that make the v1 LMM_{ins} interest rate volatilities (long term versus short term) worse than the analogous ones for v2.Note that all the differences observed between the averaged calibrations and the 12/31 ones (v1 and v2) are significant at a 95% level.

Using averaged data leads to more stability. The two economic valuations calibrated on averaged implied volatilities only differ by around 8%. One should note that such implementations have already been performed by practitioners during the most impacted years of the 2007 financial crises. But the process has been left aside to refocus on market data at 12/3.

The values in Table 5.1 were obtained for a representative French life insurer. One might think that the observed differences are amplified by the portfolio structure. To be rigorous, we have run the same analysis on two other real life portfolios, with the following characteristics :

— A 2nd standard savings product, whose assets embed stocks (13%), bonds (86%) and cash (2%). The ALM model embeds a profit sharing rule, a minimum guaranteed rate, dynamic lapse, etc.

*. The choice of 1 000 scenarios is a typical number of simulations among European life actuaries. It would of course be desirable to use many more simulations.

— A 3rd saving product, whose assets embed stocks (22%), bonds (77%), a small quantity of cash, and several derivatives (< 1%). The ALM model still embeds a profit sharing rule, minimum guaranteed rates, and dynamic lapse.

We obtain similar results (see Table 5.2), which shows that this phenomenon is not limited to a particular portfolio type.

Table 5.2 – Comparison between the obtained economic own funds value – additional tests.

	October 14	Oct. & Nov. 14	12/31/14 v1	12/31/14 v2
portfolio nb 2	12 826	12 283	9 517	10 000
portfolio nb 3	12 553	12 073	6 050	10 000

The manipulation issue

Another relevant issue, illustrated in Section 5.2.2, is the opportunity for insurers to manipulate the market-consistency and economic values.

Indeed, a re-allocation of assets, and the corresponding switch in asset portfolio duration, could be an easy justification to switch from (v1) to (v2). In practice, it is very likely that this switch, justified by the new asset portfolio, and still leading to an *efficient market-consistency* of the valuation, would be accepted by the local supervisor. For the insurer, the impact is immediate: a 40% increase in its net asset value.

Other manipulations such as financial model changes, are possible. Indeed, a company can test several interest rates models that are all accepted by the control authority. It could be tempting to select the one that provides the best results.

Lastly, as the notion of market-consistency is currently based on 12/31 market data, it is imaginable that, in the future, a big company that understands market finance well enough, will optimize its relevant swaption volatilities values, through the buying and selling of swaptions. This kind of financial manipulation has already been observed (see among others Abrantes-Metz et al. (2012), Abrantes-Metz et al. (2011)).

To limit such manipulation opportunities, regulators could impose restrictions on the calibration sets and simulation models. Another possibility could be to simply use averaged data based on one or two months, such as alternatives proposed in Section 5.2.2. This kind of practice would make market manipulations more difficult because of the averaged data's inertia.

Practitioners should also better justify their valuation choices, by better linking market-consistency, models, calibration sets, financial modeling choices, and insurance product specificities. A first alternative to improve this situation is presented in Section 5.2.3.

5.2.3 The entity-specific aspect of valuation

In this section, we address a specific point regarding current actuarial practices. This could lead to an improvement in the objectification of the valuation probability measure, and enable one to better take into account the local aspect of market-consistency.

Operational valuation practices

In Section 5.1, we have seen that there exist several potentially subjective choices available due to selection of the valuation probability measure (e.g., the choice of financial model, calibration set, etc.). These choices are justified by the requirement that the valuation scheme must efficiently fit the entity's economic environment. As an example, a firm highly subject to variations in the French economy chooses the CAC40 as its reference stock index, and the corresponding at-the-money options (Calls/Puts on the CAC40, etc.) as calibration assets for its financial models. The chosen assets depend on the models to calibrate, but most companies use similar calibration sets.

The calibration framework is indeed mostly homogeneous, due to two facts. First, EIOPA has developed standardized methodologies to calibrate Standard Formula shocks (see e.g., EIOPA (2010)), based on standard calibration sets of assets. These sets have become standards and are often used by insurance stakeholders to calibrate their financial models. Second, the major economic scenarios provider, Moody's analytics (formerly known as Barrie & Hibbert,) has developed software, financial models and calibration methodologies that tend to become standards for their clients and for the European life insurance market.

From a more general point of view, current implementations lead to entity-specific valuation frameworks. Indeed, the companies calibrate, once and for all, the financial models they plan to use to valuate the economic balance sheet associated with all of their life insurance products (or one set of models for every product sold under the same economy). This results in the use of only one valuation probability measure for all life insurance products (or one per economy). The process and the values obtained are therefore more entity-specific than product-specific.

As it does not take the specific characteristics of the valuated products into account, this aspect of valuation leads to great difficulties when comparing the economic values of similar products which are estimated by different companies. Comparability (like objectivity) is however a key factor for efficient regulation.

In Sections 5.2.3 and 5.2.3, we present the potential impact of these implementation choices, and propose an alternative probability measure selection approach.

Local Market Consensus Probability Measures

In the banking industry, consider two different complex financial assets on a standard risk, for example an American Call and an American Put option on the Dow Jones index, with different maturities and moneynesses. Assume that a Monte Carlo approach is used to valuate these two hedgeable options, using similar financial models. Under these assumptions, the parameters of the two models used in the Monte Carlo processes should be quite different, because the market for such options is illiquid and incomplete, and because one part of the parameters' choice process requires (subjective) expert trader judgment.

Life insurance products are much more complicated and much less liquid than these two financial instruments. It therefore seems necessary to consider product-specific (or at least ring-fence-specific *) parameters, which leads to the consideration of a much more locally market-consistent valuation probability measure.

We therefore propose to consider a more locally-defined market-consistency criterion. Insurance undertakings should take account of the characteristics of their very own products when they are economically valuating them. As exact market-consistency is impossible to attain, more attention should be paid to the choice of the asset sets used to calibrate financial models chosen by the entity. Some standard characteristics of life insurance products can easily be taken into account. Liability duration and convexity, the portfolio used to protect the product against financial movements (if it exists, which is not frequent in the European life insurance market due, among other things, to the cost of such portfolios in terms of Solvency Capital Requirement in Solvency II), as well as the asset portfolio used to face insurance liabilities, are good examples of such data. In general, one should consider a different calibration set for each economic valuation. There exists almost one different risk neutral probability measure for each complex financial asset valuation; similarly, there should exist one different risk neutral probability measure for each life insurance liability (or ring-fenced set of liabilities) valuation.

This is a first proposition to refine the calibration process used by practitioners. This alternative approach leads to use a more locally-adapted valuation measure: a Local Market-Consensus Probability Measure (LMCPM or more simply, LMC). Through Section 5.2.3, we propose a *LMC approach* implementation, and study the difference in valuation observed between the standard framework and this alternative, for a real life insurance product.

$*$. Through the Solvency II implementation scheme, products are grouped in ring-fences, and most economic valuations are in fact applied to ring-fenced products. We always speak of valuating products, for the sake of simplicity, but the LMCPM approach developed here, and implemented below, can easily be extended, without loss of generality, to ring-fences of products/liabilities valuations, which may be more useful to practitioners. In particular, a ring-fenced LMCPM measure still leads to a more local and well-adapted market-consistency than a standard valuation approach, where each ring-fenced group of life insurance products is valuated under the same probability measure.

Case study

We adapt the LMC approach to a real savings product. This product shows all the characteristics of a standard French savings contract, embedding optionality (implemented using dynamic lapses, taking profit-sharing rules and minimal guaranteed rate rules into account), and a fine granularity of policyholders. Its asset portfolio is composed of standard and inflation-linked bonds, stocks, OPCVM, hedge fund and property indexes, in two different currencies, US dollars and Euros. As it is a French product and mostly impacted by changes in the Euro zone markets, we chose the relevant risk-free interest rates term structure to be the Euroswap curve (no CRA/VA/UFR). We do not use the regulatory-specific curve in this section, to avoid adding any noise to the true LMC impact. We believe this choice has little impact on our analysis. The valuation is processed as at 12/31/2013.

Implementation As only interest rate risk is hedged, we only project this one stochastically. The other risk factors (stock, OPCVM, hedge fund and OPCVM indexes, US dollar currency bonds and indexes) are assumed to be deterministically capitalized at the forward rate observed as at the valuation date. The only financial model used to stochastically project the risks is the Euroswap interest rates model, an LMM_{ins}. The difference between the standard calibration and the LMC approach lies in the LMM_{ins} parameters obtained through the calibration process, either replicating a standard (10 maturities x 10 tenors) matrix of at-the-money receiver swaption implied volatilities, or the ones of the protection portfolio.

Results

The LMM_{ins} enables us to estimate the prices of at-the-money swaptions, using efficient approximations (see Hull and White (2000)). Note that, in our implementation, we have chosen to price the swaptions and initialize our simulations using the Euroswap curve as at 12/31/2013, without the CRA/VA/UFR adjustment, in order to make clearer the calibration/simulation process (see Section 5.2.1). We believe that this choice has little consequence on the potential impact of the LMC approach on valuation. We have implemented and compared two approaches.

— The methodology used by practitioners to calibrate their LMM_{ins} is depicted in Section 5.2.1). In a *standard implementation*, we have implemented this methodology based on the replication of a swaption implied volatility matrix, for maturities from 1 to 10 and tenors from 1 to 10, obtained as at 12/31/2013.
— Protected life insurance products, as far as interest rate risk is concerned, use receiver swaptions, but also caps and floors. However, as most interest rate models are calibrated based on the replication of swaptions volatilities, we have chosen in an *LMC application* to replicate

only the swaption protection portfolio. The swaptions weights are integrated into the least squares calibration program.

The estimated economic own funds (OF) can be divided into two parts: the Revalued Net Asset (RNA, corresponding to the capital already earned by the firm), and the Value of In-Force (VIF, corresponding to the average of the net present value of future margins). In practice, the VIF is the only one impacted by the LMM_{ins} calibration process. We give below the decomposition of the economic own funds obtained considering the standard calibration process. The VIF is estimated according to the process developed in Section 5.1.3, based on only 1 000 economic scenarios, due to the time-consuming character of this implementation.

Table 5.3 – Comparison between obtained economic quantities

Economic values	Standard valuation	LMC approach	Relative deviation
OF	2’069	2’113	2.1%
RNA	1’000	1’000	0.0%
VIF	1’069	1’113	4.1%

Analysis of the results and limits to our proposition

Though the number of scenarios used in this case study is limited, the Monte Carlo estimators of the VIF have low volatility (lower than 30% of the estimated expectations), and the LMC VIF is significantly higher (at a 5% confidence level) than the standard VIF. The observed deviation is small, but this is associated with the fact that this product’s asset portfolio is not fully composed with interest rate products. It also embeds stock, OPCVM, property and hedge fund indexes. Note that due to the complex link between the calibration process and the assessed economic values, it is almost impossible to predict the impact of switching from the standard approach to the LMC approach. In this LMC implementation, we can see that the LMC method has a small but significant impact. However, the locality in space depicted in Section 5.2.2 illustrates well the potential impact induced by the LMC methodology.

We have chosen several simplification hypotheses that may have reduced the impact of the LMC approach. For example, we have applied the methodology in a scheme where only interest rates are stochastic. In general, one should use a more general protection portfolio. Due to cumulative effects, it is likely that the difference in valuation would have been bigger if all financial risks (stock, hedge fund, OPCVM, property risks, etc.) were stochastically projected and calibrated under our alternative methodology.

Conclusion and policy recommendations

In this paper, we have placed the European actuarial risk-neutral (*economic*) valuation scheme in its context. We have shown, in particular, how the incompleteness of financial markets and the associated non-uniqueness of the risk-neutral probability measure, drive actuaries to add a complex selection criterion when defining their valuation probability measures, *market-consistency*. The final valuation process makes use of the risk-neutral paradigm in a poorly-adapted context. For this framework, we have presented some of the main pitfalls in actuarial practice. This has led us to re-focus on the true goals of European regulations. The first goal is to control the systemic risk embedded in European life insurance. The second is to provide an objective valuation scheme adaptable to a large set of different countries and market specificities, and most of all, one that is easily controllable. The third is to enable full-comparability of valuations. The second point, though the current scheme leads to theoretical issues and control difficulties, seems to have been more or less fulfilled. Concerning the third point, we think that more could be done because of the great subjectivity in the final economic values. European authorities could give additional orders; alternative implementation methods could be proposed; the regulation scheme could be simplified. Although the new directive charts very interesting new ideas (including the willingness to estimate the time-value of insurance products, and develop a scheme adaptable to the specific characteristics of companies), one must limit the sources of possible manipulation induced by the current process, and the importance given to external, unsupervised data providers.

The main conclusions of our study can be summarized in the two following warnings.

First, risk neutrality used for long term projections simply leads to a lack of realism, which can strongly impact the real accounting solvency of undertakings. Not taking market trends into account, it considers a case that artificially over-penalizes asset management and protection. The major problem is that these implementations deal with solvency, systemic risk, and customers.

Second, market-consistency leads to potential manipulation of values. It allows many entity-specific choices, such as models and calibration sets, leading to high volatility and non-comparability of economic values (Solvency Capital Requirement included). In addition, the choice of the *turn-of-the-year* to select the data used to assess regulatory capital and other economic valuations, is dangerous due to the untypical and volatile data available on financial markets at this date. In this framework, the work of supervisors is extremely challenging.

Concerning these two points, our policy recommendations are the following.

Risk-neutrality is hardly adaptable to life insurance liabilities. The will to impose and legitimate economic valuation has led to successive adjustments (e.g., CRA/VA/UFR). These are difficult to justify in practice, and lead to over-complexification of the directive application. This should not allow market actors forget the directive's true aim. Realism is a key element to evaluate the efficiency of any market to face systemic risk. As far as life insurance is concerned, this can be applied simply by fitting more realistic models (accepting getting out of the risk neutral paradigm, and creating an insurance-specific one) or limiting the projection of data to only 5 or 10 years. The idea behind these refinements is clear: catch the real impact of risks. Further work is clearly necessary to quantify the impact of this lack of realism and propose alternative implementations.

We have presented some first possibilities to set the market-consistent character in a more transparent way, and raise the need to restore the link between insurance product properties and Economic Scenarios Generator choices (models and calibration sets used). The current processes used to calibrate models provide, once fully understood, unstable and manipulable results. Concerning this point, a first improvement could be to use new, more stable calibration data sets (averaged data based on 1 or 2 months) or methodologies (the LMCPM process), or new models that produce more realistic simulations.

Lastly, this study points towards various research directions. Concerning the issues raised in the paper, we have simply pointed out some initial alternative possibilities to standard economic valuation implementations, but it will be necessary to analyze more deeply the implementation of these propositions and their operational impact. It would also be interesting to consider more general alternatives, and in particular refine and reshape the market-consistency criterion. It would also be useful to refocus the regulator's mind on the very aim of the directive: a reduction of systemic risk. The complexity of the defined scheme seems to have left this goal aside for the sake of legitimacy. In practice, it is likely that sensitivity analysis under realistic conditions could have greater impact than this 99.5% - Value-at-Risk estimation, especially under unrealistic assumptions.

Appendix: The LIBOR Market Model - actuarial version (LMM_{ins})

The LIBOR Market Model provides forward rates dynamic modeling. The following presentation assumes that the considered swaptions are options on yearly swaps. Therefore, only annual payment dates are used. This assumption is often chosen in practice when calibrations are made on yearly forward rates, which is generally the case.

Let $0 \leq j \leq J$ be a set of $J+1$ successive dates. We get the 1y-forward zero-coupon prices from, $\forall j \in [\![1;J]\!]$, $0 \leq t \leq j$,

$$F_j(t) = \left(\frac{P(t,j)}{P(t,j+1)} - 1\right).$$

Let n be the ceiling function defined by:

$$\begin{cases} n(0) = 1 \\ n(t) = \lceil t \rceil \text{ if } t \geq 0. \end{cases}$$

Let $(g_j)_{j \in [\![1;J]\!]}$ and $\Phi(t)$ be the following deterministic functions: $\forall j \in [\![1;J]\!]$, $0 \leq t \leq j$,

$$g_j(t) = [a + b(j - n(t) + 1)]e^{-c(j-n(t)+1)} + d,$$

and

$$\Phi(t) = \theta + (1-\theta) \times e^{-\kappa t}.$$

In a general framework, the 1y-forward diffusion is given, under the spot LIBOR measure, by the following equation (see Brigo and Mercurio (2007)): for $0 \leq t \leq j$,

$$dF_j(t) = F_j(t)\left(\Phi(t)^2 \sum_{i=n(t)}^{j} \left[\frac{F_i(t)}{1+F_i(t)} g_i(t) g_j(t) \rho_{i,j}(t)\right] dt + \Phi(t) g_j(t) dZ_j^M(t)\right),$$

with Z^M a M-dimensional Brownian motion, and $\rho_{i,j}(t)dt = d < Z_i^M, Z_j^M >_t$.

Practitioners suppose Z^M can be decomposed through a two-dimensional standard Brownian motion:

$$W(t) = \begin{pmatrix} W^1(t) \\ W^2(t) \end{pmatrix}.$$

In addition, the specification of the correlation structure is given by the user through a set of parameters

$$\beta_j = \begin{pmatrix} \beta_j^1 \\ \beta_j^2 \end{pmatrix}_{j \in [\![1;M]\!]},$$

such that $\rho_{i,j}(t) = \beta_{i-n(t)+1}\beta_{j-n(t)+1}$. The β parameters are therefore meta-parameters, often *a priori* assessed based on a historical Forward data set.

Denoting $\eta_j(t) = \Phi(t)g_j(t)\beta_{j-n(t)+1}$, the Forward diffusion is assumed to be, for $0 \le t \le j$,

$$dF_j(t) = F_j(t)\left(\sum_{i=n(t)}^{j}\left[\frac{F_i(t)}{1+F_i(t)}\eta_i(t)\eta_j(t)\right]dt + \eta_j(t).dW(t)\right).$$

The calibration therefore aims to estimate, in an optimal fashion, the six parameters $P = (a,b,c,d,\kappa,\theta)$.

Suppose a user wants to fit the at-the-money receiver swaption Black implied volatilities of maturities $(m_i)_{i\in[\![1;I]\!]}$ and tenors $(n_i)_{i\in[\![1;I]\!]}$, denoted as $(\sigma_i)_{i\in[\![1;I]\!]}$. Using the *Rebonato formula* (see Rebonato (1999), Hull and White (2000), Brigo and Mercurio (2007)), the LMM model implied volatility $\sigma_i^{LMM}(P)$ can be approximated by:

$$\sigma_i^{LMM}(P) \approx \sqrt{\frac{1}{m_i}\sum_{k=1}^{m_i-1}\sum_{q=1}^{2}\left(\sum_{j=m_i}^{m_i+n_i-1} y_j^{m_i,n_i}\frac{F_j(0)}{S^{m_i,n_i}}\eta_j^q(i)\right)^2},$$

with

$$y_k^{m_i,n_i} = \frac{P(0,k+1)}{\sum_{j=m_i+1}^{m_i+n_i} P(0,j)}$$

and

$$S^{m_i,n_i} = \sum_{k=1}^{n_i-1} y_k^{m_i,n_i} F_k(0).$$

The parameters are estimated by:

$$\hat{P} = \text{Argmin}\left[\sum_{i=1}^{I}\left(\frac{\sigma_i - \sigma_i^{LMM}(P)}{\sigma_i}\right)^2\right].$$

This calibration approach is standard in day-to-day trading on the swaption market using the specific daily data. The Black implied volatilities are indeed very sensitive to the yield curve. To project the interest rates in the economic scenarios table, actuaries simply identify the spot LIBOR measure to the risk neutral measure. This LMM_{ins}, used for very long term projections, is simply not justified by market standards.

Bibliography

Abrantes-Metz, R. M., Kraten, M., Metz, A. D. and Seow, G. S. (2012), 'Libor manipulation?', *Journal of Banking & Finance* **36**(1), 136–150.

Abrantes-Metz, R. M., Villas-Boas, S. B. and Judge, G. (2011), 'Tracking the Libor rate', *Applied Economics Letters* **18**(10), 893–899.

Black, F. and Scholes, M. (1973), 'The pricing of options and corporate liabilities', *The journal of political economy* **81**(3), 637–654.

Bonnin, F., Planchet, F. and Juillard, M. (2014), 'Best estimate calculations of savings contracts by closed formulas: application to the orsa', *European Actuarial Journal* **4**(1), 181–196.

Brigo, D. and Mercurio, F. (2007), *Interest rate models-theory and practice: with smile, inflation and credit*, Springer.

CFO Forum (2009), 'Market Consistent Embedded Value Principles'.

Chen, H. and Singal, V. (2003), 'A December effect with tax-gain selling?', *Financial Analysts Journal* **59**(4), 78–90.

Christensen, J. H., Diebold, F. X. and Rudebusch, G. D. (2009), 'An arbitrage-free generalized Nelson-Siegel term structure model', *The Econometrics Journal* **12**(3), C33–C64.

Christensen, J. H., Diebold, F. X. and Rudebusch, G. D. (2011), 'The affine arbitrage-free class of Nelson–Siegel term structure models', *Journal of Econometrics* **164**(1), 4–20.

Coroneo, L., Nyholm, K. and Vidova-Koleva, R. (2011), 'How arbitrage-free is the Nelson-Siegel model?', *Journal of Empirical Finance* **18**(3), 393–407.

Danielsson, J., de Jong, F., Laeven, R., Laux, C., Perotti, E. and Wüthrich, M. V. (2011), 'A prudential regulatory issue at the heart of Solvency II'.

Delcour, I. G. (2012), On the use of risk-free rates in the discounting of insurance cash flows, PhD thesis, Katholieke Universiteit Leuven.

Devineau, L. and Loisel, S. (2009), 'Construction d'un algorithme d'accélération de la méthode des "simulations dans les simulations" pour le calcul du capital économique Solvabilité II', *Bulletin Français d'Actuariat* **10**(17), 188–221.

Dybvig, P. H., Ingersoll Jr, J. E. and Ross, S. A. (1996), 'Long forward and zero-coupon rates can never fall', *Journal of Business* **69**(1), 1–25.

EIOPA (2010), 'QIS5 technical specifications'.

EIOPA (2015), 'EIOPA publishes Solvency II relevant risk free interest rate term structures'.

El Karoui, N., Frachot, A. and Geman, H. (1997), 'A note on the behavior of long zero coupon rates in a no arbitrage framework', *Review of Derivatives Research* **1**, 351–370.

Floreani, A. (2011), 'Risk Margin Estimation through the Cost of Capital Approach: Some conceptual issues', *The Geneva Papers on Risk and Insurance-Issues and Practice* **36**(2), 226–253.

Glasserman, P. (2003), *Monte Carlo methods in financial engineering*, Springer.

Hirt, G. A. and Block, S. B. (2006), *Fundamentals of investment management*, McGraw Hill.

Hull, J. C. and White, A. (2000), 'Forward rate volatilities, swap rate volatilities, and the implementation of the Libor Market Model', *Available at SSRN 1295227* .

Kemp, M. (2009), *Market Consistency: Model Calibration in Imperfect Markets*, John Wiley & Sons.

Kreps, D. M. (1981), 'Arbitrage and equilibrium in economies with infinitely many commodities', *Journal of Mathematical Economics* **8**(1), 15–35.

Kroese, D. P., Taimre, T. and Botev, Z. I. (2013), *Handbook of Monte Carlo Methods*, John Wiley & Sons.

Lakonishok, J. and Smidt, S. (1988), 'Are seasonal anomalies real? A ninety-year perspective', *Review of Financial Studies* **1**(4), 403–425.

Malamud, S., Trubowitz, E. and Wüthrich, M. V. (2008), 'Market consistent pricing of insurance products', *ASTIN Bulletin* **38**(2), 483–526.

Markowitz, H. (1952), 'Portfolio selection', *The Journal of Finance* **7**(1), 77–91.

Martin, P. and Rey, H. (2004), 'Financial super-markets: size matters for asset trade', *Journal of International Economics* **64**(2), 335–361.

Merton, R. C. (1973), 'Theory of rational option pricing', *The Bell Journal of Economics and Management Science* **4**(1), 141–183.

Moehr, C. (2011), 'Market-Consistent valuation of insurance liabilities by Cost of Capital', *ASTIN Bulletin* **41**(2), 315–341.

Mukerji, S. and Tallon, J.-M. (2001), 'Ambiguity aversion and incompleteness of financial markets', *The Review of Economic Studies* **68**(4), 883–904.

Rebonato, R. (1999), 'On the simultaneous calibration of multi-factor log-normal interest-rates models to black volatilities and to the correlation matrix', *Journal of Computational Finance* **2**, 5–27.

Schubert, T. and Grießmann, G. (2004), 'Solvency II = Basel II+ X', *Versicherungswirtschaft* **18**, 1399–1402.

Sheldon, T. J. and Smith, A. D. (2004), 'Market consistent valuation of life assurance business', *British Actuarial Journal* **10**(3), 543–605.

Söderlind, P. and Svensson, L. (1997), 'New techniques to extract market expectations from financial instruments', *Journal of Monetary Economics* **40**(2), 383–429.

Sum, V. (2010), 'The january and size effects on stock returns: more evidence', *International Journal of Applied Accounting and Finance* **1**(1), 47–52.

Turc, J. and Ungari, S. (2009), *Filtering the interest rate curve, the MENIR framework*.

Vedani, J. and Devineau, L. (2013), 'Solvency assessment within the ORSA framework: issues and quantitative methodologies', *Bulletin Français d'Actuariat* **13**(25), 35–71.

Wüthrich, M. V., Bühlmann, H. and Furrer, H. (2008), *Market-Consistent actuarial valuation*, Springer.

Conclusion

Aujourd'hui la mise en œuvre de la directive Solvabilité II, imparfaite en 2012, est une réalité. Le processus ORSA étudié, formalisé et exemplifié dans ma thèse est au cœur de la directive et permet d'améliorer fortement le lien des opérationnels au suivi, à la quantification et à la gestion des risques inhérents à leurs pratiques. D'autre part, au travers d'outils innovants ou améliorés, cette thèse a donné des moyens aux actuaires de réduire la complexité induite par l'utilisation du cadre de valorisation économique introduit par la directive. Il apparait ainsi que l'utilisation d'approches proxies telles que le Least Squares Monte-Carlo (LSMC) pour la projection de bilans économiques peut s'avérer particulièrement pertinente pour l'application de la directive et de l'ORSA. De par sa très grande simplicité et son utilisation intuitive, le LSMC est en effet particulièrement désigné pour l'utilisation de processus de projections économiques très avancés (pluriannuel, infra-annuel, sur ratios de solvabilité,...). Cette approche reste un outil n'apportant que des valeurs approximées et il convient de les utiliser avec circonspection. L'utilisation d'une approche conservatrice de correction par métamodèle permet de limiter cette difficulté.

Si les proxies sont particulièrement pertinents pour la mise en œuvre de l'ORSA, leur utilisation est toutefois questionnable pour le calcul du *SCR* où c'est la solvabilité réglementaire qui est estimée. En effet, la problématique de l'erreur de proxy, même corrigée par métamodèle, n'est jamais totalement maîtrisable. L'utilisation d'une approche totalement simulatoire est toutefois trop lourde à implémenter. Dans ma thèse je propose donc une application de l'accélérateur SdS de Devineau et Loisel (2009a) couplé aux bonnes propriétés de la *VAN forward* pour ce calcul. Cet outil apparait comme une possibilité efficace d'estimation du *SCR*, dans un cadre de modèle interne, sans mise en œuvre de type *proxy*.

D'autre part, le cadre de valorisation économique, atypique dans sa difficulté d'implémentation comme dans son suivi, est particulièrement peu objectif. Dans sa mise en œuvre actuelle il est possible que l'asymétrie d'information entre les compagnies et leurs autorités nationales de contrôle puisse mener à des manipulations particulièrement délicates à localiser et pourtant néfastes pour le marché assurantiel. Le dernier article présenté dans cette thèse a clairement pour objectif de sensibiliser ses acteurs à ces enjeux et de les orienter vers une réflexion poussée sur ces sujets.

Concernant les utilisations d'approches proxies pour l'ORSA, nombreuses sont les perspectives ouvertes concernant les approches développées dans les articles présentés. Les implémentations doivent être testées sur d'autres produits d'assurance afin d'affiner l'analyse de leur efficacité. D'autre part, la problématique du suivi et du recalibrage des proxies n'est pas évoquée ici. Enfin, de nouveaux travaux sur l'approche par méta-modèle, proposée pour corriger l'erreur de proxy, sont en cours actuellement, et consistent à affiner l'hypothèse de loi des résidus et à étudier l'efficacité d'approches non-paramétriques.

Pour ce qui est du calcul du capital réglementaire, si l'accélérateur SdS adapté à la *VAN forward* donne de très bons résultats, il reste toutefois à contrôler l'erreur d'estimation du quantile à 0.5%. Cette problématique est pour le moment mal prise en compte par les opérationnels. Pourtant, il est certain que l'estimation d'un tel niveau d'adversité de la variable FP_1, avec un contrôle de l'erreur efficace, nécessite un nombre beaucoup plus important de scénarios primaires que ce qui est utilisé actuellement.

Au niveau des réflexions déjà menées et à venir sur la valorisation économique, de nombreux travaux restent à faire. L'analyse des problématiques de choix et de calibrage des modèles de taux sous une contrainte de *market-consistency* doit être approfondie afin de proposer un cadre objectif de valorisation du risque tenant compte des spécificités des produits valorisés. La proposition de l'approche LMCPM (partie 3 du dernier article) reste en effet perfectible. Elle manque encore de stabilité et n'est pas adaptée à l'intégralité du marché européen de l'assurance vie.

Bibliographie

AARTS, E. et KORST, J. (1988). *Simulated annealing and Boltzmann machines*. New York, NY ; John Wiley and Sons Inc.

ABRANTES-METZ, R. M., KRATEN, M., METZ, A. D. et SEOW, G. S. (2012). Libor manipulation ? *Journal of Banking & Finance*, 36(1):136–150.

ABRANTES-METZ, R. M., VILLAS-BOAS, S. B. et JUDGE, G. (2011). Tracking the Libor rate. *Applied Economics Letters*, 18(10):893–899.

AHLGRIM, K. C., D'ARCY, S. P., GORVETT, R. W. et ARM, F. (2004). Modeling of economic series coordinated with interest rate scenarios.

ALBIZZATI, M.-O. et GEMAN, H. (1994). Interest rate risk management and valuation of the surrender option in life insurance policies. *Journal of Risk and Insurance*, 61(4):616–637.

ALBRECHER, H., CONSTANTINESCU, C. et LOISEL, S. (2011). Explicit ruin formulas for models with dependence among risks. *Insurance : Mathematics and Economics*, 48(2):265–270.

ALGORITHMICS (2011). *Curve Fitting for Calculating Solvency Capital Requirements under Solvency II : Practical insights and best practices from leading European Insurers*.

ANKENMAN, B., NELSON, B. L. et STAUM, J. (2010). Stochastic kriging for simulation metamodeling. *Operations research*, 58(2):371–382.

AROUNA, B. (2004). Adaptative monte carlo method, a variance reduction technique. *Monte Carlo Methods and Applications mcma*, 10(1):1–24.

ARROW, K. J. et DEBREU, G. (1954). Existence of an equilibrium for a competitive economy. *Econometrica*, 22(3):265–290.

ASMUSSEN, S. et ALBRECHER, H. (2010). *Ruin probabilities*. World scientific.

BABBEL, D. F. et MERRILL, C. (1998). Economic valuation models for insurers. *North American actuarial journal*, 2(3):1–15.

BALDVINSDÓTTIR, E. K. et PALMBORG, L. (2011). On constructing a market consistent economic scenario generator. *In Sweden : Seminars Matematical Statistics KTH*.

BARDOU, O., FRIKHA, N. et PAGÈS, G. (2009). Computing var and cvar using stochastic approximation and adaptive unconstrained importance sampling. *Monte Carlo Methods and Applications*, 15(3):173–210.

BARRIE & HIBBERT (2007). *Smart nested simulations : learning from option traders*.

BARRIE & HIBBERT (2011). *A Least Squares Monte Carlo approach to liability proxy modelling and capital calculation*.

BARRIEU, P. M. et EL KAROUI, N. (2007). Pricing, hedging and optimally designing derivatives via minimization of risk measures.

BARTH, A., SCHWAB, C. et ZOLLINGER, N. (2011). Multi-level Monte Carlo finite element method for elliptic PDEs with stochastic coefficients. *Numerische Mathematik*, 119(1):123–161.

BAUER, D., BERGMANN, D. et REUSS, A. (2010). Solvency II and nested simulations–a least-squares Monte Carlo approach. *In Proceedings of the 2010 ICA congress*.

BECHERER, D. et DAVIS, M. H. (2010). Arrow-Debreu prices.

BERGMANN, D. (2011). *Nested simulations in life insurance*. IFA.

BLACK, F. et KARASINSKI, P. (1991). Bond and option pricing when short rates are lognormal. *Financial Analysts Journal*, 47(4):52–59.

BLACK, F. et SCHOLES, M. (1973). The pricing of options and corporate liabilities. *The journal of political economy*, 81(3):637–654.

BONNIN, F., PLANCHET, F. et JUILLARD, M. (2014). Best estimate calculations of savings contracts by closed formulas : application to the orsa. *European Actuarial Journal*, 4(1):181–196.

BÖRGER, M. (2010). Deterministic shock vs. stochastic value-at-risk—an analysis of the Solvency II standard model approach to longevity risk. *Blätter der DGVFM*, 31(2):225–259.

BOUYCHOU, L. et ATTAL, J. (2010). Le risque de taux en épargnr assurance vie.

BRIGO, D. et MERCURIO, F. (2007). *Interest rate models-theory and practice : with smile, inflation and credit*. Springer.

BROADIE, M., DU, Y. et MOALLEMI, C. C. (2011). Efficient risk estimation via nested sequential simulation. *Management Science*, 57(6):1172–1194.

CARR, P., GEMAN, H. et MADAN, D. B. (2001). Pricing and hedging in incomplete markets. *Journal of financial economics*, 62(1):131–167.

CARR, P. et WU, L. (2004). Time-changed Lévy processes and option pricing. *Journal of Financial economics*, 71(1):113–141.

CFO FORUM (2009). Market Consistent Embedded Value Principles.

CHAIRE MANAGEMENT DE LA MODÉLISATION (2014). *Attraits et limites de la probabilité "risque neutre" pour valoriser le passif d'un assureur*. BNP Paribas Cardiff. slides.

CHARRIER, J., SCHEICHL, R. et TECKENTRUP, A. L. (2013). Finite element error analysis of elliptic PDEs with random coefficients and its application to multilevel monte carlo methods. *SIAM Journal on Numerical Analysis*, 51(1):322–352.

CHEN, H. et SINGAL, V. (2003). A December effect with tax-gain selling ? *Financial Analysts Journal*, 59(4):78–90.

CHEN, X., NELSON, B. L. et KIM, K.-K. (2012). Stochastic kriging for conditional value-at-risk and its sensitivities. *In Simulation Conference (WSC), Proceedings of the 2012 Winter*, pages 1–12. IEEE.

CHERNOV, M., GALLANT, A. R., GHYSELS, E. et TAUCHEN, G. (2003). Alternative models for stock price dynamics. *Journal of Econometrics*, 116(1):225–257.

CHRISTENSEN, J. H., DIEBOLD, F. X. et RUDEBUSCH, G. D. (2009). An arbitrage-free generalized Nelson-Siegel term structure model. *The Econometrics Journal*, 12(3):C33–C64.

CHRISTENSEN, J. H., DIEBOLD, F. X. et RUDEBUSCH, G. D. (2011). The affine arbitrage-free class of Nelson–Siegel term structure models. *Journal of Econometrics*, 164(1):4–20.

COCHRANE, J. H. (2001). Asset Pricing.

CORONEO, L., NYHOLM, K. et VIDOVA-KOLEVA, R. (2011). How arbitrage-free is the Nelson-Siegel model ? *Journal of Empirical Finance*, 18(3):393–407.

CRAINE, R., LOCHSTOER, L. A. et SYRTVEIT, K. (2000). Estimation of a stochastic-volatility jump-diffusion model. *Revista de Analisis Economico*, 15(1):61–87.

CRAMÉR, H. (1930). *On the mathematical theory of risk*. Centraltryckeriet.

DANIELSSON, J., de JONG, F., LAEVEN, R., LAUX, C., PEROTTI, E. et WÜTHRICH, M. V. (2011). A prudential regulatory issue at the heart of Solvency II.

DAVIS, L. (1987). *Genetic algorithms and simulated annealing*. Pitman Publishing.

DAVIS, M. H. (1997). Option pricing in incomplete markets.

de BÂLE, C. (1988). *Convergence internationale de la mesure et des normes de fonds propres*.

DECUPÈRE, S. (2011). Agrégation des risques et allocation de capital sous Solvabilité II.

DELBAEN, F., GRANDITS, P., RHEINLÄNDER, T., SAMPERI, D., SCHWEIZER, M. et STRICKER, C. (2002). Exponential hedging and entropic penalties. *Mathematical finance*, 12(2):99–123.

DELCOUR, I. G. (2012). *On the use of risk-free rates in the discounting of insurance cash flows*. Thèse de doctorat, Katholieke Universiteit Leuven.

DEREICH, S. (2011). Multilevel Monte Carlo algorithms for Lévy-driven SDEs with gaussian correction. *The Annals of Applied Probability*, 21(1):283–311.

DEVINEAU, L. (2011). *La méthode des simulations dans les simulations. Mise en œuvre d'un modèle actif / passif en assurance-vie : quelles techniques ?, Part 2*. Milliman. slides.

DEVINEAU, L. et CHAUVIGNY, M. (2011). Replicating portfolios : calibration techniques for the calculation of the Solvency II economic capital. *Bulletin Français d'Actuariat*, 11(21):59–97.

DEVINEAU, L. et LOISEL, S. (2009a). Construction d'un algorithme d'accélération de la méthode des "simulations dans les simulations" pour le calcul du capital économique Solvabilité II. *Bulletin Français d'Actuariat*, 10(17):188–221.

DEVINEAU, L. et LOISEL, S. (2009b). Risk aggregation in Solvency II : How to converge the approaches of the internal models and those of the standard formula ? *Bulletin Français d'Actuariat*, 9(18):107–145.

DIEBOLD, F. X. et LI, C. (2006). Forecasting the term structure of government bond yields. *Journal of Econometrics*, 130(2):337–364.

DRAPER, N. R., SMITH, H. et POWNELL, E. (1966). *Applied regression analysis*. Wiley New York.

DUFFIE, D. (1996). *Dynamic asset pricing theory*. Princeton University Press.

DYBVIG, P. H., INGERSOLL JR, J. E. et ROSS, S. A. (1996). Long forward and zero-coupon rates can never fall. *Journal of Business*, 69(1):1–25.

EGLOFF, D., LEIPPOLD, M. *et al.* (2010). Quantile estimation with adaptive importance sampling. *The Annals of Statistics*, 38(2):1244–1278.

EIOPA (2010a). QIS5 technical specifications.

EIOPA (2010b). The underlying assumptions in the standard formula for the Solvency Capital Requirement calculation.

EIOPA (2015). EIOPA publishes Solvency II relevant risk free interest rate term structures.

EL KAROUI, N., FRACHOT, A. et GEMAN, H. (1997). A note on the behavior of long zero coupon rates in a no arbitrage framework. *Review of Derivatives Research*, 1:351–370.

EL KAROUI, N., LOISEL, S., PRIGENT, J.-L. et VEDANI, J. (2015). Market inconsistencies of the market-consistent European life insurance economic valuations : pitfalls and practical solutions.

EL KAROUI, N. et QUENEZ, M.-C. (1995). Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM journal on Control and Optimization*, 33(1):29–66.

EMBRECHTS, P., KLÜPPELBERG, C. et MIKOSCH, T. (1997). *Modelling extremal events*. Springer Science & Business Media.

EMBRECHTS, P. et VERAVERBEKE, N. (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance : Mathematics and Economics*, 1(1):55–72.

EUROPEAN COMMISSION (2009). *Solvency II - EC directive 2009/138*.

EUROPEAN COMMISSION (2011). Draft Implementing measures Solvency II.

FÉDÉRATION FRANÇAISE DES SOCIÉTÉS D'ASSURANCE (2011). L'assurance vie, pilier de l'épargne longue.

FLOREANI, A. (2011). Risk Margin Estimation through the Cost of Capital Approach : Some conceptual issues. *The Geneva Papers on Risk and Insurance-Issues and Practice*, 36(2):226–253.

FÖLLMER, H. et SCHIED, A. (2011). *Stochastic finance : an introduction in discrete time*. Walter de Gruyter.

FROMENTEAU, M. et PETAUTON, P. (2012). *Théorie et pratique de l'assurance vie. : Cours et exercices corrigés*. Dunod.

GATZERT, N. et WESKER, H. (2012). A comparative assessment of Basel II/III and Solvency II. *The Geneva Papers on Risk and Insurance-Issues and Practice*, 37(3):539–570.

GERBER, H. U. (1974). On additive premium calculation principles. *Astin Bulletin*, 7(3):215–222.

GERBER, H. U. (1979). *An introduction to mathematical risk theory*.

GERBER, H. U. et LOISEL, S. (2012). Why ruin theory should be of interest for insurance practitioners and risk managers nowadays. *In Actuarial and Financial Mathematics Conference*, page 17.

GILES, M. B. (2008). Multilevel Monte Carlo path simulation. *Operations Research*, 56(3):607–617.

GLASSERMAN, P. (2003). *Monte Carlo methods in financial engineering*. Springer.

GORDY, M. B. et JUNEJA, S. (2010). Nested simulation in portfolio risk measurement. *Management Science*, 56(10):1833–1848.

GROSEN, A. et JØRGENSEN, P. L. (2000). Fair valuation of life insurance liabilities : the impact of interest rate guarantees, surrender options, and bonus policies. *Insurance : Mathematics and Economics*, 26(1):37–57.

GUIBERT, Q., PLANCHET, F. et JUILLARD, M. (2010). Un cadre de référence pour un modèle interne partiel en assurance de personnes. *Bulletin Français d'Actuariat*, 10(20):5–34.

HAMILTON, J. D. (1990). Analysis of time series subject to changes in regime. *Journal of econometrics*, 45(1):39–70.

HAMILTON, J. D. (2005). Regime-switching models.

HAMILTON, J. D. et LIN, G. (1996). Stock market volatility and the business cycle. *Journal of Applied Econometrics*, 11(5):573–593.

HAMILTON, J. D. et SUSMEL, R. (1994). Autoregressive conditional heteroskedasticity and changes in regime. *Journal of Econometrics*, 64(1):307–333.

HARDY, M. R. (2001). A regime-switching model of long-term stock returns. *North American Actuarial Journal*, 5(2):41–53.

HARRIS, G. R. (1995). Low frequency statistical interest rate models. *In 5th AFIR International Colloquium*, pages 799–831.

HARRIS, G. R. (1999). Markov chain Monte Carlo estimation of regime switching vector autoregressions. *ASTIN bulletin*, 29(1):47–79.

HASTIE, T., TIBSHIRANI, R. et FRIEDMAN, J. (2009). The elements of statistical learning.

HAUSMAN, J. A. (1978). Specification tests in econometrics. *Econometrica*, 46(6):1251–1271.

HEINRICH, S. (2001). Multilevel Monte Carlo methods. *In Large-scale scientific computing*, pages 58–67. Springer.

HENDERSON, V. et HOBSON, D. (2004). Utility indifference pricing-an overview.

HESTON, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of financial studies*, 6(2):327–343.

HIRT, G. A. et BLOCK, S. B. (2006). *Fundamentals of investment management*. McGraw Hill.

HODGES, S. D. et NEUBERGER, A. (1989). Optimal replication of contingent claims under transaction costs. *Review of futures markets*, 8(2):222–239.

HORN, R. A. et JOHNSON, C. R. (1990). *Matrix analysis.* Cambridge University Press.

HORST, U. et MÜLLER, M. (2007). On the spanning property of risk bonds priced by equilibrium. *Mathematics of Operations Research*, 32(4):784–807.

HORTON, J., MACVE, R. H. et SERAFEIM, G. (2007). Market Consistent Embedded Values as "Fair Value" Measurements for Life Insurance Accounting : A step too far with finance theory ? *Available at SSRN 956104*.

HULL, J. et WHITE, A. (1988). The use of the control variate technique in option pricing. *Journal of Financial and Quantitative analysis*, 23(3):237–251.

HULL, J., WHITE, A. *et al.* (1993). One-factor interest-rate models and the valuation of interest-rate derivative securities. *Journal of financial and quantitative analysis*, 28(2):235–254.

HULL, J. C. et WHITE, A. (2000). Forward rate volatilities, swap rate volatilities, and the implementation of the Libor Market Model. *Available at SSRN 1295227*.

HWANG, C.-R. (1988). Simulated annealing : theory and applications. *Acta Applicandae Mathematicae*, 12(1):108–111.

IGNATOV, Z. G., KAISHEV, V. K. et KRACHUNOV, R. S. (2001). An improved finite-time ruin probability formula and its mathematica implementation. *Insurance : Mathematics and Economics*, 29(3):375–386.

INGERSOLL, J. E. (1987). *Theory of financial decision making*. Rowman & Littlefield.

JAMSHIDIAN, F. (1989). An exact bond option formula. *The journal of Finance*, 44(1):205–209.

JARROW, R. A., LANDO, D. et TURNBULL, S. M. (1997). A Markov model for the term structure of credit risk spreads. *Review of Financial Studies*, 10(2):481–523.

JUNEJA, S. et RAMPRASATH, L. (2009). Nested simulation for estimating portfolio losses within a time horizon. *In Winter Simulation Conference*, pages 434–443. Winter Simulation Conference.

KALBERER, T. (2012). Stochastic determination of the value at risk for a portfolio of assets and liabilities. *Der Aktuar*, 1:7–14.

KEMP, M. (2009). *Market Consistency : Model Calibration in Imperfect Markets*. John Wiley & Sons.

KORN, R., KORN, E. et KROISANDT, G. (2010). *Monte Carlo methods and models in finance and insurance*. CRC press.

KREPS, D. M. (1981). Arbitrage and equilibrium in economies with infinitely many commodities. *Journal of Mathematical Economics*, 8(1):15–35.

KROESE, D. P., TAIMRE, T. et BOTEV, Z. I. (2013). *Handbook of Monte Carlo Methods*. John Wiley & Sons.

LAKONISHOK, J. et SMIDT, S. (1988). Are seasonal anomalies real ? A ninety-year perspective. *Review of Financial Studies*, 1(4):403–425.

LAN, H., NELSON, B. L. et STAUM, J. (2007). Two-level simulations for risk management. *In Proceedings of the 2007 INFORMS Simulation Society Research Workshop*, pages 102–107.

LEMAIRE, V. *et al.* (2014). Multilevel richardson-romberg extrapolation. *Available at SSRN 2539114*.

LEMAIRE, V., PAGES, G., PANLOUP, F. *et al.* (2015). Invariant measure of duplicated diffusions and application to Richardson-Romberg extrapolation. *In Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 51, pages 1562–1596. Institut Henri Poincaré.

LIU, M. et STAUM, J. (2010). Stochastic kriging for efficient nested simulation of expected shortfall. *Journal of Risk*, 12(3):3–27.

LOISEL, S. et GERBER, H.-U. (2012). Why ruin theory should be of interest for insurance practitioners and risk managers nowadays. *In Proceedings of the AFMATH Conference 2012*, pages 17–21.

LONGSTAFF, F. A., MITHAL, S. et NEIS, E. (2005). Corporate yield spreads : Default risk or liquidity ? new evidence from the credit default swap market. *The Journal of Finance*, 60(5):2213–2253.

LONGSTAFF, F. A. et SCHWARTZ, E. S. (2001). Valuing american options by simulation : A simple least-squares approach. *Review of Financial studies*, 14(1):113–147.

LUNDBERG, F. (1903). *I. Approximerad framställning af sannolikhetsfunktionen. II. Återförsäkring af kollektivrisker.* Almqvist och Wiksells boktryckeri.

MAK, C. H. (1992). Stochastic method for real-time path integrations. *Physical review letters*, 68(7): 899.

MALAMUD, S., TRUBOWITZ, E. et WÜTHRICH, M. V. (2008). Market consistent pricing of insurance products. *ASTIN Bulletin*, 38(2):483–526.

MARKOWITZ, H. (1952). Portfolio selection. *The Journal of Finance*, 7(1):77–91.

MARTIAL, A.-C. et GARNIER, J.-B. (2013). Optimisation du calcul du capital économique d'une compagnie d'assurance vie par la méthode des simulations dans les simulations.

MARTIN, P. et REY, H. (2004). Financial super-markets : size matters for asset trade. *Journal of International Economics*, 64(2):335–361.

MERTON, R. C. (1973). Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, 4(1):141–183.

MILLS, T. C. et MARKELLOS, R. N. (2008). *The econometric modelling of financial time series*. Cambridge University Press.

MOEHR, C. (2011). Market-Consistent valuation of insurance liabilities by Cost of Capital. *ASTIN Bulletin*, 41(2):315–341.

MUERMANN, A. (2003). Actuarially consistent valuation of catastrophe derivatives.

MUKERJI, S. et TALLON, J.-M. (2001). Ambiguity aversion and incompleteness of financial markets. *The Review of Economic Studies*, 68(4):883–904.

NTEUKAM, O. et PLANCHET, F. (2012). Stochastic evaluation of life insurance contracts : Model point on asset trajectories and measurement of the error related to aggregation. *Insurance : Mathematics and Economics*, 51(3):624–631.

O'BRIEN, C. (2009). Valuation of life insurance liabilities on a market-consistent basis : experience from the United Kingdom. *In Actuarial Practice Forum*.

PALERM, T. (2006). Tarification de garanties plancher en cas de vie.

PASKOV, S. H. et TRAUB, J. F. (1995). Faster valuation of financial derivatives. *The Journal of Portfolio Management*, 22(1):113–123.

PELSSER, A. (2000). *Efficient methods for valuing interest rate derivatives*. Springer Science & Business Media.

PELSSER, A. (2010). Time-consistent and market-consistent actuarial valuations. *Available at SSRN 1551323*.

PENTIKÄINEN, T. (1982). *Solvency of insurers and equalization reserves. Volume I : General aspects*. Helsinki : Insurance Publishing Company Ltd.

PENTIKÄINEN, T., BONSDORFF, H., PESONEN, M., RANTALA, J. et RUOHONEN, M. (1989). *Insurance solvency and financial strength*. Finnish Insurance Training and Publishing Company Helsinki.

PETAUTON, P. (2002). *Théorie et pratique des opérations de l'assurance vie*. Dunod, Paris.

PFEIFER, D. et STRASSBURGER, D. (2008). Solvency II : stability problems with the SCR aggregation formula. *Scandinavian actuarial journal*, 1:61–77.

PICARD, P. et LEFÈVRE, C. (1997). The probability of ruin in finite time with discrete claim size distribution. *Scandinavian Actuarial Journal*, 1997(1):58–69.

PLACKETT, R. L. (1950). Some theorems in least squares. *Biometrika*, 37(1):149–157.

PLANCHET, F. (2006). Le risque neutre.

PLANCHET, F., GUIBERT, Q. et JUILLARD, M. (2010). Un cadre de référence pour un modèle interne partiel en assurance de personnes. *Bulletin Français d'Actuariat*, 10(20):pp–5.

PLANCHET, F., THÉROND, P.-E. et KAMEGA, A. (2009). *Scénarios économiques en assurance-Modélisation et simulation*. Économica.

PLOMP, M. (2013). Economic Scenario Generators.

RAFFOURNIER, B. (2005). *Les normes comptables internationales (IFRS/IAS)*. Economica.

REBONATO, R. (1999). On the simultaneous calibration of multi-factor log-normal interest-rates models to black volatilities and to the correlation matrix. *Journal of Computational Finance*, 2:5–27.

REVELEN, J. (2009). Replicating Portoflio et capital économique en assurance vie.

RIBEREAU, P. et RULLIÈRE, D. (2011). Agrégation d'informations et alternative au krigeage en environnement aléatoire.

ROUGE, R. et EL KAROUI, N. (2000). Pricing via utility maximization and entropy. *Mathematical Finance*, 10(2):259–276.

RUBINSTEIN, R. Y. (1981). *Simulation and the Monte Carlo method*. John Wiley & Sons.

RULLIERE, D. et LOISEL, S. (2004). Another look at the Picard-Lefèvre formula for finite-time ruin probabilities. *Insurance : Mathematics and Economics*, 35(2):187–203.

SAPORTA, G. (2006). *Probabilités, analyses des données et statistiques*. editions Technip.

SCHMOCK, U. (1999). Estimating the value of the Wincat coupons of the Winterthur insurance convertible bond : a study of the model risk. *Astin Bulletin*, 29(1):101–163.

SCHRAGER, D. (2008). Replicating portfolios for insurance liabilities.

SCHUBERT, T. et GRIESSMANN, G. (2004). Solvency II = Basel II+ X. *Versicherungswirtschaft*, 18:1399–1402.

SHELDON, T. J. et SMITH, A. D. (2004). Market consistent valuation of life assurance business. *British Actuarial Journal*, 10(3):543–605.

SÖDERLIND, P. et SVENSSON, L. (1997). New techniques to extract market expectations from financial instruments. *Journal of Monetary Economics*, 40(2):383–429.

SUM, V. (2010). The january and size effects on stock returns : more evidence. *International Journal of Applied Accounting and Finance*, 1(1):47–52.

SUN, Y., APLEY, D. W. et STAUM, J. (2011). Efficient nested simulation for estimating the variance of a conditional expectation. *Operations research*, 59(4):998–1007.

TOSETTI, A., PARIS, B., PASLKY, P. et LE VALLOIS, F. (2003). *Gestion Actif Passif en Assurance vie : réglementation, outils, méthodes*. Économica.

TSANAKAS, A. et CHRISTOFIDES, N. (2006). Risk exchange with distorted probabilities. *Astin Bulletin*, 36(1):219–243.

TURC, J. et UNGARI, S. (2009). *Filtering the interest rate curve, the MENIR framework*.

VEDANI, J. et DEVINEAU, L. (2013). Solvency assessment within the ORSA framework : issues and quantitative methodologies. *Bulletin Français d'Actuariat*, 13(25):35–71.

VEDANI, J. et VIREPINTE, P.-A. (2011). Modèle structurel de crédit dans une optique assurantielle.

WALRAS, L. (1874). *Éléments d'économie politique pure*. F. Rouge.

WHITE, H. (1980). A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity. *Econometrica*, 48(4):817–838.

WILKIE, A. D. (1984). A stochastic investment model for actuarial use. *Transactions of the Faculty of Actuaries*, 39:341–403.

WÜTHRICH, M. V., BÜHLMANN, H. et FURRER, H. (2008). *Market-Consistent actuarial valuation*. Springer.