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Explicit Föllmer-Schweizer decomposition of life insurance liabilities through Malliavin calculus

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Abstract

This paper addresses the problem of finding hedging strategies in an incomplete market. We study the Föllmer-Schweizer decomposition of a life insurer's liabilities (involving stochastic mortality and financial profit sharing), splitting them into a component that can be hedged on the financial market and a component that cannot. Using Malliavin calculus tools, we obtain an explicit formula for each part, and therefore a complete probabilistic description of both components of the liabilities (the hedgeable and non-hedgeable). In addition to the optimal risk-minimizing "hedging" strategy, it allows to compute various business-related quantities, such as e.g. the risk margin (as defined in Solvency II) associated with the balance sheet. Two different models for the financial asset available for investment are considered.

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1 Introduction

The problem of hedging the balance sheet of a financial institution has drawn the scientific community's interest for many decades. In very simple cases, the classical theory describes a methodology allowing to build a dynamic portfolio of financial instruments that perfectly hedges the balance sheet. Such a portfolio does unhappily not exist when the framework becomes a little bit more complex, that is when the market is incomplete. This question is meaningful, as the pricing of financial assets is straightforwardly linked to the existence of these hedging strategies.

The market that we consider in this paper belongs to the problematic category: we study hedging strategies associated with the liabilities of an insurer selling life insurance contracts. There is no hope in perfectly hedging these liabilities with financial instruments, as they generate mortality risk (and as the mortality-linked securities market is not sufficiently liquid).

The scientific community has proposed many different methods to overcome the incompleteness problem. Among them, we consider in this paper the Föllmer-Schweizer quadratic risk-minimization approach. It starts with the following idea: it may not be possible to find a perfect hedging strategy, but one can select among all imperfect strategies the less risky. Föllmer and Sondermann (in [7]) and Föllmer and Schweizer (in [6]) showed that under some assumptions, for any conditional asset H there exists a unique risk-minimizing strategy attaining H. This very nice result is however only an existence result, it does not give the explicit expression of this strategy.

The point of this paper is to explicitly compute the Föllmer-Schweizer optimal strategy in the case of life insurer's liabilities. The tool we use to achieve this goal is the Malliavin calculus. Our main result is the explicit decomposition of the liabilities into two components, the first one being the hedgeable part and the second one being the non-hedgeable part. Additionally to the optimal strategy, it gives us a complete description of both parts and therefore grants us useful information about the non-headgeable component of the liabilities. Hence we can compute, among many other quantities, the *risk margin* of the insurer's balance sheet as defined by the Solvency II directive.

This paper is structured as follows. In Section 2 we describe the liabilities we consider throughout the analysis. We then present the mortality model and the underlying assumptions in Section 4. Our main decomposition result comes in two distinct variants. The first one, considering stocks as available financial instruments, is described in Section 5, while the second one, considering bonds as available instruments, is described in Section 6. A recapitulation of the assumptions made all along the paper is given in Section 7. In Section 8, the error generated by our main mortality assumption is analysed. A final conclusion is given in the last section, while the Appendix contains a satellite technical lemma.

Notations. Throughout this paper we consider a probability space $(\Omega, \mathbb{P})$. We will denote by W, W^1 and W^2 standard Brownian motions on it. The space $(\Omega, \mathbb{P})$ is equipped with the filtration $\mathscr{F}$ which is associated with the values of W^1 and W^2: $\mathscr{F}_t = \sigma\left(\{W_s^1 : s \in [0,1]\} \cup \{W_s^2 : s \in [0,1]\}\right)$. We will denote as usual the normal density and cumulative distribution functions by φ and Φ respectively. The notation $\mathbb{D}^{1,2}$ stands for the set of all Malliavin-differentiable random variables, and D, D^1 and D^2 are the Malliavin derivative operators along W, W^1 and W^2 respectively. More details about these definitions and more generally about the Malliavin calculus can be found in [11] and [12].

2 Expression of the liabilities

2.1 Pure endowments with profit sharing

We consider the liabilities of an insurance company selling life insurance contracts, or more precisely 1 year pure endowments. We consider contracts running from $t = 0$ to $t = 1$ and offering to the insured a rate i on a capital C_0. The amount payable on survival is thus $C_0 e^i = C_1$. The liabilities of the insurer consist of two components: on one hand the guaranteed capital C_1 paid on survival, on the other hand the profit sharing. We consider here a profit sharing based on the financial results of the insurer only, i.e. we do not incorporate the mortality benefits in it, assuming that the premium paid by the insured has been fairly computed.
Let us begin by determining the amount the insured will receive at $t = 1$. Denoting by ℓ_x the proportion of surviving insured in the portfolio with age x (that we will assume to be continuously differentiable, the force of mortality is defined as $\mu_x = -\frac{\partial \ln \ell_x}{\partial x}$. We consider a stochastic mortality model, meaning that the force of mortality randomly changes with time: $\mu_{x+s}(s) = \mu_{x+s}(s, w)$ (where $w \in \Omega$ and $s \in [0, 1]$). This is the actual modelled quantity, i.e. it is for μ that we shall choose a model later on.
As the preceding definition implies

$$\ell_x(t) = \exp\left(-\int_0^t \mu_{x+s}(s)\,\mathrm{d}s\right),$$

the total number of surviving individuals in the portfolio is

$$P_t = \sum_{x=\omega_{\text{min}}}^{\omega_{\text{max}}} N_x \ell_x(t) = \sum_{x=\omega_{\text{min}}}^{\omega_{\text{max}}} N_x \exp\left(-\int_0^t \mu_{x+s}(s)\,\mathrm{d}s\right),$$

denoting by N_x the number of insured with age x in the initial portfolio and by ω_{min} (resp. ω_{max}) the age of the youngest (resp. oldest) person in the portfolio.
If our study did not include the profit sharing, the amount payable by the insurer at the end at the year would thus be $L_1 = P_1 C_0 e^i$. However we consider a framework where the insurer shares his profit with his clients, giving them a capital surplus, which is proportional to the difference between the returns generated by the assets and the liabilities (when this difference is positive). The total amount to be paid by the company at the end of the year is hence

$$L_1 = C_0 e^i P_1 \left(1 + \beta \left(\frac{S_1}{S_0} - \frac{P_1 e^i}{P_0}\right)_+\right),$$

where β is the PS rate, i.e. the proportion of shared benefit, and S_t is the market value of the financial asset held by the insurer.

2.2 Reference age assumption

Unhappily, the plurality of the ages contained in the portfolio raises technical problems. As will be seen in the following, the considered mortality model will assess to $\ell_x(t)$ a log-normal distribution.

In the case of a portfolio containing insured with different ages, we would thus work with a sum of several log-normally distributed stochastic processes, a framework which is very difficult to handle (even in the case of the sum of only two log-normally distributed processes, no closed formula exists for the distribution, and the numerical methods to overcome the difficulty are not very efficient, see [2]).
For this reason, we assume that the effect of time is equal for every age, i.e. that we can write

$$\mu_{x+s}(s) = \kappa(x) + \mu_{\hat{x}+s}(s),$$

where κ is a deterministic function of the age, and $\hat{x}$ is a reference age chosen among the age spectrum of the portfolio. Hence we obtain

$$\begin{aligned} P_t &= \sum_{x=\omega_{\text{min}}}^{\omega_{\text{max}}} N_x \exp\left(-\int_0^t \mu_{x+s}(s)\,\mathrm{d}s\right) \\ &= \sum_{x=\omega_{\text{min}}}^{\omega_{\text{max}}} N_x \exp\left(-t\kappa(x)\right) \exp\left(-\int_0^t \mu_{\hat{x}+s}(s)\,\mathrm{d}s\right) \\ &= G(t) \exp\left(-\int_0^t \mu_{\hat{x}+s}(s)\,\mathrm{d}s\right), \end{aligned}$$

where G is a deterministic function. The impact of the assumption on the modelling is now very understandable: as $\ell_{\hat{x}}(t)$ is log-normally distributed, so is P_t. It is clear that if the portfolio contains individuals with the same age, choosing $\kappa \equiv 0$ and $\hat{x} = x$ does the job. In the case where $\kappa \equiv 0$, $\hat{x}$ can really be viewed as the age with average survival index:

$$\sum_{x=\omega_{\text{min}}}^{\omega_{\text{max}}} N_x \exp\left(-\int_0^t \mu_{x+s}(s)\,\mathrm{d}s\right) = \sum_{x=\omega_{\text{min}}}^{\omega_{\text{max}}} N_x \exp\left(-\int_0^t \mu_{\hat{x}+s}(s)\,\mathrm{d}s\right)$$

whence

$$\exp\left(-\int_0^t \mu_{\hat{x}+s}(s)\,\mathrm{d}s\right) = \sum_{x=\omega_{\text{min}}}^{\omega_{\text{max}}} \frac{N_x}{\sum_x N_x} \exp\left(-\int_0^t \mu_{x+s}(s)\,\mathrm{d}s\right).$$

Let us finally mention that this assumption has never been formulated in the literature to our best knowledge. Section 8 is devoted to the analysis of the error it introduces.

3 Liabilities decomposition

The crux of this paper is the following decomposition of the insurer's liabilities. As explained supra, we will apply to L_1 the Föllmer-Schweizer decomposition and then compute the resulting terms using the Malliavin calculus.
From now on, we work under a particular probability measure. More precisely, we select one of the martingale equivalent measures (being the risk neutral measure in Section 5, and the forward neutral measure in Section 6). The stochastic process $\tilde{S}$ we will handle, which is related to the financial asset S of the insurer (in the first case it is the actualized stock price, in the second case

it is the forward price of the bond) is therefore a martingale under this measure: there exists a process K and a (maybe multidimensional) standard Brownian motion W^S such that

$$\tilde{S}_t = \tilde{S}_0 + \int_0^t K_u \,\mathrm{d}W_u^S.$$

On one hand, the Föllmer-Schweizer decomposition (which actually reduces to the Kunita-Watanabe decomposition as $\tilde{S}$ is a martingale, see [14]) allows to write

$$L_1 = \mathbb{E}[L_1] + \int_0^1 \tilde{\theta}_s \,\mathrm{d}\tilde{S}_s + I_1 = \mathbb{E}[L_1] + \int_0^1 \theta_s \,\mathrm{d}W_s^S + I_1,$$

where $\tilde{\theta}$, θ and I_1 are martingales enjoying the following properties: $\tilde{\theta}$ is square-integrable with respect to $\tilde{S}$, θ is square-integrable with respect to W^S and I_1 is square-integrable and orthogonal to all integrals of the form $\int \theta \,\mathrm{d}W^S$. The term I_1, only different from 0 when the market is incomplete, stands for the component of L_1 which is not hedgeable on the financial markets (thanks to its orthogonality property).
We now make a crucial hypothesis: we assume that the orthogonal term I_1 lives in a Gaussian universe, i.e. that there exists a process ξ and a (maybe multidimensional) standard Brownian motion W^I such that

$$I_1 = \int_0^1 \xi_s \,\mathrm{d}W_s^I.$$

We thus obtain the decomposition

$$L_1 = \mathbb{E}[L_1] + \int_0^1 \theta_s \,\mathrm{d}W_s^S + \int_0^1 \xi_s \,\mathrm{d}W_s^I. \tag{1}$$

On the other hand, we can consider the random variable L_1 as a functional of the n-dimensional Brownian motion $W = (W^1, \ldots, W^n)$. Assuming at this point that L_1 is sufficiently regular (this fact will be a straightforward consequence of the chosen models for S and μ, see infra), we can apply to it one of the master results of the Malliavin calculus, the Clark-Ocone formula (see [11]):

$$L_1 = \mathbb{E}[L_1] + \int_0^1 \mathbb{E}[D_t L_1 | \mathscr{F}_t] \,\mathrm{d}W_t = \mathbb{E}[L_1] + \sum_{i=1}^n \int_0^1 \mathbb{E}\left[D_t^i L_1 | \mathscr{F}_t\right] \mathrm{d}W_t^i. \tag{2}$$

As the Kunita-Watanabe decomposition is unique, there exists an integer k such that, upon relabelling of the indices,

$$\begin{cases} W^S = (W^1, \ldots, W^k), \\ W^I = (W^{k+1}, \ldots, W^n). \end{cases}$$

The computation of the Malliavin derivatives therefore gives the explicit expression of the integrands of the decomposition into hedgeable and non-hedgeable parts.

4 Stochastic mortality model

Stochastic mortality models drive a lot of attention since the last decade. Among other authors considering this topic, see [4] and [5].
We assume that the force of mortality process follows the same dynamics under the historical probability measure as under the martingale equivalent probability measure. This means that the mortality is not affected by the evolution of the financial market, its behaviour being the same regardless the world (real, risk-neutral of forward-neutral) in which it is observed. This assumption seems rather natural on short and midterms: the market conditions should not influence the population's rate of death. However it becomes much more doubtful when considering the long term. One could indeed argue that bad economic conditions lead to bad life quality and thus bad health care, which of course impacts the mortality. Everything is fine since we only consider here one year contracts, but this limit should be kept in mind when trying to generalize our results to longer contracts.
We impose now a dynamics to μ in order to explicitly compute the decomposition terms. The model we have chosen for the force of mortality is a one factor Vasicek model (for a discussion about the mortality models, see e.g. [4]). As $\hat{x}$ has been previously chosen and won't change, we write $\mu_{\hat{x}+t}(t) = \mu_t$ when the context is clear.

Vasicek model for the force of mortality

$$\mathrm{d}\mu_t = a(\theta - \mu_t)\,\mathrm{d}t + \sigma_\mu \,\mathrm{d}W_t^2,$$

$$\mu_t = \mu_s e^{-a(t-s)} + \theta\left(1 - e^{-a(t-s)}\right) + \sigma_\mu \int_s^t e^{-a(t-u)}\,\mathrm{d}W_u^2,$$

$$\mu_t|\mu_s \sim \mathcal{N}\left(\mu_s e^{-a(t-s)} + \theta\left(1 - e^{-a(t-s)}\right), \frac{\sigma_\mu^2}{2a}\left(1 - e^{-2a(t-s)}\right)\right),$$

$\mu \in \mathbb{D}^{1,2}$ and $D_s\mu_t = \sigma_\mu e^{-a(t-s)}\chi_{s\leqslant t}$.

Parameters: $\mu_0, a, \theta, \sigma_\mu > 0$.

Remark. The Vasicek dynamics allows the modelled process to become negative. This would result in an upward jump for the number of living individuals, which of course does not make any sense. However the probability of such an event is rather low, and can be explicitly computed as in the case of interest rates (see e.g. [3]). The scenarios giving rise to non-monotone populations will thus be excluded from the numerical computations.
The force of mortality μ only appears in the decomposition through another quantity, the process $\ell_t = \exp\left(-\int_0^t \mu_s\,\mathrm{d}s\right)$. The following result gives the probability distribution of the latter.

Lemma 1. *One has, for every* $0 \leqslant v \leqslant t$

$$\ell_t = \ell_v \exp\left(\frac{\mu_0 - \theta}{a}(e^{-at} - e^{-av}) - \theta(t - v)\right)\exp\left(-\sigma_\mu \int_v^t \frac{1 - e^{-a(t-u)}}{a}\,dW_u^2\right)$$

and therefore

$$\frac{\ell_t}{\ell_v}\Big|\ell_v \sim \log\mathcal{N}\left(\frac{\mu_0-\theta}{a}(e^{-at}-e^{-av})-\theta(t-v), \int_v^t \frac{\sigma_\mu^2}{a^2}\left(1-e^{-a(t-u)}\right)^2 du\right).$$

Moreover $\ell \in \mathbb{D}^{1,2}$ *and*

$$D_s\ell_t = \ell_t\frac{\sigma_\mu}{a}(e^{-a(t-s)}-1)\chi_{s\leqslant t}.$$

Proof. One can write

$$\begin{aligned}\ell_t &= \exp\left(-\int_0^t \mu_s\,\mathrm{d}s\right)\\ &= \ell_v\exp\left(-\int_v^t \mu_s\,\mathrm{d}s\right)\\ &= \ell_v\exp\left(-\int_v^t \left(\mu_0 e^{-as}+\theta\left(1-e^{-as}\right)\right)\mathrm{d}s\right)\\ &\qquad\qquad\cdot\exp\left(-\sigma_\mu\int_v^t\int_0^s e^{-a(s-u)}\,\mathrm{d}W_u^2\,\mathrm{d}s\right).\end{aligned}$$

The multi-dimensional Ito formula for the product of two processes applied to $\int_0^s e^{au}\,\mathrm{d}W_u^2$ and $-\frac{e^{-as}}{a}$ gives

$$\begin{aligned}-\frac{e^{-at}}{a}\int_0^t e^{au}\,\mathrm{d}W_u^2 &= -\frac{e^{-av}}{a}\int_0^v e^{au}\,\mathrm{d}W_u^2+\int_v^t\int_0^s e^{au}\,\mathrm{d}W_u^2\,\mathrm{d}\left(-\frac{e^{-as}}{a}\right)\\ &\quad -\int_v^t\frac{e^{-as}}{a}\,\mathrm{d}\left(\int_0^s e^{au}\,\mathrm{d}W_u^2\right)\\ &= \int_v^t\int_0^s e^{-a(s-u)}\,\mathrm{d}W_u^2\,\mathrm{d}s-\int_v^t\frac{e^{-as}}{a}e^{as}\,\mathrm{d}W_s^2.\end{aligned}$$

Hence one obtains

$$\int_v^t\int_0^s e^{-a(s-u)}\,\mathrm{d}W_u^2\,\mathrm{d}s=\int_v^t\frac{1-e^{-a(t-u)}}{a}\,\mathrm{d}W_u^2=\frac{1}{a}\left(W_t-W_v-\int_v^t e^{-a(t-u)}\,\mathrm{d}W_u^2\right)$$

which allows to write

$$\ell_t=\ell_v\exp\left(\frac{\mu_0-\theta}{a}(e^{-at}-e^{-av})-\theta(t-v)\right)\exp\left(-\sigma_\mu\int_v^t\frac{1-e^{-a(t-u)}}{a}\,\mathrm{d}W_u^2\right).$$

We thus know the distribution of ℓ_t/ℓ_v conditionally to ℓ_v at every instant $t\geqslant v$:

$$\frac{\ell_t}{\ell_v}\Big|\ell_v \sim \log\mathcal{N}\left(\frac{\mu_0-\theta}{a}(e^{-at}-e^{-av})-\theta(t-v), \int_v^t \frac{\sigma_\mu^2}{a^2}\left(1-e^{-a(t-u)^2}\right)\mathrm{d}u\right).$$

The Malliavin derivative gives:

$$\begin{aligned}D_s^2\left(\int_0^t(1-e^{-a(t-u)})\,\mathrm{d}W_u^2\right) &= D_s^2W_t-e^{-at}D_s^2\int_0^t e^{au}\,\mathrm{d}W_u^2\\ &= \chi_{s\leqslant t}-e^{-at}e^{as}\chi_{s\leqslant t},\end{aligned}$$

whence the result follows using the chain rule (see e.g. [11] for details about Malliavin calculus results). □

5 First model: the asset is a stock

5.1 Model specification

In this section, we only work under the risk-neutral measure. We assume that the insurer's financial asset is a stock modelled by a geometric Brownian motion.

GBM model for the financial asset (in the risk-neutral world)

$$\begin{aligned}
&\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma_S S_t\,\mathrm{d}W_t^1,\\
&S_t = S_s \exp\left((r-\frac{1}{2}\sigma_S^2)(t-s) + \sigma_S(W_t^1 - W_s^1)\right),\\
&\frac{S_t}{S_s}\Big| S_s \sim \log\mathcal{N}\left((r-\frac{1}{2}\sigma_S^2)(t-s), \sigma_S^2(t-s)\right),\\
&S \in \mathbb{D}^{1,2} \text{ and } D_s S_t = \sigma_S S_t \chi_{s\le t}.
\end{aligned}$$

Parameters: $S_0, r, \sigma_S > 0$.

As the asset S itself is not a martingale, the quantity of interest in the following will be the actualization of S:

$$T_t = e^{-rt} S_t.$$

It is straightforward to check that

$$\begin{aligned}
&\mathrm{d}T_t = \sigma_S S_t\,\mathrm{d}W_t^1, && T_0 = S_0,\\
&T_t = T_s \exp\left(-\frac{1}{2}\sigma_S^2(t-s) + \sigma_S(W_t^1 - W_s^1)\right), && \frac{T_t}{T_s}\Big| T_s \sim \log\mathcal{N}\left(-\frac{1}{2}\sigma_S^2(t-s), \sigma_S^2(t-s)\right),\\
&T \in \mathbb{D}^{1,2}, && D_s T_t = \sigma_S T_t \chi_{s\le t}.
\end{aligned}$$

The insurer's liabilities can then be rewritten as

$$C_0 e^i P_1 \left(1 + \beta\left(\frac{e^r T_1}{S_0} - \frac{P_1 e^i}{P_0}\right)_+\right).$$

It is easy to show that $L_1 \in \mathbb{D}^{1,2}$.

5.2 Explicit decomposition

The following decomposition is rather technical. Corollary 3 shows the practical benefits of it. In the following, we denote by $\mathbb{E}$ the expectation under the risk-neutral probability $\mathbb{P}^*$.

Theorem 2. *The Föllmer-Schweizer decomposition of L_1 is given by*

$$L_1 = \mathbb{E}[L_1] + \int_0^1 \mathbb{E}\left[D_s^1 L_1 | \mathscr{F}_s\right] dW_s^1 + \int_0^1 \mathbb{E}\left[D_s^2 L_1 | \mathscr{F}_s\right] dW_s^2$$

$$
\begin{aligned}
&= C_0 e^i G(1)\Bigg(\exp\left(\nu_0 + \frac{1}{2}\tau_0^2\right) \\
&\quad + \beta e^r \exp\left(-\nu_0 - \frac{1}{2}\tau_0^2\right)\Phi\left(\frac{\beta_0 + \alpha_0(\nu_0+\tau_0^2)}{\sqrt{1+2\alpha_0^2\tau_0^2}}\right) \\
&\quad - \beta e^i \exp\left(-2\nu_0 - 2\tau_0^2\right)\Phi\left(\frac{\beta_0 + 2\alpha_0(2\nu_0+4\tau_0^2)}{\sqrt{1+32\alpha_0^2\tau_0^2}}\right)\Bigg) \\
&\quad + \int_0^1 \Bigg[\frac{C_0\beta\sigma_S e^{i+r} G(1) T_s \ell_s}{S_0} \\
&\qquad\qquad \cdot \exp\left(-\nu_s - \frac{1}{2}\tau_s^2\right)\Phi\left(\frac{\beta_s + \alpha_s(\nu_s+\tau_s^2)}{\sqrt{1+2\alpha_s^2\tau_s^2}}\right)\Bigg] dW_s^1 \\
&\quad + \int_0^1 \Bigg[\frac{C_0 e^i G(1)\sigma_\mu(e^{-a(1-s)}-1)}{a}\Bigg(\ell_s \exp\left(\nu_s + \frac{1}{2}\tau_s^2\right) \\
&\qquad\qquad + \frac{\beta e^r T_s \ell_s}{S_0}\exp\left(-\nu_s - \frac{1}{2}\tau_s^2\right)\Phi\left(\frac{\beta_s + \alpha_s(\nu_s+\tau_s^2)}{\sqrt{1+2\alpha_s^2\tau_s^2}}\right) \\
&\qquad\qquad - 2\beta e^i \ell_s^2 \exp\left(-2\nu_s - 2\tau_s^2\right) \\
&\qquad\qquad\qquad \cdot \Phi\left(\frac{\beta_s + 2\alpha_s(2\nu_s+4\tau_s^2)}{\sqrt{1+32\alpha_s^2\tau_s^2}}\right)\Bigg)\Bigg] dW_s^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_s &= \frac{-1}{\sigma_S\sqrt{1-s}}, \\
\beta_s &= \frac{\ln \frac{T_s}{S_0 e^{i-r} G(1)\ell_s} + \frac{\sigma_S^2}{2}(1-s)}{\sigma_S\sqrt{1-s}}, \\
\nu_s &= \frac{\mu_0-\theta}{a}(e^{-a} - e^{-as}) - \theta(1-s), \\
\tau_s^2 &= \int_s^1 \frac{\sigma_\mu^2}{a^2}\left(1 - e^{-a(1-u)}\right)^2 du.
\end{aligned}
$$

Proof. Let us compute successively every term of the decomposition. The liabilities L_1 can be viewed as a functional of the two Brownian motions W^1 and W^2, so that we will consider Malliavin derivatives in two different “directions".

Derivative along $\mathbf{W^1}$. As the function $x \mapsto (x-K)_+$ is Lipschitz, the chain rule allows to compute

$$\begin{aligned} D_s^1 L_1 &= D_s^1\left(C_0 e^i P_1\left(1+\beta\left(\frac{e^r T_1}{S_0}-\frac{P_1 e^i}{P_0}\right)_+\right)\right) \\ &= C_0 e^i P_1 \beta D_s^1\left(\frac{e^r T_1}{S_0}-\frac{P_1 e^i}{P_0}\right)_+ \\ &= \frac{C_0 e^i P_1 \beta e^r}{S_0}\chi_M D_s^1 T_1 \\ &= \frac{C_0 e^i P_1 \beta e^r}{S_0}\chi_M \sigma_S T_1 \chi_{s\leqslant 1} \end{aligned}$$

denoting by M the set $\{x \in \Omega : T_1(x) \geqslant S_0 P_1(x) e^{i-r}\}$. Let $s \leqslant 1$. The derivative appears in the Clark-Ocone formula through its conditional expectation:

$$\mathbb{E}\left[D_s^1 L_1 | \mathscr{F}_s\right] = \frac{C_0 e^i \beta \sigma_S G(1) e^r}{S_0}\mathbb{E}\left[\ell_1 S_1 \chi_M | \mathscr{F}_s\right]. \tag{3}$$

In order to treat successively the two sources of randomness, let us define

$$\tilde{\mathscr{F}}_{(s,1)} = \sigma\left(\{W_t^1 : t \in [0,s]\} \cup \{W_t^2 : t \in [0,1]\}\right).$$

This new σ-algebra stands for the information available about W^1 until the instant $t = s$ and the information available about W^2 until the instant $t = 1$. By the law of iterated expectations, we have, as $\mathscr{F}_s \subset \tilde{\mathscr{F}}_{(s,1)}$,

$$\mathbb{E}\left[D_s^1 L_1 | \mathscr{F}_s\right] = \mathbb{E}\left[\mathbb{E}\left[D_s^1 L_1 \middle| \tilde{\mathscr{F}}_{(s,1)}\right] \middle| \mathscr{F}_s\right]. \tag{4}$$

We therefore first consider the expectation with respect to the new filtration. The crucial point is of course that P_1 is $\tilde{\mathscr{F}}_{(s,1)}$-measurable:

$$\mathbb{E}\left[D_s^1 L_1 \middle| \tilde{\mathscr{F}}_{(s,1)}\right] = \mathbb{E}\left[\frac{C_0 e^i P_1 \beta}{S_0}\chi_M \sigma_S e^r T_1 \middle| \tilde{\mathscr{F}}_{(s,1)}\right] = \frac{C_0 e^i P_1 \beta \sigma_S e^r}{S_0}\mathbb{E}\left[\chi_M T_1 \middle| \tilde{\mathscr{F}}_{(s,1)}\right]. \tag{5}$$

As in the classical derivation of the Black-Scholes formula (see e.g. [10, Theorem 3.1.1]), we define an auxiliary measure on $(\Omega, \mathscr{F})$ with the help of the Radon-Nikodym density: $\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}^*} = \lambda_1$, where the process λ is given by

$$\lambda_s = \exp\left(-\frac{\sigma_S^2}{2}s + \sigma_S W_s^1\right).$$

By Girsanov Theorem, the process $\tilde{W}_t^1 = W_t^1 - \sigma_S t$ is a standard Brownian motion under the new measure $\tilde{\mathbb{P}}$. Note that the asset's dynamics is now

$$\mathrm{d}T_t = T_t\left(\sigma_S^2 \mathrm{d}t + \sigma_S \mathrm{d}\tilde{W}_t^1\right),$$

or

$$T_t = T_s \exp\left(\frac{\sigma_S^2}{2}(t-s) + \sigma_S(\tilde{W}_t^1 - \tilde{W}_s^1)\right). \tag{6}$$

We can now compute the conditional expectation:

$$\begin{aligned}\mathbb{E}\left[T_1\chi_M|\tilde{\mathscr{F}}_{(s,1)}\right] &= \mathbb{E}\left[T_s\exp\left(-\frac{\sigma_S^2}{2}(1-s)+\sigma_S(W_1^1-W_s^1)\right)\chi_M\Big|\tilde{\mathscr{F}}_{(s,1)}\right]\\ &= T_s\mathbb{E}\left[\exp\left(-\frac{\sigma_S^2}{2}(1-s)+\sigma_S W_{1-s}^1\right)\chi_M\Big|\tilde{\mathscr{F}}_{(s,1)}\right]\\ &= T_s\mathbb{E}\left[\lambda_1\lambda_s^{-1}\chi_M\Big|\tilde{\mathscr{F}}_{(s,1)}\right]\\ &= T_s\mathbb{E}_{\tilde{\mathbb{P}}}\left[\chi_M\big|\tilde{\mathscr{F}}_{(s,1)}\right]\\ &= T_s\tilde{\mathbb{P}}\left[T_1\geqslant S_0P_1e^{i-r}|T_s,P_1\right].\end{aligned}\tag{7}$$

as the Bayes Theorem implies (see e.g. [10, Lemma 9.6.2])

$$\mathbb{E}_{\mathbb{P}^*}\left[\lambda_1\lambda_s^{-1}\chi_M\big|\tilde{\mathscr{F}}_{(s,1)}\right]=\frac{\mathbb{E}_{\mathbb{P}^*}\left[\chi_M\lambda_1\big|\tilde{\mathscr{F}}_{(s,1)}\right]}{\mathbb{E}_{\mathbb{P}^*}\left[\lambda_1\big|\tilde{\mathscr{F}}_{(s,1)}\right]}=\mathbb{E}_{\tilde{\mathbb{P}}}\left[\chi_M\big|\tilde{\mathscr{F}}_{(s,1)}\right].$$

We now use the dynamics of T under $\tilde{\mathbb{P}}$ given by (6):

$$\begin{aligned}\tilde{\mathbb{P}}\left[T_1\geqslant S_0P_1e^{i-r}|T_s,P_1\right] &= \tilde{\mathbb{P}}\left[T_s\exp\left(\frac{\sigma_S^2}{2}(1-s)+\sigma_S(\tilde{W}_1^1-\tilde{W}_s^1)\right)\geqslant S_0P_1e^{i-r}\Big|T_s,P_1\right]\\ &= \tilde{\mathbb{P}}\left[\frac{\tilde{W}_{1-s}^1}{\sqrt{1-s}}\geqslant\frac{\ln\frac{S_0P_1e^{i-r}}{T_s}-\frac{\sigma_S^2}{2}(1-s)}{\sigma_S\sqrt{1-s}}\Bigg|T_s,P_1\right]\\ &= 1-\Phi\left(\frac{\ln\frac{S_0P_1e^{i-r}}{T_s}-\frac{\sigma_S^2}{2}(1-s)}{\sigma_S\sqrt{1-s}}\right)\\ &= \Phi\left(\frac{\ln\frac{T_s}{S_0P_1e^{i-r}}+\frac{\sigma_S^2}{2}(1-s)}{\sigma_S\sqrt{1-s}}\right),\end{aligned}$$

as $\tilde{W}_1$ is a standard Brownian motion under $\tilde{\mathbb{P}}$. Gathering (5), (7) and the last equation, we obtain

$$\mathbb{E}\left[D_s^1L_1\big|\tilde{\mathscr{F}}_{(s,1)}\right]=\frac{C_0e^iP_1\beta\sigma_ST_se^r}{S_0}\Phi\left(\frac{\ln\frac{T_s}{S_0P_1e^{i-r}}+\frac{\sigma_S^2}{2}(1-s)}{\sigma_S\sqrt{1-s}}\right).$$

Let us come back to the conditional expectation with respect to $\mathscr{F}_s$ by recalling (4):

$$\mathbb{E}\left[D_s^1L_1\big|\mathscr{F}_s\right]=\mathbb{E}\left[\frac{C_0e^iP_1\beta\sigma_ST_se^r}{S_0}\Phi\left(\frac{\ln\frac{T_s}{S_0P_1e^{i-r}}+\frac{\sigma_S^2}{2}(1-s)}{\sigma_S\sqrt{1-s}}\right)\Bigg|\mathscr{F}_s\right]$$

$$= \frac{C_0 e^i \beta \sigma_S T_s e^r}{S_0} \mathbb{E}\left[P_1 \Phi\left(\frac{\ln \frac{T_s}{S_0 P_1 e^{i-r}} + \frac{\sigma_S^2}{2}(1-s)}{\sigma_S \sqrt{1-s}} \right) \middle| \mathscr{F}_s \right].$$

We can rewrite this expectation as

$$\mathbb{E}\left[P_1 \Phi\left(\frac{\ln \frac{T_s}{S_0 P_1 e^{i-r}} + \frac{\sigma_S^2}{2}(1-s)}{\sigma_S \sqrt{1-s}} \right) \middle| \mathscr{F}_s \right]$$
$$= G(1)\ell_s \mathbb{E}\left[\frac{\ell_1}{\ell_s} \Phi\left(\frac{-1}{\sigma_S\sqrt{1-s}} \ln\left(\frac{\ell_1}{\ell_s}\right) + \frac{\ln \frac{T_s}{S_0 e^{i-r} G(1) \ell_s} + \frac{\sigma_S^2}{2}(1-s)}{\sigma_S \sqrt{1-s}} \right) \middle| \mathscr{F}_s \right].$$

Lemma 1 (see the Appendix) ensures that ℓ_1/ℓ_s is log-normally distributed conditionally to ℓ_s. We thus have to compute an expression looking like

$$\mathbb{E}\left[R\Phi(\alpha \ln R + \beta) \right], \tag{8}$$

where R is log-normally distributed and $\alpha, \beta > 0$. Lemma 9 allows to obtain a closed formula for this expectation. Gathering all the factors, we obtain

$$\mathbb{E}\left[D_s^1 L_1 \middle| \mathscr{F}_s \right] = \frac{C_0 e^i \beta \sigma_S e^r G(1) T_s \ell_s}{S_0} \cdot \exp\left(-\nu_s - \frac{1}{2}\tau_s^2 \right) \Phi\left(\frac{\beta_s + \alpha_s(\nu_s + \tau_s^2)}{\sqrt{1 + 2\alpha_s^2 \tau_s^2}} \right). \tag{9}$$

Derivative along W^2. As the function $x \mapsto (K-x)_+$ is Lipschitz, the chain rule implies

$$\begin{aligned}
D_s^2 L_1 &= C_0 e^i D_s^2 \left(P_1 \left(1 + \beta \left(\frac{e^r T_1}{S_0} - \frac{P_1 e^i}{P_0} \right)_+ \right) \right) \\
&= C_0 e^i \left(D_s^2(P_1) \left(1 + \beta \left(\frac{e^r T_1}{S_0} - \frac{P_1 e^i}{P_0} \right)_+ \right) - \frac{P_1 \beta e^i D_s^2(P_1) \chi_M}{P_0} \right) \\
&= C_0 e^i D_s^2(P_1) \left(1 + \beta \left(\frac{e^r T_1}{S_0} - \frac{P_1 e^i}{P_0} \right)_+ - \frac{P_1 \beta e^i \chi_M}{P_0} \right) \\
&= C_0 e^i D_s^2(P_1) \left(1 + \beta \left(\frac{e^r T_1}{S_0} - \frac{P_1 e^i}{P_0} \right)_+ - \frac{P_1 \beta e^i \chi_M}{P_0} \right) \\
&= \frac{C_0 e^i G(1) \sigma_\mu (e^{-a(1-s)} - 1)}{a} \ell_1 \left(1 + \beta \left(\frac{e^r T_1}{S_0} - 2\ell_1 e^i \right) \chi_M \right) \\
&= \frac{C_0 e^i G(1) \sigma_\mu (e^{-a(1-s)} - 1)}{a} \left(\ell_1 + e^r \ell_1 T_1 \frac{\beta}{S_0} \chi_M - 2\beta e^i \ell_1^2 \chi_M \right) \\
&= \frac{C_0 e^i G(1) \sigma_\mu (e^{-a(1-s)} - 1)}{a} (A + B - C).
\end{aligned}$$

Let us successively compute the conditional expectation of the three non-deterministic terms A, B and C.
The first one is not a problem, since it simply is the expectation of a log-normal random variable:

$$\mathbb{E}\left[A|\mathscr{F}_s\right] = \ell_s \mathbb{E}\left[\frac{\ell_1}{\ell_s}\middle|\mathscr{F}_s\right] = \ell_s \exp\left(\nu_s + \frac{1}{2}\tau_s^2\right).$$

The second one brings back to the computation performed for the derivative along W^1 (see Equation (3)):

$$\mathbb{E}\left[B\middle|\mathscr{F}_s\right] = \frac{\beta e^r}{S_0}\mathbb{E}\left[\ell_1 T_1 \chi_M|\mathscr{F}_s\right] = \frac{\beta e^r T_s \ell_s}{S_0}\exp\left(-\nu_s - \frac{1}{2}\tau_s^2\right)\Phi\left(\frac{\beta_s + \alpha_s(\nu_s + \tau_s^2)}{\sqrt{1+2\alpha_s^2\tau_s^2}}\right).$$

The third one can be treated in a way similar to the derivative along W^1, i.e. using the law of iterated expectations and the technical lemma of the appendix. We write

$$\mathbb{E}\left[C|\mathscr{F}_s\right] = 2\beta e^i \ell_s^2 \mathbb{E}\left[\left(\frac{\ell_1}{\ell_s}\right)^2 \chi_M\middle|\mathscr{F}_s\right] = 2\beta e^i \ell_s^2 \mathbb{E}\left[\mathbb{E}\left[\left(\frac{\ell_1}{\ell_s}\right)^2 \chi_M\middle|\tilde{\mathscr{F}}_{(s,1)}\right]\middle|\mathscr{F}_s\right].$$

Considering the expectation with respect to the “partial" σ-algebra $\tilde{\mathscr{F}}_{(s,1)}$:

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\ell_1}{\ell_s}\right)^2 \chi_M\middle|\tilde{\mathscr{F}}_{(s,1)}\right] &= \left(\frac{\ell_1}{\ell_s}\right)^2 \mathbb{E}\left[\chi_M\middle|\tilde{\mathscr{F}}_{(s,1)}\right] \\
&= \left(\frac{\ell_1}{\ell_s}\right)^2 \mathbb{P}\left[T_1 \geqslant S_0 e^{i-r} P_1 | T_s, P_1\right] \\
&= \left(\frac{\ell_1}{\ell_s}\right)^2 \mathbb{P}\left[\frac{W^1_{1-s}}{\sqrt{1-s}} \geqslant \frac{\ln\frac{S_0 P_1 e^{i-r}}{T_s} + \frac{\sigma_S^2}{2}(1-s)}{\sigma_S\sqrt{1-s}}\middle| T_s, P_1\right] \\
&= \left(\frac{\ell_1}{\ell_s}\right)^2 \Phi\left(\frac{\ln\frac{T_s}{S_0 P_1 e^{i-r}} - \frac{\sigma_S^2}{2}(1-s)}{\sigma_S\sqrt{1-s}}\right).
\end{aligned}$$

Hence the expectation with respect to $\mathscr{F}_s$ gives

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{E}\left[\left(\frac{\ell_1}{\ell_s}\right)^2 \chi_M\middle|\tilde{\mathscr{F}}_{(s,1)}\right]\middle|\mathscr{F}_s\right] \\
&\qquad = \mathbb{E}\left[\left(\frac{\ell_1}{\ell_s}\right)^2 \Phi\left(\frac{\ln\frac{T_s}{S_0 P_1 e^{i-r}} - \frac{\sigma_S^2}{2}(1-s)}{\sigma_S\sqrt{1-s}}\right)\middle|\mathscr{F}_s\right] \\
&\qquad = \mathbb{E}\left[\left(\frac{\ell_1}{\ell_s}\right)^2 \Phi\left(\frac{-1}{2\sigma_S\sqrt{1-s}}\ln\left(\frac{\ell_1}{\ell_s}\right)^2 + \frac{\ln\frac{T_s}{S_0 e^{i-r} G(1)\ell_s} - \frac{\sigma_S^2}{2}(1-s)}{\sigma_S\sqrt{1-s}}\right)\middle|\mathscr{F}_s\right].
\end{aligned}$$

As ℓ_1/ℓ_s is log-normally distributed, so is $(\ell_1/\ell_s)^2$ (with mean $2\nu_s$ and variance $4\tau_s^2$). This expression is thus of the form of (8), and we can apply Lemma 9:

$$\mathbb{E}\left[\mathbb{E}\left[\left(\frac{\ell_1}{\ell_s}\right)^2\chi_M\middle|\tilde{\mathscr{F}}_{(s,1)}\right]\middle|\mathscr{F}_s\right]=\exp\left(-2\nu_s-2\tau_s^2\right)\Phi\left(\frac{\beta_s+2\alpha_s(2\nu_s+4\tau_s^2)}{\sqrt{1+32\alpha_s^2\tau_s^2}}\right) \tag{10}$$

with the same notations as previously.
Gathering the conditional expectations of the three terms A, B and C, we obtain the conditional expectation of the Malliavin derivative:

$$\begin{aligned}\mathbb{E}\left[D_s^2L_1|\mathscr{F}_s\right]=\frac{C_0e^iG(1)\sigma_\mu(e^{-a(1-s)}-1)}{a}&\left\{\ell_s\exp\left(\nu_s+\frac{1}{2}\tau_s^2\right)\right.\\&+\frac{\beta e^rT_s\ell_s}{S_0}\exp\left(-\nu_s-\frac{1}{2}\tau_s^2\right)\Phi\left(\frac{\beta_s+\alpha_s(\nu_s+\tau_s^2)}{\sqrt{1+2\alpha_s^2\tau_s^2}}\right)\\&\left.+2\beta e^i\ell_s^2\exp\left(-2\nu_s-2\tau_s^2\right)\Phi\left(\frac{\beta_s+2\alpha_s(2\nu_s+4\tau_s^2)}{\sqrt{1+32\alpha_s^2\tau_s^2}}\right)\right\}.\end{aligned}$$

Expectation. Remark first that combining (9) and (3), we have proved that

$$\mathbb{E}\left[\ell_1T_1\chi_M|\mathscr{F}_s\right]=T_s\ell_s\exp\left(-\nu_s-\frac{1}{2}\tau_s^2\right)\Phi\left(\frac{\beta_s+\alpha_s(\nu_s+\tau_s^2)}{\sqrt{1+2\alpha_s^2\tau_s^2}}\right). \tag{11}$$

Compute the expectation of L_1:

$$\begin{aligned}\mathbb{E}\left[L_1\right]&=\mathbb{E}\left[C_0e^iP_1\left(1+\beta\left(\frac{e^rT_1}{S_0}-\frac{P_1e^i}{P_0}\right)_+\right)\right]\\&=C_0e^iG(1)\mathbb{E}\left[\ell_1\left(1+\beta\left(\frac{e^rT_1}{S_0}-\ell_1e^i\right)_+\right)\right]\\&=C_0e^iG(1)\left(\mathbb{E}\left[\ell_1\right]+\frac{\beta e^r}{S_0}\mathbb{E}\left[\ell_1T_1\chi_M\right]-\beta e^i\mathbb{E}\left[\ell_1^2\chi_M\right]\right).\end{aligned}$$

The first expectation is easy to compute: it is the expectation of a log-normally distributed random variable. The processing of the second and third expectations consists only in taking $s=0$ in (11) and (10) respectively. Hence we obtain

$$\begin{aligned}\mathbb{E}\left[L_1\right]=C_0e^iG(1)&\left(\exp\left(\nu_0+\frac{1}{2}\tau_0^2\right)\right.\\&+\frac{\beta}{S_0}e^rS_0\exp\left(-\nu_0-\frac{1}{2}\tau_0^2\right)\Phi\left(\frac{\beta_0+\alpha_0(\nu_0+\tau_0^2)}{\sqrt{1+2\alpha_0^2\tau_0^2}}\right)\end{aligned}$$

$$-\beta e^{i}\exp\left(-2\nu_0-2\tau_0^2\right)\Phi\left(\frac{\beta_0+2\alpha_0(2\nu_0+4\tau_0^2)}{\sqrt{1+32\alpha_0^2\tau_0^2}}\right)\Bigg).$$

□

5.3 Optimal strategy

The preceding result being rather technical, it is necessary to underline its practical interests. The strategies are given here as 2-dimensional processes $\varphi = (\theta, \eta)$. In such a strategy, θ_t describes the number of units of risky asset held at instant t, and η_t is the amount invested in risk-free asset at time t.

Corollary 3. *Let $g = g(W^1, W^2)$ and $h = h(W^1, W^2)$ be the two integrands of the preceding theorem, i.e. the previous decomposition of L_1 can be expressed as*

$$L_1 = \mathbb{E}[L_1] + \int_0^1 g_s\, dW_s^1 + \int_0^1 h_s\, dW_s^2.$$

The unique risk minimizing strategy (attaining L_1) $\varphi^* = (\theta^*, \eta^*)$ is given by

$$\theta_t^* = \frac{e^{2rt} g_t}{\sigma_S S_t},$$
$$\eta_t^* = e^{-rt}\left(V_t(\varphi^*) - \theta_t^* S_t\right),$$

where the value process V_t writes

$$V_t(\varphi^*) = \mathbb{E}[L_1] + \int_0^t g_s\, dW_s^1 + \int_0^t h_s\, dW_s^2.$$

Moreover the residual risk associated to φ^* is

$$J = \mathbb{V}\left[\int_0^1 h_s\, dW_s^2\right] = \mathbb{E}\left[\int_0^1 (h_s)^2\, ds\right].$$

Proof. The only detail that is not a straightforward consequence of the Föllmer-Sondermann theorem and Theorem 2 and that is left to prove is the expression for θ^*. Recall that we have applied the decomposition to the actualization of the asset process, i.e. $T_t = e^{-rt} S_t$. Using T's dynamics, we have thus

$$\int_0^1 g_s\, \mathrm{d}W_s^1 = \int_0^1 \frac{g_s}{T_s \sigma_S}\, \mathrm{d}T_s.$$

The quantity of the asset T that the insurer should hold at instant s is therefore

$$\frac{g_s}{\sigma_S T_s} = \frac{e^{rs} g_s}{\sigma_S S_s}, \qquad \text{i.e.} \quad \frac{e^{2rs} g_s}{\sigma_S S_s}$$

in terms of S numeraire. □

Remark that, as we have an explicit expression for the non-hedgeable component of the liabilities, it is straightforward to compute various business-related quantities, as e.g. the risk margin defined by the Solvency II directive: it suffices to compute the quantiles of this component by Monte-Carlo simulations, asses the solvency capital (SCR) and finally compute the cost of capital.

5.4 Numerical verification

We have run numerical verifications of the equality of Theorem 2 with the statistical software R. The chosen parameters are those of Table 1.

Contract parameters		Asset parameters		Mortality parameters	
C_0	100	r	0.05	μ_0	0.000797
β	0.75	σ_S	0.1	θ	0
i	0.06	S_0	1	a	-0.051085
G_1	1			σ_μ	0.001343

Table 1: Parameters used in numerical computations

The parameters for the force of mortality μ come from [13, p. 20], where the authors calibrate a Vasicek model for the force of mortality using Italian life insurance premiums. The integrals have been computed with discretization steps of 0.01.
We first present the result of two simulations (i.e. a couple of trajectories of (W^1, W^2)), the first one involving no insurer's profit, the second involving positive profit sharing. The computation of the two sides of the equality is given in Table 2.

Simulated quantity	Simulation 1	Simulation 2
Left hand side (direct computation of L_1)	106.0980414	107.3920
Right hand side (decomposition of L_1)	104.2494689	109.4549
ℓ_1	0.9991937	0.9985033
S_1	0.9465239	1.077441
$\int \ldots \mathrm{d}W^1$	-2.9243965	2.314875
$\int \ldots \mathrm{d}W^2$	-0.0862554	-0.1200882
$SP = (1 + \beta(\ldots)_+)$	0.00000000	0.01289534

Table 2: Results of numerical computations for two chosen simulations

Next we show the evolution of the optimal strategy $\varphi^* = (\theta^*, \eta^*)$ across time in the case of Simulation 2. The results are given in Figure 1, showing the trajectories of the stock price, of the mortality and of the value process V_t.

Finally we present the result of 100 simulations. The R software gives 107.2601 for the right hand side expectation, while it gives 107.0981 for the average value of the left hand side. The empiric mean of the two Ito integrals is, as expected, very close to 0. Remark that approximations are made in the computation of both sides of the equality: stochastic integrals are indeed also present in the left hand side, as one needs to compute ℓ_1. The results, simulation by simulation, are given in Figure 2.

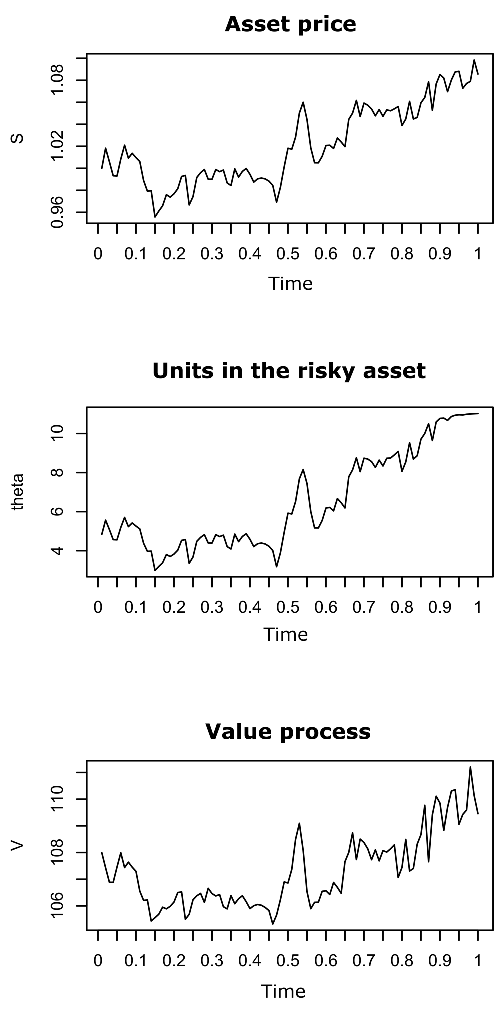

Figure 1: One simulation of the modelled quantities and the evolution of the optimal strategy

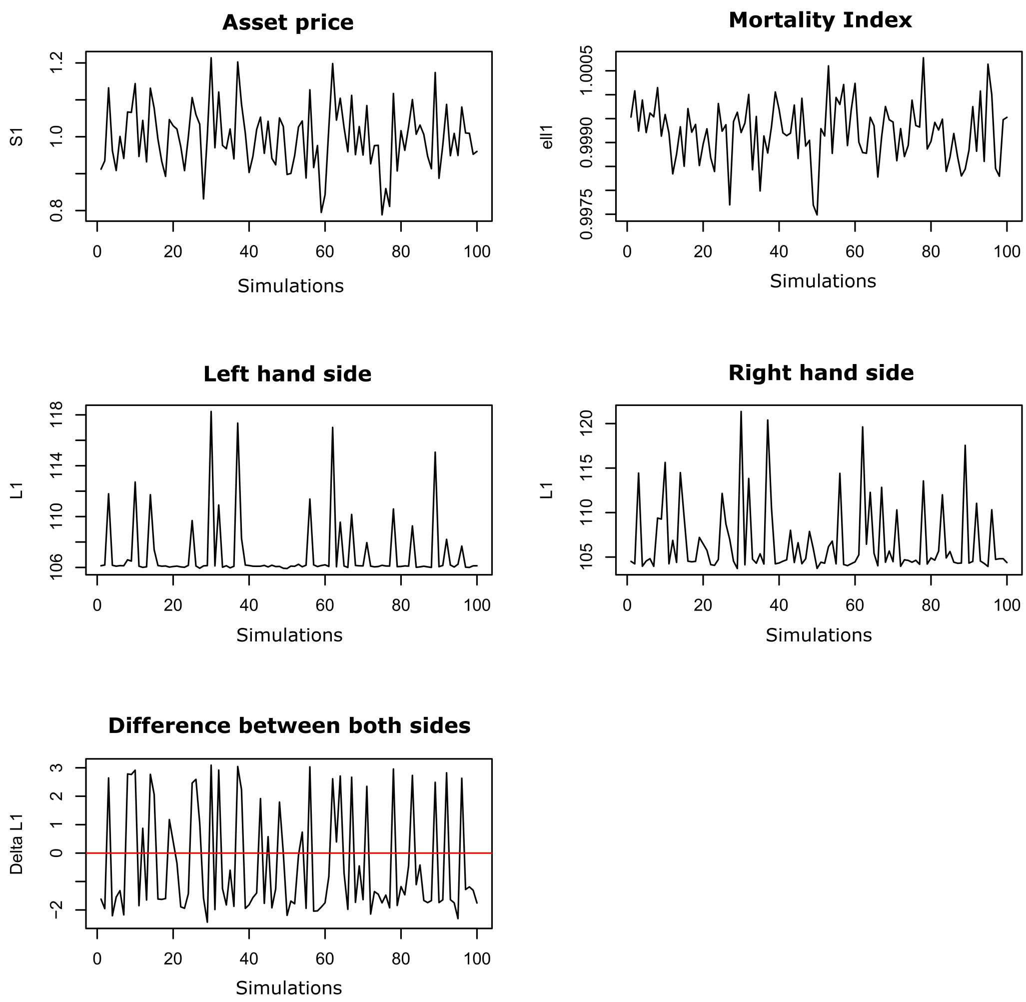

Figure 2: Hundred simulations of the modelled quantities in the first model

6 Second model: the asset is a bond

6.1 Model specification

We now assume that the risked asset available for investment is a zero-coupon bond, and that the interest rate follows a one factor Hull-White dynamics.

Hull-White model for the interest rate (in the risk-neutral world)

$$\mathrm{d}r_t = b(\xi(t) - r_t)\,\mathrm{d}t + \sigma_r\,\mathrm{d}W_t^1,$$
$$r_t = r_s e^{-b(t-s)} + b\int_0^t \xi_u e^{-b(t-u)}\,\mathrm{d}u + \sigma_r \int_0^t e^{-b(t-u)}\,\mathrm{d}W_u^1,$$
$$r_t\Big|r_s \sim \mathcal{N}\left(r_s e^{-b(t-s)} + b\int_0^t \xi_u e^{-b(t-u)}\,\mathrm{d}u, \sigma_r^2 \int_0^t e^{-2b(t-u)}\,\mathrm{d}u\right),$$

Parameters: $r_0, b, \sigma_r > 0$ and ξ, a function allowing a calibration fitting perfectly to the initial yield curve (see e.g. [3]).

As the quantity bearing interest for us is the bond price, we recall its distribution as it can be driven from the dynamics of the interest rate (see e.g. [3, p. 76]).

Lemma 4. *Under the Hull-White model, the price of the zero-coupon bond at instant t is equal to*

$$S_t(M) = \exp\left(A(t,M) - C(t,M)r_t\right),$$

where

$$C(t,M) = \frac{1}{b}\left(1 - e^{-b(M-t)}\right),$$
$$A(t,M) = \ln\frac{P^m(0,M)}{P^m(0,t)} - C(t,M)\frac{\partial \ln P^m(0,t)}{\partial t} - \frac{\sigma_r^2}{4b}C^2(t,M)(1-e^{-2bt}).$$

Note that the bond that we consider as an investment opportunity for the insurer matures after the end of the insurance contract, i.e. that $1 < M$.
In order to properly apply the Kunita-Watanabe decomposition, it is necessary to handle a martingale, so that we will consider the forward price (with respect to date $t = 1$) of the bond:

$$F_t(M,1) = \frac{S_t(M)}{S_t(1)}$$

under the forward-neutral measure $\bar{\mathbb{P}}$ defined by the Radon-Nikodym density

$$\frac{\mathrm{d}\bar{\mathbb{P}}}{\mathrm{d}\mathbb{P}^*} = \exp\left(-\sigma_r \int_0^1 C(u,1)\,\mathrm{d}W_u^1 - \frac{1}{2}\sigma_r^2 \int_0^1 C(u,1)^2\,\mathrm{d}u\right)$$

as shown in the following result, namely Lemma 11.3.1 of [10].

Lemma 5. *The process*

$$\bar{W}_t^1 = W_t^1 + \sigma_r \int_0^t C(u,1)\,du$$

is a standard Brownian motion under the forward-neutral measure. Moreover, the forward price then follows the dynamics

$$dF_t(M,1) = F_t(M,1)\gamma(t,M,1)\, d\bar{W}_t^1,$$

where

$$\gamma(t,M,1) = \sigma_r\left(C(t,1) - C(t,M)\right),$$

hence

$$F_1(M,1) = F_s(M,1)\exp\left(-\int_s^1 \gamma(u,M,1)\, d\bar{W}_u^1 - \frac{1}{2}\int_s^1 \gamma(u,M,1)^2\, du\right).$$

As M is fixed, we will simply write $F_t = F_t(M,1)$ and $\gamma(t) = \gamma(t,M,1)$ when the context is clear.

6.2 Explicit decomposition

In the following, the Malliavin derivative D^1 has to be understood as a derivative along $\bar{W}^1$, i.e. we leave the risk-neutral world to enter the forward-neutral world. We will write $\mathbb{E}$ for the expectation under the forward-neutral measure $\bar{\mathbb{P}}$.

Theorem 6. *The Föllmer-Schweizer decomposition of L_1 is given by*

$$\begin{aligned}
L_1 &= \mathbb{E}[L_1] + \int_0^1 \mathbb{E}\left[D_s^1 L_1 | \mathscr{F}_s\right] d\bar{W}_s^1 + \int_0^1 \mathbb{E}\left[D_s^2 L_1 | \mathscr{F}_s\right] dW_s^2 \\
&= C_0 e^i G(1)\Bigg(\exp\left(\nu_0 + \frac{1}{2}\tau_0^2\right) \\
&\quad + \frac{\beta}{S_0(1)}\exp\left(-\nu_0 - \frac{1}{2}\tau_0^2\right)\Phi\left(\frac{\beta_0 + \alpha_0(\nu_0 + \tau_0^2)}{\sqrt{1 + 2\alpha_0^2\tau_0^2}}\right) \\
&\quad - \beta e^i \exp\left(-2\nu_0 - 2\tau_0^2\right)\Phi\left(\frac{\beta_0 + 2\alpha_0(2\nu_0 + 4\tau_0^2)}{\sqrt{1 + 32\alpha_0^2\tau_0^2}}\right)\Bigg) \\
&\quad - \int_0^1 \Bigg[\frac{C_0 e^i G(1)\gamma(s) S_s(M)\ell_s \beta}{S_0(M) S_s(1)} \\
&\qquad \cdot \exp\left(-\nu_s - \frac{1}{2}\tau_s^2\right)\Phi\left(\frac{\beta_s + \alpha_s(\nu_s + \tau_s^2)}{\sqrt{1 + 2\alpha_s^2\tau_s^2}}\right)\Bigg] d\bar{W}_s^1 \\
&\quad + \int_0^1 \Bigg[\frac{C_0 e^i G(1)\sigma_\mu(e^{-a(1-s)} - 1)}{a}\Bigg(\ell_s \exp\left(\nu_s + \frac{1}{2}\tau_s^2\right) \\
&\qquad + \frac{\beta S_s(M)\ell_s}{S_0(M) S_s(1)}\exp\left(-\nu_s - \frac{1}{2}\tau_s^2\right)\Phi\left(\frac{\beta_s + \alpha_s(\nu_s + \tau_s^2)}{\sqrt{1 + 2\alpha_s^2\tau_s^2}}\right)
\end{aligned}$$

$$
\begin{aligned}
&-2\beta e^{i}\ell_s^2 \exp\left(-2\nu_s-2\tau_s^2\right)\\
&\qquad\qquad \cdot\Phi\left(\frac{\beta_s+2\alpha_s(2\nu_s+4\tau_s^2)}{\sqrt{1+32\alpha_s^2\tau_s^2}}\right)\Bigg)\Bigg]\,dW_s^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_s &= \frac{-1}{\sqrt{\int_s^1(\gamma(u))^2\,du}},\\
\beta_s &= \frac{\ln\frac{S_s(M)}{S_0(M)e^{i}G(1)\ell_s S_s(1)}+\frac{1}{2}\int_s^1(\gamma(u))^2\,du}{\sqrt{\int_s^1(\gamma(u))^2\,du}},\\
\gamma_s &= \sigma_r(C(s,1)-C(s,M)),\\
\nu_s &= \frac{\mu_0-\theta}{a}(e^{-a}-e^{-as})-\theta(1-s),\\
\tau_s^2 &= \int_s^1\frac{\sigma_\mu^2}{a^2}\left(1-e^{-a(1-u)}\right)^2\,du.
\end{aligned}
$$

The proof of this second decomposition theorem is similar to the one of Theorem 2, so we omit it.

6.3 Optimal strategy

As in the first model, the practical interest of the preceding theorem has to be emphasized.

Corollary 7. *Let $g=g(W^1,W^2)$ and $h=h(W^1,W^2)$ be the two integrands of the preceding theorem, i.e. the decomposition of l_1 can be expressed as*

$$
L_1=\mathbb{E}[L_1]+\int_0^1 g_s\,dW_s^1+\int_0^1 h_s\,dW_s^2.
$$

The unique admissible and risk minimizing strategy $\varphi^*=(\theta^*,\eta^*)$ is given by

$$
\begin{aligned}
\theta_t^* &= \frac{S_t^2(1)g_t}{\sigma_r S_t(M)(C(t,1)-C(t,M))},\\
\eta_t^* &= e^{-rt}\left(V_t(\varphi^*)-\theta_t^*\frac{S_t(M)}{S_t(1)}\right),\\
V_t(\varphi^*) &= \mathbb{E}[L_1]+\int_0^t g_s\,dW_s^1+\int_0^t h_s\,dW_s^2.
\end{aligned}
$$

Moreover the residual risk associated to φ^* is

$$
J=\mathbb{V}\left[\int_0^1 h_s\,dW_s^2\right]=\mathbb{E}\left[\int_0^1(h_s)^2\,ds\right].
$$

7 Summary of our assumptions

We recall here all the assumptions we have made throughout our modelling methodology:

(H_1) The considered contract is a 1 year pure endowment with purely financial profit sharing,

(H_2) The mortality index of the whole insured portfolio can be represented by an “average age" mortality index,

(H_3) The non-hedgeable part of the liabilities is normally distributed,

(H_4) The force of mortality follows the same Vasicek model in the real, risk-neutral and forward-neutral worlds (i.e. its dynamics is not affected by the changes of measure),

(H_{5a}) The financial risky asset is a stock, and its price follows a geometric Brownian motion in the risk-neutral word,

(H_{5b}) The financial risky asset is a zero-coupon bond, the interest rate following a Hull-White model in the risk-neutral world.

8 Succinct analysis of the mortality model error

We present in this section a succinct analysis of the error generated by Assumption (H_2).

8.1 Numerical assessment of the error

We consider a portfolio containing three insured individuals, respectively aged 20, 40 and 60 years. In [13], the authors give parameters for the three associated forces of mortality, as shown in Table 3.

Age	μ_0	θ	a	σ_μ
20	0.000797	0	-0.051085	0.001343
40	0.001217	0	-0.106695	0.000199
60	0.010054	0	-0.095001	0.001071

Table 3: Parameters of the force of mortality for the three considered ages

With these parameters, we compare the two following variables: on one hand the real number of surviving insured after one year (i.e. not using the approximation)

$$\tilde{P}_1 = \sum_{x=20,40,60} \exp\left(-\int_0^1 \mu_{x+s}(s)\,\mathrm{d}s\right)$$

and on the other hand the approximated number of surviving individuals

$$P_1 = 3\exp\left(-\int_0^1 \mu_{40+s}(s)\,\mathrm{d}s\right),$$

meaning that we have chosen $\hat{x} = 40$. Figure 3 shows the comparison of 1000 simulations. The error seems, in the very simple case, to be rather reasonable. The two first empirical moments of the difference between the approximated and non-approximated quantities are equal to

$$\hat{\mu}\left(\tilde{P}_1 - P_1\right) = -0.008355366 \quad \text{and} \quad \hat{\sigma}\left(\tilde{P}_1 - P_1\right) = 0.00383535004.$$

Table 4 gives the average number of surviving persons from an initial portfolio of 30.000 insured that is uniformly distributed to the three ages, i.e. composed of 10.000 individuals of each age class.

Age class	Surviving insured	Deceased insured
20 years	9991	9
40 years	9987	13
60 years	9894	106
Non approximated total	29872	128
Approximated total	29961	39

Table 4: Average number of surviving individuals in a portfolio of 30.000 contracts

8.2 Upper bound on the error

Here we present a theoretical upper bound on the mortality error considered above. As we will see,this result can be used to optimally calibrate the mortality parameters.
We write

$$\tilde{P}_1 = \sum_{x=\omega_{min}}^{\omega_{max}} N_x \exp\left(-\int_0^1 \mu_{x+s}(s)\,\mathrm{d}s\right)$$

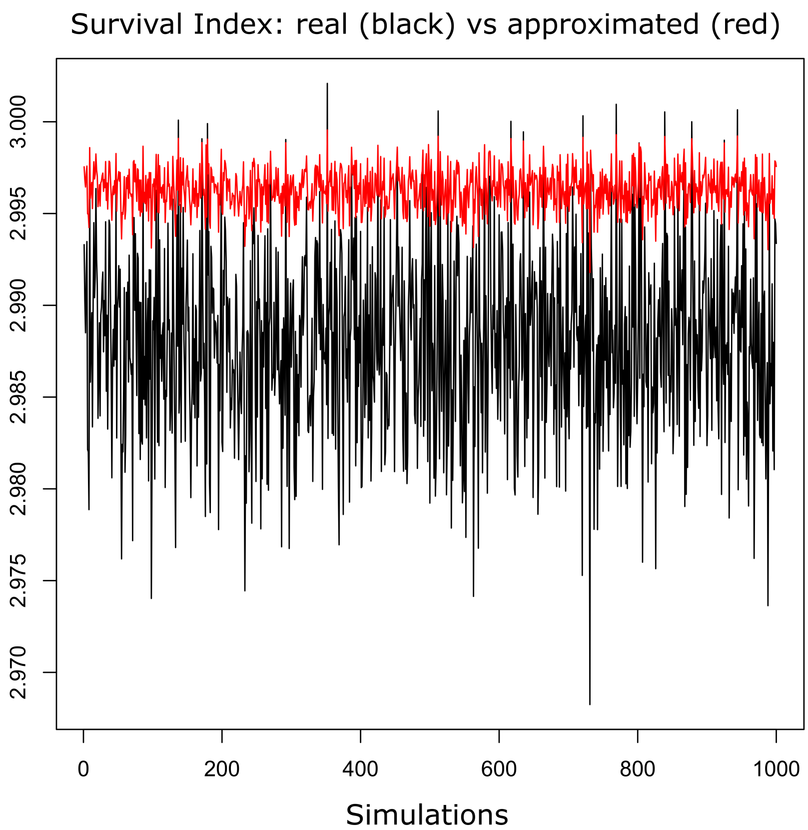

Figure 3: Error due to Assumption (H_2) on 1000 simulations

for the real mortality (where N_x is the number of insured with age x in the portfolio at the beginning of the contract) and

$$P_1 = \sum_{x=\omega_{min}}^{\omega_{max}} N_x \exp\left(-\int_0^1 \mu_{\hat{x}+s}(s)\,\mathrm{d}s\right) = N \exp\left(-\int_0^1 \mu_{\hat{x}+s}(s)\,\mathrm{d}s\right)$$

for the approximated mortality (where N is the total size of the portfolio, regardless the ages).

Proposition 8. *Assume that the integral of the force of mortality (i.e. the quantity $-\ln \ell_s$) is always positive. Then one has*

$$\mathbb{V}\left[L_1(\tilde{P}_1, S_1) - L_1(P_1, S_1)\right] \leqslant C_0^2 e^{2i} \sum_{x=\omega_{min}}^{\omega_{max}} N_x^2 \Bigg(\mathbb{V}\left[\beta\frac{S_1}{S_0}\right]\mathbb{V}\left[|m_x - m|\right] + \mathbb{V}\left[\beta\frac{S_1}{S_0}\right]\mathbb{E}\left[|m_x - m|\right]^2 + \mathbb{E}\left[1+\beta\frac{S_1}{S_0}\right]^2 \mathbb{V}\left[|m_x - m|\right]\Bigg)^{1/2},$$

where

$$m_x = \int_0^1 \mu_{x+s}(s)\, ds \qquad \textit{and} \qquad m = \int_0^1 \mu_{\hat{x}+s}(s)\, ds.$$

Proof. First remark that for every $y, m, \tilde{m} \geqslant 0$,

$$\tilde{x}\left(1+\left(y-\tilde{x}\right)_+\right) - x\left(1+\left(y-x\right)_+\right) \leqslant (1+y)\,|\tilde{x}-x|\,.$$

Indeed, applying the mean value Theorem to the function $x \mapsto x\left(1+\left(y-x\right)_+\right) = L(x,y)$, we obtain

$$\tilde{x}\left(1+\left(y-\tilde{x}\right)_+\right) - x\left(1+\left(y-x\right)_+\right) = \frac{\partial L}{\partial x}(z,y)\,|\tilde{x}-x| \leqslant (1+y)\,|\tilde{x}-x|$$

where z is between $\tilde{x}$ and x.
Getting back to the liabilities, we thus have

$$\begin{aligned}
\mathbb{V}&\left[C_0 e^i \tilde{P}_1\left(1+\beta\left(\frac{S_1}{S_0}-e^i\tilde{P}_1\right)_+\right) - C_0 e^i P_1\left(1+\beta\left(\frac{S_1}{S_0}-e^i P_1\right)_+\right)\right] \\
&= C_0^2 e^{2i}\mathbb{V}\left[\tilde{P}_1\left(1+\beta\left(\frac{S_1}{S_0}-e^i\tilde{P}_1\right)_+\right) - P_1\left(1+\beta\left(\frac{S_1}{S_0}-e^i P_1\right)_+\right)\right] \\
&\leqslant C_0^2 e^{2i}\mathbb{V}\left[\left(1+\beta\frac{S_1}{S_0}\right)\left|\tilde{P}_1-P_1\right|\right] \\
&\leqslant C_0^2 e^{2i}\mathbb{V}\left[\sum_{x=\omega_{min}}^{\omega_{max}} N_x\left(1+\beta\frac{S_1}{S_0}\right)\left|e^{-m_x}-e^{-m}\right|\right] \\
&\leqslant C_0^2 e^{2i}\mathbb{V}\left[\sum_{x=\omega_{min}}^{\omega_{max}} N_x\left(1+\beta\frac{S_1}{S_0}\right)|m_x-m|\right]
\end{aligned}$$

applying the mean value Theorem again (to $x \mapsto e^{-x}$). The result of the proposition is then obtained by applying the two following classical equalities from the theory of probability:

- for random variables $X_1, X_2, \ldots, X_n$ one has

$$\mathbb{V}\left[\sum_{i=1}^n X_i\right] \leqslant \left(\sum_{i=1}^n \mathbb{V}\left[X_i\right]^{1/2}\right)^2;$$

- for two independent random variables X_1, X_2, one has

$$\mathbb{V}\left[X_1 X_2\right] = \mathbb{V}\left[X_1\right]\mathbb{V}\left[X_2\right] + \mathbb{V}\left[X_1\right]\mathbb{E}\left[X_2\right]^2 + \mathbb{V}\left[X_2\right]\mathbb{E}\left[X_1\right]^2.$$

□

This upper bound on the error possesses an interest from the theoretical point of view, but also allows to perform an “optimal" calibration. Indeed, in order to minimize the error made with the approximation (H_2), it is possible to choose the parameters for the force of mortality $\mu_{\hat{x}}$ used in the previous sections' computations so that the preceding bound is as small as possible. We do not present this computation here, because it is very heavy to write down but very easy on a conceptual point of view. Remark that, as the force of mortality follows a Vasicek model, $m_x - m$ is normally distributed, and thus $|m_x - m|$ is folded-normally distributed. The folded-normal distribution's moments are known in closed form (but their expression is rather heavy), so that the determination of the optimal set of parameters should be an easy exercise.

9 Conclusion and prospects

In this paper we have shown how to blend Malliavin calculus with the Föllmer-Schweizer decomposition to obtain explicit results about evaluation and hedging in an incomplete market framework. In particular, we have applied these techniques to the liabilities of a life insurer.
We have used this methodology in two different modelling frameworks. It is clear that they are only examples of what could be made. Other models could clearly be considered. Notice that our proof can be straightforwardly extended to any model such that the asset price process X is adapted, log-normally distributed and with deterministic logarithmic Mallaivin derivative (i.e. such that $D \ln X = DX/X$ is a deterministic function).
Similarly, it should be possible to use different models for the force of mortality. We conjecture that it is feasible for models of the *affine term structure* class, i.e. for models involving a survival probability function which has the form $\exp(a\mu + b)$, where a, b are real constants and μ is the force of mortality (as the Vasicek model we have chosen in this paper). For example, it should be possible to apply our methodology to the Cox-Ingersoll-Ross model, even if the resulting Malliavin derivative is rather complex (see [1]).
It is also possible to extend our results by adding a random factor, e.g. by putting randomness in the interest rate or in the asset's volatility. Our attempts in this direction have given results technically heavy and thus difficult to handle, but not infeasible.
In another direction, it is possible to generalize our theorem with the treatment of more complex insurance contracts. This will certainly lead to heavier expressions, so that obtaining closed formula as we did would only result from a miracle. Remark that the Föllmer-Schweizer decomposition, and thus our entire approach, is only valid for conditional assets H paying only once, at the terminal date of the contract. A generalization to more general financial cash-flows could perhaps be driven from the results of [9], where the author treats insurance contracts having a more complex payment structure.
One can also turn from the life insurance domain and apply these techniques to other incomplete markets, such as damage insurance, handling e.g. inflation and IBNR amounts.

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Appendix

A Technical result

Lemma 9. *Let $N \sim \mathcal{N}(\mu, \sigma^2)$. Then one has, for every $\alpha, \beta \in \mathbb{R}$,*

$$\mathbb{E}\left[\exp(N)\Phi(\alpha N+\beta)\right] = \exp\left(-\mu - \frac{1}{2}\sigma^2\right)\Phi\left(\frac{\beta+\alpha(\mu+\sigma^2)}{\sqrt{1+2\alpha^2\sigma^2}}\right).$$

The proof of this technical lemma is based on the following result given in [8, p. 891]: for every $p > 0$ and $a, b \in \mathbb{R}$,

$$\int_{\mathbb{R}} e^{-px^2}\Phi(a+bx)\,\mathrm{d}x = \sqrt{\frac{\pi}{p}}\Phi\left(\frac{a\sqrt{p}}{\sqrt{b^2+p}}\right). \tag{12}$$

Proof. We compute

$$\begin{aligned}
\mathbb{E}\left[\exp(N)\Phi(\alpha N+\beta)\right] &= \frac{1}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}} e^x\Phi(\alpha x+\beta)e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x \\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}} \Phi(\alpha x+\beta)e^{\frac{-x^2-2(\mu+\sigma^2)x+\mu^2}{2\sigma^2}}\,\mathrm{d}x \\
&= \frac{e^{\frac{(\mu+\sigma^2)^2-\mu^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}} \Phi(\alpha x+\beta)e^{\frac{-(x-(\mu+\sigma^2))^2}{2\sigma^2}}\,\mathrm{d}x \\
&= \frac{e^{\frac{(\mu+\sigma^2)^2-\mu^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}} \Phi\left(\alpha y+\beta+\alpha(\mu+\sigma^2)\right)e^{\frac{-y^2}{2\sigma^2}}\,\mathrm{d}x \\
&= \frac{e^{\frac{(\mu+\sigma^2)^2-\mu^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\sqrt{2\pi\sigma^2}\Phi\left(\frac{(\beta+\alpha(\mu+\sigma^2))(2\sigma^2)^{-1}}{\sqrt{\alpha^2+(2\sigma^2)^{-1}}}\right) \\
&= e^{-\mu-\frac{1}{2}\sigma^2}\Phi\left(\frac{\beta+\alpha(\mu+\sigma^2)}{\sqrt{1+2\alpha^2\sigma^2}}\right)
\end{aligned}$$

using the change of variables $y = x-(\mu+\sigma^2)$ and then Equation (12). □