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Spatial Risk Measures and Applications to Max-Stable Processes

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Abstract

The risk of extreme environmental events is of great importance for both the authorities and the insurance industry. This paper concerns risk measures in a spatial setting, in order to introduce the spatial features of damages stemming from environmental events into the measure of the risk. We develop a new concept of spatial risk measure, based on the spatially aggregated loss over the region of interest, and propose an adapted set of axioms which quantify the sensitivity of the risk measure with respect to space and are linked to spatial diversification in particular. In order to model the loss underlying our definition of spatial risk measure, we apply a damage function to the environmental variable considered. In our examples, the latter is assumed to follow a max-stable process, very well suited to the modeling of extreme spatial events. The damage function considered is adapted to heatwaves. The theoretical properties of the resulting examples of spatial risk measures are studied and some interpretations in terms of insurance are provided.

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1 Introduction

It is of prime importance for both authorities and (re)insurance companies to take the spatial features of environmental risks into account. For the authorities, it is crucial to be able to detect the areas at risk: is it safe to build houses in a given area or would it be better somewhere else? Similarly, an insurance/reinsurance company has to choose its geographical zone of activity and its portfolio size, which is obviously related to spatial diversification. Thus, tools (and especially risk measures) capable of quantitatively dealing with spatial diversification are needed.

The notion of risk measure has been widely studied in the literature. A risk measure Π is defined as a mapping from a set of bounded random variables (typically a cone) to the real numbers, such that certain axioms are satisfied. The seminal paper by Artzner et al. (1999) introduced the concept of coherent risk measure, which was then generalized to the convex case by Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002). This static framework for risk measures was then extended to the conditional and dynamic setting. For a detailed review of conditional and dynamic risk measures, we refer to Acciaio and Penner (2011). The most often used risk measure in the regulatory context is Value-at-Risk (VaR).[1]

The above-mentioned risk measures are univariate. In $\mathbb{R}$, the natural order allows the notion of quantile and therefore VaR to be easily defined. However, in dimensions higher than 2, the lack of such a natural order makes straightforward generalizations non-trivial. This is why many different definitions of multivariate quantiles have emerged in the literature. In particular, these include multivariate quantile functions based on depth functions (see e.g. Zuo and Serfling, 2000), multivariate quantiles based on norm minimization, also called geometric quantiles (see e.g. Chaudhuri, 1996) and multivariate quantiles as inversions of mappings (see e.g. Koltchinskii, 1997). For a detailed review of multivariate quantiles, we refer to Serfling (2002). For the extension of VaR to a multivariate setting, see e.g. Embrechts and Puccetti (2006) and Cousin and Di Bernardino (2013).

To the best of our knowledge, only Föllmer (2014) and Föllmer and Klüppelberg (2014) use the expression *spatial risk measure*. At each node of a network of financial institutions, they carry out a local conditional risk assessment in the sense that the risk measure applied takes into account the situation at the other nodes. The main issue they raise is whether the local risk assessments can be aggregated consistently in order to provide a global risk measure.

Since risk measures were initially developed to deal with financial risks, they do not make explicit, in an insurance context, the influence of the region where the contracts were underwritten. However, in an insurance or a reinsurance portfolio, this particular region has an obvious impact on the risk undertaken by the company. In this paper, we introduce a new notion of spatial risk measure by explicitly disentangling the spatial region and the hazard that generates losses in this region. Then, we study how the measure of the risk is expected to evolve with respect to some of the features of the spatial region, such as its location and size. This leads to a set of axioms adapted to the spatial context. Contrary to the axioms proposed by Artzner et al. (1999), we study the sensitivity of the

[1] Although it would be more rigorous to call this risk measure a quantile, we will mainly use the term VaR since it now entered in the common language of applications in banking and insurance.

measure of the risk with respect to the space variable.

The analogy in a times-series context is the sensitivity of the measure of risk with respect to the time horizon and is referred to as the term structure of risk measures. It is linked to temporal diversification and is, of course, of interest to banks as well (re)insurance companies. Let $L_{t,t+h}$ be the loss of a financial institution within the period $[t, t+h]$. For example, the homogeneity property with respect to the time horizon involves comparing $\Pi(L_{t,t+\lambda h})$ and $\Pi(L_{t,t+h})$, for all $\lambda > 0$. The literature gives some results for this term structure in the particular cases of VaR and Expected Shortfall (ES). If the price variations are independent, identically distributed and Gaussian, we have $\text{VaR}_{t,t+\lambda}(\alpha) = \sqrt{\lambda}\,\text{VaR}_{t,t+1}(\alpha)$, where $\text{VaR}_{t,t+h}(\alpha)$ is the Value-at-Risk relative to the loss $L_{t,t+h}$, at level $\alpha > 0$. If the price variations follow an autoregressive process of order 1, an analytic expression is also available. Apart from these two cases, only a few offer a closed formula. Guidolin and Timmermann (2006) carry out a comparison of the term structure of risk measures such as VaR and ES under different econometric models; see also Embrechts et al. (2005). Except for quite simple models, closed formulas are not available and simulation methods are necessary.

Let us denote by $A \subset \mathbb{R}^2$ the region under consideration and by $\{C_P(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2}$ the process of the economic (or insured) cost due to a particular environmental hazard (e.g. wind). The easiest approach to build spatial risk measures is to integrate the classical existing risk measures (e.g. the variance or VaR) over A, i.e. to consider $\frac{1}{|A|}\int_A \Pi[C_P(\mathbf{x})]\, d\mathbf{x}$, where $|.|$ denotes the Lebesgue measure in $\mathbb{R}^2$. However, if the process C_P is stationary[2], then the distribution of $C_P(\mathbf{x})$ is independent of $\mathbf{x}$ and thus, for any risk measure Π, the previous quantity is equal to $\Pi[C(\mathbf{0})]$. The corresponding spatial risk measure reduces to the classical risk measure associated with a single site, meaning that this approach does not account for the spatial dependence structure of the cost process. In order to overcome this defect, we define our spatial risk measure by applying a univariate (static) risk measure to the normalized aggregated loss over A.

After having defined our notion of spatial risk measure and the corresponding set of axioms, we introduce a model for the cost process. This model involves a mapping of the environmental variable under consideration to an economic (or insured) loss via a damage function. In a context of climate change some extreme events tend to be more and more frequent; see e.g. SwissRe (2014). It is of prime importance for the authorities as well as for the insurance industry to assess the risk of natural disasters. A precise assessment of the risk of extreme events is crucial in order to satisfy capital requirements under the Solvency II regulatory framework. Therefore, due to the spatial feature of the environmental events, we model the process of the environmental variable using max-stable processes, which constitute an extension of the extreme value theory to the level of stochastic processes (de Haan, 1984; de Haan and Pickands, 1986; Resnick, 1987).

The remainder of the paper is organized as follows. Our concept of spatial risk measure and its corresponding set of axioms are introduced in Section 2. Then Section 3 describes our model for the economic (or insured) loss. Section 4 studies concrete examples of spatial risk measures adapted to heatwaves. Section 5 concludes. All proofs are gathered in the Appendix.

[2] In the whole paper, we consider strict stationarity.

2 Spatial risk measures

Let us denote by $\mathscr{A}$ the set of all measurable subsets of $\mathbb{R}^2$ with a positive and finite Lebesgue measure: $\mathscr{A} = \{A \text{ measurable} : A \subset \mathbb{R}^2, |A| > 0 \text{ and } |A| < \infty\}$. Denote by $\mathscr{P}$ a family of distributions of stochastic processes on $\mathbb{R}^2$ having continuous sample paths. Each process represents the economic or insured cost (also referred to as loss) caused by the events belonging to specified classes and occurring during a given time period, say $[0, T_L]$. In the following, T_L is considered as fixed and does not appear subsequently in the notations for reasons of neatness. The events considered here have a spatial extent and thus it is natural to consider a loss process on $\mathbb{R}^2$. Each class of events (e.g. heatwave, hurricane, earthquake, hail storm) will be referred to as a hazard in the following. Let $\mathscr{L}$ be the set of all positive-valued and bounded random variables and $\mathscr{L}_\Pi$ the set of all real-valued and bounded random variables, both defined on a measurable space $(\Omega, \mathscr{F})$. A risk measure typically will be a function of the type $\Pi : \mathscr{L}_\Pi \mapsto \mathbb{R}$.

2.1 Definitions

We first give the definition of the spatially aggregated loss, which allows the contribution of space and that of hazards to be disentangled. In the case of an insurance company, the total loss in a portfolio of risks depends on both the region where the policies have been underwritten and the hazards covered in these policies.

Definition 1 (Spatially aggregated loss)**.** *For $A \in \mathscr{A}$ and $P \in \mathscr{P}$, the spatially aggregated loss over A associated with the hazards generating financial costs characterized by P is defined as follows:*

$$L(A, P) = \int_A C_P(\mathbf{x})\, d\mathbf{x}, \tag{1}$$

where the stochastic process $\{C_P(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2}$ has distribution P.

The integral (1) exists since $|A| < \infty$ and the process C_P has continuous sample paths. Moreover, $L(A, P) \in \mathscr{L}$ due to the stochastic and positive nature of the process C_P. The random variable $L(A, P)$ corresponds to the total economic (or insured) loss over region A due to specified hazards and is therefore of interest for spatial risk management. It seems more relevant, for both theoretical study and practical interpretation, to consider the normalized version of the spatially aggregated loss.

Definition 2 (Normalized spatially aggregated loss)**.** *For $A \in \mathscr{A}$ and $P \in \mathscr{P}$, the normalized spatially aggregated loss is defined by*

$$L_N(A, P) = \frac{\int_A C_P(\mathbf{x})\, d\mathbf{x}}{|A|}, \tag{2}$$

where the stochastic process $\{C_P(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2}$ has distribution P.

The normalized spatially aggregated loss is a loss per surface unit and can be interpreted in a discrete setting as the loss per insurance policy.
Using the concept introduced in Definition 2 , we now define our notion of spatial risk measure, which makes the contribution of space to risk measurement explicit.

Definition 3 (Spatial risk measure)**.** *A spatial risk measure is a function $\mathcal{R}_\Pi$ that associates a real number with any region $A \in \mathscr{A}$ and any distribution $P \in \mathscr{P}$:*

$$\begin{array}{rccc} \mathcal{R}_\Pi & : & \mathscr{A} \times \mathscr{P} & \rightarrow & \mathbb{R} \\ & & (A,P) & \mapsto & \mathcal{R}_\Pi(A,P) = \Pi[L_N(A,P)], \end{array}$$

where $L_N(A,P)$ is defined in (2)*.*

If the distribution P of the economic (or insured) loss process is given, then the function $\mathcal{R}_\Pi(.,P)$ summarizes the risk caused by the hazards characterized by P for any region belonging to $\mathscr{A}$. In the following, $\mathcal{R}_\Pi(.,P)$ will be referred to as the spatial risk measure induced by P. The above definition takes the spatial dependence structure of the process C_P into account, except in the trivial case of the expectation.
It now appears natural to analyze how $\mathcal{R}_\Pi(A,P)$ evolves with respect to A for a given P. Several desirable properties of $\mathcal{R}_\Pi(.,P)$ are described in the set of axioms presented below. The spatial properties of $\mathcal{R}_\Pi(.,P)$ depend on both the risk measure Π (variance, VaR, ES, ...) and the probabilistic properties of the economic loss process characterized by P.

2.2 A set of axioms for spatial risk measures

This section provides a set of axioms in the context of the spatial risk measures introduced above. These axioms concern the spatial risk measure properties with respect to space and not to economic loss distribution, the latter being considered as given by the problem at hand.

Definition 4 (Set of axioms for spatial risk measures)**.** *For a fixed $P \in \mathscr{P}$, we define the following axioms for the spatial risk measure induced by P:*

1. ***Spatial invariance under translation:***
   For all $\mathbf{v} \in \mathbb{R}^2$ and $A \in \mathscr{A}$, $\mathcal{R}_\Pi(A+\mathbf{v},P) = \mathcal{R}_\Pi(A,P)$, where $A+\mathbf{v}$ denotes the region A translated by the vector $\mathbf{v}$;

2. ***Spatial sub-additivity:***
   For all $A_1, A_2 \in \mathscr{A}$, $\mathcal{R}_\Pi(A_1 \cup A_2, P) \leq \min[\mathcal{R}_\Pi(A_1,P), \mathcal{R}_\Pi(A_2,P)]$;

3. ***Asymptotic spatial homogeneity of order $-\boldsymbol{\alpha}, \boldsymbol{\alpha} > \mathbf{0}$:***
   For all $A \in \mathscr{A}$,
   $$\mathcal{R}_\Pi(\lambda A, P) \underset{\lambda \to \infty}{=} K_1 + \frac{K_2}{\lambda^\alpha} + o\left(\frac{1}{\lambda^\alpha}\right), \tag{3}$$
   where λA is the area obtained by applying a homothety of rate λ to A with respect to its center and $K_1, K_2 \in \mathbb{R}$. Note that K_1 and K_2 can depend on A;

4. ***Spatial anti-monotonicity:***
   For all $A_1, A_2 \in \mathscr{A},\ A_1 \subset A_2 \Rightarrow R_\Pi(A_2, P) \le R_\Pi(A_1, P)$.

It is easy to derive the two following statements.

Proposition 1. *The properties of spatial sub-additivity and spatial anti-monotonicity are equivalent.*

Proposition 2. *In the case of a stationary process* C_P*, there is spatial invariance under translation.*

As stated in Proposition 2, the axiom of spatial invariance under translation is natural when the process $\{C_P(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2}$ is stationary. Our spatial sub-additivity axiom means that the risk associated with the normalized spatially aggregated loss is lower when considering the union of two regions instead of only one of these. It indicates that there is spatial diversification, which appears as a natural property when the cost process is stationary. If this axiom is satisfied, an insurance company would be well advised to underwrite policies in both regions A_1 and A_2 since it decreases its risk per policy. Obviously, the spatial anti-monotonicity axiom is also linked to the concept of spatial diversification. As we will see below, the axiom of asymptotic spatial homogeneity of order $-\alpha$ can be satisfied especially if Π is the variance or VaR. This axiom constitutes a suggestion of spatial diversification behavior but other types of homogeneity properties could be introduced.
Although there are some links between our notion of spatial risk measures and financial risk measures as for instance summarized in Föllmer and Schied (2004), the inclusion of space and the C_p process in Definition 3 sets our approach rather aside.
In order to propose concrete examples of spatial risk measures, we need a model for the economic loss process C_P. Such a model is developed in the following section.

3 A model for the economic (or insured) loss process

3.1 The model

Our model for the economic loss process $\{C_P(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2}$ requires two components. The first concerns the loss generating hazards. We assume that the economic loss is only due to a unique class of events, i.e. to a unique hazard. In the following, we consider a natural hazard (e.g. a heatwave, a windstorm or an earthquake) described by the stochastic process of an environmental variable (temperature, wind speed and magnitude respectively), denoted by $\{Z(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2}$. We assume that Z is representative of the risk during the entire period $[0, T_L]$.
The second component involves a model mapping the natural hazard to damage and thus an economic cost. This model requires both the destruction percentage and the exposure at each location. The destruction percentage is obtained by applying a damage function (also referred to as a vulnerability curve in the literature), denoted by $D(.)$, to the natural hazard. The damage function is specific to the type of hazard considered. The exposure process, denoted by $\{E(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2}$, can be considered as deterministic and involves the demographic, economic and topographic conditions in particular. If insured losses are of interest, then the penetration rate of insurance should also be

taken into account. Finally, the destruction percentage must be multiplied by the exposure, yielding the following model for the economic loss at location $\mathbf{x}$:

$$C_P(\mathbf{x}) = E(\mathbf{x})\ D\left[Z(\mathbf{x})\right]. \tag{4}$$

Note that the model in (4) is only a special case of a wider issue. Many generalizations are possible such as the introduction of a dependence of $D(.)$ with respect to the location in order to account for the type of building at location $\mathbf{x}$ as well as the explicit introduction of the temporal aspect. These sophistications will be considered in subsequent work.

Remark 1. *The presence of P in the right-hand term of (4) is implicit : the distribution P of the process $C_P(\mathbf{x})$ indeed depends on the three components of the right-hand term.*

Remark 2. *The model in (4) involves only one environmental variable. However, in the case of hurricanes, damage to buildings depends on both the wind speed and the rainfall. Therefore, in this case, two stochastic processes $\{Z_1(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2}$ and $\{Z_2(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2}$ should be considered.*

Due to the complexity of computations in the following, we consider that the exposure is uniformly equal to unity. Finally, the model for the economic loss reduces to

$$C_P(\mathbf{x}) = D\left[Z(\mathbf{x})\right]. \tag{5}$$

In Section 4, we will consider the following damage function:

$$D[Z(\mathbf{x})] = \mathbf{I}_{\{Z(\mathbf{x})>u\}}, \text{ where } u > 0, \tag{6}$$

which is adapted to heatwaves. Note that this function does not correspond exactly to a destruction percentage in the sense that there is not complete destruction when $Z(\mathbf{x}) > u$. It should be scaled in order to represent a realistic destruction percentage. We could consider $D[Z(\mathbf{x})] = \dfrac{\mathbf{I}_{\{Z(\mathbf{x})>u\}}}{10}$ for instance. However, there is no loss of generality if the scaling factor is taken to be equal to 1.
An important focus of our paper is the modeling of economic losses stemming from extreme events. This is particularly relevant for both the authorities and insurance and reinsurance companies. Hence, at each point in space, we consider $Z(\mathbf{x})$ to be the temporal maximum of the considered environmental variable at location $\mathbf{x}$. As explained in the next section, in this case, max-stable processes (de Haan, 1984; de Haan and Pickands, 1986; Resnick, 1987) are ideally suited for modeling purposes. For the remainder of the paper, we will assume the process $\{Z(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2}$ to be max-stable.

Definition 5. *(Max-stable process). For $d \in \mathbb{N}^*$, a stochastic process $\{G(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}$ with continuous sample paths is said to be max-stable if there are sequences of continuous functions $\{a_T(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d} > 0$ and $\{b_T(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}$ such that if $G_t(\mathbf{x})$, $t = 1, \ldots, T$, are independent replications of $G(\mathbf{x})$,*

$$\left\{\frac{\max_{t=1}^{T} G_t(\mathbf{x}) - b_T(\mathbf{x})}{a_T(\mathbf{x})}\right\}_{\mathbf{x}\in\mathbb{R}^d} \overset{d}{=} \{G(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d},$$

where $\overset{d}{=}$ denotes equality in distribution.

Max-stable processes play an important role at the crossroads of geostatistics and extreme value theory. The next section provides a description of their main features.

3.2 A short introduction to max-stable processes

3.2.1 Motivation

Let us consider independent replications $T_i(\mathbf{x}), i = 1,\ldots,n,$ of a stochastic process $\{T(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}$ having continuous sample paths. In the case where $d \in \{1,2,3\}$, $T(\mathbf{x})$, for instance, can be an environmental variable. Let $\{c_n(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d} > 0$ and $\{d_n(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}$ be sequences of continuous functions. It can then be shown that if there exists a non degenerate process $\{G(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}$ such that

$$\left\{\frac{\max_{i=1}^n T_i(\mathbf{x}) - d_n(\mathbf{x})}{c_n(\mathbf{x})}\right\}_{\mathbf{x}\in\mathbb{R}^d} \xrightarrow{d} \{G(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}, \text{ for } n\to\infty, \tag{7}$$

then $G(\mathbf{x})$ is necessarily max-stable. Therefore, max-stable processes are very well suited to characterize the joint behavior of the temporal maxima at all points in space. Choosing our environmental process $\{Z(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2}$ to be max-stable implies that the loss given by (1) corresponds to the spatial aggregation of the losses caused by the worst events happening at each location during the time period considered.
For practical purposes, the number n of observations for which the maxima are taken depends on the length T_L of the time period considered. We assume that the limit in (7) has been reached. This assumption does, of course, need statistical justification in the examples analyzed. Classically, $T_L = 1$ year and $n = 365$. However, an insurer can be more interested in the losses due to a specific event than that corresponding to the worst event of the year. In that case, we can for instance take $T_L = 1$ week.

3.2.2 Comments on the definition

As a direct consequence of Definition 5, the one-dimensional marginal distributions are max-stable and hence belong to the class of generalized extreme value distributions (GEV):

$$\forall \mathbf{x}\in\mathbb{R}^d,\ G(\mathbf{x}) \sim GEV\left[\mu(\mathbf{x}),\sigma(\mathbf{x}),\xi(\mathbf{x})\right],$$

where $\{\mu(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}$, $\{\sigma(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}$ and $\{\xi(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}$ are respectively the deterministic processes of location, scale and shape parameters. For a detailed review of extreme value theory, we refer to Resnick (1987), Embrechts et al. (1997), Coles (2001), Beirlant et al. (2006) and de Haan and Ferreira (2007).
In this paper, the margins are assumed to be standard Fréchet, i.e. for all $\mathbf{x}\in\mathbb{R}^d$, for all $z > 0$, $\mathbb{P}(Z(\mathbf{x}) \le z) = \exp\left(-\frac{1}{z}\right)$; see e.g. Smith (1990). A max-stable process having standard Fréchet margins will be referred to as a simple max-stable process in the following.

3.2.3 Spectral representations and max-stable models

There are currently two main types of spectral representation of max-stable processes. The first, essentially due to de Haan (1984), is based on moving maxima processes. The second, essentially due to Penrose (1992) and Schlather (2002), involves maxima on stochastic processes. For an arbitrary set B, we denote by $C^+(B)$ the set of positive processes having continuous sample paths on B.

First representation:

Theorem 1 (Theorem 9.6.7 in de Haan and Ferreira (2007), p.316)**.** *Let $\{(\xi_i, c_i)\}_{i\geq 1}$ be the points of a Poisson point process on $(0,\infty] \times [0,1]$ with intensity measure $d\Lambda(\xi, c) = \xi^{-2} d\xi dc$, where ν is the Lebesgue measure on $[0,1]$. If Z is a simple max-stable process in $C^+(\mathbb{R})$, then there exists a family of functions $f_x(c)$ $(x \in \mathbb{R}, c \in [0,1])$ with*

- *for each $c \in [0,1]$, we have a non-negative continuous function $f_x(c) : \mathbb{R} \to [0,\infty)$;*
- *for each $x \in \mathbb{R}$, $\int_0^1 f_x(c)\ dc = 1$,*
- *for each compact interval $K \subset \mathbb{R}$, $\int_0^1 \sup_{x\in K} f_x(c)\ dc < \infty$,*

such that

$$\{Z(x)\}_{x\in\mathbb{R}} \stackrel{d}{=} \left\{\max_{i\geq 1} \xi_i f_x(c_i)\right\}_{x\in\mathbb{R}}.$$

Theorem 2 (Smith (1990))**.** *Let S be an arbitrary index set. Let $\{(\xi_i, \mathbf{c}_i)\}_{i\geq 1}$ be the points of a Poisson point process on $(0,\infty) \times C$ with intensity measure $d\Lambda(\xi, \mathbf{c}) = \xi^{-2} d\xi \nu(d\mathbf{c})$, where C is an arbitrary measurable set and ν is a σ-finite measure on C. Let $\{f_{\mathbf{x}}(\mathbf{c}), \mathbf{x} \in S, \mathbf{c} \in C\}$ be a non-negative function for which*

$$\forall \mathbf{x} \in S,\ \int_0^1 f_{\mathbf{x}}(\mathbf{c})\ d\mathbf{c} = 1. \tag{8}$$

Then, the process

$$\{Z(\mathbf{x})\}_{\mathbf{x}\in S} = \left\{\max_{i\geq 1} \xi_i f_{\mathbf{x}}(\mathbf{c}_i)\right\}_{\mathbf{x}\in S}$$

is a simple max-stable process.

Second representation:

Theorem 3 (Corollary 9.4.5 in de Haan and Ferreira (2007), p.307)**.** *Let $\{\xi_i\}_{i\geq 1}$ be the points of a Poisson point process on $(0,\infty]$, with intensity $d\Lambda(\xi) = \xi^{-2} d\xi$ and $Y_1, Y_2, \ldots$ independent replications of a stochastic process Y in $C^+[0,1]$ satisfying $\mathbb{E}[Y(x)] = 1$ for all $x \in [0,1]$ and $\mathbb{E}[\sup_{0\leq x\leq 1} Y(x)] < \infty$. Let the point process and the sequence Y_1, Y_2, ... be independent. Then, if Z is a simple max-stable process in $C^+[0,1]$, we have*

$$\{Z(x)\}_{x\in[0,1]} \stackrel{d}{=} \left\{\max_{i\geq 1} \xi_i Y_i(x)\right\}_{x\in[0,1]}.$$

Theorem 4 (Schlather (2002)). *Let* $\{\xi_i\}_{i\geq 1}$ *be the points of a Poisson point process on* $(0,\infty)$*, with intensity* $d\Lambda(\xi) = \xi^{-2}d\xi$ *and* $Y_1, Y_2, \ldots$ *independent replications of a stationary stochastic process* Y *on* $\mathbb{R}^d$ *satisfying* $\mathbb{E}\{\max[0, Y(\mathbf{0})]\} = 1$. *Then*

$$\{Z(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d} = \left\{\max_{i\geq 1} \xi_i Y_i(\mathbf{x})\right\}_{\mathbf{x}\in\mathbb{R}^d}$$

is a stationary simple max-stable process.

These two representations have led to different models for max-stable processes, presented in the following.

The Smith model:
Smith (1990) uses Theorem 2 to provide a parametric model for max-stable processes. He considers a particular setting where $S = C = \mathbb{R}^d$, ν is the Lebesgue measure on $\mathbb{R}^d$ and $f_{\mathbf{x}}(\mathbf{c}) = f_\Sigma(\mathbf{x}-\mathbf{c})$, where f_Σ is the density of a d-variate normal law with mean $\mathbf{0}$ and covariance matrix Σ:

$$f_{\mathbf{x}}(\mathbf{c}) = f_\Sigma(\mathbf{x}-\mathbf{c}) = (2\pi)^{-\frac{d}{2}}|\Sigma|^{-\frac{d}{2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{c})'\Sigma^{-1}(\mathbf{x}-\mathbf{c})\right].$$

The parameter is the covariance matrix Σ, which contains all the information about the spatial dependence structure. A nice feature of this model lies in its interpretation in terms of rainfall-storm processes (Smith, 1990), the shape of these storms being driven by the covariance matrix. Moreover, in the case $d = 2$, the trivariate density (the density of an observation at 3 sites) can be explicitly written (see e.g. Genton et al., 2011) unlike the Schlather model below.

The Schlather model:
Schlather (2002) proposes to set $Y(\mathbf{x}) = \sqrt{2\pi}\epsilon(\mathbf{x})$ in Theorem 4, where $\{\epsilon(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}$ is a stationary standard Gaussian process with any correlation function $\rho(.)$. All correlation functions stemming from the geostatistical literature can be used, allowing for a rich diversity of behaviors. We will contrast and compare the following correlation families:

Whittle-Matern:	$\rho(h) = \frac{2^{1-c_2}}{\Gamma(c_2)}\left(\frac{h}{c_1}\right)^{c_2} K_{c_2}\left(\frac{h}{c_1}\right),\ c_1 > 0,\ c_2 > 0,$
Exponential:	$\rho(h) = \exp\left[-\frac{h}{c_1}\right],\ c_1 > 0,$
Cauchy:	$\rho(h) = \left[1 + \left(\frac{h}{c_1}\right)^2\right]^{-c_2},\ c_1 > 0,\ c_2 > 0,$
Powered exponential:	$\rho(h) = \exp\left[-\left(\frac{h}{c_1}\right)^{c_2}\right],\ c_1 > 0,\ 0 < c_2 < 2,$

where c_1 and c_2 are the range and smoothing parameters, Γ is the Gamma function and K_{c_2} is the modified Bessel function of the third kind of order c_2.

The geometric Gaussian model:
Independence is unreachable in the case of the Schlather model (see Section 4). To deal with this issue, Davison (2003) introduces the geometric Gaussian model. In Theorem 4, he takes a log normal process and not a Gaussian process: $Y(\mathbf{x}) = \exp\left(\sigma\epsilon(\mathbf{x}) - \frac{\sigma^2}{2}\right)$, where $\{\epsilon(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}$ is a standard Gaussian process with variance σ^2 and correlation function $\rho(.)$.

The Brown-Resnick model:
The geometric Gaussian process is a particular case of a model introduced by Brown and Resnick (1977). Kabluchko et al. (2009) introduce a generalization of the latter model, which they refer to as the Brown-Resnick model, by taking in Theorem 4 $Y(\mathbf{x}) = \exp\left(W(\mathbf{x}) - \frac{\sigma^2(\mathbf{x})}{2}\right)$, where $\{W(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}$ is a zero-mean Gaussian process with stationary increments and $\sigma^2(\mathbf{x}) = \text{Var}[W(\mathbf{x})]$, for all $\mathbf{x}\in\mathbb{R}^d$, where Var is the variance. The process W and therefore the resulting Brown-Resnick process are completely characterized by the variance $\sigma(\mathbf{x})$ and the semi-variogram, defined by

$$\gamma(\mathbf{h}) = \frac{1}{2}\text{Var}[W(\mathbf{x}+\mathbf{h}) - W(\mathbf{x})], \forall \mathbf{h}\in\mathbb{R}^d.$$

It should be noted that both the Smith process and the geometric Gaussian process are particular cases of the Brown-Resnick process. This is clear in the case of the geometric Gaussian process since the standard Gaussian process has stationary increments. For the Smith process, see e.g. Yuen and Stoev (2013).

3.2.4 Extremal coefficient

The extremal coefficient (Schlather and Tawn, 2003) is a measure of spatial dependence for max-stable processes and will play an important role in the study of concrete examples of spatial risk measure in Section 4. In the case of M locations $(\mathbf{x}_1,\ldots,\mathbf{x}_M)$, it is denoted $\Theta(\mathbf{x}_1,\ldots,\mathbf{x}_M)$ and is defined by

$$\mathbb{P}\left(Z(\mathbf{x}_1)\le u,\ldots,Z(\mathbf{x}_M)\le u\right) = \exp\left(-\frac{\Theta(\mathbf{x}_1,\ldots,\mathbf{x}_M)}{u}\right).$$

In the case $M=2$, if Z is stationary, then $\Theta(\mathbf{x}_1,\mathbf{x}_2)$ only depends on the vector $\mathbf{h}=\mathbf{x}_1-\mathbf{x}_2$ and is denoted by $\Theta(\mathbf{h})$. If Z is isotropic, $\Theta(\mathbf{x}_1,\mathbf{x}_2)$ only depends on $h=\|\mathbf{h}\|$, the Euclidean distance between sites $\mathbf{x}_1$ and $\mathbf{x}_2$, and is denoted by $\Theta(h)$.

3.2.5 Ergodicity, mixing properties and extremal coefficient

The spatial diversification results presented in Section 4 are expressed in terms of the extremal coefficient. Moreover, spatial diversification is linked with the notions of ergodicity and mixing. Thus, in order to link our results with the existing literature concerning the mixing properties of max-stable processes, let us briefly discuss the established links between the extremal coefficient behavior and the properties of ergodicity and mixing.

We first recall basic concepts of ergodicity and mixing. Let us consider a stationary stochastic process $\{R(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}$ such that $\mathbb{E}[R(\mathbf{x})] < \infty$ and denote $\mathscr{F}_A$ the σ-field generated by the random variables $\{R(\mathbf{x}) : \mathbf{x} \in A\}$, for a region $A \subset \mathbb{R}^d$.

Definition 6 (Mean-ergodicity)**.** *The stochastic process $\{R(\mathbf{x},\omega)\}_{\mathbf{x}\in\mathbb{R}^d}$ is said to be mean-ergodic if, for $A \subset \mathbb{R}^d$,*

$$\lim_{|A|\to\infty} \mathbb{E}\left[\left(\frac{1}{|A|}\int_A R(\mathbf{x},\omega)\ d\mathbf{x} - \mu\right)^2\right] = 0,$$

where $\mu = \mathbb{E}(R)$.

Definition 7 (Strong mixing)**.** *The α-mixing coefficient (or strong mixing coefficient according to Rosenblatt (1956)) between the σ-fields $\mathscr{F}_{A_1}$ and $\mathscr{F}_{A_2}$ is defined by*

$$\alpha(A_1, A_2) = \sup\{|\mathbb{P}(S_1 \cap S_2) - \mathbb{P}(S_1)\mathbb{P}(S_2)| : S_1 \in \mathscr{F}_{A_1}, S_2 \in \mathscr{F}_{A_2}\}.$$

The process R is said to be strongly mixing if $\alpha(A_1, A_2)$ tends to 0 as $d(A_1, A_2)$ tends to ∞, where $d(A_1, A_2)$ is the distance between A_1 and A_2.

Other equivalent definitions can be found, e.g. in Kabluchko and Schlather (2010) (Definition 2.1, Bullet 3). Their definition in the case of $\mathbb{R}$ can be easily extended to $\mathbb{R}^d$.

Stoev (2010) uses the extremal integral representation to derive necessary and sufficient conditions for mixing of max-stable processes. Kabluchko and Schlather (2010) extend these results to the class of max-infinitely divisible processes, that encompasses max-stable processes. In the max-stable case, they introduce the dependence coefficient $\{r(h)\}_{h\in\mathbb{R}}$ defined by $r(h) = 2 - \Theta(h)$, where $\Theta(h)$ is the extremal coefficient. Their Theorem 3.1 states that for a stationary measurable simple max-stable process $\{Z(x)\}_{x\in\mathbb{R}}$, Z is strongly mixing if and only if $\lim_{h\to\infty} r(h) = 0$. In their Theorem 3.2, they show that Z is ergodic if and only if $\lim_{l\to\infty} \dfrac{1}{l}\displaystyle\int_1^l r(h)\ dh = 0$. These results could be extended to $\mathbb{R}^d$.

To close this section, let us consider the process $\{H(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d} = \{D[Z(\mathbf{x})]\}_{\mathbf{x}\in\mathbb{R}^d}$, where D is any function.

Lemma 1. *If the process Z is strongly mixing (respectively ergodic), then the process H is strongly mixing (respectively ergodic).*

This means that the mixing (respectively ergodic) properties of the environmental process are also valid for the economic loss process.

4 Examples based on the threshold damage function

In this section, we consider the normalized spatially aggregated loss obtained by combining (2), (5) and (6):

$$L_N(A, P) = \frac{1}{|A|}\int_A \mathbf{I}_{\{Z(\mathbf{x})>u\}}\ d\mathbf{x}, \tag{9}$$

where $\{Z(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2}$ is a simple max-stable process and u is a positive threshold. This quantity is particularly interesting when analyzing, for instance, the impact of high temperatures on populations and on the distribution network of electricity, typically as in the case of the European heatwave in 2003. If the threshold is well chosen, $L_N(A,P)$ represents indeed the proportion of the surface area at which the temperature exceeds a dangerous threshold for populations and electric cables.

Note that the spatially aggregated loss $\int_A \mathbf{I}_{\{Z(\mathbf{x})>u\}}\, d\mathbf{x}$ corresponds to the area, or the so-called intrinsic volume, of the excursion set $E_u(Z,A) = \{\mathbf{x}\in A : Z(\mathbf{x}) \geq u\}$. Excursion sets of stochastic processes have been widely studied in the literature. In particular, Lévy processes, diffusions, stable and Gaussian processes have been investigated; see e.g. Berman (1992), references in Ivanov et al. (2013) and Spodarev (2014) for an overview.

The dependence of $L_N(.,.)$ with respect to P lies in the distribution of Z. In the following, the different max-stable models introduced in Section 3.2.3 will be considered. In each case, the model used will be explicitly indicated. As a result, we make the dependence in P implicit in $L_N(A,P)$: from now on, $L_N(A,P)$ will be denoted $L_N(A)$. Likewise, the risk measure $R_\Pi(A,P)$ will be denoted $R_\Pi(A)$.

The case of expectation $R_1(A) = \mathbb{E}[L_N(A)]$ is trivial. Here only high losses are considered, so $R_1(.)$ can, for instance, be the actuarial premium of a reinsurance contract. Using the linearity property of the expectation, we immediately show that for all $A\in\mathscr{A}$, $R_1(A) = 1-\exp\left(-\frac{1}{u}\right)$, meaning that the premium does not depend on the region considered (and thus on its size). It stems directly from the fact that the process has standardized margins. This is similar to the case of an insurance portfolio composed of homogeneous risks. The expectation is not a very useful risk measure since it does not involve any information that is relative to the variability. Furthermore, due to linearity, it does not account for the spatial dependence of the loss process.

In the following, we study the case of variance in detail, before providing some insights concerning VaR.

4.1 The variance

In this section, we consider the quantity $R_2(A) = \mathrm{Var}[L_N(A)]$. The variance allows part of the spatial dependence to be taken into account in the risk assessment. Hence, its study is interesting for both the risk management of extreme spatial events and the understanding of some properties of max-stable processes. Moreover, variance is of prime interest for (re)insurance companies since it controls the variability of the normalized portfolio's loss around the expected one. As we will see, $R_2(.)$ is linked with the notions of spatial diversification, ergodicity and mixing.

In the following, our aim is to study whether $R_2(.)$ satisfies the axioms presented in Definition 4. Before doing so, we will study the function $\lambda \mapsto R_2(\lambda A)$ in detail. It is, of course, related to the property of asymptotic spatial homogeneity. However, we are also interested in the behavior of $R_2(\lambda A)$ for finite values of λ, which we refer to as spatial homogeneity. This study can be of practical relevance for the (re)insurance industry and the results obtained will be used to prove the axioms of spatial anti-monotonicity and spatial sub-additivity.

4.1.1 General results relating to spatial homogeneity

The expression of $R_2(\lambda A)$ in the general case is given in the following theorem.

Theorem 5. *In the case of a simple max-stable process, for all $A \in \mathscr{A}$ and $\lambda > 0$, we have*

$$R_2(\lambda A) = \frac{1}{\lambda^4 |A|^2} \int_{\lambda A} \int_{\lambda A} \left[\exp\left(-\frac{\Theta(\mathbf{x},\mathbf{y})}{u}\right) - \exp\left(-\frac{2}{u}\right) \right] \, d\mathbf{x} \, d\mathbf{y}. \tag{10}$$

From Theorem 5, we can derive the behavior of $R_2(\lambda A)$ in the case of isotropic max-stable processes, when A is either a disk or a square. The result is given in the next corollary.

Corollary 1. *Consider an isotropic simple max-stable process having $\{\Theta(h)\}_{h\in\mathbb{R}_+}$ as extremal coefficient function and $A \in \mathscr{A}$. Then:*

1. ***If $\mathbf{A}$ is a disk** with radius R, for all $\lambda > 0$, we have*

$$R_2(\lambda A) = -\exp\left(-\frac{2}{u}\right) + \int_0^{2R} f_d(h,R) \exp\left(-\frac{\Theta(\lambda h)}{u}\right) \, dh, \tag{11}$$

where f_d is the density of the distance between two points uniformly distributed on A, given by

$$f_d(h,R) = \frac{2h}{R^2}\left(\frac{2}{\pi}\arccos\left(\frac{h}{2R}\right) - \frac{h}{\pi R}\sqrt{1-\frac{h^2}{4R^2}}\right).$$

2. ***If $\mathbf{A}$ is a square** with side R, for all $\lambda > 0$, we have*

$$R_2(\lambda A) = -\exp\left(-\frac{2}{u}\right) + \int_0^{\sqrt{2}R} f_s(h,R) \exp\left(-\frac{\Theta(\lambda h)}{u}\right) \, dh, \tag{12}$$

where f_s is the density of the distance between two points uniformly distributed on A, given by:
For $h \in [0,R]$,

$$f_s(h,R) = \frac{2\pi h}{R^2} - \frac{8h^2}{R^3} + \frac{2h^3}{R^4}.$$

For $h \in [R, R\sqrt{2}]$,

$$f_s(h,R) = \left(-2 - b + 3\sqrt{b-1} + \frac{b+1}{\sqrt{b-1}} + 2\arcsin\left(\frac{2-b}{b}\right) - \frac{4}{b\sqrt{1-\frac{(2-b)^2}{b^2}}}\right)\frac{2h}{R^2},$$

where $b = \dfrac{h^2}{R^2}$.

3. ***In both cases***, $R_2(\lambda A)$ *converges as* $\lambda \to \infty$ *to the limiting risk measure given by*

$$-\exp\left(-\frac{2}{u}\right) + \lim_{\lambda\to\infty} \exp\left(-\frac{\Theta(\lambda h)}{u}\right). \tag{13}$$

Remark 3. *Expressions* (11) *and* (12) *could have the same structure for other types of region A (see the proof of Corollary 1). However, the densities f_d and f_s should be replaced with the appropriate density which can be computed using the approach described in Moltchanov (2012). However, this may not be obvious in some cases.*

The corollary below directly follows on from Corollary 1.

Corollary 2. *In the case of perfect dependence, i.e. for all $h \geq 0$, $\Theta(h) = 1$, we have*

$$R_2(A) = \exp\left(-\frac{1}{u}\right) - \exp\left(-\frac{2}{u}\right).$$

In the case of the max-stable models introduced in Section 3.2.3, the function $\Theta(\lambda h)$ is strictly increasing with respect to λ, giving that $R_2(\lambda A)$ is strictly decreasing with respect to λ in the cases of the disk and the square. Consequently, there is spatial diversification. Corollary 1 offers an interesting tool for the insurance industry since it allows the dimension of the geographical area required to reach a low variance level to be determined. It can be seen in the following corollary that, in some cases, diversification can be total.

Corollary 3. *In the case of asymptotic independence, i.e.* $\lim_{h\to\infty} \Theta(h) = 2$, *we have*

$$\lim_{\lambda\to\infty} R_2(\lambda A) = 0.$$

This result is not surprising. Indeed, as mentioned in Section 3.2.5, $\lim_{h\to\infty} \Theta(h) = 2$ implies that the process $\{Z(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2}$ is mixing, if we accept the extension from $\mathbb{R}$ to $\mathbb{R}^d$. Thus every transformation of Z is also mixing (see Lemma 1), giving that $\mathbf{I}_{\{Z(\mathbf{x})>u\}}$ is mixing and therefore mean-ergodic, which is equivalent to Corollary 3. In terms of insurance, Corollary 3 states that the spatial diversification can be total. If there is asymptotic spatial independence and if the insurance company can underwrite policies in a sufficiently large region, then the corresponding portfolio is "equivalent" to a portfolio containing i.i.d. risks.
Corollary 1 shows that the decrease of $R_2(\lambda A)$ as λ increases is mainly driven by the extremal coefficient $\Theta(.)$; the latter naturally depends on the max-stable model under consideration. Our aim in the following is to study the influence of the factor λ for different max-stable models. However, the integrals in Corollary 1 have no closed form so we use a Riemann approximation.

4.1.2 Spatial homogeneity for max-stable models already introduced in the literature

Using the results of Corollary 1, we study the behavior of the function $\lambda \mapsto R_2(\lambda A)$ in the case of the parametric models of max-stable processes introduced in Section 3.2.3. Without loss of generality, we set $R = 1$. The choice of the threshold u has no influence on the shape of the function $\lambda \mapsto R_2(\lambda A)$ and we choose $u = 1$.

The Smith model

For two locations $\mathbf{x}_1$ and $\mathbf{x}_2$, the extremal coefficient is given by

$$\Theta(\mathbf{x}_1 - \mathbf{x}_2) = 2\Phi\left(\frac{\sqrt{(\mathbf{x}_1 - \mathbf{x}_2)'\Sigma^{-1}(\mathbf{x}_1 - \mathbf{x}_2)}}{2}\right),$$

where Φ denotes the distribution function of the standard Gaussian random variable. The Smith process is isotropic only if Σ is proportional to the identity matrix. Without loss of generality, let Σ be the identity matrix. In this case, we have $\Theta(h) = 2\Phi\left(\frac{h}{2}\right)$. Hence, $\lim_{h\to\infty}\Theta(h) = 2$ and Corollary 3 gives that $\lim_{\lambda\to\infty} R_2(\lambda A) = 0$, meaning that the spatial diversification is total. We observe in Figure 1 that $R_2(\lambda A)$ rapidly decreases to the limiting risk measure when λ increases.

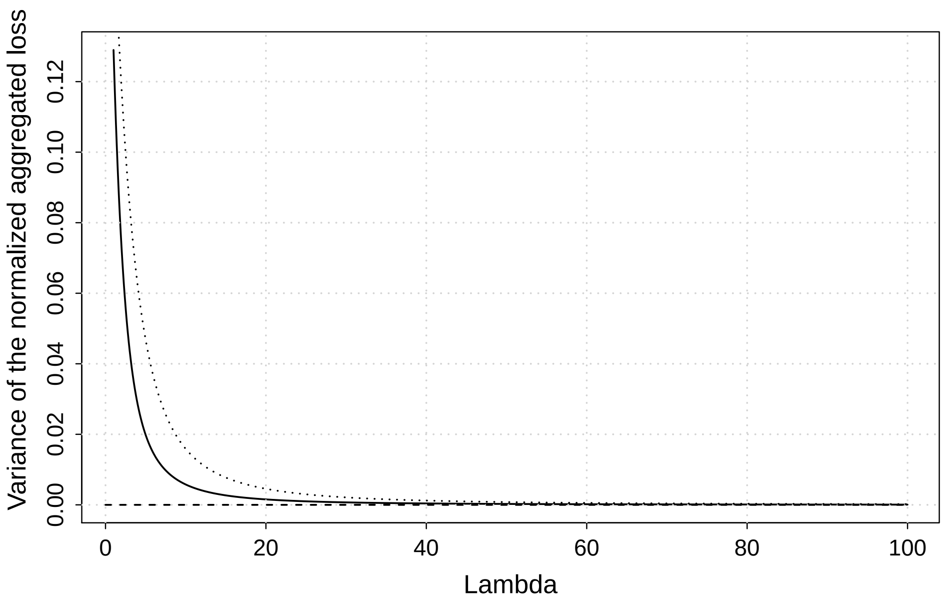

Figure 1: The solid (respectively dotted) line depicts the evolution of $R_2(\lambda A)$ with respect to λ in the case of the Smith model, where A is a disk (respectively a square). The dashed line represents the limiting risk measure.

The Schlather model

The extremal coefficient is given by

$$\Theta(\mathbf{x}_1 - \mathbf{x}_2) = 1 + \sqrt{\frac{1 - \rho(\mathbf{x}_1 - \mathbf{x}_2)}{2}}.$$

There is isotropy if and only if the correlation function ρ is isotropic. In that case, $\Theta(h) = 1 + \sqrt{\frac{1-\rho(h)}{2}}$. Thus, if $\lim_{h\to\infty}\rho(h) = 0$, we have $\lim_{\lambda\to\infty}\Theta(\lambda h) = 1 + \sqrt{\frac{1}{2}}$ and (13) yields that

$$\lim_{\lambda\to\infty} R_2(\lambda A) = -\exp\left(-\frac{2}{u}\right) + \exp\left(-\frac{1+\sqrt{\frac{1}{2}}}{u}\right),$$

which is different from zero.
The limiting risk measure is positive, showing that the process $\mathbf{I}_{\{Z(\mathbf{x})>u\}}$ is not mean-ergodic. This is consistent with the fact that the Schlather process is neither mixing nor ergodic (see Section 3.2.5). In terms of insurance, this result means that the spatial diversification is never total: there is always some kind of residual common risk factor.
We set the range parameter $c_1 = 1$ and the smoothing parameter $c_2 = 0.5$. Figure 2 shows that the

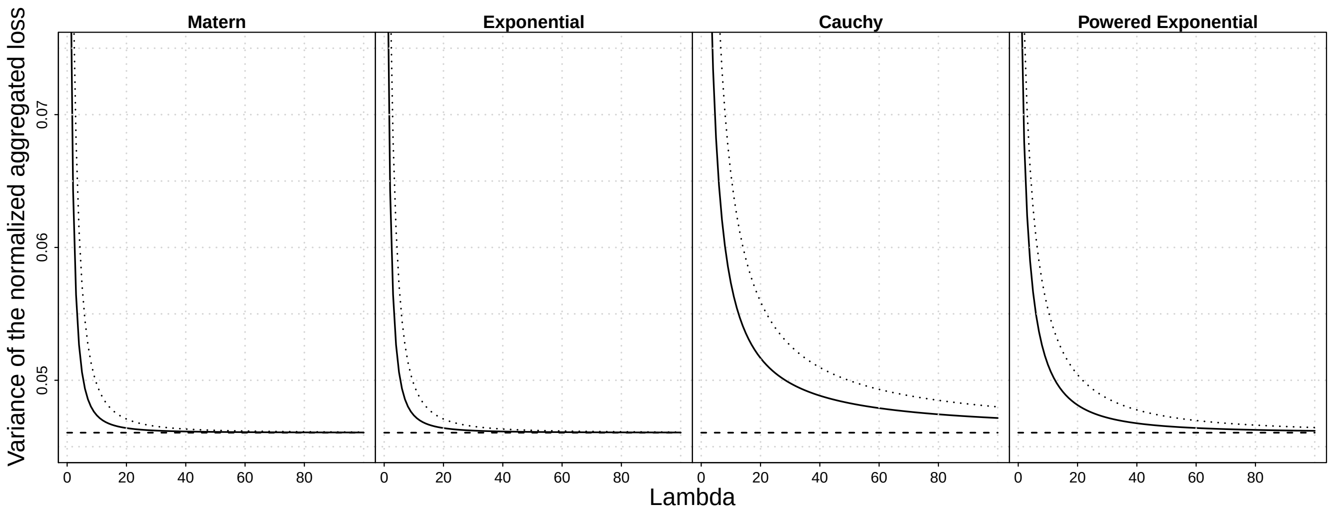

Figure 2: Each panel corresponds to a different correlation function. The solid (respectively dotted) line depicts the evolution of $R_2(\lambda A)$ with respect to λ in the case of the Schlather model, where A is a disk (respectively a square). The dashed line represents the limiting risk measure.

speed of decrease of $R_2(\lambda A)$ to the limiting risk measure depends on the type of correlation function. This decrease is much slower in the case of the Cauchy and powered exponential functions. Hence, the Schlather process allows for a large variety of spatial diversification behaviors.
On the whole, we observe that the decrease to the limiting risk measure is slower than in the case of the Smith model. This comparison must be made with equivalent characteristic distances of the spatial correlation, meaning that the eigenvalues of Σ must be equal to c_1, which is the case. Note that the decrease is obviously slower when increasing the range parameter c_1.

The geometric Gaussian model

The extremal coefficient is given by

$$\Theta(\mathbf{x}_1 - \mathbf{x}_2) = 2\Phi\left(\sqrt{\frac{\sigma^2[1-\rho(\mathbf{x}_1 - \mathbf{x}_2)]}{2}}\right).$$

Consequently, if the function ρ is isotropic, the extremal coefficient becomes $\Theta(h) = 2\Phi\left(\sqrt{\frac{\sigma^2[1-\rho(h)]}{2}}\right)$.

Therefore, $\lim_{\lambda\to\infty}\Theta(\lambda h) = 2\Phi\left(\sqrt{\frac{\sigma^2}{2}}\right)$, giving

$$\lim_{\lambda\to\infty} R_2(\lambda A) = -\exp\left(-\frac{2}{u}\right) + \exp\left(-\frac{2\Phi\left(\sqrt{\frac{\sigma^2}{2}}\right)}{u}\right),$$

which is different from zero.
As previously, we set $c_1 = 1$ and $c_2 = 0.5$. From Figure 3, we draw very similar conclusions to those related to the Schlather model. The only difference consists in the value of the limiting risk measure.

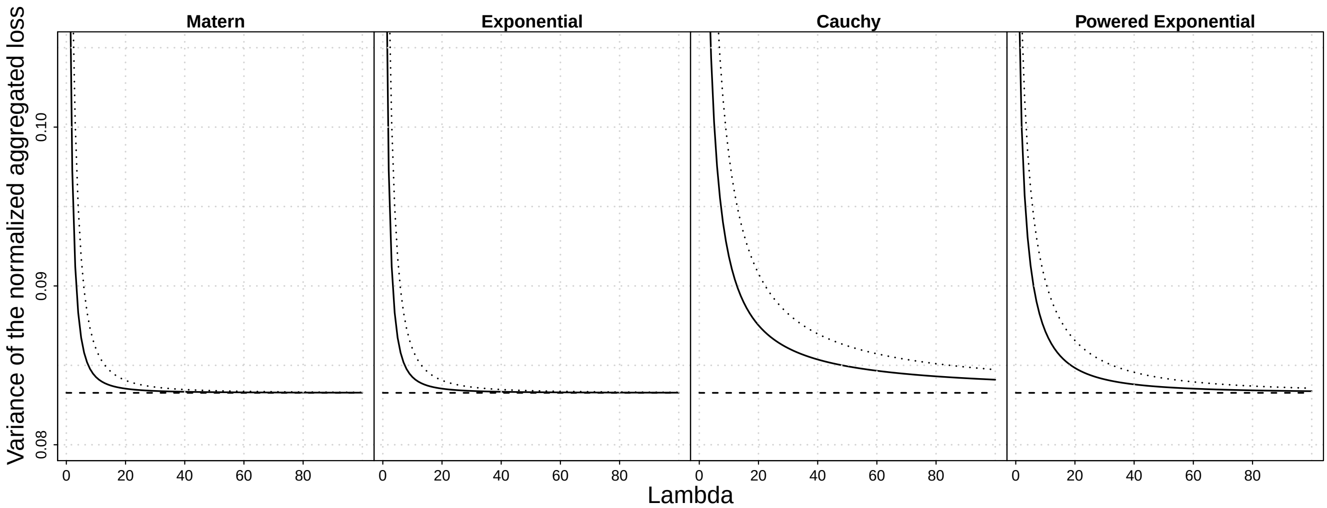

Figure 3: Each panel corresponds to a different correlation function. The solid (respectively dotted) line depicts the evolution of $R_2(\lambda A)$ with respect to λ in the case of the geometric Gaussian model, where A is a disk (respectively a square). The dashed line represents the limiting risk measure.

4.1.3 Spatial homogeneity for a new max-stable model: the tube model

In order to allow for faster spatial diversification, we introduce a new max-stable model, the tube model, defined below.

Definition 8 (The tube model). *The tube model is defined using Theorem 2, with ν being the Lebesgue measure:*

$$\{Z(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^2} = \left\{\max_{i\geq 1} \xi_i f_0(\mathbf{c}_i, \mathbf{x})\right\}_{\mathbf{x}\in\mathbb{R}^2},$$

where $f_0(\mathbf{c},\mathbf{x}) = f_0(\mathbf{c}-\mathbf{x}) = h_b\, \mathbf{I}_{\{\|\mathbf{c}-\mathbf{x}\|<R_b\}}$, *with* $R_b > 0$ *and* $h_b = \dfrac{1}{\pi R_b^2}$.

The last condition stems from (8), which imposes $\pi R_b^2 h_b = 1$. The density f_0 has the shape of a tube of height h_b centered at point $\mathbf{c}$ and with radius R_b. The extremal coefficient is given in the next proposition and depicted in Figure 4 for $R_b = 1$.

Proposition 3. *The extremal coefficient of the tube model is given by*

$$\Theta(h) = \begin{cases} 2\left[1 - h_b\left(R_b^2 \arcsin\left(\dfrac{\sqrt{4R_b^2 - h^2}}{2R_b}\right) - \dfrac{h}{4}\sqrt{4R_b^2 - h^2}\right)\right] & \text{if } h \leq 2R_b, \\ 2 & \text{if } h > 2R_b. \end{cases}$$

We can easily show that for all $h > 0$, the function $\lambda \mapsto \theta(\lambda h)$ is strictly increasing for $\lambda \leq \dfrac{2R_b}{h}$ and constant above that. Thus, (11) and (12) show that the function $\lambda \mapsto R_2(\lambda A)$ is strictly decreasing in the case of the disk and the square. Hence, there is spatial diversification. An interesting property stems from the fact that $\Theta(h)$ reaches 2 whenever $h \geq 2R_b$, meaning that there is spatial independence at a finite distance. This explains why the spatial diversification is faster than in the case of the previously introduced models, as can be observed in Figure 5 (obtained with $R_b = 1$). Obviously, $\lim_{h\to\infty}\Theta(h) = 2$, and Corollary 3 gives $\lim_{\lambda\to\infty} R_2(\lambda A) = 0$.
Furthermore, due to the spatial independence at finite distance, we have the following proposition.

Proposition 4. *In the case of the tube model, we have*

$$\forall \lambda \geq 1, \lim_{R_b \to 0} R_2(\lambda A) = R_2(A) = 0.$$

The limit process arising as R_b tends to 0 corresponds to the case of perfect independence.

4.1.4 Spatial homogeneity: a comparison between previous models

By way of summary, a comparison of the 4 models considered (Smith, Schlather, geometric Gaussian and tube) is provided in Figure 6. In the case of the Schlather and the geometric Gaussian

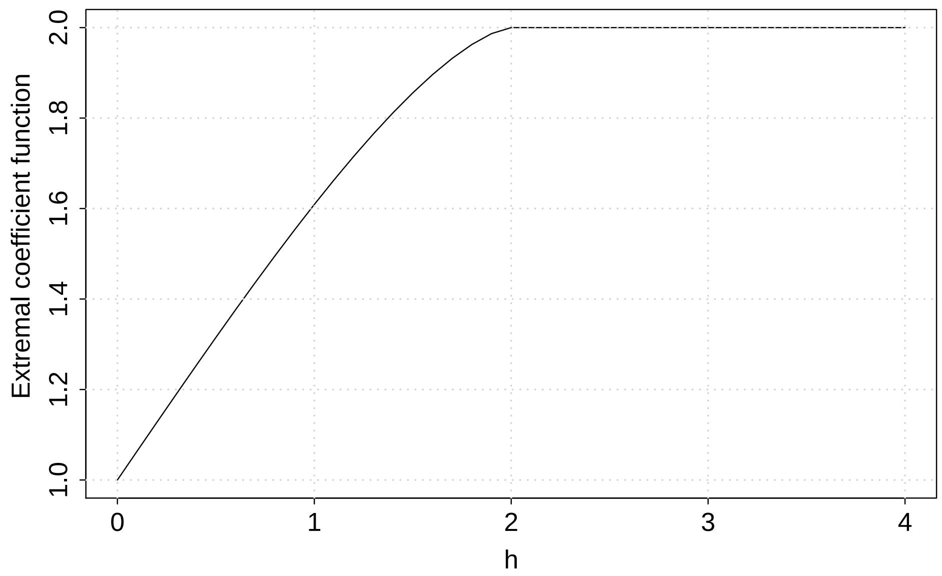

Figure 4: Extremal coefficient function of the tube model.

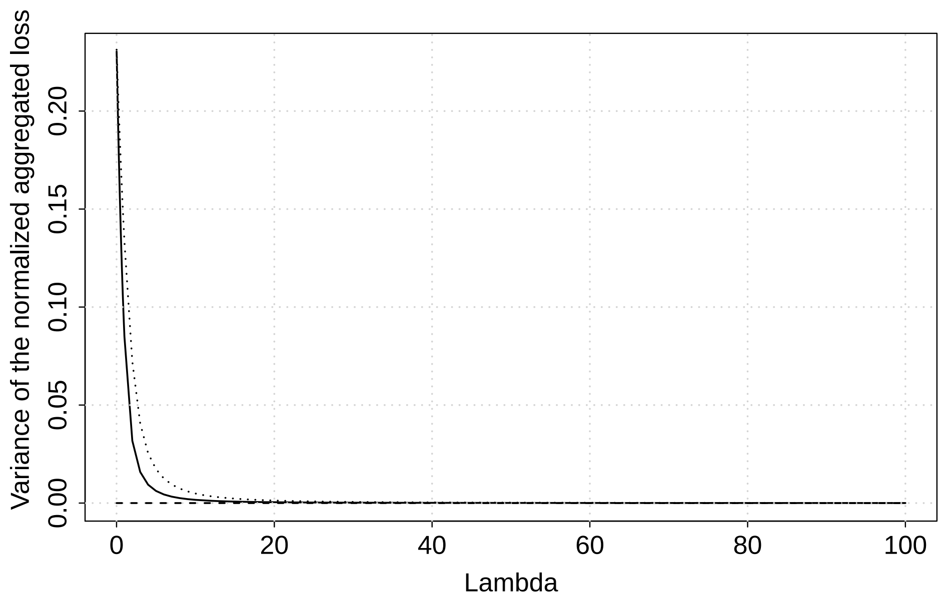

Figure 5: The solid (respectively dotted) line depicts the evolution of $R_2(\lambda A)$ with respect to λ in the case of the tube model, where A is a disk (respectively a square). The dashed line represents the limiting risk measure.

models, a Cauchy correlation function has been used. These different processes show a large variety of behaviors, both in terms of speed and "completeness" of spatial diversification. The Brown-Resnick model itself includes various types of behaviors. The two particular cases considered here (the Smith model and the geometric Gaussian model) are very different. In terms of spatial diversification, the optimal strategy for an insurance company depends to a large extent on the type of max-stable process driving extreme events.

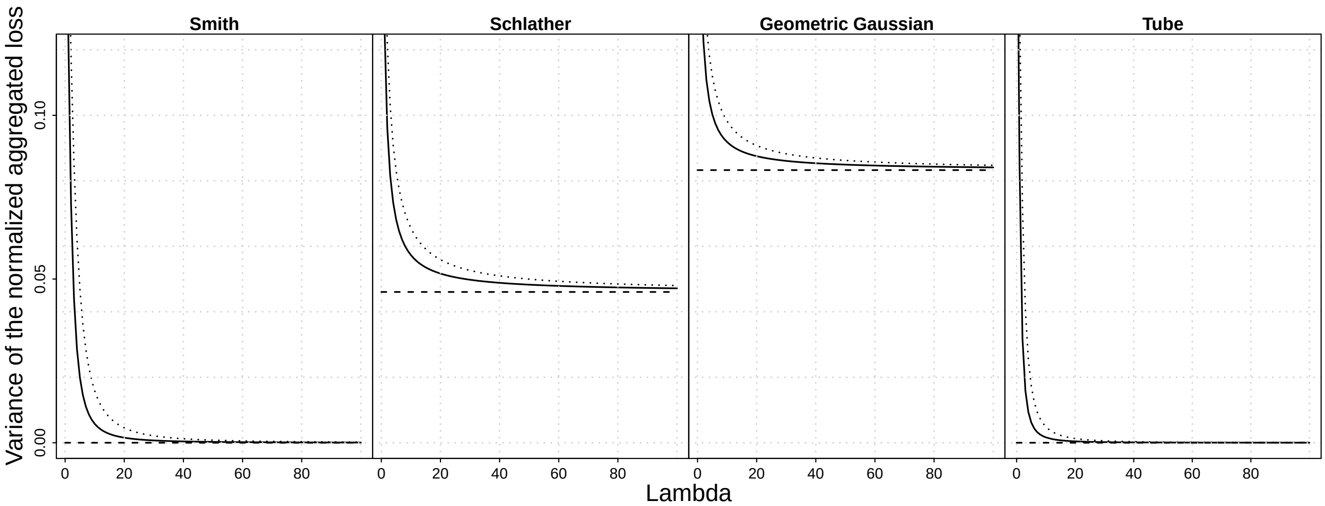

Figure 6: Each panel corresponds to a different max-stable model. The solid (respectively dotted) line depicts the evolution of $R_2(\lambda A)$ with respect to λ, where A is a disk (respectively a square). The dashed line represents the limiting risk measure.

In all cases, the behavior of the function $\lambda \mapsto R_2(\lambda A)$ is very similar in the case of the disk and the square. The only difference consists in the fact that the spatial diversification is slightly slower. Hence, the optimal strategy for an insurance company also depends on the type of region.

Remark 4. *It is important to note that our analysis has been carried out with a standardized dependence structure: the eigenvalues of Σ, the range parameter c_1 and the radius of the tube R_b are equal to 1. However, in a real case study, the characteristic dimension of the area required to reach a given level of variance depends on the real values of Σ, c_1 and R_b on the region of interest.*

4.1.5 Central limit theorem and axioms

Since mixing conditions are generally rather difficult to check, Spodarev (2014) proposes a central limit theorem for excursion sets based on the concept of association. Using his result, we can derive the following theorem.

Theorem 6. *In the case of the Smith model, the Brown-Resnick model with a variogram satisfying $\gamma(\mathbf{h}) \underset{\|\mathbf{h}\|\to\infty}{\sim} k\|\mathbf{h}\|^a$ where $k \in \mathbb{R}$ and $a > 0$, and the tube model, we have, for all $A \in \mathscr{A}$,*

$$\lambda\left(L_N(\lambda A) - \left[1 - \exp\left(-\frac{1}{u}\right)\right]\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2), \text{ for } \lambda \to \infty,$$

where

$$\sigma^2 = \int_{\mathbb{R}^2} \left[\exp\left(-\frac{\Theta(\mathbf{x})}{u}\right) - \exp\left(-\frac{2}{u}\right)\right] d\mathbf{x}.$$

The following theorem shows that the different axioms introduced in Definition 4 are satisfied in the case of some max-stable processes.

Theorem 7. *1. For any stationary max-stable process, the spatial risk measure R_2 satisfies the axiom of spatial invariance under translation.*

2. For any stationary and isotropic simple max-stable process such that the extremal coefficient function $h \mapsto \theta(h)$ increases, the spatial risk measure R_2 satisfies the following axioms:

(a) Spatial sub-additivity when the two regions are both a disk or a square;

(b) Spatial anti-monotonicity when the two regions are both a disk or a square.

3. In the cases of the Smith model, the Brown-Resnick model with a variogram satisfying $\gamma(\mathbf{h}) \underset{\|\mathbf{h}\|\to\infty}{\sim} k\|\mathbf{h}\|^a$ where $k \in \mathbb{R}$ and $a > 0$, and the tube model, we have asymptotic spatial homogeneity of order -2, with

$$K_1 = 0 \quad \text{and} \quad K_2 = \int_{\mathbb{R}^2} \left[\exp\left(-\frac{\Theta(\mathbf{x})}{u}\right) - \exp\left(-\frac{2}{u}\right)\right] d\mathbf{x}.$$

4.2 The Value-at-Risk

Here we focus on $R_{3,1-\alpha}(A) = \text{VaR}_{1-\alpha}[L_N(A)]$, where $\text{VaR}_{1-\alpha}$ is the VaR at the level $1-\alpha$, for α small. It seems impossible to derive formulas (even up to an integral) for the VaR of $L_N(A)$. Therefore, we evaluate it using Monte-Carlo techniques. The process $Z(\mathbf{x})$ is simulated on a grid containing different locations $\mathbf{x}_m \in A \in \mathscr{A}, m = 1, \ldots, M$. As a result, realization of the normalized loss can be approximated using a Riemann sum. Different approximation methods are available, the most efficient of which is probably the trapeze method. The Riemann-based approach has the advantage of providing the convergence rate of the approximated realization to the real one, using classical results for the discretization error.
Then, by generating a number S of independent approximated replications of the random variable $L_N(A)$, an approximation of its distribution is obtained. Finally, an approximation of VaR can be obtained by taking the empirical quantile of this distribution. The uncertainty on VaR stemming from this second step can be quantified using classical bootstrap methods. The VaR can be computed on

many sub-samples of the sample $s = 1, \ldots, S$, yielding the empirical variance of the corresponding estimates. Under classical regularity conditions, confidence intervals on VaR can be obtained.
To illustrate the procedure, the evolution of $R_{3,0.9}(\lambda A)$ with respect to λ in the case of the Smith model is shown in Figure 7, for $M = 49$ and $S = 10000$, where M is the number of sites in region A. These sites are located on a regular grid. Note that region λA contains $\lambda^2 M$ sites. Of course, the

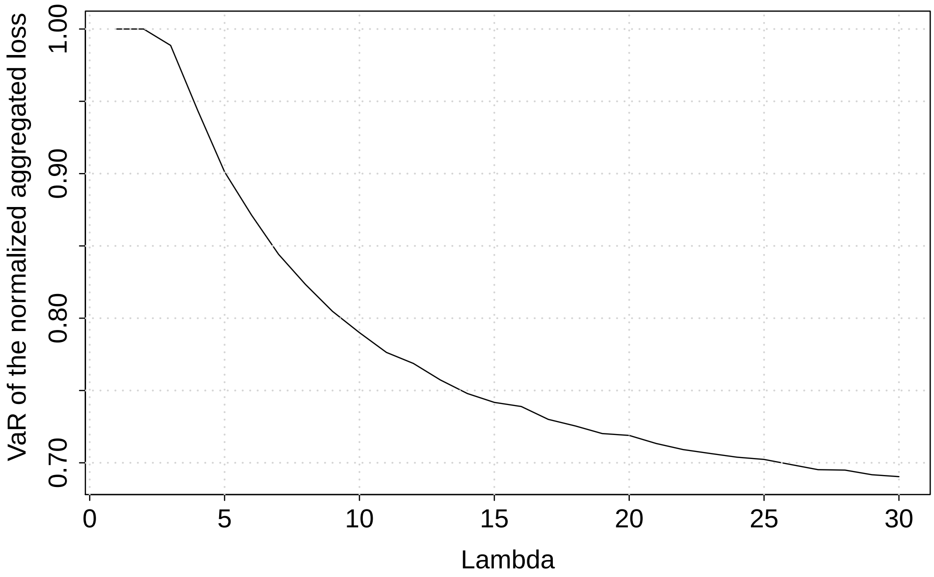

Figure 7: $R_{3,0.9}(\lambda A)$ with respect to λ in the case of the Smith process, where A is a square of side 1.

larger M and S, the smoother the curve. Figure 7 shows a rapid diversification.

Although we do not have an explicit formula for $R_{3,1-\alpha}(\lambda A)$ for finite values of λ, by using Theorem 6, we know the asymptotic behavior (when $\lambda \to \infty$) of $R_{3,1-\alpha}(\lambda A)$ for some max-stable models. This is presented in the following theorem.

Theorem 8. *In the case of the Smith model, the Brown-Resnick model with a variogram satisfying* $\gamma(\mathbf{h}) \underset{\|\mathbf{h}\| \to \infty}{\sim} k\|\mathbf{h}\|^a$ *where* $k \in \mathbb{R}$ *and* $a > 0$*, and the tube model,* $R_{3,1-\alpha}$ *satisfies the axiom of asymptotic spatial homogeneity of order -1, with*

$$K_1 = 1 - \exp\left(-\frac{1}{u}\right) \quad \text{and} \quad K_2 = q_{1-\alpha}\sqrt{\int_{\mathbb{R}^2}\left[\exp\left(-\frac{\Theta(\mathbf{x})}{u}\right) - \exp\left(-\frac{2}{u}\right)\right] d\mathbf{x}},$$

where $q_{1-\alpha}$ *is the quantile at the level* $1-\alpha$ *of the standard Gaussian distribution.*

Regarding the other axioms, based on Proposition 2, we can state that for any stationary max-stable process, $R_{3,1-\alpha}$ is invariant under translation. But since we do not have a formula for the VaR for non-asymptotic values of λ, there is nothing we can say about spatial sub-additivity and spatial anti-monotonicity.

Remark 5. *Other spatial risk measures of interest are based on*

$$L_N(A) = \frac{\int_A Z(\mathbf{x})^\beta \, d\mathbf{x}}{|A|}, \text{ where } \beta > 0. \tag{14}$$

Indeed, the damage function $Z(\mathbf{x})^\beta$ is particularly adapted in the case of wind hazard; see for instance Klawa and Ulbrich (2003), Pinto et al. (2007) and Donat et al. (2011), where the damage is proportional to the third power of the wind speed.
In a subsequent publication, we introduce a dependence measure for the damage related to wind and study the variance and the VaR of $L_N(A)$ defined in (14) *where Z is the Smith process. We show that $R_2(A) = Var[L_N(A)]$ satisfies the axioms of spatial invariance under translation, spatial-sub-additivity and anti-monotonicity when the two regions are both a disk or a square as well as asymptotic spatial homogeneity of order -2. Moreover, $R_{3,1-\alpha}(A) = VaR_{1-\alpha}[L_N(A)]$ satisfies the axiom of asymptotic spatial homogeneity of order -1. The way in which these results are established is similar to that used here but is more technical.*

5 Conclusion

This paper introduces a new notion of spatial risk measure, based on the normalized spatially aggregated loss, and proposes a set of axioms adapted to the spatial context. The latter appears natural under the assumption of stationarity for the cost process. Contrary to the classical literature, our axiomatic approach aims at quantifying the sensitivity of the risk measurement with respect to the space variable. The idea is to propose a new framework for risk measures as well as relevant tools for public authorities and the (re)insurance industry. Characterizing all risk measures and processes that satisfy the proposed axioms and even providing other adapted axioms could be relevant and useful.
In order to develop concrete examples of spatial risk measures, we propose a model that maps the process of the environmental variable generating loss into an economic damage, via a damage function. In this paper, we are mainly interested in risks related to extreme environmental events. Hence, we model the environmental process using max-stability. A damage function adapted to heatwaves is considered. Theoretical properties of our spatial risk measures are derived for both classical max-stable models and a max-stable model introduced in this article, the tube model. We show that in the case of variance, these risk measures satisfy the axioms proposed, for some underlying models. Furthermore, an interpretation in terms of insurance is provided.
From a practical viewpoint, the construction of the spatial risk measures introduced here involves the following steps:

1. Fit several max-stable models (Smith, Schlather, Brown-Resnick, tube) to historical data, using, for instance, the composite likelihood approach (see e.g. Padoan et al., 2010);

2. Choose the best max-stable model via a model selection based, for instance, on the Akaike Information Criterion (Akaike, 1974) or the likelihood ratio statistic (Davison, 2003);

3. Choose the model that converts the environmental hazard into economic losses;

4. Compute the risk measure as explained.

It should be noted that we have considered processes with standard Fréchet margins, which is a classical assumption in the literature. However, considering more realistic margins would be an improvement. An extension to non stationary max-stable processes (involving varying margin parameters) could also be of interest. This would require a new set of axioms.
Examples of spatial risk measures developed in our paper include only one environmental hazard (e.g. a heatwave or a windstorm). However, it would be possible to extend this approach to more sources of hazard. Let us denote by $Z_1, \ldots, Z_k$ the spatial processes of the maxima of k environmental variables. In this case, the loss on region A would be given by

$$L(A, P) = \int_A C_P(\mathbf{x}) \, d\mathbf{x} = \int_A D[Z_1(\mathbf{x}), \ldots, Z_k(\mathbf{x})] \, d\mathbf{x},$$

where the damage function D is k-variate.

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Appendix

A Appendix: Proofs

A.1 For Proposition 1

Proof. We first show that spatial sub-additivity implies spatial anti-monotonicity. Let $A_1, A_2 \in \mathscr{A}$, with $A_1 \subset A_2$. We have

$$R_{\Pi}(A_2, P) = R_{\Pi}(A_1 \cup A_2, P) \leq \min[R_{\Pi}(A_1, P), R_{\Pi}(A_2, P)] \leq R_{\Pi}(A_1, P).$$

We now prove that spatial anti-monotonicity implies spatial sub-additivity. Let $A_1, A_2 \subset \mathscr{A}$. We have $A_1 \subset A_1 \cup A_2$ and $A_2 \subset A_1 \cup A_2$, giving that

$$R_{\Pi}(A_1 \cup A_2, P) \leq R_{\Pi}(A_1, P) \text{ and } R_{\Pi}(A_1 \cup A_2, P) \leq R_{\Pi}(A_2, P).$$

Therefore, $R_{\Pi}(A_1 \cup A_2, P) \leq \min[R_{\Pi}(A_1, P), R_{\Pi}(A_2, P)]$. □

A.2 For Proposition 2

Proof. Using the fact that $|A + \mathbf{v}| = |A|$ and the change of variable $\mathbf{y} = \mathbf{x} - \mathbf{v}$, we have

$$R_{\Pi}(A + \mathbf{v}, P) = \Pi\left[\frac{1}{|A + \mathbf{v}|}\int_{A+\mathbf{v}} C_P(\mathbf{x})\ d\mathbf{x}\right] = \Pi\left[\frac{1}{|A|}\int_{A} C_P(\mathbf{y} + \mathbf{v})\ d\mathbf{y}\right]. \tag{15}$$

Due to the stationarity of C_P, we have for all $\mathbf{x}, \mathbf{y}, \mathbf{v} \in \mathbb{R}^2, C_P(\mathbf{x}) \stackrel{d}{=} C_P(\mathbf{y} + \mathbf{v}),$ yielding

$$\Pi\left[\frac{1}{|A|}\int_{A} C_P(\mathbf{y} + \mathbf{v})\ d\mathbf{y}\right] = \Pi\left[\frac{1}{|A|}\int_{A} C_P(\mathbf{x})\ d\mathbf{x}\right] = R_{\Pi}(A, P). \tag{16}$$

The combination of (15) and (16) provides the result. □

A.3 For Lemma 1

Proof. Mixing and ergodicity are properties of the σ-algebra generated by the underlying process. Since the σ-algebra associated to a function of the process Z is smaller than that associated to Z, H is mixing (respectively ergodic) whatever the function D. □

A.4 For Theorem 5

Proof. We have

$$\begin{aligned}\mathbb{E}\left[(L(A))^2\right] = \mathbb{E}\left[\left(\int_A \mathbf{I}_{\{Z(\mathbf{x})>u\}}\, d\mathbf{x}\right)^2\right] &= \mathbb{E}\left[\int_A \mathbf{I}_{\{Z(\mathbf{x})>u\}}\, d\mathbf{x} \int_A \mathbf{I}_{\{Z(\mathbf{y})>u\}}\, d\mathbf{y}\right] \\ &= \mathbb{E}\left[\int_A\int_A \mathbf{I}_{\{Z(\mathbf{x})>u, Z(\mathbf{y})>u\}}\, d\mathbf{x}\, d\mathbf{y}\right] \\ &= \int_A\int_A \mathbb{P}(Z(\mathbf{x})>u, Z(\mathbf{y})>u)\, d\mathbf{x}\, d\mathbf{y}.\end{aligned}$$

Moreover, using the fact that Z has standard Fréchet margins, we have

$$\mathbb{P}(Z(\mathbf{x})>u, Z(\mathbf{y})>u) = 1 + \mathbb{P}(Z(\mathbf{x})\le u, Z(\mathbf{y})\le u) - 2\exp\left(-\frac{1}{u}\right), \text{ yielding}$$

$$\mathbb{E}\left[(L(A))^2\right] = |A|^2\left(1 - 2\exp\left(-\frac{1}{u}\right)\right) + \int_A\int_A \exp\left(-\frac{\Theta(\mathbf{x},\mathbf{y})}{u}\right)\, d\mathbf{x}\, d\mathbf{y}.$$

Hence,

$$\begin{aligned}&R_2(A)\\ &= \frac{1}{|A|^2}\left[|A|^2\left(1-2\exp\left(-\frac{1}{u}\right)\right) + \int_A\int_A \exp\left(-\frac{\Theta(\mathbf{x},\mathbf{y})}{u}\right)\, d\mathbf{x}\, d\mathbf{y} - |A|^2\left(1-\exp\left(-\frac{1}{u}\right)\right)^2\right]\\ &= \frac{1}{|A|^2}\int_A\int_A \left[\exp\left(-\frac{\Theta(\mathbf{x},\mathbf{y})}{u}\right) - \exp\left(-\frac{2}{u}\right)\right]\, d\mathbf{x}\, d\mathbf{y}.\end{aligned}$$

The result is obtained by replacing A by λA. □

A.5 For Corollary 1

Proof. We first show the result in the case of A being a disk with radius R. For any function $g : (\mathbf{x},\mathbf{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ depending only on $h = \|\mathbf{x}-\mathbf{y}\|$, it is easy to show that, for all $\lambda > 0$,

$$\int_{\lambda A}\int_{\lambda A} g(\mathbf{x},\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} = \lambda^4 |A|^2 \int_0^{2\lambda R} f_d(h,\lambda R)\, g(h)\, dh. \quad (17)$$

Moreover, Moltchanov (2012) shows that

$$f_d(h,R) = \frac{2h}{R^2}\left(\frac{2}{\pi}\arccos\left(\frac{h}{2R}\right) - \frac{h}{\pi R}\sqrt{1-\frac{h^2}{4R^2}}\right), \text{ for } 0 \le h \le 2R,$$

yielding $f_d(\lambda h, \lambda R) = \frac{1}{\lambda} f_d(h,R)$. Therefore, by the change of variable $h_d = \frac{h}{\lambda}$, we obtain

$$\int_0^{2\lambda R} f_d(h,\lambda R)\, g(h)\, dh = \int_0^{2R} f_d(h_d, R)\, g(\lambda h_d)\, dh_d. \quad (18)$$

Applying (17) and (18) to $g(\mathbf{x},\mathbf{y}) = \exp\left(-\frac{\Theta(\mathbf{x}-\mathbf{y})}{u}\right) = \exp\left(-\frac{\Theta(\|\mathbf{x}-\mathbf{y}\|)}{u}\right)$ and using (10), we obtain

$$R_2(\lambda A) = -\exp\left(-\frac{2}{u}\right) + \int_0^{2R} f_d(h,R)\ \exp\left(-\frac{\Theta(\lambda h)}{u}\right)\ dh.$$

The same reasoning yields the result in the case of a square. Moltchanov (2012) shows that the distribution function of the distance between 2 points uniformly distributed on a square with side R is written

$$F_s(h,R) = \begin{cases} \frac{\pi h^2}{R^2} - \frac{8h^3}{3R^3} + \frac{h^4}{2R^4} & \text{if } h \in [0,R], \\ \frac{1}{3} - 2b - \frac{b^2}{2} + \frac{2}{3}\sqrt{(b-1)^3} + 2\sqrt{b-1} + 2b\sqrt{b-1} + 2b\arcsin\left(\frac{2-b}{b}\right) & \text{if } h \in [R, R\sqrt{2}], \end{cases}$$

where $b = \frac{h^2}{R^2}$. Hence, for $h \in [0,R]$, the density is

$$f_s(h,R) = \frac{2\pi h}{R^2} - \frac{8h^2}{R^3} + \frac{2h^3}{R^4}.$$

For $h \in [R, R\sqrt{2}]$, we obtain

$$f_s(h,R) = \left(-2 - b + 3\sqrt{b-1} + \frac{b+1}{\sqrt{b-1}} + 2\arcsin\left(\frac{2-b}{b}\right) - \frac{4}{b\sqrt{1-\frac{(2-b)^2}{b^2}}}\right)\frac{2h}{R^2}.$$

□

A.6 For Proposition 3

Proof. Since the $\{(\xi_i, \mathbf{c}_i)\}_{i\geq 1}$ are the points of a Poisson point process on $(0,\infty)\times\mathbb{R}^2$ with intensity measure $d\Lambda(\xi,\mathbf{c}) = \xi^{-2}d\xi d\mathbf{c}$, we have, for all $z_1, z_2 > 0$,

$$\begin{aligned}
&-\log[\mathbb{P}(Z(\mathbf{x}_1) \leq z_1, Z(\mathbf{x}_2) \leq z_2)] \\
&= \int_{\mathbb{R}^2}\int_{\min\left(\frac{z_1}{f_0(\mathbf{x}-\mathbf{x}_1)},\frac{z_2}{f_0(\mathbf{x}-\mathbf{x}_2)}\right)}^{\infty} \xi^{-2}d\xi\ d\mathbf{x} \\
&= \int_{\mathbb{R}^2} \max\left(\frac{f_0(\mathbf{x}-\mathbf{x}_1)}{z_1}, \frac{f_0(\mathbf{x}-\mathbf{x}_2)}{z_2}\right)\ d\mathbf{x} \\
&= \int_{\mathbb{R}^2} \frac{f_0(\mathbf{x}-\mathbf{x}_1)}{z_1}\mathbf{I}_{\left\{\frac{f_0(\mathbf{x}-\mathbf{x}_1)}{z_1} > \frac{f_0(\mathbf{x}-\mathbf{x}_2)}{z_2}\right\}}\ d\mathbf{x} + \int_{\mathbb{R}^2} \frac{f_0(\mathbf{x}-\mathbf{x}_2)}{z_2}\mathbf{I}_{\left\{\frac{f_0(\mathbf{x}-\mathbf{x}_2)}{z_2} \geq \frac{f_0(\mathbf{x}-\mathbf{x}_1)}{z_1}\right\}}\ d\mathbf{x} \\
&= \int_{\mathbb{R}^2} \frac{f_0(\mathbf{x})}{z_1}\mathbf{I}_{\left\{\frac{f_0(\mathbf{x})}{z_1} > \frac{f_0(\mathbf{x}+\mathbf{x}_1-\mathbf{x}_2)}{z_2}\right\}}\ d\mathbf{x} + \int_{\mathbb{R}^2} \frac{f_0(\mathbf{x})}{z_2}\mathbf{I}_{\left\{\frac{f_0(\mathbf{x})}{z_2} \geq \frac{f_0(\mathbf{x}+\mathbf{x}_2-\mathbf{x}_1)}{z_1}\right\}}\ d\mathbf{x}
\end{aligned}$$

$$= \frac{1}{z_1}\mathbb{P}\left(\frac{f_0(\mathbf{X})}{z_1} > \frac{f_0(\mathbf{X}+\mathbf{x}_1-\mathbf{x}_2)}{z_2}\right) + \frac{1}{z_2}\mathbb{P}\left(\frac{f_0(\mathbf{X})}{z_2} \geq \frac{f_0(\mathbf{X}+\mathbf{x}_2-\mathbf{x}_1)}{z_1}\right), \tag{19}$$

where $\mathbf{X}$ is a random vector having density f_0.
Let us denote by E_1 the event $\left\{\frac{f_0(\mathbf{X})}{z_1} > \frac{f_0(\mathbf{X}+\mathbf{x}_1-\mathbf{x}_2)}{z_2}\right\}$. We have

$$E_1 = \left\{z_2\ \mathbf{I}_{\{\|\mathbf{X}\|\leq R_b\}} > z_1\ \mathbf{I}_{\{\|\mathbf{X}-\mathbf{x}_2+\mathbf{x}_1\|\leq R_b\}}\right\} = \{\|\mathbf{X}\| \leq R_b \text{ and } z_2 > z_1 \text{ if } \|\mathbf{X}-\mathbf{x}_2+\mathbf{x}_1\| \leq R_b\}.$$

Thus, if $z_2 > z_1$, $E_1 = \{\|\mathbf{X}\| \leq R_b\}$, giving $\mathbb{P}(E_1) = 1$. Indeed $\mathbf{X}$ has density f_0 and then $\|\mathbf{X}\| \leq R_b$ almost surely.
If $z_1 \geq z_2$, $E_1 = \{\|\mathbf{X}\| \leq R_b \text{ and } \|\mathbf{X}-\mathbf{x}_2+\mathbf{x}_1\| > R_b\} = \{\|\mathbf{X}-\mathbf{x}_2+\mathbf{x}_1\| > R_b\}$ since $\|\mathbf{X}\| \leq R_b$ is necessarily satisfied almost surely. Therefore,

$$\begin{aligned}\mathbb{P}(E_1) = \int_{\mathbb{R}^2} \mathbf{I}_{\{\|\mathbf{x}-\mathbf{x}_2+\mathbf{x}_1\|>R_b\}}\ \mathbf{I}_{\{\|\mathbf{x}\|\leq R_b\}}\ d\mathbf{x} &= h_b \int_{\mathbb{R}^2} \mathbf{I}_{\{\|\mathbf{x}\|\leq R_b\ \cap\ \|\mathbf{x}-(\mathbf{x}_2-\mathbf{x}_1)\|>R_b\}}\ d\mathbf{x} \\ &= h_b\ [\pi R_b^2 - A_{int}(h)],\end{aligned}$$

where $A_{int}(h)$ is the area of the intersection between the base of the tube of center $\mathbf{0}$ and that of the tube of center $(\mathbf{x}_2-\mathbf{x}_1)$, and $h = \|\mathbf{x}_2-\mathbf{x}_1\|$. Note that by symmetry, the area of the intersection between the base of the tube of center $\mathbf{0}$ and that of the tube of center $(\mathbf{x}_1-\mathbf{x}_2)$ is also equal to $A_{int}(h)$.
Let us denote by E_2 the event $\left\{\frac{f_0(\mathbf{X})}{z_2} \geq \frac{f_0(\mathbf{X}+\mathbf{x}_2-\mathbf{x}_1)}{z_1}\right\}$. We have that

$$E_2 = \left\{z_1\ \mathbf{I}_{\{\|\mathbf{X}\|\leq R_b\}} \geq z_2\ \mathbf{I}_{\{\|\mathbf{X}-\mathbf{x}_1+\mathbf{x}_2\|\leq R_b\}}\right\} = \{(\|\mathbf{X}\| \leq R_b \text{ and } z_1 \geq z_2) \text{ if } \|\mathbf{X}-\mathbf{x}_1+\mathbf{x}_2\| \leq R_b\}.$$

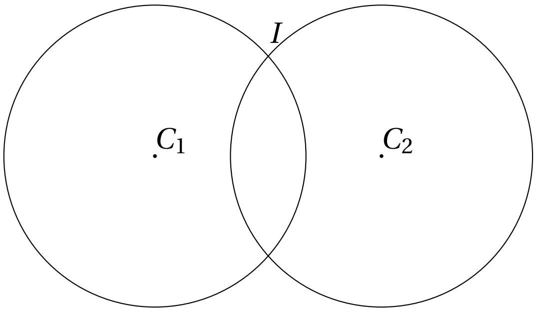
Thus, if $z_2 > z_1$, $E_2 = \{\|\mathbf{X}-\mathbf{x}_1+\mathbf{x}_2\| > R_b\}$, giving $\mathbb{P}(E_2) = h_b[\pi R_b^2 - A_{int}(h)]$.
If $z_1 \geq z_2$, $E_2 = \{\|\mathbf{X}\| \leq R_b\}$ if $\|\mathbf{X}-\mathbf{x}_1+\mathbf{x}_2\| \leq R_b$ and is always satisfied otherwise, yielding $\mathbb{P}(E_2) = 1$ since $\{\|\mathbf{X}\| \leq R_b\}$ is necessarily satisfied.
By definition of the extremal coefficient,

$$\Theta(\mathbf{x}_1-\mathbf{x}_2) = -\log[\mathbb{P}(Z(\mathbf{x}_1) \leq u, Z(\mathbf{x}_2) \leq u]\ u.$$

Hence, using (19), we obtain

$$\Theta(\mathbf{x}_1-\mathbf{x}_2) = u\left(\frac{h_b}{u}\left[\pi R_b^2 - A_{int}(h)\right] + \frac{1}{u}\right) = 2 - h_b\ A_{int}(h). \tag{20}$$

Since we consider the L_2 norm, the bases of the tubes are circular. Let us then compute the intersection between two discs, respectively with radius R_b and centers C_1 and C_2 at a distance h. This intersection is not empty if and only if $h \leq 2R_b$. We consider this case, represented in the following picture:



By using Héron's formula, the area of the triangle IC_1C_2, denoted by A_T, is given by

$$A_T = \sqrt{p(p-R_b)(p-R_b)(p-h)} \quad \text{where } p = \frac{1}{2}(2R_b + h). \tag{21}$$

Furthermore, by denoting H the height of the triangle IC_1C_2, we have $A_T = \dfrac{hH}{2}$, giving

$$H = \frac{2A_T}{h}. \tag{22}$$

Denote α and β the angles $\widehat{IC_1C_2}$ and $\widehat{IC_2C_1}$, respectively. We have $\sin\alpha = \sin\beta = \dfrac{H}{R_b}$, yielding, using (22),

$$\alpha = \beta = \arcsin\left(\frac{2A_T}{hR_b}\right). \tag{23}$$

Denote by S the area of the angular sectors delimited respectively by the angles α and β. We have

$$S = \frac{\alpha R_b^2}{2}. \tag{24}$$

Combining (21), (23) and (24), we obtain, for $h \leq 2R_b$,

$$A_{int}(h) = 2(2S - A_T) = 2\left[R_b^2 \arcsin\left(\frac{2\sqrt{p(p-R_b)^2(p-h)}}{hR_b}\right) - \sqrt{p(p-R_b)^2(p-h)}\right].$$

We finally obtain

$$A_{int}(h) = \begin{cases} 2\left(R_b^2 \arcsin\left(\frac{\sqrt{4R_b^2-h^2}}{2R_b}\right) - \frac{h}{4}\sqrt{4R_b^2 - h^2}\right) & \text{if } h \leq 2R_b, \\ 0 & \text{if } h > 2R_b. \end{cases} \tag{25}$$

The combination of (20) and (25) yields the result. $\square$

A.7 For Proposition 4

Proof. We consider the case of A being a disk; the proof is exactly the same in the case of a square. Using (11), we have

$$
\begin{aligned}
&R_2(\lambda A) \\
&= -\exp\left(-\frac{2}{u}\right) + \int_0^{2R_b} f_d(h,R)\exp\left(-\frac{\Theta(\lambda h)}{u}\right)\,dh + \int_{2R_b}^{2R} f_d(h,R)\exp\left(-\frac{\Theta(\lambda h)}{u}\right)\,dh.
\end{aligned}
$$

Moreover, for $\lambda \geq 1$ and $h \geq 2R_b$, we have $\Theta(\lambda h) = \Theta(h) = 2$. Hence, for all $\lambda \geq 1$,

$$
\begin{aligned}
&R_2(\lambda A) \\
&= \int_0^{2R_b} f_d(h,R)\exp\left(-\frac{\Theta(\lambda h)}{u}\right)\,dh - \exp\left(-\frac{2}{u}\right) + \int_0^{2R} f_d(h,R)\exp\left(-\frac{\Theta(h)}{u}\right)\,dh \\
&\quad - \int_0^{2R_b} f_d(h,R)\exp\left(-\frac{\Theta(h)}{u}\right)\,dh \\
&= R_2(A) + \frac{1}{|A|^2}\int_0^{2R_b} f_d(h,R)\left[\exp\left(-\frac{\Theta(\lambda h)}{u}\right) - \exp\left(-\frac{\Theta(h)}{u}\right)\right]dh,
\end{aligned}
$$

When R_b tends to 0, the second term vanishes, giving $\lim_{R_b\to 0} R_2(\lambda A) = R_2(A)$. Moreover, if $R_b = 0$, for all $h \geq 0$, for all $\lambda \geq 0, \Theta(\lambda h) = 2$. Thus, (11) gives that for all $\lambda \geq 0, R_2(\lambda A) = 0$. □

A.8 For Theorem 6

Proof. We denote by Cov the covariance. A random field $\{X(\mathbf{x})\}_{\mathbf{x}\in\mathbb{R}^d}$ is called associated if $\text{Cov}(f(X_I), g(X_I)) \geq 0$ for any discrete finite subset $I \subset \mathbb{R}^d$ and for any bounded coordinatewise non-decreasing functions $f : \mathbb{R}^{\text{card}(I)} \mapsto \mathbb{R}$, $g : \mathbb{R}^{\text{card}(I)} \mapsto \mathbb{R}$ (card stands for cardinality), where $X_I = \{X(\mathbf{x}) : \mathbf{x} \in I\}$. Max-stable processes are associated and therefore positively associated (see e.g. Spodarev, 2014). Moreover, we have

$$
\begin{aligned}
\sigma^2 = \int_{\mathbb{R}^2} \text{Cov}\left(\mathbf{I}_{\{Z(\mathbf{0})>u\}}, \mathbf{I}_{\{Z(\mathbf{x})>u\}}\right) d\mathbf{x} &= \int_{\mathbb{R}^2} \mathbb{P}(Z(\mathbf{0}) > u, Z(\mathbf{x}) > u) - \left[1 - \exp\left(-\frac{1}{u}\right)\right]^2 d\mathbf{x} \\
&= \int_{\mathbb{R}^2} \left[\exp\left(-\frac{\Theta(\mathbf{x})}{u}\right) - \exp\left(-\frac{2}{u}\right)\right] d\mathbf{x}.
\end{aligned}
$$

Firstly, note that in the case of the max-stable models considered, $\Theta(\mathbf{x}) < 2$, i.e. $\exp\left(-\frac{\Theta(\mathbf{x})}{u}\right) > \exp\left(-\frac{2}{u}\right)$, on a set with a positive Lebesgue measure. Hence,

$$
\int_{\mathbb{R}^2} \text{Cov}\left(\mathbf{I}_{\{Z(\mathbf{0})>u\}}, \mathbf{I}_{\{Z(\mathbf{x})>u\}}\right) d\mathbf{x} > 0.
$$

We now show that this integral converges. In the case of the Smith model, recall that

$$\Theta(\mathbf{x}) = 2\Phi\left(\frac{\|\mathbf{x}\|_\Sigma}{2}\right), \tag{26}$$

where $\|.\|_\Sigma$ stands for the norm associated to the matrix Σ, i.e. $\|\mathbf{x}\|_\Sigma = \sqrt{\mathbf{x}'\Sigma^{-1}\mathbf{x}}$. Mill's ratio gives us that the survival distribution function $\bar{\Phi}$ of the standard Gaussian random variable satisfies $\bar{\Phi}(h) \underset{h\to\infty}{\sim} \frac{\exp\left(-\frac{h^2}{2}\right)}{\sqrt{2\pi}h}$. Thus, using (26), we have

$$\begin{aligned}
\exp\left(-\frac{\Theta(\mathbf{x})}{u}\right) - \exp\left(-\frac{2}{u}\right) &= \exp\left(-\frac{2}{u}\right)\left(\exp\left(\frac{2-\Theta(\mathbf{x})}{u}\right) - 1\right) \\
&\underset{\|\mathbf{x}\|\to\infty}{\sim} \exp\left(-\frac{2}{u}\right)\left(\frac{2-\Theta(\mathbf{x})}{u}\right) \\
&= \frac{2\exp\left(-\frac{2}{u}\right)}{u}\left[1-\Phi\left(\frac{\|\mathbf{x}\|_\Sigma}{2}\right)\right] \\
&\underset{\|\mathbf{x}\|\to\infty}{\sim} \frac{2\exp\left(-\frac{2}{u}\right)}{u}\frac{\exp\left(-\frac{\|\mathbf{x}\|_\Sigma^2}{8}\right)}{\sqrt{2\pi}\frac{\|\mathbf{x}\|_\Sigma}{2}} \\
&= \frac{4\exp\left(-\frac{2}{u}\right)\exp\left(-\frac{\|\mathbf{x}\|_\Sigma^2}{8}\right)}{\sqrt{2\pi}u\|\mathbf{x}\|_\Sigma},
\end{aligned}$$

which is clearly convergent.
In the case of the Brown-Resnick model with a variogram satisfying $\gamma(\mathbf{h}) \underset{\|\mathbf{h}\|\to\infty}{\sim} k\|\mathbf{h}\|^a$ where $k \in \mathbb{R}$ and $a > 0$, the convergence is obtained in exactly the same way. Indeed, we have $\Theta(\mathbf{x}) = 2\Phi\left(\sqrt{\frac{\gamma(\mathbf{x})}{2}}\right)$.
In the case of the tube model, we have $\Theta(\mathbf{x}) = 2$ for $\|\mathbf{x}\| \geq 2R_b$. Therefore, the term $\exp\left(-\frac{\Theta(\mathbf{x})}{u}\right) - \exp\left(-\frac{2}{u}\right)$ has a compact support and is integrable.
Finally, in all cases mentioned, we have $\sigma^2 < \infty$. By applying Theorem 7 in Spodarev (2014), we obtain the result. □

A.9 For Theorem 7

Proof. 1. Spatial invariance under translation:
Since the max-stable process Z is assumed to be stationary, the same is true for the process $C_P(\mathbf{x}) = \mathbf{I}_{\{Z(\mathbf{x})>u\}}$. Hence the invariance under translation directly follows from Proposition 2.

2. (b) Spatial anti-monotonicity when the two regions are both a disk or a square:
Let us consider two regions A_1 and A_2 being both a disk or a square and such that $A_1 \subset A_2$. Due to spatial invariance under translation, the region A_2 can be translated to region A_2', where A_2' corresponds to the region obtained by an homothety of A_1, the center of which is the center of A_1 and the factor of which is denoted $\lambda \geq 1$. Thus, $R_2(A_2) = R_2(A_2') = R_2(\lambda A_1)$.
Moreover, for all $h > 0$, the function $\lambda \mapsto \theta(\lambda h)$ is increasing. Thus, (11) and (12) give that $R_2(\lambda A)$ is a decreasing function of λ. Hence, we have $R_2(\lambda A_1) \leq R_2(A_1)$, giving $R_2(A_2) \leq R_2(A_1)$.
2. (a) Spatial sub-additivity when the two regions are both a disk or a square:
Due to Proposition 1, the spatial sub-additivity directly follows from the spatial anti-monotonicity.
3. Theorem 6 gives that

$$\lambda\,(L_N(\lambda A) - m) \xrightarrow{d} \mathcal{N}(0, \sigma^2), \text{ for } \lambda \to \infty, \tag{27}$$

where

$$m = \left[1 - \exp\left(-\frac{1}{u}\right)\right] \text{ and } \sigma^2 = \int_{\mathbb{R}^2} \left[\exp\left(-\frac{\Theta(\mathbf{x})}{u}\right) - \exp\left(-\frac{2}{u}\right)\right] d\mathbf{x}.$$

Thus, $R_2(\lambda A) \underset{\lambda \to \infty}{=} \dfrac{\sigma^2}{\lambda^2} + o\left(\dfrac{1}{\lambda^2}\right)$. Hence, by setting $K_1 = 0$ and $K_2 = \sigma^2$, the axiom of spatial asymptotic homogeneity of order -2 is satisfied. □

A.10 For Theorem 8

Proof. By (27), Proposition 0.1 in Resnick (1987) and using the classical formula of the VaR of a Gaussian random variable, we obtain

$$R_{3,1-\alpha}(\lambda A) = \mathrm{VaR}_{1-\alpha}[L_N(\lambda A)] \underset{\lambda \to \infty}{=} m + \frac{\sigma}{\lambda} q_{1-\alpha} + o\left(\frac{1}{\lambda}\right) \underset{\lambda \to \infty}{=} K_1 + \frac{K_2}{\lambda} + o\left(\frac{1}{\lambda}\right),$$

where $K_1 = m$ and $K_2 = \sigma q_{1-\alpha}$ (with m and σ defined above). □

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