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Credibility balanced Bias vs. Instability trade-off within Double Chain Ladder

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Abstract

This dissertation attempts to balance the bias and instabilities trade-off observed in Double Chain Ladder (DCL) (Martinez-Miranda et al. 2012) and its extensions BDCL (Martínez-Miranda et al. 2013a), IDCL (Agbeko et al. 2014), PDCL (Hiabu et al. 2016a) using credibility theory. All the extensions of DCL have tried to stabilise the potentially unstable inflation estimates of DCL by incorporating prior information from the experts. The resulting inflation estimates, however, are susceptible to human bias. Hence, we propose to optimise such trade-off by a weighted average of inflation estimates from DCL and that from those extensions based on their relative credibility. The intention is to produce better forecasts of future claim payments for general insurance (GI) companies such that the prediction error can be reduced.

A pragmatic while untraditional approach developed from validation (Agbeko et al. 2014) is introduced to weigh the bias and instability during the estimation process. Empirical studies are conducted on three data sets with various length and characteristics. As an initial effort, there are limitations in the construction and estimation of the proposed method that affect its forecasting performance. However, it is shown that the simple estimation procedures are indeed trying to replicate the results from a more demanding credibility theory. Importantly, even under very restricted condition, we are able to produce smaller prediction error than any of DCL and its previous extensions under most circumstances, especially for longer tailed GI business. Furthermore, it is suggested that both DCL and PDCL are always credible while IDCL tends to attract zero credibility.

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1. Introduction

Accurately predicting potential future claims arising from in-force policies and hence adequately reserving is of utmost importance to a general insurance (GI) company and its stakeholders. Those potential obligations can account for a significant proportion of the total outstanding liabilities on the financial statement of a GI company. Hence, their accuracy can have serious implications on the company's financial performance and positions, pricing on future products as well as fulfilment of its obligatory duty to its existing policyholders and regulatory requirements, such as Solvency II (City University London 2014, Financial Services Authority 2011). Combining mathematical statistics and tacit knowledge to help practitioners understand the risks underpinning those potential outstanding liabilities have been highly appreciated by the professionals and organisations (City University London 2014). The Double Chain Ladder (DCL) (Martinez-Miranda et al. 2012) represents one such milestone that not only can modernise the reserving process via formal statistical science, but also encourages more engaged communications of the familiar Chain Ladder Method (CLM) (Tarbell 1934) between interested parties. This dissertation is devoted to improve predictions of future claim payments from DCL and its extensions (Martínez-Miranda et al. 2013a, Agbeko et al. 2014, Hiabu et al. 2016a) such that the prediction error, or the cost of point forecasts missing the targeted payments, is as small as possible.

The simplicity and intuitive appealing of CLM have gained its predominance amongst GI reserving techniques. The survey conducted by the Institute and Faculty of Actuaries (IFoA), UK, reveals that all the respondents from either personal lines or London Market employ CLM (MacDonnell & Labaune 2014). However, forecasting of future claim payments in CLM is not based on actual claim risk generating process. Structuring CLM into statistical models (Kremer 1982, Mack 1991, Renshaw & Verrall 1998) have not enhanced the understanding of the real risks behind the company's balance sheet. Thus, the introduction of a well formulated risk generating process that is capable of easy implementation and generalisation is expected to transform the reserving function in GI companies into a new era.

Follows the paradigm of Wright (1990) and Mack (1991), Verrall et al. (2010) introduces a micro model based on compound Poisson processes. A triangle of aggregated incurred claims counts is involved in addition to the usual triangle of aggregated claim payments. Such extra information is usually readily available within a GI company (Verrall et al. 2010, Martinez-Miranda et al. 2012). This is in contrast to Taylor & McGuire (2004), Norberg (1986, 1993, 1999) where extensive and detailed data is required. The second triangle permits the separation of reporting delay and settlement delay via a delay function. Not only can the claim tail behaviour be naturally and consistently forecasted, but also the total reserves can be decomposed into the reported but not settled (RBNS) and the incurred but not reported (IBNR) components.

The significance of knowing RBNS and IBNR individually lies in the fact that the company can be more actively engaged in risk management, which may be much appreciated in an uncertain environment. This is because the sources and magnitude of risks inherent in RBNS and IBNR are different. The former are controllable by the company to some extent, while the latter depends on various other factors unrelated to the company. Whereas the size of RBNS may account for roughly 80% of the total reserves, it contributes only approximately 20% of its total fluctuations (as measured by coefficient of variation). Nonetheless, in Verrall et al. (2010), estimation requires complex computation and prediction is performed net of claim inflation. Hence, an inquest into a more flexible and general model is desired.

Built on Verrall et al. (2010), it is in Martinez-Miranda et al. (2012) that the foundation of this dissertation, DCL, emerges. Inspired by Taylor (1977), the benefit of adding claim counts is further utilised to represent the claim severities inflation in the model. The *Double* comes from the fact that it applies the classical CLM twice. A simple regression enables the complete replication of CLM on paid data under certain conditions. Similar to CLM, this ease of application allows fast and straightforward automation that is much appreciated in GI (Clarke & Harland 1974). More than CLM, each and every parameter in DCL has a real-world interpretation that can be communicated to non-actuaries.

DCL hence becomes the more understandable version of CLM that can be explained in terms of how each individual claim aggregates to the total payments made. This enhanced communication can be expected to strengthen the connection between actuaries and other professionals. However, by inputting the potentially volatile paid data, DCL may also suffer from the instability issue of

CLM such that predictions from DCL can be far from reality. This is because the majority of claims that require prediction are those originated in more recent years where payment data is scarce and prediction is highly sensitive to outliers. It is identified in Martínez-Miranda et al. (2013a) that the claim inflation estimates are responsible for such potentially unreliable predictions that artificially impair the company's ability to meet future financial obligations. By separately representing claim inflation in the model, DCL facilitates stabilisation of results to be made by its extensions. Current extensions of DCL involve a third aggregated triangle of incurred data, which is a mixture of actual paid data and expert's prior RBNS case estimates and routinely prepared by insurance companies (MacDonnell & Labaune 2014).

Motivated by the "second-best" (MacDonnell & Labaune 2014), namely the Bornhuetter & Ferguson (1972) CLM , Martínez-Miranda et al. (2013a) suggests the Bournuetter-Ferguson DCL (BDCL) to reduce the risk of an exaggerated outstanding liabilities on GI company's balance sheet. It is recognised that by replacing the inflation parameters from DCL by that implied from the incurred data, forecasting of future payments can be more realistically and stably achieved. Similar to BF CLM, the robustness of BDCL is attested by its ability to accommodate a variety of complex and challenging business situations as well as statistical model structures. Forecasting outstanding liabilities using BDCL is as pragmatic as, while less subjective than, BF CLM.

Practitioners have been used to applying CLM on incurred data to overcome the instability issues caused by paid data. Although DCL can replicate the CLM exactly, predictions by BDCL are different from that by incurred CLM. This gap of link between CLM and DCL is filled by the Incurred DCL (IDCL) in Agbeko et al. (2014). IDCL replicates the reserves estimates from incurred CLM by appropriately scaling the inflation estimates using incurred data. This enables practitioners to not only compare DCL and CLM using both paid and incurred data but also maintain the aforementioned benefits of DCL construct.

Nevertheless, the advantage of separating IBNR and RBNS in DCL is still yet to be fully exploited. While RBNS from case department may have based on hard facts unknown to actuaries, the IBNR implied from their RBNS estimates may not be superior than DCL. Thus, the RBNS-preserving DCL (PDCL) is constructed in Hiabu et al. (2016a) to *preserve* the prior RBNS case estimates in DCL framework precisely while keeping the predictive power of a statistical model.

There has been a classical trade-off between bias and instabilities (Breiman

1996) while trying to sophisticate and develop DCL. Whereas those extensions incorporating prior knowledge may be more stable than DCL, they are vulnerable to human bias (Verrall et al. 2010, Martinez-Miranda et al. 2012). Bias, on the other hand, is expected to make predictions away from "true" future payments. Such unpleasant subjectivity is minimal in DCL where "real" data is used. Both bias and instabilities may cause the forecasts of claim costs to deviate from what will actually be paid. It is the sum of these two that will eventually determine the quality of projected outstanding liabilities from DCL and its extensions.

We propose to optimise such trade-off so that, for each unit of instabilities discarded by DCL, the smallest amount of bias is introduced, yielding a lowest possible sum of these two. Inspired by the credibility theory (Bühlmann 1967), we cast our model as a weighted average of DCL and above mentioned extensions. The proportion attached to DCL is denoted as Z (Whitney 1918), and known as the *credibility factor*. It answers the question that to what extent DCL can be relied upon relative to its extensions. The name ZDCL emphasises the fact that the credibility factor, Z, is attached to DCL, the low-biased model.

The gain of efficiency in reducing prediction error using ZDCL may be analogue to that achieved by pooling risks in insurance. Each of bias and instability represents a potential risk that causes inaccurate predictions of claims and increases costs to the companies. By diversifying away them amongst different versions of DCL, the outstanding liabilities can be determined more accurately as well as more precisely than any one of them alone. As an initial attempt to demonstrating such benefit, this dissertation focuses on the credibility estimator for the inflation parameter, which has been proven to be crucial to the correctness of forecasted payments while hardest to estimate.

This initiative further strengthens the connection between DCL and CLM since credibility theory has long been appreciated by actuaries endorsing CLM. An eminent pioneer is Bühlmann (1967). While unaware by the authors, BF CLM and its variate cape-cod CLM, aka the Stanard-Bühlmann CLM (Feldblum 2003), are also within the realm of credible reserving. This credibility interpretation of BF CLM is explicitly developed in Neuhaus (1992). A slightly different approach is adopted in Benktander (1976) that is further developed in Mack (2000) and subsequently by Hürlimann (2009). The compatibility between the credibility CLM and the classical CLM is formalised in Gisler & Wüthrich (2008). This enhances the assertion that the credibility theory is powerful and elegant that can be applied under very general conditions (Bühlmann & Gisler 2005). In other words, we can always find a credibility version for almost each and every mathematical structure. This generalisability stimulates our confidence in trying to estimate the credibility factor in an unorthodox approach.

Instead of relying on complex statistical structures, we suggest a data-driven estimation process. It takes advantage of the dynamics of the pragmatic, versatile and forward-looking validation via back-testing (Martínez-Miranda et al. 2013a, Agbeko et al. 2014). Similar to the origin of validation (Larson 1931), back-testing is envisaged to test the predictive power of a model. Its practicality and effectiveness have been attested by its popularities across various disciplines, such as by practitioners in financial econometrics (Burman & Nolan 1992, Bergmeir & Beìtez 2012, McQuarrie & Tsai 1998, Tashman 2000), and importantly by GI professionals, such as the General Insurance Research Organising (GIRO) Committee (Gibson et al. 2007, Bruce et al. 2008), and GI academics, like Meyers & Shi (2011) and Leong et al. (2012).

The objective of this dissertation is to reduce errors in predicting future claims and hence the cost for GI companies using a DCL framework. To accomplish this task, we will try to balance the trade-off between bias and instabilities. This process starts with acknowledging the intuitions behind, and limitations of, the classical CLM in Chapter 2. Chapter 3, devoted to DCL, will be more elaborated since it is the basis of our entire work. The real-world interpretation and the first moment assumptions will be given in Section 3.1, while the details of estimating those parameters and predicting future claim payments is contained in Section 3.2 and 3.3, respectively. Chapter 4 spends each of the first three sections describing the three extensions, namely BDCL, IDCL and PDCL, respectively. The last one, Section 4.4, presents the validation delineated in Agbeko et al. (2014). ZDCL is constructed in Section 5.1 with the general steps to estimate the parameters. Section 5.3 includes a discussion of how we select and form the statistics that strives to balance bias and instabilities. Based on Section 4.4 and this measures, Section 5.2 suggests some procedures to decide the credibility attached to DCL and other extensions, respectively. Empirical studies is conducted in Chapter 6 to examines ZDCL compared to the other aforementioned methods. Chapter 7 attempts some limitations and suggests some possible further studies before reaching a conclusion. We finally remark some neglected details from past papers in Chapter 8.

2. Classical Chain Ladder Method: CLM

The classical CLM has been known in actuarial profession for many decades (Tarbell 1934) and is a very general construct that have many extensions. This dissertation will refer CLM to the classical version. For a comprehensive survey, please refer to England & Verrall (2002). CLM assumes that a run-off triangle, $\mathbf{C}_m$, is available. $\mathbf{C}_m$ accommodates the aggregated incremental data, C_{ij}, in the upper-right triangle of a $m \times m$ matrix with index set

$$\mathcal{I} = \{(i, j) : i \in \{1, \ldots, m\}, j \in \{0, \ldots, m-1\}, i + j \in \{1, \ldots, m\}\}$$

where m is the total number of years available, i represents the year of origin and j records the delay in years. CLM is flexible in the sense that year of origin can be either underwriting year or accident year; C_{ij} can be actual paid data, incurred data or claim counts; more frequent intervals than year are readily applicable; it can be applied to data sets from many business lines in general insurance, and so on. CLM is intuitive and assumes that past claim experience can be reasonably expected to repeat under normal circumstances. In particular, the way in which claim arises and being settled and how claim severities change from one year to the next is relatively stable over time. Thus, CLM is conceived as a simple and straightforward method where the cumulative triangle of $\mathbf{C}_m$, $\mathbf{D}_m$, is expected to have proportionate columns (Taylor 1977), where $\mathbf{D}_m$ consists of

$$D_{ij} = \sum_{j=0}^{m-i} C_{ij}$$

Those proportion is known as the development factor, λ, with

$$\lambda = (\lambda_1, \ldots, \lambda_{m-1})$$

where λ_j represents how the cumulative data D_{ij} is expected to "develop" from its previous value $D_{i,j-1}$ in the same row i. Thus, we have

$$E\left[D_{ij}|D_{i,j-1}\right] = \lambda_j D_{i,j-1}$$

In the classical CLM, λ is estimated as

$$\widehat{\lambda}_j = \frac{\sum_{i=1}^{m-j} D_{ij}}{\sum_{i=1}^{m-j} D_{i,j-1}} = \frac{\sum_{i=1}^{m-j}\sum_{k=0}^{j} C_{ik}}{\sum_{i=1}^{m-j}\sum_{k=0}^{j-1} C_{i,k}} \qquad j \in 1, \ldots, m-1$$

Finally, the future cumulative claims is estimated as for $i > 1$ as

$$\widehat{D}_{ij} = \begin{cases} D_{ij}\widehat{\lambda}_j & \text{for} \quad j = m-i+1 \\ \widehat{D}_{ij}\widehat{\lambda}_j & \text{for} \quad j \in \{m-i+2, \ldots, m-1\} \end{cases}$$

which yields

$$\widehat{C}_{ij} = \begin{cases} \widehat{D}_{ij} - D_{i,j-1} & \text{for} \quad j = m-i+1 \\ \widehat{D}_{ij} - \widehat{D}_{i,j-1} & \text{for} \quad j \in \{m-i+2, \ldots, m-1\} \end{cases}$$

Summing over the row and diagonal of $\mathbf{C}_m$ gives the reserve estimates by year of origin and calendar year, respectively. And the total reserves are then the sum of all the estimated incremental claims.

Evidently, the development factor, λ, encompasses all the information that will be used to predict the potential outstanding liabilities. However, this is not how actual claim payments "develop". Thus, risks inherited in potential future payments are masked. The real-life risks in claim payments come from such factors, among others, as delay in reporting, delay in payments as well as the changes in claim severities. Understanding the CLM and hence the actual risk underpinning those forecasts is one of the task that can be accomplished by DCL.

3. Double Chain Ladder: DCL

3.1. Notations and assumptions

3.1.1. Aggregate and micro-level variables

DCL assumes two aggregated triangles available over the index set $\mathcal{I}$, which are observations of the following random variables (r.v.):

- $\aleph_m = \{N_{ij} : (i,j) \in \mathcal{I}\}$: the r.v. for the triangle consisting of aggregated incurred claim counts, N_{ij}
- N_{ij}: the r.v. representing aggregated incremental number of claims for year of origin i known with j years of delays
- $\Delta_m = \{X_{ij} : (i,j) \in \mathcal{I}\}$: the r.v. for the triangle accommodating aggregated paid claim amounts, X_{ij}
- X_{ij}: the r.v. denoting total amount paid with j years of delay in respect of claims incurred in year of origin i

In addition, there are also micro-level variables that describe how the risks are generated. They are usually unobserved or latent in practice:

- N_{ij}^{paid}: the r.v. representing the total number of paid claims for year of origin i and development year j, i.e.

$$N_{ij}^{\text{paid}} = N_{ij0}^{\text{paid}} + N_{i,j-1,1}^{\text{paid}} + \ldots + N_{i,0,j}^{\text{paid}} = \sum_{l=0}^{j} N_{ijl}^{\text{paid}}$$

- N_{ijl}^{paid}: the r.v. denoting the number of claims paid from the incurred claim counts N_{ij} with l years of delays, $l \in \{0, \ldots, m-1\}$
- X_{ijl}: the r.v. for total paid claim amounts for the paid claims N_{ijl}^{paid}, i.e.:

$$X_{ijl} = \sum_{k=0}^{N_{ijl}^{\text{paid}}} X_{ijl}^{(k)}$$

- $X_{ijl}^{(k)}$: the r.v. for individual paid claim amount coming from the paid claims N_{ijl}^{paid}, i.e. $k \in \{1, \ldots, N_{ijl}^{\text{paid}}\}$

3.1.2. First moment parameters and assumptions

DCL is built on this set of parameters:

$$\begin{aligned}\eta =&\{\alpha, \beta, \widetilde{\alpha}, \widetilde{\beta}, \pi, \gamma, \mu\} \\ =&\{(\alpha_1, \ldots, \alpha_m), (\beta_0, \ldots, \beta_{m-1}), (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_m), (\widetilde{\beta}_0, \ldots, \widetilde{\beta}_{m-1}), \\ &(\pi_0, \ldots, \pi_{m-1}), (\gamma_1, \ldots, \gamma_m), \mu\}\end{aligned}$$

These parameters have real-world interpretations that are intrinsic to the claim development process. Specifically, $\alpha_i : i \in \{1, \ldots, m\}$ can be considered as the parameter for the ultimate claim in respect of year of origin i while $\beta_j : j \in \{0, \ldots, m-1\}$ is the proportion of α_i emerges in reporting year j. Whilst $\widetilde{\beta}_j : j \in \{0, \ldots, m-1\}$ has very similar meanings as β_j, $\widetilde{\alpha}_i : i \in \{1, \ldots, m\}$ may be interpreted as the cumulative claim amounts by the end of delay year $m-1$ in respect of claims incurred in year of origin i. In other words, we open to the possibility that tail may exist. We introduce $\pi_l : l \in \{0, \ldots, m-1\}$ as the delay parameter to describe the payment pattern of incurred claims N_{ij}. Furthermore, an inflation parameter $\gamma_i : i \in \{1, \ldots, m\}$ is incorporated to represent severity inflation in year of origin i. This explicit parametrisation allows the addition of prior knowledge regarding claim severity inflation from the industry. Finally, μ is the common severity mean factor for all individual claims.

With these parameters, we follow Martinez-Miranda et al. (2012) and present the three first moment assumptions critical to the model:

M1. The mean of N_{ij} is in multiplicative parametrisation as $E[N_{ij}] = \alpha_i \beta_j$ with identification $\sum_{j=0}^{m-1} \beta_j = 1$ (Kremer 1982).

M2. The mean of N_{ijl}^{paid} conditional on the number of incurred claims is $E[N_{ijl}^{\text{paid}} | \aleph_m] = N_{ij}\pi_l, (i,j) \in \mathcal{I}, l \in \{0, \ldots, m-1\}$.

M3. The conditional mean of the individual payment size is $E[X_{ijl}^{(k)} | N_{ijl}^{\text{paid}}, \aleph_m] = \mu\gamma_i$ with the identification $\gamma_1 = 1$.

Kremer's identification scheme (Kremer 1982) in **M1** implies that the $\aleph_m$ has run off, i.e. the total number of incurred claim in year of origin 1 is completely known. This identification is either explicitly assumed in Hiabu et al. (2016a) or implied in Martinez-Miranda et al. (2012). However, it is named as Mack's identification there. The formulation in **M2** assumes that payment delay is independent of the year of origin i and reporting delay j. Similarly, **M3** states that the conditional mean of $X_{ijl}^{(k)}$ depends only on the year of origin i but not on the reporting delay j and the delay in payment l.

To reduce complications while concentrate on the essential feature of DCL, we have restricted ourselves to a smaller model than in Martinez-Miranda et al. (2012) and Hiabu et al. (2016a), and collected the identification $\gamma_1 = 1$ in the **M3**. This is because **M1-M3** are what we can actually work on in the absence of external knowledge. They have been eventually assumed in all the developing papers of DCL. The more general assumptions can be found in Appendix A.1. Most importantly, lessons learnt from using this smaller model can be easily generalised when more information is available, for example, as suggested in Verrall et al. (2010) or Miranda et al. (2015).

Martinez-Miranda et al. (2012) shows that, by **M2** and **M3**, the conditional mean of X_{ij} is

$$E[X_{ij}|\aleph_m] = \gamma_i \mu \sum_{l=0}^{j} N_{i,j-l}\pi_l \tag{3.1}$$

Adding **M1** yields the unconditional mean of X_{ij}:

$$E[X_{ij}] = \alpha_i \gamma_i \mu \sum_{l=0}^{j} \beta_{j-l}\pi_l = \widetilde{\alpha}_i \widetilde{\beta}_j \tag{3.2}$$

where

$$\alpha_i \gamma_i \mu = \widetilde{\alpha}_i \tag{3.3}$$

$$\sum_{l=0}^{j} \beta_{j-l}\pi_l = \widetilde{\beta}_j \tag{3.4}$$

It is worth noting that, without prior knowledge, $\widetilde{\beta}_j$ have been (Martinez-Miranda et al. 2012, Verrall et al. 2010, Martinez-Miranda et al. 2011, Martínez-Miranda et al. 2013a, Hiabu et al. 2016a) and is assumed here to follow Kremer's identification, i.e. $\sum_{j=0}^{m-1} \widetilde{\beta}_j = 1$. Again, similar to above discussion, this simplification is hardly a constraint and can be easily relaxed later.

3.2. First-moment parameter estimation

The first application of CLM is to the number of incurred claims. Under **M1**, Kremer (1982) shows that

$$\beta_j = \begin{cases} \frac{1}{\prod_{l=1}^{m-1} \lambda_l} & \text{for} \quad j = 0 \\ \frac{\lambda_j - 1}{\prod_{l=j}^{m-1} \lambda_l} & \text{for} \quad j \in \{1, \ldots, m-1\} \end{cases}$$

and

$$\alpha_i = \sum_{j=0}^{m-i} N_{ij} \prod_{j=m-i+1}^{m-1} \lambda_j \quad \forall i \in \{1, \ldots, m\}$$

In a similar fashion, $\{\widetilde{\alpha}, \widetilde{\beta}\}$ can be obtained via the second application of CLM on the paid triangle. Since Equation (3.3) can be solved independently from Equation (3.4), π is then the solution to this system of equations:

$$\begin{pmatrix} \widetilde{\beta}_0 \\ \vdots \\ \vdots \\ \widetilde{\beta}_{m-1} \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 & \cdots & 0 \\ \beta_1 & \beta_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \beta_{m-1} & \cdots & \beta_1 & \beta_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \vdots \\ \vdots \\ \pi_{m-1} \end{pmatrix} \tag{3.5}$$

The identification $\gamma_1 = 1$ in **M3** yields $\mu = \frac{\widetilde{\alpha}_1}{\alpha_1}$.Rearranging Equation (3.3) yields:

$$\gamma_i = \frac{\widetilde{\alpha}_i}{\mu \alpha_i} \quad \text{for} \quad i \in \{2, \ldots, m-1\}$$

These are the relationships between parameters, their estimates will be distinguished from the estimators by placing a $\widehat{}$ on top. When predicting reserves, only a subset of estimated η, namely $\widehat{\theta} = \left(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\mu}, \widehat{\pi}\right)$, is sufficient.

3.3. Point forecast of RBNS and IBNR

The index over which the reserves are estimated are:

$$\mathcal{J}_1 = \{(i,j) : i \in \{2, \ldots, m\}, j \in \{0, \ldots, m-1\}, i+j \in \{m+1, \ldots, 2m-1\}\}$$
$$\mathcal{J}_2 = \{(i,j) : i \in \{1, \ldots, m\}, j \in \{0, \ldots, 2m-1\}, i+j \in \{m+1, \ldots, 2m-1\}\}$$
$$\mathcal{J}_3 = \{(i,j) : i \in \{2, \ldots, m\}, j \in \{0, \ldots, 2m-1\}, i+j \in \{2m, \ldots, 3m-2\}\}$$

One of the advantages of the DCL framework over CLM is its ability to estimate reserves over $\mathcal{J}_2 \cup \mathcal{J}_3$.

RBNS is defined over the index $\mathcal{J}_1 \cup \mathcal{J}_2$ (Martinez-Miranda et al. 2011). There are two versions of the best estimate of the RBNS component in cell (i, j), $\widehat{X}_{ij}^{\text{RBNS}}$, which differs in whether the actual or estimated number of incurred claims is used (Martinez-Miranda et al. 2012). By Equation (3.1), the first version is $\widehat{X}_{ij}^{\text{RBNS}(1)} = \widehat{E\left[X_{ij}^{\text{RBNS}}|\aleph_m\right]}$ and can be obtained by

$$\widehat{X}_{ij}^{\text{RBNS}(1)} = \sum_{l=i-m+j}^{j} N_{i,j-l}\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i \tag{3.6}$$

Alternatively, (3.2) gives the second version $\widehat{X}_{ij}^{\text{RBNS}(2)} = \widehat{E\left[X_{ij}^{\text{RBNS}}\right]}$ and

$$\widehat{X}_{ij}^{\text{RBNS}(2)} = \sum_{l=i-m+j}^{j} \widehat{N}_{i,j-l}\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i = \sum_{l=i-m+j}^{j} \widehat{\alpha}_i\widehat{\beta}_j\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i \tag{3.7}$$

Similarly, IBNR spins over the index set $\mathcal{J}_1 \cup \widetilde{\mathcal{J}}_2 \cup \mathcal{J}_3$, where $\widetilde{\mathcal{J}}_2$ is all index contained in $\mathcal{J}_2$ except for $(i, j) : i = 1, j \in \{m, \ldots, 2m-1\}$. Previously, this small detail have been ignored by authors such as Martinez-Miranda et al. (2012). IBNR always uses Equation (3.2) to arrive at its best estimate $\widehat{X}_{ij}^{\text{IBNR}} = \widehat{E\left[X_{ij}^{\text{IBNR}}\right]}$:

$$\widehat{X}_{ij}^{\text{IBNR}} = \sum_{l=0}^{i-m+j-1} \widehat{\alpha}_i\widehat{\beta}_j\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i \tag{3.8}$$

Regardless whether Equation (3.6) or (3.7) is used, the reserve estimate in cell (i, j), $\widehat{X}_{ij}$, can be obtained by:

$$\widehat{X}_{ij} = \begin{cases} \widehat{X}_{ij}^{\text{RBNS}} & \text{for} \quad (i,j) \in \{i = 1, j \in \{m, \ldots, 2m-1\}\} \\ \widehat{X}_{ij}^{\text{RBNS}} + \widehat{X}_{ij}^{\text{IBNR}} & \text{for} \quad (i,j) \in \mathcal{J}_1 \cup \widetilde{\mathcal{J}}_2 \\ \widehat{X}_{ij}^{\text{IBNR}} & \text{for} \quad (i,j) \in \mathcal{J}_3 \end{cases} \tag{3.9}$$

It is proved in Martinez-Miranda et al. (2012) that

$$\widehat{X}_{ij}^{\text{CLM}} = \widehat{X}_{ij}^{\text{RBNS}(2)} + \widehat{X}_{ij}^{\text{IBNR}} \quad \text{for} \quad (i,j) \in \mathcal{J}_1 \tag{3.10}$$

where $\widehat{X}_{ij}^{\text{CLM}}$ is the reserve estimates given by CLM. And when only $\mathcal{J}_1$ is considered, which is the only region where classical CLM is able to predict reserve estimates, the reserves from CLM equal that from DCL in each cell. In other words, when the expected number of incurred claims, $\widehat{N}_{ij}$, and raw delay parameter, $\widehat{\pi}$, are used, the reserve estimates from DCL over the index set $\mathcal{I}$ are exactly the same as the reserve estimates from classical CLM. However, Martinez-Miranda et al. (2012) suggests the use of actual observed N_{ij} to estimate RBNS and hence Equation (3.6) may be preferred.

Outstanding RBNS by year of origins are found by summing over the rows of RBNS estimates within $\mathcal{J}_1 \cup \mathcal{J}_2$, while summing along the diagonal of $\mathcal{J}_1 \cup \mathcal{J}_2$ gives the RBNS cash flow estimates by calendar year. Similar summing can be done for IBNR, except that it is over the index set $\mathcal{J}_1 \cup \widetilde{\mathcal{J}}_2 \cup \mathcal{J}_3$. The final reserve estimate in cell (i, j) is the sum of both RBNS and IBNR reserves over the relevant index set.

4. Extensions and validation of DCL

Current published extensions to the classical DCL all assume an additional triangle of the aggregated incurred claim amounts available. While this triangle also sits in the same index as $\aleph_m$ and Δ_m, it is not real-data in the sense that it contains RBNS estimates from expert's knowledge. To be consistent, we will regard this set of data as observations of the following r.v.

- $\mathbf{I}_m = \{I_{ij} : (i,j) \in \mathcal{I}\}$: the r.v. for the triangle of aggregated incurred claim amounts, I_{ij}
- I_{ij}: the r.v. follows this relationship

  $$I_{ij} = X_{ij} + X_{ij}^{\text{RBNS.case.estimates}}$$

  where $X_{ij}^{\text{RBNS.case.estimates}}$ is the r.v. for RBNS coming from experts who are familiar with characteristics of the claims

This chapter only introduces the first three published extensions while the more developed Haibu et al. (2016b) will be left to interested readers.

4.1. Bornhuetter-Ferguson Double Chain Ladder: BDCL

DCL shares the same limitation of CLM that it is very sensitive to outliers, especially for the more recent years where data is sparse (Martínez-Miranda et al. 2013a). However, the more recent years are also where the majority of reserves is intended for. Thus, BDCL shares the approach of the Bornhuetter & Ferguson (1972) to CLM by incorporating prior information. However, BDCL is more systematic and less objective than BF CLM. Specifically, the inflation parameter in DCL is replaced with more stable one estimated from the incurred data.

It is shown in Martínez-Miranda et al. (2013a) that under **M1**-**M3** in

Section 3.1.2,

$$E[I_{ij}] = \sum_{l=0}^{m-1} E\left[\sum_{k=0}^{N_{ijl}^{\text{paid}}} X_{ijl}^{(k)}\right] = \alpha_i \mu \gamma_i \beta_j = \widetilde{\alpha}_i \beta_j \tag{4.1}$$

where $\widetilde{\alpha}_i$ is cumulative paid claim for year of origin i and β_j is the proportion of number of incurred claims reported in year j. Hence, if the incurred data represents the "true" underlying payment process, the same inflation parameter can be extracted from either paid data or incurred data.

The estimated parameter in BDCL, $\widehat{\theta}^{\text{BDCL}} = \{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}^{\text{BDCL}}, \widehat{\mu}, \widehat{\pi}\}$ changes from $\widehat{\theta}$ in Section 3.2 only by the estimated inflation $\widehat{\gamma}^{\text{BDCL}}$. The steps as detailed in Martínez-Miranda et al. (2013a) are

Step 1 Estimate θ as in Section 3.2 using Δ_m and $\aleph_m$, yielding $\widehat{\theta} = \{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\mu}, \widehat{\pi}\}$

Step 2 Repeat **Step 1** with Δ_m replaced by $\mathbf{I}_m$ to obtain the estimate $\widehat{\gamma}^{\text{BDCL}}$

Step 3 Replace $\widehat{\gamma}$ by $\widehat{\gamma}^{\text{BDCL}}$ in $\widehat{\theta}$ to arrive at $\widehat{\theta}^{\text{BDCL}}$

It may be worth informing future users of the `DCL` package provided by Martinez-Miranda et al. (2013b) that the BDCL estimation processes there do not follow the steps enlisted in Martínez-Miranda et al. (2013a). Such effect on inflation estimates will be negligible if the first row of $\mathbf{I}_m$ and Δ_m are almost the same, while noticeable if that is not the case. The resulting $\widehat{\gamma}^{\text{BDCL}}$ may not confirm to **M3** and contradict to Equation 4.1. To be consistent, in this study, we strictly follow the steps in Martínez-Miranda et al. (2013a).

4.2. Incurred Double Chain Ladder: IDCL

As suggested in Section 3.3, under certain conditions, DCL is able to replicate CLM exactly. However, practitioners may sometimes prefer applying CLM on incurred data due to its potentially closer resemblance to reality. Thus, IDCL is created by Agbeko et al. (2014) mainly for the purpose of linking DCL to CLM by replicating the results from incurred CLM. By virtue of the construct of DCL, practitioners not only can relate the incurred CLM to IDCL, but also receive the added benefit of separating RBNS and IBNR and natural tail estimates. IDCL starts with exactly the same **Step 1** and ends with almost the same **Step 3** as in Section 4.1. From **Step 2**, the estimation becomes

Step 2 Estimate the IDCL inflation parameter by rescaling the inflation estimates from DCL, $\widehat{\gamma}$, with the ratio of two reserve estimates as

$$\widehat{\gamma}^{\text{IDCL}} = \frac{\widehat{R}_i^*}{\widehat{R}_i}\widehat{\gamma} \tag{4.2}$$

where $\widehat{R}_i^*$ is estimated reserves from CLM using incurred data in respect of year of origin i and $\widehat{R}_i$ is the reserve estimates from DCL as in Section 3.3 for the same year.

Step 3 Replace $\widehat{\gamma}$ by $\widehat{\gamma}^{\text{IDCL}}$ in $\widehat{\theta}$ to arrive at $\widehat{\theta}^{\text{IDCL}}$

For $\widehat{\gamma}_i^{\text{IDCL}}$ to be meaningful, it will take the value of $\widehat{\gamma}_i$ if the incurred CLM suggests that the claims for year i has been fully run-off, i.e. $\widehat{R}_i^* = 0$.

4.3. RBNS-Preserve Double Chain Ladder:RBNS-PDCL or PDCL

If we believe that case department in the company has done a reasonably good job at estimating the RBNS case reserves based on facts and prior knowledges, then it may be appropriate that we can fully utilise them. On the other hand, the IBNR implied from their RBNS estimates may entail too much uncertainty to be considered better than that estimated from DCL framework. Thus, the RBNS-preserve DCL intends to fully take advantage of the experts knowledge while keeping the predictive power of a mathematical model. The full steps leading to the set of parameter $\widehat{\theta}^{\text{PDCL}}$ are as follows (Hiabu et al. 2016a, Haibu et al. 2016b):

Step 1 Find the RBNS case estimate for each year of origin by

$$X_i^{\text{RBNS.case.estimate}} = \sum_{j=0}^{m-i} I_{ij} - \sum_{j=0}^{m-i} X_{ij} \tag{4.3}$$

Step 2 Perform a DCL estimation and predict the cash flows of RBNS $\widehat{X}_{ij}^{\text{RBNS.DCL}}(i,j) \in \mathcal{J}_1 \cup \mathcal{J}_2$ and IBNR $\widehat{X}_{ij}^{\text{IBNR.DCL}}(i,j) \in \mathcal{J}_1 \cup \widetilde{\mathcal{J}}_2 \cup \mathcal{J}_3$

Step 3 The RBNS implied from incurred data is preserved by construct the component

$$\widehat{X}_{ij}^{\text{RBNS.pres}} = \frac{X_i^{\text{RBNS.case.estimate}}}{\sum_{j \in \mathcal{J}_1(i) \cup \mathcal{J}_2(i)} X_{ij}^{\text{RBNS.DCL}}} \widehat{X}_{ij}^{\text{RBNS.DCL}} \tag{4.4}$$

Step 4 Construct the preliminary square (S_{ij}) by

$$S_{ij} = \begin{cases} X_{ij}, & \text{if} \quad (i,j) \in \mathcal{I} \\ \widehat{X}_{ij}^{\text{RBNS.pres}} + \widehat{X}_{ij}^{\text{IBNR.DCL}}, & \text{if} \quad (i,j) \in \mathcal{J}_1 \end{cases} \tag{4.5}$$

Step 5 Estimate the pair of PDCL parameters $\{\widetilde{\alpha}^{\text{PDCL}}, \widetilde{\beta}^{\text{PDCL}}\}$ by

$$\widehat{\widetilde{\alpha}}_i^{\text{PDCL}} = \textstyle\sum_{j=0}^{m-1} S_{ij}, \quad \widehat{\widetilde{\beta}}_i^{\text{PDCL}} = \frac{\sum_{i=1}^{m} S_{ij}}{\sum_{(i,j)\in\mathcal{I}\cup\mathcal{J}_1} S_{ij}} \tag{4.6}$$

where we used the shorthand $\widetilde{\alpha}^{\text{PDCL}} = (\widetilde{\alpha}_1^{\text{PDCL}}, \ldots, \widetilde{\alpha}_m^{\text{PDCL}})$ and $\widetilde{\beta}^{\text{PDCL}} = (\widetilde{\beta}_1^{\text{PDCL}}, \ldots, \widetilde{\beta}_m^{\text{PDCL}})$

Step 6 Estimate other parameters via the procedure detailed in Section 3.2 and obtain the set $\{\widehat{\widetilde{\alpha}}^{\text{PDCL}}, \widehat{\widetilde{\beta}}^{\text{PDCL}}, \widehat{\pi}^{\text{PDCL}}, \widehat{\gamma}^{\text{PDCL}^*}, \widehat{\mu}^{\text{PDCL}}\}$

Step 7 The properly defined PDCL inflation parameter γ^{PDCL} is estimated by

$$\widehat{\gamma}_i^{\text{PDCL}} = \frac{X_i^{\text{RBNS.case.estimate}}}{\sum_{j\in\mathcal{J}_1(i)\cup\mathcal{J}_2(i)} X_{ij}^{\text{RBNS.PDCL}^*}} \widehat{\gamma}_i^{\text{PDCL}^*} \tag{4.7}$$

where $\widehat{X}_{ij}^{\text{RBNS.PDCL}^*}$ is estimated using this set $\widehat{\theta}^{\text{PDCL}^*} = \{\widehat{\alpha}^{\text{PDCL}}, \widehat{\beta}^{\text{PDCL}}, \widehat{\pi}^{\text{PDCL}}, \widehat{\gamma}^{\text{PDCL}^*}, \widehat{\mu}^{\text{PDCL}}\}$

Step 8 Replace $\widehat{\gamma}^{\text{PDCL}^*}$ in $\widehat{\theta}^{\text{PDCL}^*}$ by $\widehat{\pi}^{\text{PDCL}}$ and obtain $\widehat{\theta}^{\text{PDCL}} = \{\widehat{\alpha}^{\text{PDCL}}, \widehat{\beta}^{\text{PDCL}}, \widehat{\pi}^{\text{PDCL}}, \widehat{\gamma}^{\text{PDCL}}, \widehat{\mu}^{\text{PDCL}}\}$

In **Step 2** of the estimation process, the RBNS and IBNR estimates can be estimated via any variate of DCL (Hiabu et al. 2016a, Haibu et al. 2016b), and different methods usually yield different PDCL parameters. It is deduced that the more developed Haibu et al. (2016b) has employed DCL. To be consistent with the research direction, this dissertation will use the steps described in Haibu et al. (2016b) rather than that in Hiabu et al. (2016a). In addition, it is observed that, in both Hiabu et al. (2016a) and Haibu et al. (2016b), the counterpart of Equation (4.7) is missing the $\widehat{\gamma}_i^{\text{PDCL}^*}$ in the formula.

4.4. Validation via back-testing

Validation provide some guidance as to which model predicts the future payments that are more closer to reality. Since it is always that payments are predicted, DCL and its extensions can be compared on the same paid-data scale. This is contrary to CLM, where incurred CLM and paid CLM are incomparable.

The validation process assumes that the last few c years are unknown (i.e. is cut-off from the available sample) while exactly the same estimation and prediction is carried out on this smaller sample consists of $m-c$ years of history. Then, the predictions over the last c years are compared against the observed values to estimate the prediction errors. In Agbeko et al. (2014), comparisons are made based on three error measures that considers the error within the cell, the error across the calendar year and the total error for all the cells, respectively. A full account of the procedure can be found in Agbeko et al. (2014). Hiabu et al. (2016a) adds another measure similar to the cell error suggested by Agbeko et al. (2014). All the error measures in Hiabu et al. (2016a) and Agbeko et al. (2014) have been scaled down by appropriately transformed observed values to form a percentage measure.

For those errors, let us denote $\widehat{X}_{ij}$, X_{ij} as the expected and actual paid claims over the omitted cells region $\mathcal{B}_c$, respectively, where c is the cut off year.

- Cell error proposed by Agbeko et al. (2014) and is called point error in Martinez-Miranda et al. (2013b)

$$\text{Point error} = \sqrt{\frac{\sum_{\forall(i,j)\in\mathcal{B}_c}(X_{ij}-\widehat{X}_{ij})^2}{\sum_{\forall(i,j)\in\mathcal{B}_c}X_{ij}^2}} \tag{4.8}$$

- Calendar error suggested by Agbeko et al. (2014)

$$\text{Calendar error} = \sqrt{\frac{\sum_{k=1}^{c}\left(\sum_{i+j=m-k}(X_{ij}-\widehat{X}_{ij})\right)^2}{\sum_{k=1}^{c}(\sum_{i+j=m-k}X_{ij})^2}} \tag{4.9}$$

- Total error suggested by Agbeko et al. (2014)

$$\text{Total error} = \frac{|\sum_{\forall(i,j)\in\mathcal{B}_c}X_{ij}-\widehat{X}_{ij}|}{\sum_{\forall(i,j)\in\mathcal{B}_c}X_{ij}} \tag{4.10}$$

- Relative error defined in Hiabu et al. (2016a)

$$\text{Relative error} = \frac{\sum_{(i,j)\in\mathcal{B}_c}|X_{ij}-\widehat{X}_{ij}|}{\sum_{(i,j)\in\mathcal{B}_c}|X_{ij}|} \tag{4.11}$$

5. Z-balanced Double Chain Ladder: ZDCL

5.1. Credible construction of ZDCL

It is verified in Martínez-Miranda et al. (2013a) that, amongst all the parameters that will be used to forecast the future, inflation parameter for more recent years are the hardest to estimate. We start with this parameter as a venture into exploring the predictive capacity within a credibility DCL framework. It has been established that inflation parameter from DCL is sensitive to volatilities in the data while that from prior information may be tamed with human bias. The credibility inflation parameter γ^{ZDCL} is defined as a linear combination of those two, i.e.

$$\gamma^{\text{ZDCL}} = Z\gamma + (1 - Z)\gamma^0 \tag{5.1}$$

where $Z \in [0, 1]$ is a real-valued scalar, γ is the inflation parameter from DCL in Chapter 3 and γ^0 is a prior inflation estimator. Equation 5.1 states that γ^{ZDCL} is a weighted average of γ and γ^0 with the weights being Z and $1-Z$, respectively. In credibility theory, Z represents how credible γ is compared to γ^0, while $1-Z$ measures how much low-bias we have to sacrifice in exchange for low-instability.

Equation 5.1 has reduced the problem by assuming that the same credibility is attached to each element of γ. This may not be the case in practice since inflation for older years have more data to estimate and hence may be more credible. In other words, we could have various values of Z_i attached to different elements of γ. However, as indicated in Chapter 1 and above, the aim of this dissertation is more modest than devising a complete credibility model. Similar to the discussion in Section 3.1.2, this simplification aids in grasping the essential features of ZDCL.

The general steps to arrive at ZDCL estimates are similar to that in Section 4.1 and 4.2.

Step 1 Estimate θ as in Section 3.2 using Δ_m and $\aleph_m$, yielding $\widehat{\theta} = \{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\mu}, \widehat{\pi}\}$

Step 2 Obtain relevant inflation estimates $\widehat{\gamma}^{\text{BDCL}}$, $\widehat{\gamma}^{\text{IDCL}}$ and $\widehat{\gamma}^{\text{PDCL}}$ as in Section 4.1, 4.2 or 4.3, respectively, and estimate $\widehat{\gamma}^{\text{ZDCL}}$ by

$$\widehat{\gamma}^{\text{ZDCL}} = \widehat{Z}\widehat{\gamma} + (1 - Z)\widehat{\widehat{\gamma}}^0 \tag{5.2}$$

where $\widehat{\widehat{\gamma}}^0 = \sum_{\forall k} \widehat{w}^k \widehat{\gamma}^k$ with $\widehat{w}^k \in [0, 1]$ and $\sum_{\forall k} \widehat{w}^k = 1$. w^k is the weights for one of currently available DCL variate k.

Step 3 Replace $\widehat{\gamma}$ by $\widehat{\gamma}^{\text{ZDCL}}$ in $\widehat{\theta}$ to arrive at $\widehat{\theta}^{\text{ZDCL}}$

Since prior knowledge can hardly be exact, we open to the possibility that more than one form of prior information may be necessary in **Step 2**. In this dissertation, we limited k to be in $\{\text{BDCL}, \text{IDCL}, \text{PDCL}\}$. It is highly possible that only one variates is needed in Equation 5.2 so that $\widehat{\widehat{\gamma}}^0 = \widehat{\gamma}^k$ for some k.

To properly apply the credibility formula to estimate Z, we need to know or estimate the instability of each of the inflation parameters. However, it is usually the case that only one data set will be available, which does not permit the acquisition of such knowledge. Mathematical statistics may help with the problem lack of data by imposing distributional assumptions. Nonetheless, this level of mathematical sophistication is outside the scope of this dissertation. Fortunately, it is argued in Bühlmann & Gisler (2005) that there *exits* a credibility estimator for almost each and every parameter under very general conditions. This credibility estimator can be expected to reduce prediction errors from both of its components. Thus, we devise a pragmatic and data-driven approach in Section 5.2 that fully exploiting the potential of validation in an unconventional manner. It allows the estimation of γ^{ZDCL} to be made under the very generalisable assumptions **M1-M3**.

5.2. ZDCL procedures to balance Bias vs. instability trade-off

In Martínez-Miranda et al. (2013a), Agbeko et al. (2014) and Hiabu et al. (2016a), back-testing is envisaged as a model selection tool to compare models. In this section, we view back-testing as a strategy to arrive at $\widehat{Z}$. These procedures will employ the combined error in Section 5.3 that is designed as a proxy to represent both bias and instability equally regardless how many years have been cut-off.

Step 1: Choose a range of cut-off years c. For each candidate $\widehat{c}$, estimate each error measure at vary values of $\widehat{\gamma}^{\text{ZDCL}}$ as follows:

1. Estimate γ, γ^{BDCL}, γ^{IDCL} and γ^{PDCL} using the procedure detailed in Chapter 3 and Chapter 4
2. Varying each of the coefficients on $\widehat{\gamma}$, $\widehat{\gamma}^{\text{BDCL}}$, $\widehat{\gamma}^{\text{IDCL}}$ and $\widehat{\gamma}^{\text{PDCL}}$ systematically, for example by ϵ each time, between 0 and 1 inclusive, to arrive at different values of $\widehat{\gamma}^{\text{ZDCL}}$ per Equation (5.2)
3. For each $\widehat{\gamma}^{\text{ZDCL}}$, calculate and register the empirical combined error per Equation (5.8) and the corresponding coefficients on $\widehat{\gamma}$, $\widehat{\gamma}^{\text{BDCL}}$, $\widehat{\gamma}^{\text{IDCL}}$ and $\widehat{\gamma}^{\text{PDCL}}$, respectively

Step 2: Find the minimum value of combined error at each cut-off year and plot it against the corresponding cut-off year. We call this Minimum Error Path graph.

Step 3: Choose $\widehat{c}^{*}$ such that there is a clear trough at this point in the MEP plotted in **Step 2**, i.e. the MEP forms a $\vee$ at $\widehat{c}^{*}$, subject to

$$\widehat{c}^{*} \lessapprox 25\% \text{ of the total years available}$$

$\widehat{Z}$ is selected as the one that minimised the combined error at $\widehat{c}^{*}$. The set of coefficients associated with $\widehat{Z}$ yields $\widehat{\gamma}^{\text{ZDCL}}$ by Equation 5.2.

The benefit of going through **Step 1** rather than some optimisation techniques is that we will be able to see how the errors develop to the minimum at each cut-off year by creating a Minimum Error Development (MED) graph as suggested in Appendix B.2. The shape of MED will empirically test our ZDCL model and suggests its viability. Specifically, a convex MED implies that there exists a unique set of inflation estimates which is expected to best predict future claim payments. And ZDCL is able to obtain such estimates and potentially results in better forecasts than any other methods.

Step 3 is critical to the entire credibility estimation in trying to balance the bias and instabilities. The main rational underpinning back-testing is that using past to predict current can reasonably resemble the situation where current is incorporated to predict the future. When deciding on how many years to cut off, there is again a trade-off between bias and volatility. The crude adjustment in combined error in Section 5.3 can only mitigate this trade-off rather than eliminate it. Cutting too little years off may cause instability problem as we need to compare errors on a very small sample of payment data that is known to be potentially volatile. Thus, for a single company, we are reluctant to take

$\widehat{c}^* = 1$, especially when the hiatory of data is sufficiently long and **Step 1-Step 3** indeed suggests otherwise. However, by cutting more years off to reduce volatility, bias increases. This is because more data will be used to estimate the error measures and less data is available for parameter estimation. By the back-testing algorithm, it means that the estimated error is expected to be a less relevant indication to the future prediction performance of the model but rather biased towards the current experience. This is particular important here, since the estimated $\widehat{Z}$ is supposed to be applied to predict the future. Hence, we constrain the maximum years that can afford to be cut-off for any data set. It is merely a rough measure and should be further investigated.

In sum, we want to find an acceptable level of cut-off that is as large as possible so that we can reduce the instability problem. At the same time, we also want to cut the data off as little as possible in order to ensure that the estimated errors will remain representative of the future errors that is expected from the model.

Furthermore, we have intentionally designed the combined error such that its estimates from different cut-off years can be on an approximately equal ground. Therefore, a $\vee$ shape in the MEP suggests that before this point we have been able to capture most of the information contained in the original data. The combined effect of bias and volatilities from the model are decreasing. A sudden increase signals that either bias or instability or both have risen sharply and hence the forecasts does not resemble the true payments.

Evidently, validation is the driver while the error measure play an essential role in this procedure. Cutting years off as in Agbeko et al. (2014) is the "best" we can do in real-life situation, which is similar to the real-data test in Bruce et al. (2008). Hence, for an individual company, we intend to improve the estimation process by modifying evaluation statistics for more fair and unified comparison.

5.3. Representing Bias and Instability in errors

The error measures in Agbeko et al. (2014) employ both square and absolute loss function, while the relative error in Hiabu et al. (2016a) adopts the absolute loss function exclusively. Although the original loss functions have been transformed, their properties remain. The estimator that minimises the prediction errors is mean and median for square loss and absolute loss, respectively (Domingos 2000). Since DCL is based on the specification of means as seen in Section 3.1.2, the errors that we can measure are also based on the estimated means. It seems reasonable to estimate the expected claim inflation using the quartic loss function, which is also favoured by Bühlmann & Gisler (2005).

The denominator in error measures proposed by Agbeko et al. (2014) and Hiabu et al. (2016a) are based on observed values. However, we believe that they should be replaced by forecasted counterparts and suggest to form error measure based on Equation 5.3:

$$\frac{\text{Sum of squared error}}{\text{Sum of squared predictions}} \tag{5.3}$$

We may regard Equation (5.3) as a competition between signal and noises. The predicted value is the signal given by the model while the errors are noises in the data uncounted for given the model. The smaller the noise compared to signal, the clear signal we can receive so that prediction can be more accurate. By focusing on the model output in the error, we can be more confident that we are assessing the future predictive power of the model instead of concentrating on the data we already have.

Nonetheless, Equation 5.3 may be incomparable between different cut-off years. A larger error may result simply due to the artefact of cutting more years off. Thus, we propose Equation (5.4)-(5.7) to assess the error in each cell, within an year of origin, across a calendar year and in total, respectively, where Equation (5.3) will be defined accordingly. GI practitioners are able to assess the model performance in the dimension that is most important to their business. Their respective counterpart in absolute loss is supplied in Section B.1.

- The mean point error is more tailored to measure the volatility of errors since each difference will count. There is no possibility for negative and positive errors to cancel each other over any dimensions.

$$\text{Mean point error} = \sqrt{\frac{\sum_{\forall(i,j)\in\mathcal{B}_c}(X_{ij}-\widehat{X}_{ij})^2}{\sum_{\forall(i,j)\in\mathcal{B}_c}\widehat{X}_{ij}^2}} \times \frac{1}{\sqrt{c(m-c)}} \tag{5.4}$$

- The mean origin error allows such cancellation within an year of origin and is more comparable to stochastic models developed in CLM such as Mack (1993). It will be exactly the same as the mean point error when only 1 year is cut off. With more years being cut off, more bias detection ability is incorporated.

$$\text{Mean origin error} = \sqrt{\frac{\sum_{i=1}^{m}\left(\sum_{j=m-i-c+1}^{m-i+1}(X_{ij}-\widehat{X}_{ij})\right)^2}{\sum_{i=1}^{m}(\sum_{j=m-i-c+1}^{m-i+1}\widehat{X}_{ij})^2}} \times \frac{1}{\sqrt{m-c}} \quad (5.5)$$

- The mean calendar error may be of interest in a business environment where knowing cash flow over a calendar year can assist business planning. Positive and negative errors in cells compensate each other over a calendar year and does not affect the total cash flows in a year. Its ability to detect volatility increases as the number of cut-off years increases.

$$\text{Mean calendar error} = \sqrt{\frac{\sum_{k=1}^{c}\left(\sum_{i+j=m-k}(X_{ij}-\widehat{X}_{ij})\right)^2}{\sum_{k=1}^{c}(\sum_{i+j=m-k}\widehat{X}_{ij})^2}} \times \frac{1}{\sqrt{c}} \quad (5.6)$$

- The total error allows volatilities to be cancelled to the largest extent. However, it is sensitive to bias where the forecasts are away from the actual claim payments in roughly the same direction. The less years being cutting off, the more alike will be between Equation (5.6) and (5.7).

$$\text{Total error} = \sqrt{\frac{(\sum_{\forall(i,j)\in\mathcal{B}_c} X_{ij}-\widehat{X}_{ij})^2}{(\sum_{\forall(i,j)\in\mathcal{B}_c}\widehat{X}_{ij})^2}} \quad (5.7)$$

The chosen error in Section 5.2 is intended to represent bias and volatility as equally as possible. This is particularly beneficial for data that does not have sufficient history to allow the bias component in the mean origin error or the volatility part in the mean calendar error to surface. A simple strategy is to combine the error detecting power on both horizontal and diagonal direction:

$$\text{Combined error} = \sqrt{\text{Mean origin error}^2 + \text{Mean calendar error}^2} \quad (5.8)$$

When the cut-off year is small, the first components will error more on volatility while the other will focus on bias. As more years being cut-off, both component will try to capture bias and volatility. Unlike the other two measures, both components of combine error are usually on roughly equal scale. Hence, Equation 5.8 is approximately balanced and may be compared among different cut-off years.

It is important to keep in mind that the consideration in this section is the bias and volatilities embedded in the deviations between the observed and the estimated values.

6. Empirical study

6.1. Exploratory analyses

It is suggested that a practical method needs to be capable of being applied to data sets across different business lines and lengths of periods available (Verrall et al. 2010). Thus, we illustrate our method via three data sets with various lengths and claim characteristics that have been aggregated in yearly format. The first two data sets are provided by the company developing the professional software implementing DCL, Res-timator. The first one collects claims from bodily injury (BI) insurance policies and the second data sets is for material damage (MD) insurance claims. Both data sets contain 11-year of data and are aggregated by accident years. The third data set is from Martínez-Miranda et al. (2013a) and also available in *R* package `DCL` by Martinez-Miranda et al. (2013b). It records 19 underwriting years of personal accident (PA) claims. The analyses are conducted in *R* (R Core Team 2016) with modification on the package `DCL` by Martinez-Miranda et al. (2013b). We follow Martínez-Miranda et al. (2013a) and denote the estimated ultimate claim count as the exposure measure.

The cumulative paid data, exposure, cumulative paid data adjusted by exposure only as well as by both exposure and estimated inflation parameter from DCL is plotted in Figure C.1, Figure C.2 and Figure C.3 for BI, MD, and PA, respectively. The paid data is plotted against development year while exposure is against year of origin. If the data does follow the DCL assumptions, parallel curves should be expected in the left bottom panel for the exposure-adjusted cumulative paid data where the vertical difference diminishes after inflation adjustment in the right bottom panel. The value of the prior information materialises when the volatility in the paid data has triggered the violations of **M1-M3**.

As intended, those plots demonstrate that these three data are in very different nature.

The exposure plotted in right top panel for both BI and MD suggest that both businesses experienced a strong growth during first 4 years while witnessed a drastic decline in the following years. For BI, the volume of businesses in year 11 seems to be even smaller than in year 1. It is highly likely that these two data sets come from the same company. In comparison, the right top panel in Figure C.3 indicates that the company who supplied the 19-year PA data appears to have sustained a relative stable volume of businesses after a significant expansion in the first 9 years.

The curves for BI claim payments in Figure C.1 for various accident years become closer and closer with each additional adjustment. However, there are still some discernible differences between the curves in the right bottom panel after both adjustments have been applied. The cumulative paid curves for MD in the top left panel of Figure C.2, however, have already in parallel. By the time both adjustments are made in the right bottom panel, the 11 curves become overlapping and hardly differentiable. Hence, it would appear that MD remarkably confirms to the DCL assumptions. Figure C.3 suggests that PA is much more volatile than any other two data sets. The rate of change in the cumulative paid curves still varies considerably in the right bottom panel with both exposure and inflation adjustments applied.

Above observations confirm to the fact that material damage claims are generally short-tailed while both bodily injury and personal accident claims tend to be medium- to long-tailed. In addition, we expect that, for MD, the predictions by DCL should be sufficiently reliable and hence the credibility we will attach to DCL in ZDCL is close to 1. In contrast, there may be significant portion of prior knowledges is necessary in ZDCL to stabilise the future claims projected by DCL for BI and PA.

6.2. Parameter estimates and prediction

This section presents the parameter estimated by employing the procedures in Section 5.2 and resulting point forecasts of outstanding liabilities for three data sets. We will discuss the validation results in the immediate subsequent section.

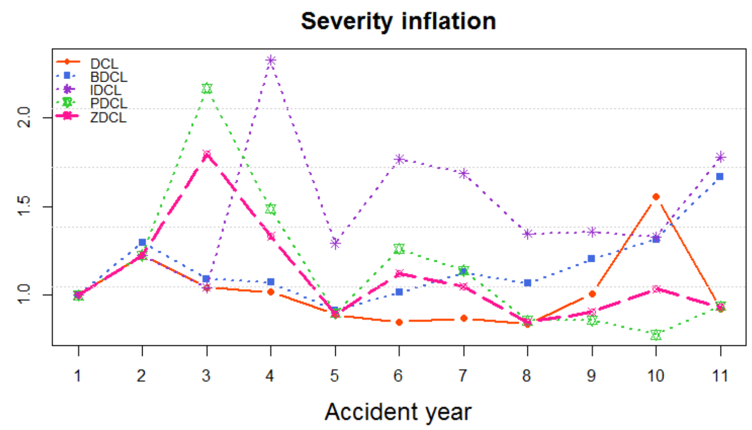
6.2.1. 11-year BI

The sharp turn in MEP graph in Figure C.4 suggests that not only $\widehat{c}^* = 3$ but also the potential of reducing prediction errors in ZDCL is significant. Table C.1 records the minimum combined error and the corresponding coefficient on each of the estimated inflation parameters by DCL, BDCL, IDCL and PDCL for each cut-off year. When $\widehat{c} = 3$, the weights given to $\widehat{\gamma}$ and $\widehat{\gamma}^{\text{PDCL}}$ is 0.33 and 0.67, respectively. Comparatively, the pure data-driven $\widehat{\gamma}$ is less reliable than $\widehat{\gamma}^{\text{PDCL}}$ by about a half. However, the other two candidates, $\widehat{\gamma}^{\text{BDCL}}$ and $\widehat{\gamma}^{\text{IDCL}}$, have been entirely absent in the formulation of $\widehat{\gamma}^{\text{ZDCL}}$. Then, $\widehat{\gamma}^{\text{ZDCL}}$ is estimated by

$$\widehat{\gamma}^{\text{ZDCL}} = 0.33\widehat{\gamma} + 0.67\widehat{\gamma}^{\text{PDCL}} \tag{6.1}$$

The resulting estimates is tabulated in Table C.2 and visualised in Figure 6.1.

Figure 6.1.: Inflation parameter estimates for 11-year BI



To reveal the tail differences, The magnitude of both $\widehat{\gamma}^{\text{IDCL}}$ and $\widehat{\gamma}^{\text{PDCL}}$ may have been adjusted for the first couple of years where reserves estimates are negligible.

The red line for $\widehat{\gamma}$ in Figure 6.1 suggests that the average bodily injury claim payment does not always increase with years but rather contracted during the middle of this 11 years and followed by a sharp peak at year 10. We acknowledge the fact that the reliability for both $\widehat{\gamma}^{\text{IDCL}}$ and $\widehat{\gamma}^{\text{PDCL}}$ may have been negatively affected by the small amount of outstanding liabilities in the first couple of years (Hiabu et al. 2016a). Nonetheless, much higher severity inflations are implied from the case department's views than paid data. This overstatement is present in each element of $\widehat{\gamma}^{\text{IDCL}}$ and later surfaces in $\widehat{\gamma}^{\text{BDCL}}$ as time approaches the end of this period. In contrast, $\widehat{\gamma}^{\text{PDCL}}$ start with a moderately high value and then drops to its lowest value in year 10. $\widehat{\gamma}^{\text{ZDCL}}$, as a credibility estimates, achieves its goal of balancing bias and instability by completely disregarding those overly high value implied from incurred data and relying on that estimated from paid data and a mixture of payments and RBNS case estimates. It is worth emphasising that DCL itself earns more creditability than that given to $\widehat{\gamma}$ since

$\widehat{\gamma}^{\text{PDCL}}$ contains IBNR obtained from DCL as well.

The predicted outstanding liabilities are shown in Table C.4. Although balances between DCL and PDCL by accident year, ZDCL yields the lowest total reserves estimate amongst all methods with 5% lower than that from DCL. This seemly unexpected result may be explained by the divergences between the delay parameter estimates for DCL and PDCL, respectively, in Table C.3.

6.2.2. 11-year MD

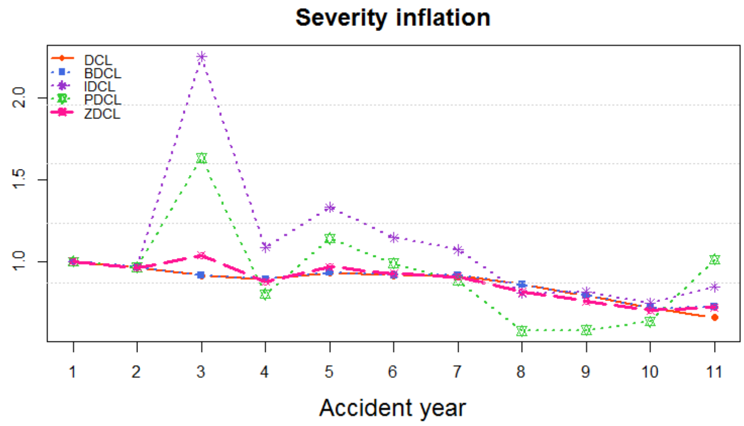
Performing the procedures in Section 5.2 on 11-year MD suggests that $\widehat{c}^* = 2$, according to the MEP graph in Figure C.5. In consistent with the observation and statement in Section 6.1, the gain by adding in prior knowledge seems slim. This is supported by Table C.5 where the reduction from 1-year to 2-year minimum combined error is insignificant and highly likely to be coincidental. To finish the exercise, we give 83% of trust to $\widehat{\gamma}$ and the rest to $\widehat{\gamma}^{\text{PDCL}}$ as suggested by Table C.5. Again, both $\widehat{\gamma}^{\text{BDCL}}$ and $\widehat{\gamma}^{\text{IDCL}}$ have not earned any credibility and attracted weights of 0. Then, ZDCL can estimate the inflation for MD as

$$\widehat{\gamma}^{\text{ZDCL}} = 0.83\widehat{\gamma} + 0.17\widehat{\gamma}^{\text{PDCL}} \tag{6.2}$$

which gives Table C.6 and Figure 6.2. Payment per material damage claims seems to have experienced steady decreases. The red line for $\widehat{\gamma}$ and blue line for $\widehat{\gamma}^{\text{BDCL}}$ almost overlaps with each other except for year 11 where a small departure between two lines is revealed. To the contrary, it is those estimates from IDCL and PDCL that fluctuate with $\widehat{\gamma}^{\text{IDCL}}$ always above $\widehat{\gamma}^{\text{PDCL}}$ except for year 11. From year 8, the line for $\widehat{\gamma}^{\text{IDCL}}$ joins that for $\widehat{\gamma}^{\text{BDCL}}$ and $\widehat{\gamma}$, leaving the green line for $\widehat{\gamma}^{\text{PDCL}}$ alone away from the other lines. With the majority of credibility given to $\widehat{\gamma}$, $\widehat{\gamma}^{\text{ZDCL}}$ has almost the same value as $\widehat{\gamma}$. Given the fact that $\widehat{\gamma}^{\text{PDCL}}$ also contains information from DCL and that MD reasonably confirms to DCL assumptions, there may be a plausible case for ZDCL to assume that $\widehat{Z} = 1$, which is also contained in Equation (5.1).

Other parameters in Table C.7 from different methods have similar values. Hence, it is not surprising that the forecasts of outstanding liabilities by ZDCL in Table C.8 is roughly 30% higher than that from DCL and about 13% lower than that from PDCL. Furthermore, we see a consistency here, the average payment of material damage claims in Table C.8 accounts for less than 5% of that from the bodily injury claims in Table C.4.

Figure 6.2.: Inflation parameter estimates for 11-year MD



To reveal the tail differences, The magnitude of both $\widehat{\gamma}^{\mathrm{IDCL}}$ and $\widehat{\gamma}^{\mathrm{PDCL}}$ may have been adjusted for the first couple of years where reserves estimates are negligible.

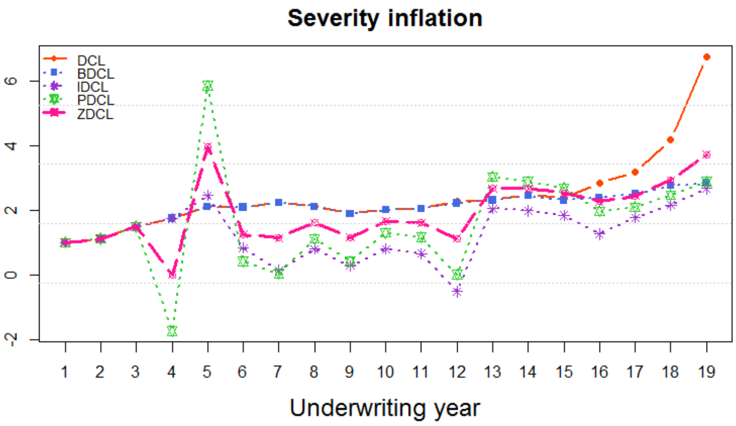
6.2.3. 19-year PA

The turn at $\widehat{c}^{*} = 4$ in Figure C.6 is material and hence is selected. For PA, DCL seems to give less reliable predictions by itself alone and we have to rely more on prior information given by case department. Table C.9 suggests that $\widehat{\gamma}$ is only 28% credible while $\widehat{\gamma}^{\mathrm{ZDCL}}$ has to draw another 22% from $\widehat{\gamma}^{\mathrm{BDCL}}$ and a further 50% from $\widehat{\gamma}^{\mathrm{PDCL}}$. This gives

$$\widehat{\gamma}^{\mathrm{ZDCL}} = 0.28\widehat{\gamma} + 0.22\widehat{\gamma}^{\mathrm{BDCL}} + 0.5\widehat{\gamma}^{\mathrm{PDCL}} \tag{6.3}$$

Table C.10 tabulates the estimated inflation parameters by underwriting years, which are illustrated in Figure 6.3. It is similar to that found in Hiabu et al. (2016a) except that different adjustments may have been applied to the first couple of years to reveal tail differences. Both $\widehat{\gamma}$ and $\widehat{\gamma}^{\mathrm{BDCL}}$ are in congruence and depict a stable increasing tend in the average bodily injury settlement until year 15. For the last 4 years, $\widehat{\gamma}$ rises sharply and almost approaches a value of 7. On the other hand, both $\widehat{\gamma}^{\mathrm{IDCL}}$ and $\widehat{\gamma}^{\mathrm{PDCL}}$ drop to near zero for the middle of this 19 years before it started to converges to $\widehat{\gamma}^{\mathrm{BDCL}}$ from year 13. It seems that the unstable values of $\widehat{\gamma}$ in the last few years have detrimentally affected its credibility, so that $\widehat{\gamma}^{\mathrm{ZDCL}}$ has to stabilises it with very large proportion of human judgement. For the last few years, where considerable part of outstanding liabilities come from, $\widehat{\gamma}^{\mathrm{ZDCL}}$ is only half of $\widehat{\gamma}$. Meanwhile, ZDCL is able to correct potential downward bias of inflation estimates from IDCL (Hiabu et al. 2016a) and PDCL, which allows $\widehat{\gamma}^{\mathrm{ZDCL}}$ to give much more sensible value of change in claim severities between year 6 and 13. The future payments are predicted in Table C.12 with additional inputs in Table C.11. The forecasted total outstanding liabilities from ZDCL is roughly 16.7% higher than that predicted by BDCL while approximately 31.8% lower than that suggested by DCL.

Figure 6.3.: Inflation parameter estimates for 19-year PA



To reveal the tail differences, The magnitude of both $\widehat{\gamma}^{\mathrm{IDCL}}$ and $\widehat{\gamma}^{\mathrm{PDCL}}$ may have been adjusted for the first couple of years where reserves estimates are negligible..

6.3. Prediction improvement examination

We will compare the accuracy of point forecasts from ZDCL with that from other methods using all the errors in Section 5.3 and for all cut-off years up to the decided $\widehat{c}^{*}$.

6.3.1. 11-year BI

Table 6.1 tabulates the change in prediction errors by adopting ZDCL as opposed to other methods for each cut-off year up to $\widehat{c}^{*}$. Columns of Table 6.1 represent the methods other than ZDCL and rows indicate errors suggested in Section 5.3. It is derived from Table C.13, which registers the estimated error measures by cut-off year for each method. Negative figure in Table 6.1 indicates a reduction in the relevant error.

For BI, ZDCL is able to reduce almost all forms errors suggested in Section 5.3 for all cut-off years except for two occasions. The two positive values in Table 6.1 are the 2-year-cut-off mean calendar and total error where DCL seems to outperform. This may not so disappointing if we recall that the sudden peak in $\widehat{\gamma}$ in Figure 6.1 at year 10. Overall, ZDCL seems to be the best in forecasting outstanding liabilities, followed by DCL. IDCL seems to be the worst amongst all the methods considered.

Table 6.1.: % change in prediction errors by adopting ZDCL as oppose to DCL, BDCL, IDCL, PDCL for 11-year BI

c	Error	DCL	BDCL	IDCL	PDCL
	Mean point	-18.58	-27.28	-40.54	-21.52
	Mean accident	-18.58	-27.28	-40.54	-21.52
1	Mean calendar	-6.14	-30.43	-54.32	-9.20
	Combined	-8.45	-29.95	-52.77	-11.49
	Total	-6.14	-30.43	-54.32	-9.20
	Mean point	-5.52	-33.59	-46.58	-19.68
	Mean accident	-14.64	-56.56	-66.70	-33.75
2	Mean calendar	2.60	-53.64	-70.73	-19.39
	Combined	-2.33	-54.37	-69.92	-23.57
	Total	43.81	-59.78	-75.67	-29.11
	Mean point	-8.49	-42.35	-55.89	-26.37
	Mean accident	-41.44	-70.98	-77.92	-39.17
3	Mean calendar	-11.44	-71.31	-81.69	-18.60
	Combined	-26.89	-71.19	-80.57	-28.27
	Total	-17.67	-75.64	-84.40	-17.97

It measures the percentage changes in error estiamtes tabulated in Table C.13 after adopting ZDCL from other methods. Negatives in this table means ZDCL is able to reduce the error measures.

6.3.2. 11-year MD

As already be alluded to by discussions in both Section 6.1 and 6.2, DCL can fit this data set sufficiently well, which again is proved in Table 6.2. With 1 year being cut off, only the mean point error can be reduced by ZDCL by roughly 8% while the total error is more than 100% smaller in DCL. DCL still outperforms ZDCL in total error by more than 90% when 2 years has been cut though ZDCL is able reduce other error measures this time. Similar to the BI, ZDCL is able to forecast future payments better than BDCL, IDCL and PDCL under all circumstances. However, unlike BI, both IDCL and PDCL have produced worse predictions than BDCL.

Table 6.2.: % change in prediction errors by adopting ZDCL as oppose to DCL, BDCL, IDCL, PDCL for 11-year MD

c	Error	DCL	BDCL	IDCL	PDCL
1	Mean point	-8.24	-13.09	-56.42	-70.11
	Mean accident	-8.24	-13.09	-56.42	-70.11
	Mean calendar	101.15	-16.01	-67.32	-67.97
	Combined	68.06	-15.66	-66.45	-68.20
	Total	101.15	-16.01	-67.32	-67.97
2	Mean point	-25.12	-5.29	-14.13	-36.99
	Mean accident	-21.08	-12.00	-41.04	-57.64
	Mean calendar	-4.28	-11.04	-72.45	-65.17
	Combined	-12.72	-11.42	-66.14	-62.43
	Total	92.88	-17.86	-77.03	-69.79

It measures the percentage changes in error estiamtes tabulated in Table C.14 after adopting ZDCL from other methods. Negatives in this table means ZDCL is able to reduce the error measures.

6.3.3. 19-year PA

Comparing Table 6.3 to Table 6.1 and Table 6.2, it seems that the case department in this company have been able to collectively correct their miscalculation of reserves over a two-year interval since BDCL has produced very small total error when the 2 or 4 years have been cut off from the data. For the 1-year and 3-year cut-off in Table 6.3, ZDCL can always yield smaller errors than any other methods for any error in question. Again, the predictions from IDCL is the worst in Table C.15.

Table 6.3.: % change in prediction errors by adopting ZDCL as oppose to DCL, BDCL, IDCL, PDCL for 19-year PA

c	Error	DCL	BDCL	IDCL	PDCL
1	Mean point	-17.43	-45.03	-58.91	-38.60
	Mean accident	-17.43	-45.03	-58.91	-38.60
	Mean calendar	-87.53	-46.26	-91.59	-70.07
	Combined	-78.76	-45.49	-85.94	-57.24
	Total	-87.53	-46.26	-91.59	-70.07
2	Mean point	-5.92	-17.66	-27.53	-3.16
	Mean accident	-11.24	-19.97	-28.59	-4.11
	Mean calendar	-54.50	25.58	-58.75	24.47
	Combined	-46.76	1.37	-52.56	11.42
	Total	-66.99	225.00	-63.85	383.11
3	Mean point	19.79	-26.30	-38.73	-13.19
	Mean accident	13.05	-33.15	-43.68	-18.33
	Mean calendar	-37.87	-12.42	-67.53	-25.02
	Combined	-27.31	-22.47	-62.33	-22.66
	Total	-88.14	-51.10	-92.24	-72.50
4	Mean point	-13.16	-37.47	-54.17	-11.61
	Mean accident	-50.57	-58.18	-68.92	-25.85
	Mean calendar	-46.15	-3.54	-71.56	-20.97
	Combined	-46.89	-24.50	-71.23	-21.83
	Total	-73.29	293.75	-79.67	7.85

It measures the percentage changes in error estiamtes tabulated in Table C.15 after adopting ZDCL from other methods. Negatives in this table means ZDCL is able to reduce the error measures.

6.4. Illustration of the validity of estimation procedure

A full account of how and why credibility theory is able to help GI companies in predicting their outstanding liabilities is beyond the scope of this dissertation. However, we intend to practically demonstrate the plausibility of the steps in Section 5.2. This has implication on whether we have approximated desired credibility estimates.

In Section 5.2, a Minimum Error Development (MED) graph is suggested to inform the ability of those procedures to find the value of Z that minimise the prediction errors. The MED graph for the BI, MD, and PA is supplied in Figure C.7, C.8 and C.9, respectively. With the exception of 1-year cut error for BI and MD, which is only based on 10 data points, all the MED graphs exhibit a convex shape. It is evident that there indeed *exists* a credibility inflation estimates that can balances the bias and instabilities so as to achieve the minimum amount of prediction error. From Table C.1, C.5 and C.9, we learnt that those estimated credibility factors are non-trivial. Therefore, by going through the estimation process, it seems that we can be confident that the estimated credibility estimates, and hence the predicted future claims, are largely valid.

7. Limitation, further studies and conclusion

7.1. Limitations and further studies

We have based our work on intuitive reasoning throughout while not laid down a solid theoretical background for credibility theory. As expressed by professionals (MacDonnell & Labaune 2014), being pragmatic is the key of a model and that is the central theme here. The beloved practicality, however, is attempted at the expense of the results being unable to sustain rigorous scrutiny. In particular, we have used back-testing to estimate the parameters, validating on the same data sets may suffer from the critique that we have re-engineered the validation results. To mitigate such effect, we use only one error measure in one cut-off year to estimate parameter while validate results using all the error statistics and for more than one cut-off years. Although not completely satisfactory, we have shown that the prediction in ZDCL is better under most scenarios. Given the fact that we have applied the same credibility to all the years of origin, the potential of reducing prediction errors as suggested in Table 6.1, 6.2, and 6.3 are encouraging. We believe that with differentiated credibility factor, more accurate predictions can be expected by employing ZDCL. This is one possible route to fully explore the potential of credibility theory.

The estimation procedures are demonstrated in a rather subjective manner since we want to explain the rational underpinning each step. It may be desirable if the entire process can be done automatically, which is relatively simple to achieve. Furthermore, the adjustments to form the combined error measure may be too crude. It is suggested as a proxy to incorporate both bias and volatility as fair as possible for all data sets regardless of their (reasonable) length of history. Future studies may consider to devise measures with more insights from mathematical statistics.

Finally, we have not extended this framework into a stochastic model while endeavours to obtain the best point forecasts. Nevertheless, as it is evident from the richness of stochastic CLM literatures, see, for example, Schmidt (2015), England & Verrall (2002), Wüthrich & Merz (2008), there should be a wealth of stochastic assumptions that can be accommodated by DCL. In particular, DCL is rigorously articulated from basic principle of claim risk generation. We briefly introduce one such possible structure from Martinez-Miranda et al. (2012) in Section 8.1. In the future, a stochastic version of credibility balanced DCL may be explored.

7.2. Conclusion

We have attempted a credibility balanced DCL to achieve the optimal trade-off between bias and instability so as to reduce prediction errors. The empirical studies on three data sets with various background and characteristics are in congruence on:

1. For each cut-off year, there is an "optimal" trade off between bias and instability such that the prediction error is minimised.
2. While unconventional and heuristic, the simple and straightforward procedures in Section 5.2 have strived to mimic the estimation results from a more complex credibility theory and attach appropriate credibility factors to each of $\widehat{\gamma}$, $\widehat{\gamma}^{\mathrm{BDCL}}$, $\widehat{\gamma}^{\mathrm{IDCL}}$ and $\widehat{\gamma}^{\mathrm{PDCL}}$ to achieve the overall "optimal" trade-off for all cut-off years. Even under inflexible assumptions, the results are motivating.
3. It is always the case that the credibility given to $\widehat{\gamma}$ and $\widehat{\gamma}^{\mathrm{PDCL}}$, respectively, are non-zero while to $\widehat{\gamma}^{\mathrm{IDCL}}$ is zero.
4. ZDCL is always able to outperform IDCL regardless which error measure and how many years are cut off.
5. ZDCL can outperform DCL, BDCL, and PDCL under most situations, especially for odd cut-off years and for longer-tailed business.

8. Remarks on forecasting scenarios in past papers

While studying papers, we found some inconsistencies in the empirical studies. Specifically, the outstanding liabilities forecasted by different methods have not been conducted under the same model assumptions. For example, in both Hiabu et al. (2016a) and Haibu et al. (2016b), DCL generates outstanding liability forecasts under the first moment assumption while BDCL have applied the stochastic assumptions in Martinez-Miranda et al. (2012) to predicts future payments. This seems rather unusual since both papers have not mentioned any stochastic assumptions and the validity of BDCL is proved indeed under the **M1-M3** in Hiabu et al. (2016a). To facilitate the further discussion, the stochastic DCL proposed in Martinez-Miranda et al. (2012) is introduced with added comments.

8.1. A possible stochastic model

One of the flexibilities of DCL is its ability to extend to various stochastic models by imposing different distributional assumptions. This would allow the generation of predictive distributions of reserves via the bootstrap detailed in Martinez-Miranda et al. (2011), which may assist the fulfilment of regulatory requirement, such as Solvency II. Synthesised from Verrall et al. (2010) and Martinez-Miranda et al. (2011), Martinez-Miranda et al. (2012) suggests a possible mathematical structure:

D1. N_{ij}'s are independent Poisson variables with mean $E[N_{ij}] = \alpha_i\beta_j$ and Kremer's identification(Kremer 1982) $\sum_{j=0}^{m-1}\beta_j = 1$. In other words,

$$N_{ij} \sim \text{Poisson}(\alpha_i\beta_j) \text{ with } \sum_{j=0}^{m-1}\beta_j = 1 \ \forall(i,j) \in \mathcal{I} \cup \mathcal{J}_1$$

D2. The conditional distribution of N^{paid}_{ijl} given N_{ij} is multinomial:

$$\left(N^{\text{paid}}_{i,j,0}, \ldots, N^{\text{paid}}_{i,j,m-1}\right) | N_{ij} \sim \text{Multinomial}\left(N_{ij}; p_0, \ldots, p_{m-1}\right) \;\; \forall (i,j)$$

where $(p_0, \ldots, p_{m-1})$ denotes the delay probabilities with $\sum_{l=0}^{m-1} p_l = 1$ and $p_l \in [0,1], \;\; \forall l \in \{0, \ldots, m-1\}$

D3. The individual payment size $X^{(k)}_{i,j-l,l}$'s are independent from the number of incurred claims $N_{ij}, \forall (i, j-l, l) : i \in \{1, , m-1\}, j \in \{0, \ldots, 2m-1\}, l \in \{0, \ldots, \min(j, d)\}$.

D4. The $X^{(k)}_{i,j-l,l}$ are mutually independent and has marginal distribution F_{ij} with mean μ_{ij} and variance $\sigma^2_{i,j}$ for all $(i,j) \in \mathcal{I} \cup \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$. It is further assumed that μ_{ij} and σ^2_{ij} can be parametrised as $\mu_{ij} = \mu\gamma_i$ and $\sigma^2_{ij} = \sigma^2\gamma^2_i$, respectively.

Without loss of generality, **D2** assumes that the maximum years of delay is $m-1$. Alternatively, it can be informed by expert's opinion as in Verrall et al. (2010). It is by **D3** that **D4** can be discussed without regarding to number of incurred claims. Hence **D3** and **D4** are in the order different from that in both Martinez-Miranda et al. (2012) and Martínez-Miranda et al. (2013a). In **D4**, μ and σ^2 is the common mean and variance factor for all F_{ij}, respectively. It is implied that the distribution of $X^{(k)}_{i,j-l,l}$ depends only the year of origin i.

Hence, similar to Equation (3.1) and (3.2), Martinez-Miranda et al. (2012) also provides the conditional and unconditional mean of X_{ij} in the stochastic case:

$$E[X_{ij} | \aleph_m] = \gamma_i \mu \sum_{N}^{j} N_{i,j-l} p_l \tag{8.1}$$

$$E[X_{ij}] = \alpha_i \gamma_i \mu \sum_{l=0}^{j} \beta_{j-l} p_l = \widetilde{\alpha}_i \widetilde{\beta}_j \tag{8.2}$$

respectively, where

$$\alpha_i \gamma_i \mu = \widetilde{\alpha}_i \tag{8.3}$$

$$\sum_{l=0}^{j} \beta_{j-l} p_l = \widetilde{\beta}_j \tag{8.4}$$

Following from Verrall et al. (2010), it can be shown that the conditional variance of X_{ij} is given by

$$V[X_{ij} | \aleph_m] = \gamma^2_i \sum_{l=0}^{\min(j,m-1)} N_{i,j-l} p_l \left(\sigma^2 + (1-p_l)\,\mu^2\right) \tag{8.5}$$

The estimation processes only differ from that in Section 3.2 by adjusting the delay function to a properly defined probability space and an additional estimation for the variance parameter that is provided in Appendix A.2

In order for **D2** to be satisfied, some adjustment to the estimated delay function may be necessary. This is because, for some data set, Equation (3.5) may yield estimates that, for example, the total number of reported claims is more that actually incurred, which cannot happen in reality. To obtain properly defined delay probabilities, two adjustments are currently suggested, both are available in `DCL` package by Martinez-Miranda et al. (2013b) in *R* (R Core Team 2016), which we will denote as P_1 and P_2, respectively, and are described in Appendix B.3. Alternatively, one could optimise Equation (3.5) by imposing the constraint that $\widehat{\pi}_l \in [0,1], \forall l \in \{0,\ldots,m-1\}$. Martinez-Miranda et al. (2011) and Martinez-Miranda et al. (2012) claim that forecasting future payments using the adjusted $\widehat{p} = \{\widehat{p}_0,\ldots,\widehat{p}_{m-1}\}$ and raw $\widehat{\pi}$ are very similar in practice. We will empirically test and prove this statement by using three real-life data in Section 8.2. Nonetheless, if the delay function have been adjusted, as in Martínez-Miranda et al. (2013a), Kremer's identification (Kremer 1982) needs to be preserved by dividing the estimated mean factor $\widehat{\mu}$ by $\widehat{\kappa} = \sum_{j=0}^{m-1}\sum_{l=0}^{j}\widehat{\beta}_{j-l}\widehat{p}_l$, which we denote as $\widehat{\mu}_d$.

The point estimates in the stochastic model will largely the same except possibly $\widehat{p}$ and $\widehat{\mu}_d$. Hence, the point parameter estimates in this case will be $\widehat{\theta}_d = \left(\widehat{\alpha},\widehat{\beta},\widehat{\gamma},\widehat{\mu}_d,\widehat{p}\right)$ in vector notation. The corresponding estimates of RBNS and IBNR becomes:

$$\widehat{X}_{ij}^{\mathrm{RBNS}(1)} = \sum_{l=i-m+j}^{j} N_{i,j-l}\widehat{p}_l\widehat{\mu}_d\widehat{\gamma}_i \tag{8.6}$$

$$\widehat{X}_{ij}^{\mathrm{RBNS}(2)} = \sum_{l=i-m+j}^{j} \widehat{N}_{i,j-l}\widehat{p}_l\widehat{\mu}_d\widehat{\gamma}_i = \sum_{l=i-m+j}^{j} \widehat{\alpha}_i\widehat{\beta}_j\widehat{p}_l\widehat{\mu}_d\widehat{\gamma}_i \tag{8.7}$$

and

$$\widehat{X}_{ij}^{\mathrm{IBNR}} = \sum_{l=0}^{i-m+j-1} \widehat{\alpha}_i\widehat{\beta}_j\widehat{p}_l\widehat{\mu}_d\widehat{\gamma}_i \tag{8.8}$$

Equation (8.6)-(8.8) is the counterpart of Equation (3.6)-(3.8) in the stochastic model, respectively. The total reserve is then in the same manner found by summing the RBNS and IBNR component accordingly by as in Equation (3.9). Following the procedure in Martinez-Miranda et al. (2011), a predictive distribution can be obtained via bootstrap either with or without parameter uncertainty.

It may deserve to mention that by the time this report is written, both IDCL and PDCL have only been theorised and tested within the first moment assumptions. This may because DCL is a new innovation and under continuously improvements, it only seems sensible to investigate the model structure first. Constructing a stochastic version of a well-constructed model will be straightforward. Although this does affect the point forecasts, it does mean that particular care is required while bootstrapping extensions based on either IDCL or PDCL.

8.2. Summary of varying forecasting scenarios and its effects

While one of the advantage of DCL is the provision of tails, some practitioner may prefer to remove the tails in order to assess the estimation power of the model and to compare with that of the classical CLM. For example, all the reserve estimates in both Hiabu et al. (2016a) and Haibu et al. (2016b) have been reported without tails. This removal of tails, although stated explicitly in Haibu et al. (2016b), does not seem to be indicated in Hiabu et al. (2016a).

Point forecasts of outstanding liabilities have different values in the development papers of DCL. We conclude that, these differences arose because, based on the current developments, the forecasts can vary depending on three factors

- whether expected or actual number of incurred claim is used in estimating RBNS reserves
- whether tail is included; i.e whether the forecast is over $\mathcal{J}_1$ or over $\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$.
- whether raw or adjusted delay function estimates is used, and if adjusted, which adjusting method is used; i.e. whether $\widehat{\pi}$, or P_1 or P_2 is used

In total, there are $2 \times 2 \times 3 = 12$ variates in the best estimates of reserves alone. The 12 possible ways to arrive the best estimates of reserves are listed in Table 8.1.

A summary of which forecasting scenario is used together with their respective reserve predictions in Martinez-Miranda et al. (2012), Martínez-Miranda et al. (2013a), Hiabu et al. (2016a), Haibu et al. (2016b) as well as in Res-timator are provided in Table 8.2. To our best knowledge, both Hiabu et al. (2016a) and Haibu et al. (2016b) have not specify the scenario applied on each method and why different scenarios have been employed to compare reserves from DCL and its extensions.

Table 8.1.: 12 outstanding liabilities forecasting scenarios

Notation	Equations for RBNS	$\widehat{\pi}$, P_1 or P_2	N_{ij} or$\widehat{N}_{ij}$	Cell index
F1	(3.7)	$\widehat{\pi}$	$\widehat{N}_{ij}$	$\mathcal{J}_1$
F2	(3.7)	$\widehat{\pi}$	$\widehat{N}_{ij}$	$\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$
F3	(3.6)	$\widehat{\pi}$	N_{ij}	$\mathcal{J}_1$
F4	(3.6)	$\widehat{\pi}$	N_{ij}	$\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$
F5	(8.7)	P_1	$\widehat{N}_{ij}$	$\mathcal{J}_1$
F6	(8.7)	P_1	$\widehat{N}_{ij}$	$\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$
F7	(8.6)	P_1	N_{ij}	$\mathcal{J}_1$
F8	(8.6)	P_1	N_{ij}	$\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$
F9	(8.7)	P_2	$\widehat{N}_{ij}$	$\mathcal{J}_1$
F10	(8.7)	P_2	$\widehat{N}_{ij}$	$\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$
F11	(8.6)	P_2	N_{ij}	$\mathcal{J}_1$
F12	(8.6)	P_2	N_{ij}	$\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$

Table 8.2.: Forecasting methods used in DCL developing papers and in Res-timator

	DCL	BDCL	IDCL	PDCL
Martínez-Miranda et al. (2013a)	**F8**	**F8**	-	-
Point forecast (£000s)	191,918	112,233	-	-
Hiabu et al. (2016a)	**F7**	**F3**	**F7**	**Special**
Point forecast (£Millions)	191.9021	112.2385	88.5565	102.8528
Haibu et al. (2016b)	**F7**	**F3**	**F7**	**F1**
Point forecast (£Millions)	191.9021	112.2385	88.5565	101.9427
Res-timator	**F12**	**F12**	**F12**	**Special**
Point forecast (£)	192,209,489	112,510,134	88,699,690	103,410,756

PDCL have different treatment in Hiabu et al. (2016a) and Res-timator, which we do not intend to drill into detail in this report. Suffices to say that it does not followed the steps in Hiabu et al. (2016a) and have been discarded in the more developed paper Haibu et al. (2016b).

Table 8.2 can be checked with the those in Appendix D, where we have collected results for all 12 methods from DCL, BDCL and IDCL for the 19-year PA data that was used in all the relevant papers. Beside to support the results in Table 8.2, Appendix D is also intended to prove that the effects of adjusted delay probability in real-life data are negligible. The largest percentage difference between 12 scenarios is 1.3% found in BDCL forecasts. We further have applied the 12 forecasting scenarios on 11-year BI and MD. The largest percentage change is 3.1% (in IDCL forecasts) and -0.5% (in IDCL forecasts) for BI and MD, respectively.

It is also observed that the differences in BDCL point forecasts for 12 scenarios are smaller when the steps in Martínez-Miranda et al. (2013a) presented in Section 4.1 have been closely followed rather than using `DCL` package by Martinez-Miranda et al. (2013b).

A. Additional remarks

A.1. More generalised first moment assumptions

M2-M3 can be slightly generalised as in Martinez-Miranda et al. (2012)

M2. The mean of N_{ijl}^{paid} conditional on the number of incurred claims is $E[N_{ijl}^{\text{paid}}|\aleph_m] = N_{ij}\widetilde{\pi}_l, (i,j) \in \mathcal{I}, l \in \{0, \dots, m-1\}$.

M3. The conditional mean of the individual payment size is $E[X_{ijl}^{(k)}|N_{ijl}^{\text{paid}}, \aleph_m] = \widetilde{\mu}_{ijl} = \widetilde{\mu}_{jl}\gamma_i$

In other words, the severity mean depends on all three time dimensions, namely the year of origin i (through the inflation parameter γ_i), the reporting delay j as well as the payment delay l. Therefore, Equation (3.1)-(3.4) becomes

$$E[X_{ij}|\aleph_m] = \gamma_i \sum_{l=0}^{j} N_{i,j-l}\widetilde{\mu}_{j-l,l}\widetilde{\pi}_l$$

and

$$E[X_{ij}] = \alpha_i\gamma_i \sum_{l=0}^{j} \beta_{j-l}\widetilde{\mu}_{j-l,l}\widetilde{\pi}_l = \widetilde{\widetilde{\alpha}}_i\widetilde{\widetilde{\beta}}_j$$

where

$$\alpha_i\gamma_i = \widetilde{\widetilde{\alpha}}_i$$

$$\sum_{l=0}^{j} \beta_{j-l}\widetilde{\mu}_{j-l,l}\widetilde{\pi}_l = \widetilde{\widetilde{\beta}}_j$$

respectively. Here, we have used a slightly different notations from Martinez-Miranda et al. (2012) to emphasise that $\widetilde{\widetilde{\alpha}}_i$ and $\widetilde{\widetilde{\beta}}_j$ have different meanings from $\widetilde{\alpha}_i$ and $\widetilde{\beta}_j$. Whereas $\widetilde{\alpha}_i$ contains actual monetary amount $\widetilde{\widetilde{\alpha}}_i$ is merely a mixture of count and inflation effects. In contrast, $\widetilde{\beta}_j$ does not involve severity while $\widetilde{\widetilde{\beta}}_j$ does. However, this set of assumption is over-parametrised and the proposed solution is to reduce dependence of severity mean to only the reporting delay j, i.e. $\widetilde{\mu}_{jl} = \widetilde{\mu}_j$ and form the common mean factor $\mu = \sum_{l=0}^{m-1} \widetilde{\pi}\widetilde{\mu}_l$ with the adjustment $\pi_l = \widetilde{\pi}\widetilde{\mu}_l/\mu$, which yield Equation (3.1)-(3.4) exactly.

A.2. Estimation of variance

The estimation of σ^2 is facilitated by the observation on Equation (8.5) that the conditional variance of X_{ij} is approximately proportional to its conditional mean. Equation (8.5) can be written as (Martinez-Miranda et al. 2011):

$$V[X_{ij}|\aleph_m] \approx \gamma_i \varphi \sum_{l=0}^{\min(j.d)} N_{i,j-l} p_l \gamma_i \mu = \gamma_i \varphi E[X_{ij}|\aleph_m]$$

where φ is the dispersion parameter and the second equality is by Equation (8.1). This resembles the an over-dispersion Poisson model, hence the dispersion parameter φ can be estimated via the generalised χ^2 statistics (McCullagh & Nelder 1989). Martinez-Miranda et al. (2011) shows that

$$\varphi = \frac{\sigma^2 + \mu^2}{\mu} - \frac{\mu}{n} \sum_{\forall (i,j \in \mathcal{I})} \frac{\sum_{l=0}^{\min(j,d)} N_{i,j-l} p_l^2}{\sum_{l=0}^{\min(j,d)} N_{i,j-l} p_l} \tag{A.1}$$

From Equation (A.1), the estimator for σ^2 can be written as

$$\sigma^2 = \mu\varphi - \mu^2 + \frac{\mu^2}{n} \sum_{\forall (i,j \in \mathcal{I})} \frac{\sum_{l=0}^{\min(j,d)} N_{i,j-l} p_l^2}{\sum_{l=0}^{\min(j,d)} N_{i,j-l} p_l}$$

However, Verrall et al. (2010) suggests to further approximate φ by

$$\varphi_{VNJ} \approx \frac{\sigma^2 + \mu^2}{\mu}$$

This yields the approximate variance factor σ^2_{VNJ} as

$$\sigma^2_{VNJ} \approx \mu\varphi_{VNJ} - \mu^2$$

The justification in Verrall et al. (2010) and Martinez-Miranda et al. (2011) is that the difference between σ^2 and σ^2_{VNJ} will be small if φ is large. The dispersion parameter is estimated as:

$$\widehat{\varphi}_{VNJ} = \frac{1}{n - (d+1)} \sum_{(i,j) \in \mathcal{I}} \frac{(X_{ij} - \widehat{X}_{ij})^2}{\widehat{X}_{ij} \widehat{\gamma}_i}$$

where $n = (m+1) * m/2$ is the total number of cells in $\mathcal{I}$, X_{ij} and $\widehat{X}_{ij}$is the observed and estimated value in cell (i,j), respectively, and γ_i is the inflation estimate for year of origin i.

B. Formulae and details

B.1. Error measures based on absolute error loss

- $$\text{Mean relative errors}=\frac{\sum_{(i,j)\in\mathcal{B}_c}|X_{ij}-\widehat{X}_{ij}|}{\sum_{(i,j)\in\mathcal{B}_c}|\widehat{X}_{ij}|}\times\frac{1}{\sqrt{c(m-c)}}$$

- $$\text{Mean relative origin error}=\frac{\sum_{i=1}^{m}|\sum_{j=m-i-c+1}^{m-i+1}(X_{ij}-\widehat{X}_{ij})|}{\sum_{i=1}^{m}|\sum_{j=m-i-c+1}^{m-i+1}\widehat{X}_{ij}|}\times\frac{1}{\sqrt{m-c}}$$

- $$\text{Mean relative calendar error}=\frac{\sum_{k=1}^{c}|\sum_{i+j=m-k}(X_{ij}-\widehat{X}_{ij})|}{\sum_{k=1}^{c}|\sum_{i+j=m-k}\widehat{X}_{ij}|}\times\frac{1}{\sqrt{c}}$$

- $$\text{Relative total error}=\frac{|\sum_{\forall(i,j)\in\mathcal{B}_c}X_{ij}-\widehat{X}_{ij}|}{\sum_{\forall(i,j)\in\mathcal{B}_c}\widehat{X}_{ij}}$$

B.2. ZDCL estimation procedures details

Assume that the output from **Step 2** in Section 5.2 is collected in Table B.1.

Table B.1.: Illustrative output of Step 1

Z	$(1-Z)*w^{\text{BDCL}}$	$(1-Z)*w^{\text{IDCL}}$	$(1-Z)*w^{\text{PDCL}}$	Error measure
0	0	0	1	error_1
0	0	ϵ	$1-\epsilon$	error_2
$\vdots$		$\dots$		$\vdots$
$\vdots$		$\ddots$		$\vdots$
$\vdots$		$\dots$		$\vdots$
$1-\epsilon$	ϵ	0	0	error_{s-1}
1	0	0	0	error_s

The first four columns contains coefficients on each of $\widehat{\gamma}$, $\widehat{\gamma}^{\text{BDCL}}$, $\widehat{\gamma}^{\text{IDCL}}$ and $\widehat{\gamma}^{\text{PDCL}}$, respectively, while the last column contains the value of the combined error. s is the total length of the matrix

where s is the total number of rows of the matrix. These matrices will have first four columns recording the coefficients and the last column registering the corresponding error value. With the error measure in Section 5.3, we will have c matrices. The MED graph could be plotted as follows:

1. Start from error_1: error_1 is on the graph
2. error_2 is on the graph if and only if $\text{error}_2 \leqslant \text{error}_1$
3. error_3 is on the graph if and only if $\text{error}_3 \leqslant \text{error}_1 \cap \text{error}_3 \leqslant \text{error}_2$, and so forth until it reaches the minimum
4. Reverse the direction and start from error_s: error_s is on the graph
5. error_{s-1} is on the graph if and only if $\text{error}_{s-1} \leqslant \text{error}_s$, and so on until it reaches the minimum again

This graph will be a single point if the first error is the minimum, i.e. when full credibility is given to $\widehat{\gamma}^{\text{PDCL}}$.

B.3. Delay parameter adjustment details

P_1: The default adjustment method in the `DCL` package.

- Find the minimum d^* such that $\widehat{\pi}_{d^*} < 0$ otherwise $d^* = m$
- Among the positive values, find the minimum d such that $\sum_{k=0}^{d^*} \widehat{\pi}_k < 1$
- Therefore,

$$\widehat{p}_k = \begin{cases} \widehat{\pi}_k & \text{if} \quad 0 < k < d \\ \widehat{p}_k = 1 - \sum_{k=0}^{d-1} \widehat{p}_k & \text{if} \quad k = d \\ \widehat{p}_k = 0 & \text{if} \quad d < k \leqslant m-1 \end{cases}$$

P_2: The professional software Res-timator uses this method and is optional in`DCL` package.

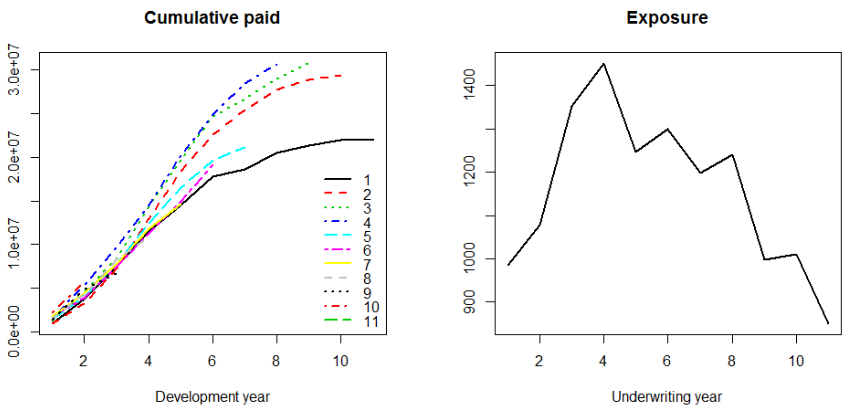
- Let $\widehat{p}_k^* = 0$ if $\widehat{\pi}_k < 0$ otherwise $\widehat{p}_k^* = \widehat{\pi}_k \quad \forall k \in \{0, \dots, m-1\}$
- Then define the offset quantity $r^* = 1 - \sum_{k=0}^{m-1} \widehat{p}_k^*$
- The finally we arrive at

$$\widehat{p}_k = \widehat{p}_k^* + r^* \times \frac{\widehat{p}_k^*}{\sum_{k=0}^{m-1} \widehat{p}_k^*} \qquad \forall k$$

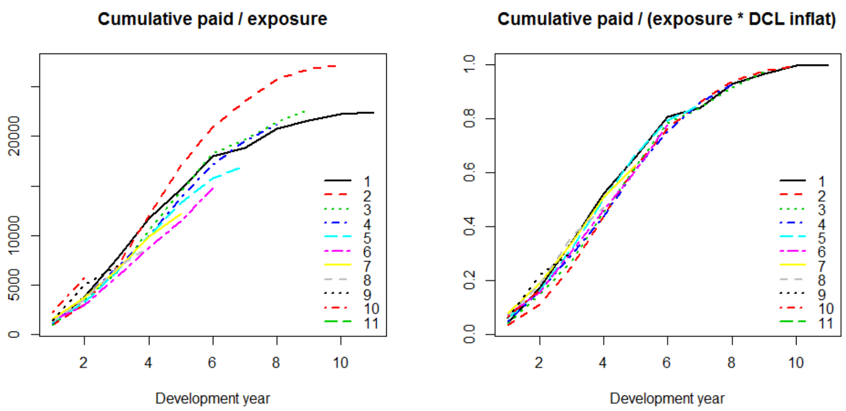
C. Empirical study tables and figures

C.1. Preliminary analyses

Figure C.1.: Exploratory analyses for 11-year BI

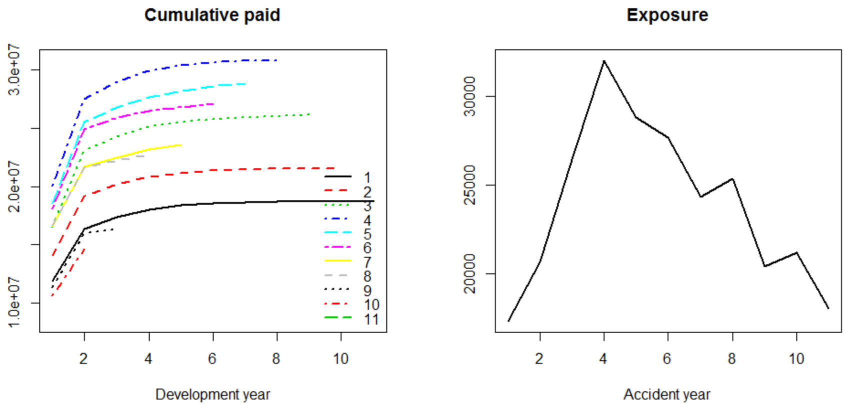


From left to right, the panel dipicts the cumulative paid data and exposure, respectively. Each curve in the left panel represents one accident year changing by development year while the exposure is against accident year.

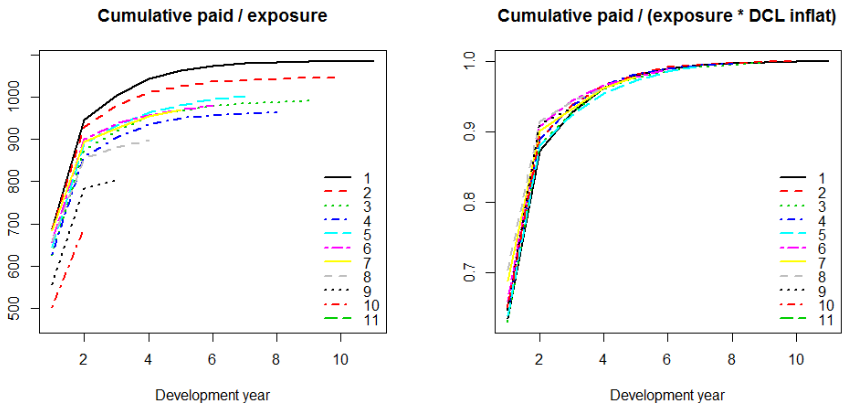


From left to right the panel dipicts the cumulative paid adjusted by exposure and adjusted by exposure and inflation paratmers estimated from DCL, respectively. Each curve represents one accident year changing by development year.

Figure C.2.: Exploratory analyses for 11-year MD

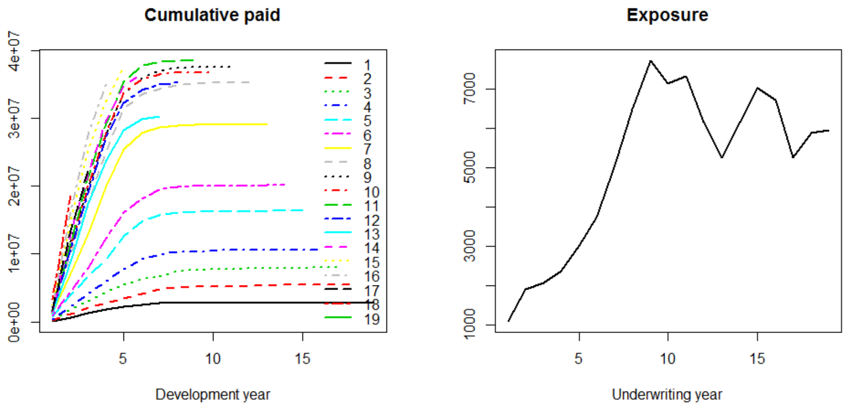


From left to right, the panel dipicts the cumulative paid data and exposure, respectively. Each curve in the left panel represents one accident year changing by development year while the exposure is against accident year.

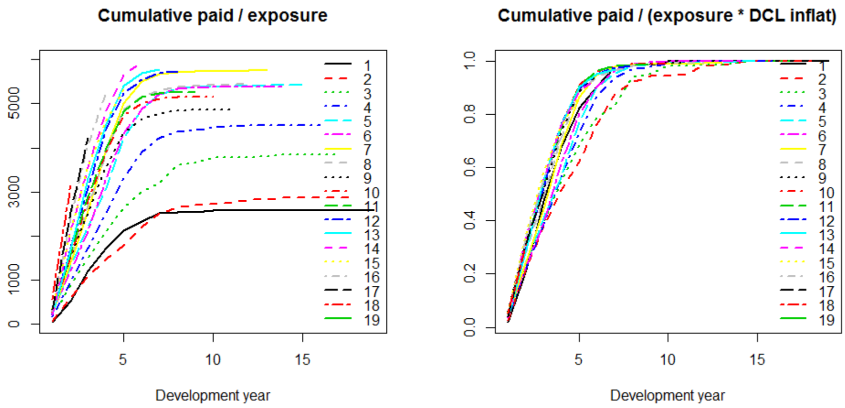


From left to right the panel dipicts the cumulative paid adjusted by exposure and adjusted by exposure and inflation paratmers estimated from DCL, respectively. Each curve represents one accident year changing by development year.

Figure C.3.: Exploratory analyses for 19-year PA



From left to right, the panel dipicts the cumulative paid data and exposure, respectively. Each curve in the left panel represents one underwriting year changing by development year while the exposure is against underwriting year.



From left to right the panel dipicts the cumulative paid adjusted by exposure and adjusted by exposure and inflation paratmers estimated from DCL, respectively. Each curve represents one underwriting year changing by development year.

C.2. Parameters and outstanding liabilities

Figure C.4.: Minimum Error Path (MEP) graph for 11-year BI

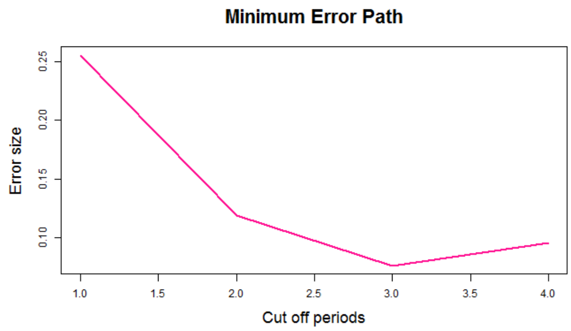


Table C.1.: Minimum combined error and associated Z values by cut-off years for 11-year BI

c	Z	$(1-Z)*w^{\text{BDCL}}$	$(1-Z)*w^{\text{IDCL}}$	$(1-Z)*w^{\text{PDCL}}$	Minimum
1	0.05	0	0.00	0.95	0.2555
2	0.61	0	0.00	0.39	0.1186
3	0.33	0	0.00	0.67	0.0756
4	0.15	0	0.14	0.71	0.0954

The error is found by minimising errors by simultaneously considering weights in three dimensions. The other dimension is constructed by the constraint that weights sum up to 1.

Table C.2.: Inflation parameter estimates from DCL, BDCL, IDCL, and PDCL for 11-year BI

Accident year	DCL	BDCL	IDCL	PDCL	ZDCL
1	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.22510	1.29754	22.28266	6.55447	4.79577
3	1.04509	1.09127	4.15413	2.16907	1.79816
4	1.01772	1.07248	2.32745	1.48924	1.33364
5	0.89078	0.91529	1.29097	0.89804	0.89564
6	0.84804	1.01513	1.76810	1.26161	1.12513
7	0.86920	1.13007	1.68601	1.13999	1.05063
8	0.83692	1.06651	1.34755	0.85727	0.85055
9	1.00557	1.20424	1.35844	0.85949	0.90770
10	1.55816	1.31559	1.32883	0.77684	1.03468
11	0.92514	1.67010	1.78222	0.93883	0.93431

Table C.3.: Delay parameters from DCL and PDCL and common mean factor for 11-year BI

	π	
A/Year	DCL/BDCL/IDCL/ZDCL	PDCL
1	0.06231	0.06205
2	0.11642	0.11605
3	0.14818	0.13809
4	0.16996	0.15718
5	0.15900	0.14827
6	0.14847	0.14423
7	0.06587	0.06928
8	0.07389	0.07847
9	0.04005	0.04874
10	0.01808	0.02956
11	0.00205	0.01432
μ (£)	22403.62	

Table C.4.: Outstanding liabilities by accident year from DCL, BDCL, IDCL, PDCL and ZDCL for 11-year BI in £000s

A/Y	CLM	DCL	BDCL	IDCL	PDCL	ZDCL
1	0.000	91.475	91.475	91.475	136.380	91.475
2	172.040	314.341	332.928	5717.384	3878.270	1230.521
3	899.506	997.578	1041.660	3965.287	3703.841	1716.414
4	2430.744	2578.987	2717.744	5897.954	5358.176	3379.542
5	3718.574	3818.435	3923.489	5533.878	4771.060	3839.278
6	5531.675	5697.859	6820.462	11879.594	9910.983	7559.563
7	8692.075	8757.947	11386.403	16987.953	12539.988	10585.967
8	12340.557	12425.414	15834.128	20006.629	13293.982	12627.885
9	15614.738	15732.043	18840.217	21252.648	13667.644	14200.812
10	29508.541	29588.480	24982.103	25233.624	14796.846	19647.858
11	16633.950	16708.991	30163.686	32188.835	16995.646	16874.593
Total	95542.399	96711.550	116134.294	148755.261	99052.816	91753.908

Prediction is performed under **F4** in Table 8.1.

Figure C.5.: Minimum Error Path (MEP) graph for 11-year MD

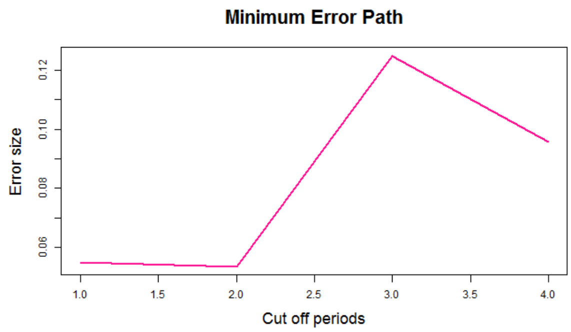


Table C.5.: Minimum combined error and associated Z values by cut-off years for 11-year MD

c	Z	$(1-Z)*w^{\text{BDCL}}$	$(1-Z)*w^{\text{IDCL}}$	$(1-Z)*w^{\text{PDCL}}$	Minimum
1	1.00	0.00	0.00	0.00	0.0551
2	0.83	0.00	0.00	0.17	0.0535
3	0.00	1.00	0.00	0.00	0.1247
4	0.00	0.14	0.86	0.00	0.0957

The error is found by minimising errors by simultaneously considering weights in three dimensions. The other dimension is constructed by the constraint that weights sum up to 1.

Table C.6.: Inflation parameter estimates from DCL, BDCL, IDCL and PDCL for 11-year MD

Accident year	DCL	BDCL	IDCL	PDCL	ZDCL
1	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.96372	0.96476	16.54482	6.29607	1.87022
3	0.91453	0.91503	2.25602	1.63450	1.03692
4	0.89192	0.89172	1.08723	0.80006	0.87630
5	0.92995	0.93142	1.33189	1.14104	0.96583
6	0.91412	0.91581	1.14507	0.98909	0.92686
7	0.91452	0.91694	1.07017	0.88175	0.90895
8	0.86100	0.85789	0.80502	0.57460	0.81231
9	0.79312	0.79366	0.81511	0.57939	0.75679
10	0.71283	0.71535	0.74431	0.63181	0.69906
11	0.65699	0.72102	0.84559	1.01295	0.71750

Table C.7.: Delay parameters from DCL and PDCL and common mean factor for 11-year MD

	π	
A/Year	DCL/BDCL/IDCL/ZDCL	PDCL
1	0.68207	0.67797
2	0.22198	0.22705
3	0.03231	0.03250
4	0.02767	0.02747
5	0.01454	0.01391
6	0.01005	0.00960
7	0.00533	0.00514
8	0.00192	0.00192
9	0.00313	0.00312
10	0.00092	0.00106
11	0.00009	0.00028
μ (£)	1085.035	

Table C.8.: Outstanding liabilities by accident year from DCL, BDCL, IDCL, PDCL and ZDCL for 11-year MD in £000s

A/Y	CLM	DCL	BDCL	IDCL	PDCL	ZDCL
1	0.000	0.209	0.209	0.209	0.385	0.209
2	2.986	3.563	3.567	61.170	51.081	6.915
3	30.610	31.702	31.719	78.204	71.820	35.945
4	132.442	132.734	132.704	161.799	127.606	130.410
5	184.431	184.919	185.212	264.845	237.655	192.055
6	327.497	330.612	331.225	414.142	361.049	335.222
7	540.561	537.232	538.655	628.668	510.021	533.960
8	891.979	886.484	883.286	828.848	576.592	836.355
9	1161.574	1160.769	1161.554	1192.945	833.419	1107.591
10	1741.275	1704.974	1710.999	1780.258	1499.072	1672.031
11	4412.212	4412.395	4842.398	5679.020	6881.033	4818.801
Total	9425.566	9385.594	9821.528	11090.109	11149.731	9669.494

Prediction is performed under **F4** in Table 8.1.

Figure C.6.: Minimum Error Path (MEP) graph for 19-year PA

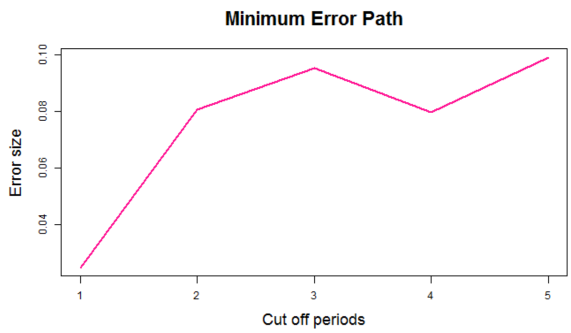


Table C.9.: Minimum combined error and associated Z values by cut-off years for 19-year PA

c	Z	$(1-Z)*w^{\text{BDCL}}$	$(1-Z)*w^{\text{IDCL}}$	$(1-Z)*w^{\text{PDCL}}$	Minimum
1	0.46	0.00	0.54	0.00	0.0248
2	0.26	0.00	0.25	0.49	0.0805
3	0.42	0.00	0.02	0.56	0.0953
4	0.28	0.22	0.00	0.50	0.0795
5	0.71	0.00	0.00	0.29	0.0990

The error is found by minimising errors by simultaneously considering weights in three dimensions. The other dimension is constructed by the constraint that weights sum up to 1.

Table C.10.: Inflation parameter estimates from DCL, BDCL, IDCL, PDCL and ZDCL for 19-year PA

Underwriting year	DCL	BDCL	IDCL	PDCL	ZDCL
1	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.11729	1.11729	1.11729	1.11729	1.11729
3	1.49473	1.49549	1.49473	126.98337	64.23926
4	1.74609	1.74452	1.74609	-29.25359	-13.75419
5	2.10746	2.10782	2.45402	5.85723	3.98245
6	2.09357	2.09139	0.82390	0.41915	1.25575
7	2.24954	2.23962	0.14356	0.03173	1.13786
8	2.12500	2.11582	0.79262	1.10935	1.61461
9	1.90280	1.88777	0.28472	0.42208	1.15823
10	2.01967	2.00670	0.79691	1.30015	1.65628
11	2.07036	2.05038	0.65670	1.15901	1.60909
12	2.26660	2.21353	-0.52391	0.00587	1.12138
13	2.31566	2.30678	2.05092	3.03157	2.67113
14	2.47468	2.44271	1.97987	2.88490	2.67084
15	2.38288	2.31091	1.84105	2.68585	2.51421
16	2.83913	2.38747	1.26057	1.96908	2.27764
17	3.18153	2.49436	1.76960	2.09704	2.44688
18	4.17470	2.74981	2.15977	2.47035	2.92355
19	6.75014	2.85389	2.67027	2.88864	3.72844

Table C.11.: Delay parameters from DCL and PDCL and common mean factor for 19-year PA

	π	
A/Year	DCL/BDCL/IDCL/ZDCL	PDCL
1	0.05922	0.06673
2	0.30977	0.31709
3	0.20318	0.20356
4	0.19964	0.19332
5	0.13884	0.13178
6	0.04403	0.04494
7	0.02268	0.02232
8	0.00949	0.00987
9	0.00176	0.00157
10	0.00288	0.00252
11	0.00020	0.00022
12	0.00259	0.00196
13	0.00189	0.00141
14	0.00319	0.00216
15	-0.00017	-0.00005
16	0.00125	0.00084
17	-0.00042	-0.00024
18	0.00004	0.00000
19	-0.00004	-0.00001
μ (£)	2579.002	

Table C.12.: Outstanding liabilities by accident year from DCL, BDCL, IDCL, PDCL, and ZDCL for 19-year PA in £000s

A/Y	CLM	DCL	BDCL	IDCL	PDCL	ZDCL
1	0.000	-0.033	-0.033	-0.033	-0.015	-0.033
2	0.000	0.634	0.634	0.634	0.699	0.634
3	0.000	-0.367	-0.368	-0.367	-23.573	-15.792
4	0.000	-1.398	-1.397	-1.398	7.935	11.013
5	17.312	15.090	15.093	17.572	31.097	28.515
6	34.620	31.702	31.669	12.476	4.962	19.017
7	138.113	142.219	141.592	9.076	1.475	71.974
8	244.858	250.324	249.243	93.370	97.482	190.265
9	352.247	360.057	357.213	53.876	60.875	219.337
10	394.267	383.014	380.554	151.128	191.418	314.247
11	552.383	525.768	520.694	166.770	237.157	408.933
12	683.887	632.035	617.237	-146.092	1.386	313.581
13	1050.408	977.482	973.732	865.728	1178.343	1127.755
14	2536.084	2549.764	2516.823	2039.939	2845.111	2753.852
15	5736.951	5449.868	5285.262	4210.649	5979.288	5760.115
16	14088.912	15438.845	12982.752	6854.829	10325.431	12532.901
17	21005.736	21741.878	17045.895	12093.031	13910.876	17003.170
18	44687.657	44459.344	29284.613	23000.944	25825.154	32045.473
19	98972.310	98973.805	41845.072	39152.831	42101.569	58095.880
Total	190495.745	191930.031	112246.280	88574.963	102776.669	130880.837

Prediction is performed under **F4** in Table 8.1.

C.3. Estimated errors

Table C.13.: Prediction errors for DCL, BDCL, IDCL, PDCL and ZDCL up to $\widehat{c}^*$ for 11-year BI

c	Error	DCL	BDCL	IDCL	PDCL	ZDCL
	Mean point	0.1254	0.1404	0.1717	0.1301	0.1021
	Mean accident	0.1254	0.1404	0.1717	0.1301	0.1021
1	Mean calendar	0.2524	0.3405	0.5186	0.2609	0.2369
	Combined	0.2818	0.3683	0.5463	0.2915	0.2580
	Total	0.2524	0.3405	0.5186	0.2609	0.2369
	Mean point	0.0743	0.1057	0.1314	0.0874	0.0702
	Mean accident	0.0690	0.1356	0.1769	0.0889	0.0589
2	Mean calendar	0.1037	0.2295	0.3635	0.1320	0.1064
	Combined	0.1245	0.2665	0.4043	0.1591	0.1216
	Total	0.0872	0.3118	0.5155	0.1769	0.1254
	Mean point	0.0601	0.0954	0.1247	0.0747	0.0550
	Mean accident	0.0777	0.1568	0.2061	0.0748	0.0455
3	Mean calendar	0.0682	0.2105	0.3299	0.0742	0.0604
	Combined	0.1034	0.2624	0.3890	0.1054	0.0756
	Total	0.1081	0.3654	0.5706	0.1085	0.0890

The errors are defined in Section 5.3 and predicitons is performed under **F2** in Table 8.1.

Table C.14.: Prediction errors for DCL, BDCL, IDCL, PDCL and ZDCL up to $\widehat{c}^*$ for 11-year MD

c	Error	DCL	BDCL	IDCL	PDCL	ZDCL
1	Mean point	0.0340	0.0359	0.0716	0.1044	0.0312
	Mean accident	0.0340	0.0359	0.0716	0.1044	0.0312
	Mean calendar	0.0433	0.1037	0.2665	0.2719	0.0871
	Combined	0.0551	0.1098	0.2760	0.2912	0.0926
	Total	0.0433	0.1037	0.2665	0.2719	0.0871
2	Mean point	0.0430	0.0340	0.0375	0.0511	0.0322
	Mean accident	0.0446	0.0400	0.0597	0.0831	0.0352
	Mean calendar	0.0421	0.0453	0.1463	0.1157	0.0403
	Combined	0.0613	0.0604	0.1580	0.1424	0.0535
	Total	0.0267	0.0627	0.2242	0.1705	0.0515

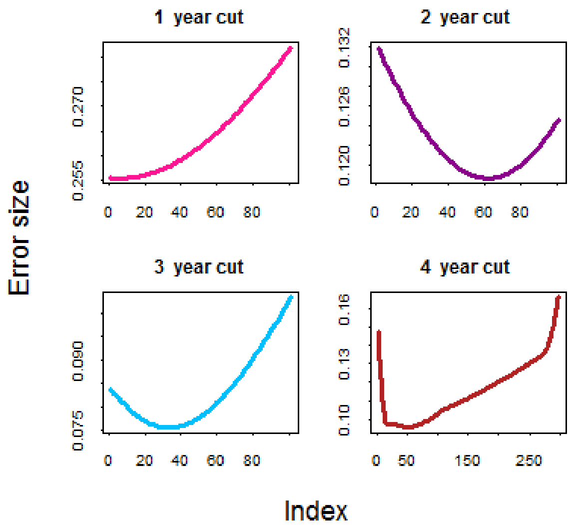
The errors are defined in Section 5.3 and predicitons is performed under **F2** in Table 8.1.

Table C.15.: Prediction errors for DCL, BDCL, IDCL, PDCL and ZDCL up to $\widehat{c}^*$ for 19-year PA

c	Error	DCL	BDCL	IDCL	PDCL	ZDCL
1	Mean point	0.0522	0.0784	0.1049	0.0702	0.0431
	Mean accident	0.0522	0.0784	0.1049	0.0702	0.0431
	Mean calendar	0.2421	0.0562	0.3591	0.1009	0.0302
	Combined	0.2476	0.0965	0.3741	0.1230	0.0526
	Total	0.2421	0.0562	0.3591	0.1009	0.0302
2	Mean point	0.0456	0.0521	0.0592	0.0443	0.0429
	Mean accident	0.0605	0.0671	0.0752	0.0560	0.0537
	Mean calendar	0.1554	0.0563	0.1714	0.0568	0.0707
	Combined	0.1668	0.0876	0.1872	0.0797	0.0888
	Total	0.2166	0.0220	0.1978	0.0148	0.0715
3	Mean point	0.0379	0.0616	0.0741	0.0523	0.0454
	Mean accident	0.0544	0.0920	0.1092	0.0753	0.0615
	Mean calendar	0.1249	0.0886	0.2390	0.1035	0.0776
	Combined	0.1362	0.1277	0.2628	0.1280	0.0990
	Total	0.2065	0.0501	0.3157	0.0891	0.0245
4	Mean point	0.0342	0.0475	0.0648	0.0336	0.0297
	Mean accident	0.0615	0.0727	0.0978	0.0410	0.0304
	Mean calendar	0.1365	0.0762	0.2584	0.0930	0.0735
	Combined	0.1497	0.1053	0.2763	0.1017	0.0795
	Total	0.2830	0.0192	0.3719	0.0701	0.0756

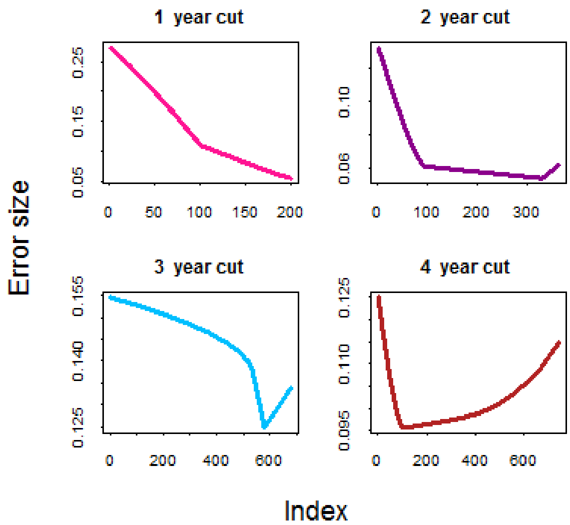
The errors are defined in Section 5.3 and predicitons is performed under **F2** in Table 8.1.

Figure C.7.: Minimum Error Development (MED) graph for 11-year BI



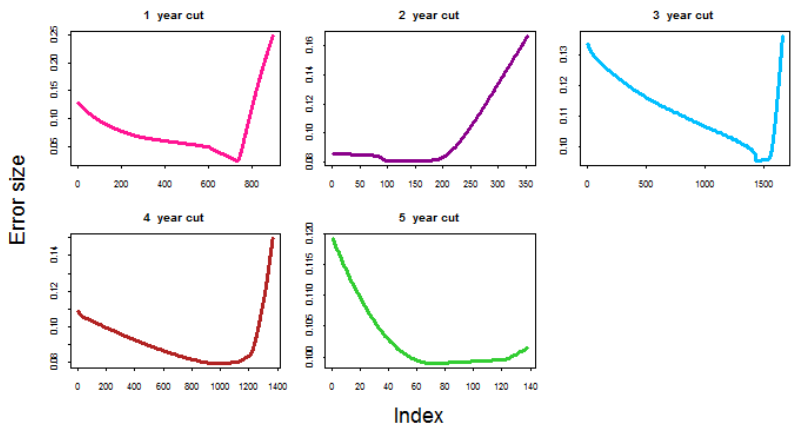
The horizontal is the index of the values rather than Z values. The trend can be interpreated by ackowledgeing the fact that as the index value increase, more credibility is given to $\widehat{\gamma}$ though not strictly. By the algorithm in Section 5.2, the first couple index have the same $\widehat{Z}$ attached to $\widehat{\gamma}$ while vary the weight given to other inflation estimates. Then the $\widehat{Z}$ increases but again hold constent when other weights varies, and so on.

Figure C.8.: Minimum Error Development (MED) graph for 11-year MD



The horizontal is the index of the values rather than Z values. The trend can be interpreated by ackowledgeing the fact that as the index value increase, more credibility is given to $\widehat{\gamma}$ though not strictly. By the algorithm in Section 5.2, the first couple index have the same $\widehat{Z}$ attached to $\widehat{\gamma}$ while vary the weight given to other inflation estimates. Then the $\widehat{Z}$ increases but again hold constent when other weights varies, and so on.

Figure C.9.: Minimum Error Development (MED) graph for 19-year PA



The horizontal is the index of the values rather than Z values. The trend can be interpreated by ackowledgeing the fact that as the index value increase, more credibility is given to $\widehat{\gamma}$ though not strictly. By the algorithm in Section 5.2, the first couple index have the same $\widehat{Z}$ attached to $\widehat{\gamma}$ while vary the weight given to other inflation estimates. Then the $\widehat{Z}$ increases but again hold constent when other weights varies, and so on.

D. Table for point forecasts under different scenarios

Table D.1.: Outstanding reserves from DCL for 19-year PA per year of origin in £000s

A/Y	CLM	**F1**(same as CLM)	**F2**	**F3**	**F4**	**F5**	**F6**
1	0.0	0.0	0.0	0.0	0.0	0.0	0.1
2	0.0	0.0	0.1	0.3	0.6	0.1	0.3
3	0.0	0.0	0.1	-0.3	-0.4	0.5	0.7
4	0.0	0.0	0.2	-1.4	-1.4	1.4	1.7
5	17.3	17.3	17.5	15.1	15.1	6.8	7.1
6	34.6	34.6	34.9	31.3	31.7	34.1	34.6
7	138.1	138.1	138.5	141.4	142.2	137.4	138.1
8	244.9	244.9	245.4	249.4	250.3	244.0	244.9
9	352.2	352.2	352.8	359.1	360.1	351.3	352.3
10	394.3	394.3	394.8	382.5	383.0	393.4	394.3
11	552.4	552.4	552.9	525.6	525.8	551.5	552.4
12	683.9	683.9	684.4	631.7	632.0	683.0	683.9
13	1050.4	1050.4	1050.9	977.1	977.5	1049.7	1050.4
14	2536.1	2536.1	2536.7	2549.3	2549.8	2535.2	2536.1
15	5737.0	5737.0	5737.6	5449.2	5449.9	5736.0	5737.1
16	14088.9	14088.9	14089.6	15438.4	15438.8	14088.1	14089.3
17	21005.7	21005.7	21006.4	21741.4	21741.9	21005.2	21006.2
18	44687.7	44687.7	44688.6	44458.4	44459.3	44687.2	44688.7
19	98972.3	98972.3	98973.8	98972.3	98973.8	98972.2	98974.7
Total	190495.7	190495.7	190505.1	191920.8	191930.0	190477.2	190492.9
Change from **F1**	0.0	0.0	9.4	1425.1	1434.3	-18.6	-2.8
% change	0.0	0.0	0.0	0.7	0.8	0.0	0.0

A/Y	**F7**	**F8**	**F9**	**F10**	**F11**	**F12**
1	0.0	0.0	0.0	0.1	0.0	0.0
2	0.5	0.8	0.2	0.4	0.5	0.9
3	0.1	0.1	1.0	1.3	0.6	0.6
4	0.7	0.8	4.8	5.1	3.4	3.5
5	3.9	4.0	25.3	25.8	23.0	23.2
6	30.9	31.4	47.0	47.6	43.7	44.4
7	140.7	141.8	156.1	156.9	159.4	160.6
8	248.5	249.8	266.6	267.6	271.0	272.6
9	358.2	359.5	375.4	376.5	382.1	383.7
10	381.8	382.5	416.9	418.0	405.0	406.2
11	524.6	525.2	576.1	577.2	549.2	550.1
12	630.9	631.5	705.6	706.6	653.5	654.3
13	976.4	977.1	1069.0	1069.9	995.8	996.6
14	2548.3	2549.3	2558.7	2559.8	2571.8	2572.9
15	5448.3	5449.4	5760.0	5761.3	5472.5	5473.7
16	15437.3	15438.5	14110.5	14112.0	15458.9	15460.4
17	21740.7	21741.8	21019.4	21020.6	21754.4	21755.7
18	44458.0	44459.5	44699.0	44700.9	44470.0	44471.8
19	98972.2	98974.7	98975.2	98978.2	98975.2	98978.2
Total	191902.1	191917.8	190766.6	190785.8	192190.3	192209.5
Change from **F1**	1406.3	1422.0	270.8	290.1	1694.6	1713.7
% change	0.7	0.7	0.1	0.2	0.9	0.9

Table D.2.: Outstanding reserves from BDCL for 19-year PA per year of origin in £000s

A/Y	**F1**	**F2**	**F3**	**F4**	**F5**	**F6**
1	0.0	0.0	0.0	0.0	0.0	0.1
2	0.0	0.1	0.3	0.6	0.1	0.3
3	0.0	0.1	-0.3	-0.4	0.5	0.7
4	0.0	0.2	-1.4	-1.4	1.4	1.7
5	17.3	17.6	15.1	15.1	6.8	7.1
6	34.6	34.9	31.3	31.7	34.1	34.6
7	137.5	137.9	140.8	141.6	136.8	137.5
8	243.8	244.3	248.3	249.2	243.0	243.8
9	349.5	350.0	356.3	357.2	348.6	349.5
10	391.7	392.3	380.0	380.6	390.9	391.7
11	547.1	547.6	520.6	520.7	546.1	547.1
12	667.9	668.4	616.9	617.2	667.0	667.9
13	1046.4	1046.8	973.3	973.7	1045.6	1046.4
14	2503.3	2503.9	2516.3	2516.8	2502.4	2503.4
15	5563.7	5564.3	5284.6	5285.3	5562.8	5563.8
16	11847.6	11848.2	12982.4	12982.8	11846.9	11847.9
17	16468.8	16469.2	17045.5	17045.9	16468.3	16469.1
18	29435.0	29435.6	29284.0	29284.6	29434.7	29435.7
19	41844.4	41845.1	41844.4	41845.1	41844.4	41845.5
Total	111098.5	111106.4	112238.5	112246.3	111080.4	111093.7
Change from **F1**	0.0	7.9	1140.0	1147.8	-18.0	-4.8
% change	0.0	0.0	1.0	1.0	0.0	0.0

A/Y	**F7**	**F8**	**F9**	**F10**	**F11**	**F12**
1	0.0	0.0	0.0	0.1	0.0	0.0
2	0.5	0.8	0.2	0.4	0.5	0.9
3	0.1	0.1	1.0	1.3	0.6	0.6
4	0.7	0.8	4.8	5.1	3.4	3.5
5	3.9	4.0	25.3	25.8	23.0	23.2
6	30.8	31.4	46.9	47.5	43.7	44.3
7	140.1	141.2	155.4	156.2	158.7	159.9
8	247.4	248.7	265.4	266.4	269.9	271.4
9	355.3	356.7	372.4	373.5	379.1	380.7
10	379.4	380.0	414.2	415.3	402.4	403.6
11	519.6	520.1	570.5	571.7	543.9	544.8
12	616.1	616.7	689.1	690.1	638.2	639.0
13	972.7	973.3	1064.9	1065.8	992.0	992.8
14	2515.4	2516.3	2525.6	2526.7	2538.6	2539.7
15	5283.8	5284.8	5586.0	5587.2	5307.2	5308.4
16	12981.5	12982.5	11865.8	11867.0	12999.6	13000.9
17	17045.0	17045.8	16479.4	16480.4	17055.7	17056.7
18	29283.7	29284.7	29442.5	29443.7	29291.6	29292.8
19	41844.4	41845.5	41845.7	41846.9	41845.7	41846.9
Total	112220.3	112233.6	111355.1	111371.3	112494.0	112510.1
Change from **F1**	1121.9	1135.1	256.6	272.8	1395.5	1411.7
% change	1.0	1.0	0.2	0.2	1.3	1.3

Table D.3.: Outstanding reserves from IDCL for 19-year PA per year of origin in £000s

A/Y	**F1**	**F2**	**F3**	**F4**	**F5**	**F6**
1	0.0	0.0	0.0	0.0	0.0	0.1
2	0.0	0.1	0.3	0.6	0.1	0.3
3	0.0	0.1	-0.3	-0.4	0.5	0.7
4	0.0	0.2	-1.4	-1.4	1.4	1.7
5	20.2	20.4	17.6	17.6	7.9	8.3
6	13.6	13.7	12.3	12.5	13.4	13.6
7	8.8	8.8	9.0	9.1	8.8	8.8
8	91.3	91.5	93.0	93.4	91.0	91.3
9	52.7	52.8	53.7	53.9	52.6	52.7
10	155.6	155.8	150.9	151.1	155.2	155.6
11	175.2	175.4	166.7	166.8	174.9	175.2
12	-158.1	-158.2	-146.0	-146.1	-157.9	-158.1
13	930.3	930.7	865.4	865.7	929.7	930.3
14	2029.0	2029.4	2039.6	2039.9	2028.3	2029.0
15	4432.5	4432.9	4210.1	4210.6	4431.8	4432.6
16	6255.5	6255.8	6854.6	6854.8	6255.1	6255.6
17	11683.6	11683.9	12092.8	12093.0	11683.3	11683.9
18	23119.1	23119.5	23000.5	23000.9	23118.8	23119.6
19	39152.2	39152.8	39152.2	39152.8	39152.2	39153.2
Total	87961.4	87965.9	88571.0	88575.0	87947.1	87954.5
Change from **F1**	0.0	4.4	609.6	613.5	-14.4	-7.0
% change	0.0	0.0	0.7	0.7	0.0	0.0

A/Y	**F7**	**F8**	**F9**	**F10**	**F11**	**F12**
1	0.0	0.0	0.0	0.1	0.0	0.0
2	0.5	0.8	0.2	0.4	0.5	0.9
3	0.1	0.1	1.0	1.3	0.6	0.6
4	0.7	0.8	4.8	5.1	3.4	3.5
5	4.5	4.7	29.5	30.0	26.8	27.0
6	12.2	12.4	18.5	18.7	17.2	17.5
7	9.0	9.0	10.0	10.0	10.2	10.3
8	92.7	93.2	99.4	99.8	101.1	101.7
9	53.6	53.8	56.2	56.3	57.2	57.4
10	150.7	150.9	164.5	164.9	159.8	160.3
11	166.4	166.6	182.7	183.1	174.2	174.5
12	-145.8	-146.0	-163.1	-163.3	-151.1	-151.2
13	864.8	865.3	946.8	947.6	882.0	882.7
14	2038.8	2039.5	2047.1	2048.0	2057.6	2058.5
15	4209.5	4210.3	4450.2	4451.2	4228.1	4229.1
16	6854.2	6854.7	6265.1	6265.7	6863.7	6864.4
17	12092.4	12093.0	11691.2	11691.9	12100.0	12100.7
18	23000.2	23001.0	23124.9	23125.9	23006.4	23007.4
19	39152.2	39153.2	39153.4	39154.6	39153.4	39154.6
Total	88556.5	88563.4	88082.3	88091.3	88691.3	88699.7
Change from **F1**	595.0	601.9	120.9	129.9	729.8	738.2
% change	0.7	0.7	0.1	0.1	0.8	0.8

Bibliography

Agbeko, T., Munir, H., Margraf, C., Martínez-Miranda, M. D., Nielsen, J. P. & Verrall, R. (2014), 'Validating the double chain ladder stochastic claims reserving model', *Variance* **8**, 138–160.

Benktander, G. (1976), 'An approach to credibility in calculating IBNR for casualty excess reinsurance', *The Actuarial Review* **3**(2), 7.

Bergmeir, C. & Beìtez, J. M. (2012), 'On the use of cross-validation for time series predictor evaluation', *Information Sciences* **191**, 192 – 213. Data Mining for Software Trustworthiness.

Bornhuetter, R. & Ferguson, R. (1972), The actuary and IBNR, *in* 'Proceedings of the Casualty Actuarial Society', number 111-112, pp. 181–195.
URL: *https://www.casact.org/pubs/proceed/proceed72/72181.pdf*

Breiman, L. (1996), Bias, variance, and arcing classifiers, Technical Report 460, Statistics Department, University of California, Berkeley, CA, USA.

Bruce, N., Chen, C., Dunne, G., Hinder, I., McMurrough, T., Meyers, G., White, A. & Wright, T. (2008), Reserving uncertainty, *in* 'ROC Working Party'.
URL: *https://www.actuaries.org.uk/research-and-resources/documents/reserving-oversight-committee-roc-working-party-paper-reserving-unc*

Bühlmann, H. (1967), 'Experience rating and credibility', *Astin Bulletin* **4**(03), 199–207.

Bühlmann, H. & Gisler, A. (2005), *A Course in Credibility Theory and its Applications*, Universitext (1979), Springer.

Burman, P. & Nolan, D. (1992), 'Data-dependent estimation of prediction functions', *Journal of Time Series Analysis* **13**(3), 189–207.

City University London (2014), 'Preventing insolvency of non-life insurance firms by understanding and quantifying the uncertainty of outstanding insurance claims', via 2014 Reserch Excellence Framework (REF) impact case studies `http://impact.ref.ac.uk/CaseStudies/CaseStudy.aspx?Id=44373#`.

Clarke, T. & Harland, N. (1974), 'A practical statistical method of estimating claims liability and claims cash flow', *ASTIN Bulletin: The Journal of the International Actuarial Association* **8**(01), 26–37.

Domingos, P. (2000), A unified bias-variance decomposition, *in* 'Proceedings of 17th International Conference on Machine Learning', pp. 231–238.

England, P. D. & Verrall, R. J. (2002), 'Stochastic claims reserving in general insurance', *British Actuarial Journal* **8**(03), 443–518.

Feldblum, S. (2003), The Stanard-Bühlmann reserving procedure: A practitioner's guide, *in* 'Proceedings of the Casualty Actuarial Society', Vol. 90, pp. 155–195.
URL: *https://www.casact.org/pubs/proceed/proceed03/03155.pdf*

Financial Services Authority (2011), 'Adequacy of year-end reserving', `http://www.fca.org.uk/your-fca/documents/finalised-guidance/fsa-fg116`.

Gibson, E., Archer-Lock, P., Bruce, N., Collins, A., Dunne, G., Felisky, K., Hamilton, A., Jewell, M., Lo, J., Locke, J. et al. (2007), Best estimates and reserving uncertainty, *in* 'ROC/GIRO Working Party'.
URL: *https://www.actuaries.org.uk/research-and-resources/documents/best-estimates-and-reserving-uncertainty*

Gisler, A. & Wüthrich, M. V. (2008), 'Credibility for the chain ladder reserving method', *Astin Bulletin* **38**(02), 565–600.

Haibu, M., Margraf, C., Martínez-Miranda, M. D. & Nielsen, J. P. (2016b), 'Cash flow generalisations of non-life insurance expert systems estimating outstanding liabilities', *Expert Systems with Applications* **45**, 400–409.

Hiabu, M., Margraf, C., Martínez-Miranda, M. D. & Nielsen, J. P. (2016a), 'The link between classical reserving and granular reserving through double chain ladder and its extensions', *British Actuarial Journal* **21**, 97–116.

Hürlimann, W. (2009), 'Credible loss ratio claims reserves: the benktander, neuhaus and mack methods revisited', *ASTIN Bulletin* **39**, 81–99.

Kremer, E. (1982), 'IBNR-claims and the two-way model of ANOVA', *Scandinavian Actuarial Journal* **1982**(1), 47–55.

Larson, S. C. (1931), 'The shrinkage of the coefficient of multiple correlation.', *Journal of Educational Psychology* **22**(1), 45.

Leong, J., Wang, S. & Chen, H. (2012), Back-testing the ODP bootstrap of the paid Chain-Ladder Model with actual historical claims data, *in* 'Casualty Actuarial Society E-Forum', pp. 1–34.
URL: *https://www.casact.org/pubs/forum/12sumforum/Leong_Wang_Chen.pdf*

MacDonnell, S. & Labaune, R. (2014), Giroc uk reserving survey 2014, *in* 'GIROC Working Party'.
URL: *https://www.actuaries.org.uk/documents/giroc-uk-reserving-survey-2014*

Mack, T. (1991), 'A simple parametric model for rating automobile insurance or estimating ibnr claims reserves', *ASTIN Bulletin: The Journal of the International Actuarial Association* **21**(01), 93–109.

Mack, T. (1993), 'Distribution-free calculation of the standard error of chain ladder reserve estimates', *Astin bulletin* **23**(02), 213–225.

Mack, T. (2000), 'Credible claims reserves: the benktander method', *Astin Bulletin* **30**(02), 333–347.

Martinez-Miranda, M. D., Nielsen, B., Nielsen, J. P. & Verrall, R. J. (2011), 'Cash flow simulation for a model of outstanding liabilities based on claim amounts and claim numbers', *ASTIN Bulletin* **41**(1), 107 – 129. Copyright 2011 Cambridge Journals.

Martinez-Miranda, M. D., Nielsen, J. P. & Verrall, R. (2013b), *DCL: Claims Reserving under the Double Chain Ladder Model*. R package version 0.1.0.
URL: *https://CRAN.R-project.org/package=DCL*

Martinez-Miranda, M. D., Nielsen, J. P. & Verrall, R. J. (2012), 'Double chain ladder', *ASTIN Bulletin* **42**(1), 59 – 76. Copyright Cambridge Journals 2012.

Martínez-Miranda, M. D., Nielsen, J. P. & Verrall, R. (2013a), 'Double chain ladder and Bornhuetter-Ferguson', *North American Actuarial Journal* **17**(2), 101–113.

McCullagh, P. & Nelder, J. (1989), *Generalized Linear Models, Second Edition*, Chapman & Hall/CRC Monographs on Statistics & Applied Probability, Taylor & Francis.

McQuarrie, A. & Tsai, C. (1998), *Regression and Time Series Model Selection*, World Scientific.

Meyers, G. G. & Shi, P. (2011), The retrospective testing of stochastic loss reserve models, *in* 'Casualty Actuarial Society E-Forum, Summer'.

Miranda, M. D. M., Nielsen, J. P., Verrall, R. & Wüthrich, M. V. (2015), 'Double chain ladder, claims development inflation and zero-claims', *Scandinavian Actuarial Journal* **2015**(5), 383–405.

Neuhaus, W. (1992), 'Another pragmatic loss reserving method or Bornhuetter-Ferguson revisited', *Scandinavian Actuarial Journal* **1992**(2), 151–162.

Norberg, R. (1986), 'A contribution to modelling of ibnr claims', *Scandinavian Actuarial Journal* **1986**(3-4), 155–203.

Norberg, R. (1993), 'Prediction of outstanding liabilities in non-life insurance', *Astin Bulletin* **23**(01), 95–115.

Norberg, R. (1999), 'Prediction of outstanding liabilities ii. model variations and extensions', *ASTIN Bulletin* **29**, 5–25.

R Core Team (2016), *R: A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria.
URL: *https://www.R-project.org/*

Renshaw, A. & Verrall, R. (1998), 'A stochastic model underlying the chain-ladder technique', *British Actuarial Journal* **4**, 903–923.

Schmidt, K. D. (2015), 'A bibliography on loss reserving'.
URL: *http://www.math.tu-dresden.de/sto/schmidt/dsvm/reserve.pdf*

Tarbell, T. F. (1934), Incurred but not reported claim reserves, *in* 'Proceedings of the Casualty Actuarial Society', Vol. 20, pp. 275–280.
URL: *http://www.casact.org/pubs/proceed/proceed33/33275.pdf*

Tashman, L. J. (2000), 'Out-of-sample tests of forecasting accuracy: an analysis and review', *International Journal of Forecasting* **16**(4), 437–450.

Taylor, G. C. (1977), 'Separation of inflation and other effects from the distribution of non-life insurance claim delays', *ASTIN Bulletin: The Journal of the International Actuarial Association* **9**(1-2), 219–230.

Taylor, G. & McGuire, G. (2004), Loss reserving with GLM: A case study, *in* '2004 Discussion Paper Program - Applying and Evaluating Generalized Linear Models Including Research', pp. 327–392.
URL: *https://www.casact.org/pubs/dpp/dpp04/04dpp.pdf*

Verrall, R., Nielsen, J. P. & Jessen, A. H. (2010), 'Prediction of rbns and ibnr claims using claim amounts and claim counts', *ASTIN Bulletin* **40**, 871–887.

Whitney, A. W. (1918), Theory of experience rating, *in* 'Proceedings of the Casualty Actuarial Society', Vol. 4, pp. 274–292.
URL: *https://www.casact.org/pubs/proceed/proceed17/17274.pdf*

Wright, T. S. (1990), 'A stochastic method for claims reserving in general insurance', *Journal of the Institute of Actuaries* **117**, 677–731.

Wüthrich, M. & Merz, M. (2008), *Stochastic claims reserving methods in insurance*, Wiley finance series, John Wiley & Sons.