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For my grandfather, Fortunato

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Financial Valuation in a Volatile Scenario of the Guaranteed Minimum Withdrawal Benefit (GMWB) Policy

Abstract

In this paper, we consider the valuation of the Guaranteed Minimum Withdrawal Benefit (GMWB) option associated with a variable annuity insurance policy. In particular, we consider a generalization of the model proposed by Milevsky and Salisbury [11], in which the volatility of the underlying portfolio is considered to be a stochastic process rather than a constant.

The results obtained for the calculation of the insurance premium are quite satisfactory as they are considerably more realistic than those obtained in [11].

1. Introduction

The past twenty years have seen a massive proliferation in insurance-linked derivative products. These kind of financial instruments are considered very interesting as they offer the possibility of addressing various risks (capital loss, mortality, natural disasters etc.) distributed over very long time horizons (for example, 30 years) by paying, in most cases, a small premium or one that is spread out over time.

One example of an insurance-linked derivative that is currently very much in vogue is the so-called Guaranteed Minimum Withdrawal Benefit (GMWB), an option that is frequently associated with variable annuity policies. This kind of contract, first introduced in the early 1970s in the United States, quickly experienced remarkable growth in Europe, especially during the last decade characterized by “bearish” financial markets and relatively low interest rates.

The GMWB option satisfies medium to long term investment needs while providing adequate hedging against market volatility risk. Indeed, based on an initial capital investment, this option guarantees the policyholder a stream of future payments, independently of the performance of the underlying policy. Specifically, upon contract signature, the policyholder pays a sum of money that is invested in a diversified asset portfolio (mainly bonds or bond funds). Movements in the investment portfolio are recorded in an account called a Variable Annuity sub-account (VA sub-account). The GMWB option guarantees the policyholder a fixed or variable sum of money on set dates until contract maturity, regardless of market performance. This sum is withdrawn from the VA sub-account if it has a positive balance, otherwise it is paid by the insurance company from its own capital. Withdrawals made by the policyholder must not exceed the original sum of money paid. In addition, the entire remaining account balance is paid to the policyholder at maturity.

It is clear that this contract involves a risk for the insurance company. Indeed, if the asset portfolio underlying the GMWB option records substantial losses and the balance of the VA sub-account is reduced to zero before contract maturity, the insurer is then obliged to repay the sum withdrawn by the policyholder out of its own pocket. Faced with this risk, the insurance company therefore requires the policyholder to pay a premium. This premium is usually spread over the entire duration of the policy in the form of periodic payments (or fees) to the VA sub-account.

Therefore, the main problem associated with the GMWB option is the valuation of the fair value of the fees payable by the policyholder.

In particular, if v_0 denotes the original amount paid at time $t = 0$, T the duration of the policy, r the interest rate, g the withdrawal rate selected by the policyholder (we assume that the interest and withdrawal rates are constant) and V_T the balance of the VA sub-account at contract maturity, then the actual value of all cash flows associated with the contract at time $t = 0$ is given by:

$$e^{-rT} E_Q[V_T] + \frac{v_0 g}{r} (1 - e^{-rT})$$

where $E_Q [.]$ denotes the expected value under risk-neutral conditions. The final balance of the account V_T is influenced, among other things, by the amount of fees paid by the policyholder (as described above, the fees are paid into the account). Therefore, the fair value of these fees is such as to establish the equivalence between the actual value of the account and the original amount paid by the policyholder, that is

$$v_0 = e^{-rT} E_Q[V_T] + \frac{v_0 g}{r} (1 - e^{-rT}) \quad (1.1)$$

The risk factor that exerts the greatest influence on the final value of the VA sub-account V_T is the performance of the asset portfolio underlying the VA policy. It is therefore obvious that, in order to assess the fair value of the fees payable by the policyholder, it is necessary to describe the dynamics of this portfolio using an accurate mathematical probabilistic model. In particular, in the analysis conducted by Milevsky and Salisbury [11] - a major work on the valuation of GMWB options - the authors assume that the evolution of the underlying portfolio over time is described by a geometric Brownian motion. Namely, if S_t denotes the value of the portfolio, we have:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where the parameters μ and σ represent, respectively, the rate of return and the volatility of the portfolio, and W_t is a standard Wiener process. In particular, μ and σ are considered to be constant in [11].

In this paper, we propose a generalization of the model presented in [11], in which the volatility of the underlying portfolio, rather than being considered as constant, is described by an additional stochastic process. Indeed, it is far more realistic to assume that volatility evolves over time as a stochastic process in its own right.

In this paper, we consider two different stochastic models to describe the complex dynamics of volatility: a generalization of the Scott model [12] and the Heston model [7]. Both of these stochastic processes are widely used to describe the evolution of different financial products over time as they offer a particularly accurate description and, moreover, in some straightforward cases (for example, in the valuation of "vanilla" European options), allow closed-form solutions to be obtained.

In the case of the valuation of the GMWB option, while the volatility of the portfolio follows the generalized Scott model or the Heston model, it is not possible to determine an exact analytical expression of the expected value of the VA sub-account required for formula (1.1). Therefore, as part of this analysis, the expected value was determined using Monte Carlo simulation [3]. Consequently, by using equation (1.1), a fairly realistic estimate of the premium (i.e. the value of the fees) payable by the policyholder was simply obtained.

Furthermore, in strictly financial terms, we were able to observe how a generalization of the model proposed by Milevsky and Salisbury allows considerably more realistic results to be obtained than the model discussed in [11]. Indeed, while the actual value of the premium is between 30 and 50 basis points, the model proposed by Milevsky and Salisbury provides premium estimates ranging between 73 and 158 basis points. This fact is specifically noted by Milevsky and Salisbury, who recognize that their model does not provide a particularly realistic estimate of market data.

In our analysis, by implementing Heston and Scott's stochastic volatility models, we obtained premium valuations of around 35 basis points, therefore achieving significantly better valuations as they are very close to the real market values of these financial products.

The analysis conducted led to the creation of an ad hoc calculation program to assess the value of the GMWB options. The software was developed in a MATLAB environment using a MAC OS X version 10.6.8 with a 2.26 GHz processor Intel Core 2 Duo processor and 4GB 1067 MHz DDR3 memory. In Section 2, we describe the model proposed by Milevsky and Salisbury for the financial valuation of the GMWB option embedded in variable annuity policies and we propose a generalization in the case of stochastic volatility, considering the Scott model and the Heston model.

In Section 3, we present the numerical results obtained, considering both the Milevsky and Salisbury model and its generalization in the case of stochastic volatility.

2. GMWB Policy Valuation Model

2.1 Assumption of deterministic volatility

The financial valuation of the GMWB option performed as part of this research is based on the article "Financial valuation of guaranteed minimum withdrawal benefits" by Moshe A. Milevsky and Thomas S. Salisbury, published in 2006 in the journal *Insurance: Mathematics and Economics* [11].

We consider in particular a variable annuity insurance policy with a life span $[0, T]$, within which the GMWB option is activated upon payment of an annual fee. Let us remind ourselves briefly of the characteristics of this product. Upon contract signature, the policyholder pays a sum of money that is invested in a diversified asset portfolio. The insurance company uses this sum of money to open an account, called a VA sub-account, in which the movements in the investment portfolio are recorded. The GMWB option guarantees the policyholder a regular sum of money until contract maturity, regardless of market performance. This sum is withdrawn from the VA sub-account if it has a positive balance, otherwise it is paid by the insurance company from its own capital. Withdrawals can be made up to the amount originally paid. Therefore, the withdrawal rate selected by the policyholder will affect the life of the policy T : for a given initial investment, the greater the amount of money withdrawn periodically by the policyholder, the lower the life span of the policy.

The fees applied for activating the GMWB option (applied annually by the insurance company) are deducted from the account balance.

The value of the option, therefore, must be such as to enable the insurance company to meet its contractual commitments whatever state of nature occurs. Therefore, the amount originally paid by the subscriber must be equal to the sum of the actual value of two figures: the cash flows arising from the policy, represented by the withdrawals made by the policyholder during the life of the contract, and the assumed value of the VA sub-account.

Milevsky and Salisbury propose a valuation of the GMWB option using two different approaches: a "static" approach and a "dynamic" approach. In the "static" approach, the authors suggest that individual investors behave passively in utilizing their guarantee, in other words they always withdraw the same amount and hold the contract to maturity. In the "dynamic" approach, however, individuals are considered to be dynamically rational, in other words they seek to maximize the value of the guarantee, either by changing the withdrawal rate during the life span the contract or by terminating the policy before maturity. In this analysis, we have given particular consideration to the static analysis conducted by these authors.

It is necessary, at this stage, to identify with greater precision the relationship between the value of the VA sub-account and uncertain market performance. Let S_t denote the market value of an asset at a given point in time t , σ the volatility of the asset price and μ the expected rate of return. The performance of the asset price is usually described by an Itô process with a drift rate of μS_t and a variance rate of σS_t :

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (2.1)$$

where W_t is a Wiener process. Following Milevsky and Salisbury's approach, it is assumed that parameters μ and σ are constant.

From the considerations made, it follows that the dynamics of the VA sub-account can be described using the following equation:

$$dV_t = -\alpha V_t dt - \gamma_t dt + V_t \frac{dS_t}{S_t} \quad (2.2)$$

where α denotes the annual fee applied by the insurance company for activating the GMWB option and γ_t denotes the withdrawals made by the policyholder at time t , with $0 < t < T$. Furthermore, if v_0 denotes the amount originally paid by the policyholder, we have:

$$V_0 = v_0$$

in other words, upon contract signature (at time $t = 0$), the balance of the VA sub-account exactly matches the initial investment made by the policyholder.

Using g_t to define the withdrawal rate allowed by the insurance company at time t , the withdrawals γ_t made at time t are given by:

$$\gamma_t = g_t v_0$$

with $0 < t < T$. It is reasonable to assume that the withdrawals made by the policyholder at a given time t can range between a minimum value equal to zero and a maximum value equal to the value of the VA sub-account at that point in time, therefore we suppose that

$$0 \leq \gamma_t \leq V_t$$

for $0 < t < T$.

From (2.1) and (2.2), we obtain the following stochastic differential equation:

$$dV_t = (\mu - \alpha)V_t dt - \gamma_t dt + \sigma V_t dW_t \quad (2.3)$$

More precisely, let $\tau := \inf \{t \in (0, T] : V_t = 0\}$ that is the first time at which the account balance is reduced to zero. The dynamics of the VA sub-account, for $0 < t < \tau$ are given by

$$\begin{cases} dV_t = (\mu - \alpha)V_t dt - \gamma_t dt + \sigma V_t dW_t \\ V_0 = v_0 \end{cases} \quad (2.4)$$

and we establish that

$$V_t := 0$$

for $\tau \leq t \leq T$.

Indeed, the initial premium paid by the policyholder is invested in the market and is subject to daily fluctuations, the size and extent of which remain a priori uncertain. Should market performance result in low or negative returns, the value of account V_t at a given point in time t may reduce to zero or even fall below this value. In this situation, however, the GMWB guarantee is activated from that point in time and the policyholder continues to be able to withdraw the same amount until the sum of money originally paid has been exhausted.

The model considered assumes that the withdrawal rate does not vary over time but remains constant, therefore

$$g_t = g$$

It also follows that withdrawals will be constant, hence we have

$$\gamma_t = g v_0 = G$$

namely, it does not consider the possibility of increasing or reducing the amount withdrawn depending on the financial needs of the policyholder.

From the above equation (2.4) which describes the dynamics of the VA sub-account, we obtain:

$$\begin{cases} dV_t = (\mu - \alpha)V_t dt - G dt + \sigma V_t dW_t \text{ for } 0 < t < \tau \\ V_0 = v_0 \end{cases}$$

and

$$V_t := 0 \quad \text{for } \tau \leq t \leq T. \quad (2.5)$$

Using Itô's lemma [9], it is possible to write the solution to problem (2.5) as follows:

$$V_T = e^{(\mu - \alpha - \frac{1}{2}\sigma^2)T + \sigma W_T} \quad (2.6)$$

$$\max \left[0, \left(v_0 - G \int_0^T e^{-(\mu - \alpha - \frac{1}{2}\sigma^2)t - \sigma W_t} dt \right) \right]$$

Ex post, it can be said that the GMWB option is activated and has a positive value only if process V_t hits zero before the maturity date of the policy $T = v_0/G$. In this case, the account balance is not sufficient to fund the withdrawals guaranteed to the policyholder and intervention by the insurance company is necessary. If, on the contrary, the dynamics of the VA sub-account are such that ruin occurs after time T , then the insurance option has a zero payout. Indeed, the account balance is in itself sufficient to guarantee the policyholder the full amount originally deposited and the guarantee therefore does not need to be activated. The guaranteed minimum withdrawal amount is then endogenously insured, even without the offer of an explicit guarantee by the insurance company.

We observe that, while taking into consideration the probability of ruin of process V_t , namely the probability that the value of the account will fall below zero before contract maturity, we should consider the following function:

$$\xi_t = P \left[\inf_{0 \leq s \leq t} V_s = 0 \right]$$

From solution (2.6) we deduce that process V_t reaches zero only if a point in time $t > 0$ exists where the following applies:

$$\max \left[0, \left(v_0 - G \int_0^t e^{-(\mu - \alpha - \frac{1}{2}\sigma^2)t - \sigma W_t} dt \right) \right] = 0$$

namely

$$\left(v_0 - G \int_0^t e^{-(\mu - \alpha - \frac{1}{2}\sigma^2)t - \sigma W_t} dt \right) \leq 0$$

therefore

$$\int_0^t e^{-(\mu - \alpha - \frac{1}{2}\sigma^2)t - \sigma W_t} dt \geq \frac{v_0}{G}$$

Therefore, we have that:

$$\begin{aligned} \xi_t &= P \left[\inf_{0 \leq s \leq t} V_s = 0 \right] \\ &= P \left[\int_0^t e^{-(\mu - \alpha - \frac{1}{2}\sigma^2)s - \sigma W_s} ds \geq \frac{v_0}{G} \right] \quad (2.7) \\ &= P \left[X_t \geq \frac{v_0}{G} \right] \end{aligned}$$

where

$$X_t = \int_0^t e^{-(\mu - \alpha - \frac{1}{2}\sigma^2)s - \sigma W_s} ds$$

Note that X_t is a monotonically increasing function of time t . Thus, if at a very precise point in time τ ($0 < \tau \leq T$), X_t exceeds v_0/G , which means that $V_\tau = 0$, it can never recover and go back above zero, so, for $t > \tau$, the account can only record a downward trend.

Under the assumption of risk neutrality normally used in finance to analyse derivatives, the value of the option, as already mentioned, is equal to the actual value of all cash flows associated with the contract itself and therefore is equal to :

$$e^{-rT} E_Q[V_T] + \frac{v_0 g}{r} (1 - e^{-rT}) \quad (2.8)$$

where $E_Q[\cdot]$ denotes the expected value under the risk-neutral measure.

In order for the price paid by the policyholder for this option to be fair, its value, as determined in expression (2.8) must be exactly equal to the original sum paid v_0 . In other words, the following must hold:

$$v_0 = e^{-rT} E_Q[V_T] + \frac{v_0 g}{r} (1 - e^{-rT}) \quad (2.9)$$

2.1 Assumption of stochastic volatility

The results obtained by Milevsky and Salisbury by applying the proposed theoretical model show that, if we consider a typical variable annuity policy offered on the market characterized by a withdrawal rate of 7% and with a volatility of between 20% and 30%, the estimated fair price of the GMWB option ranges from between 73 and 158 basis points (see Table 4 in [11]). However, this result is in contrast with what occurs in reality. Indeed, the market prices such products at between 30 and 45 basis points, thus underpricing them, according to the authors. In other words, insurance companies do not appear to charge sufficient fees to cover all of the costs associated with the option. In order to try to understand the reasons for this discrepancy, recognized as such by the authors, as part of this research we sought to change some of the assumptions introduced in the model itself. In particular, also for the purposes of considering a model that is closer to the market, we sought to weaken the constant volatility assumption and introduce a stochastic volatility process [10]. This approach certainly involves major difficulties. In fact, as in the deterministic scenario, there is no closed formula to solve the stochastic differential equation (2.5). Therefore, a financial valuation of the option cannot be performed analytically, it must be carried out using numerical calculation procedures. In particular, the expected value of the VA sub-account required in formula (2.9) was valued using Monte Carlo simulation [3].

The stochastic volatility models predict that volatility itself follows a stochastic process:

$$(\sigma_t)_{t \geq 0}$$

with

$$\sigma_t = f(Y_t)$$

where $f(y) > 0 \forall y \in \mathbb{R}$ and Y_t represents a stochastic process [6].

With this assumption, the process describing the dynamics of the price of a financial asset $(S_t)_{t \geq 0}$ given in (2.1) is characterized as follows:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Consequently, the performance of the VA sub-account and the fair price of the GMWB option are changed. In particular, the account dynamics become:

$$\begin{cases} dV_t = (\mu - \alpha)V_t dt - Gdt + \sigma V_t dW_t & \text{for } 0 < t < \tau \\ V_0 = v_0 \end{cases} \quad (2.11)$$

and we establish that

$$V_t := 0 \quad \text{for } \tau \leq t \leq T$$

where $\tau := \inf \{t \in (0, T] : V_t = 0\}$ denotes the point in time at which the account balance reduces to zero for the first time.

It is assumed that (Y_t) satisfies the following stochastic differential equation:

$$dY_t = a(m - Y_t)dt + b_1 d\hat{Z}_t$$

where a is the *mean reversion* coefficient, m denotes the long-run mean of Y , b is the rate of diffusion and $(\hat{Z}_t)_{t \geq 0}$ is a Brownian motion correlated with Wiener process W_t that appears in equation (2.10) according to a correlation coefficient $\rho \in [-1, 1]$.

From the financial data, we deduce that $\rho < 0$ and there are also economic arguments for a negative correlation or leverage effect between financial asset price and volatility shocks. Indeed, empirical studies show that asset prices tend to decrease when volatility increases. In general terms, the correlation may be time dependent, therefore it would be more correct to write $\rho(t) \in [-1, 1]$, however it is typically assumed to be constant, both to simplify the notation and because this assumption is the most widely used in most practical situations.

Among the models used in the literature to describe volatility, we have chosen to consider two in particular to describe the process followed by Y : a generalization of the Scott model [12] and the Heston model [7], two of the most widely used models in finance.

In particular, the Scott model, proposed in 1987, considers that the stochastic differential equation followed by Y_t is a mean reverting Ornstein-Uhlenbeck process:

$$dY_t = a(m - Y_t)dt + b d\hat{Z}_t \quad (2.12)$$

with $\rho = 0$ and $f(y) = e^y$. However, the assumption of no correlation between the two Brownian motions that characterize the price and volatility processes of the underlying asset, as mentioned earlier, seems unrealistic. For this reason, in this analysis we assume that $\rho < 0$.

In the Heston model, proposed in 1993, however, the driving process Y_t is described by the following stochastic differential equation:

$$dY_t = a(m - Y_t)dt + b\sqrt{Y_t} d\hat{Z}_t \quad (2.13)$$

with $f(y) = \sqrt{y}$ and $\rho \neq 0$.

3. Numerical Results

3.1 Assumption of deterministic volatility

As part of our analysis, we proceeded first to implement the theoretical model proposed in [11] passing through the discretization of the stochastic differential equations used in the valuation of the option.

In particular, by discretizing the process described by (2.5) and attributing the most realistic values possible to parameters μ , σ , α and G , we generated several possible scenarios, some of which are shown in Figures 3.1 and 3.2 below. In particular, by analyzing policies actually available on the market, offered by several major insurance companies, we discovered that the contracts typically offer a withdrawal rate $g = 7\%$ and a fee $\alpha = 40$ basis points. Assuming an initial investment $v_0 = \$100$, the policyholder then is guaranteed the ability to withdraw $\$7$ until the amount originally paid has been exhausted, namely for a period equal to $100/7 = 14.28$ years, assuming that the

policyholder always withdraws at the same rate and holds the contract to maturity. In the examples presented, we show how the guarantee offered by the GMWB option is not always activated (Figure 3.1), and when it does occur, activation can take place at different times during the life of the contract with different effects on the insurance company. In particular, in Figure 3.1 we observe how, despite the different performance of the VA sub-accounts (increasing in (a) and decreasing and then increasing in (b)), the recorded performance of the assets in the underlying portfolio, after the fees charged by the insurance company have been subtracted, is sufficient to guarantee the periodic withdrawals by the policyholder for the entire duration of the contract. In Figure 3.2 (a), however, the balance of the VA sub-account is reduced to zero at around 8 years after contract signature and therefore the guarantee is activated.

Figure 3.1: Examples of the dynamics of the VA sub-account ($\mu = 10\%$, $\sigma = 18\%$, $\alpha = 40$ basis points, $G = \$7$ and interest capitalized 250 times per annum)

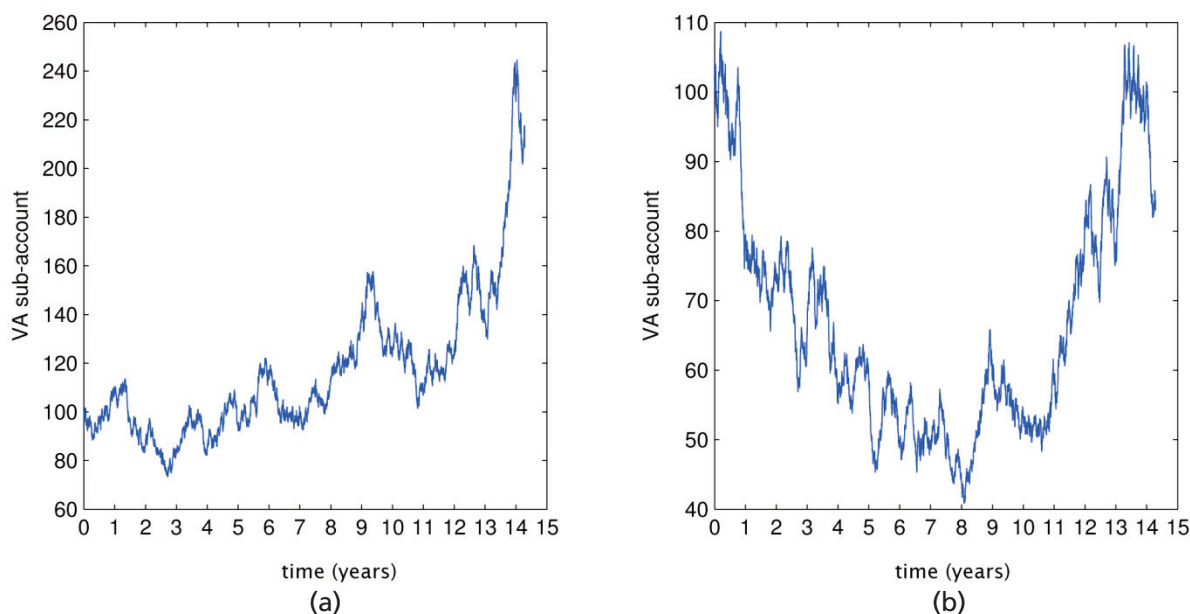
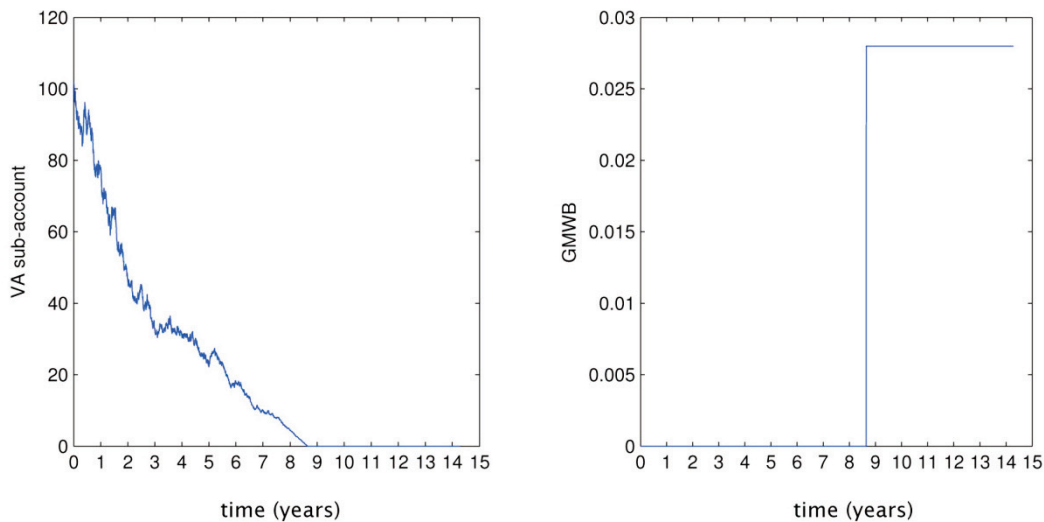


Figure 3.1: Example of the dynamics of the VA sub-account and the value of the guarantee offered by the GMWB option ($\mu=10\%$, $\sigma =18\%$, $\alpha = 40$ basis points, $G=\$7$ and 250 time periods in each year)



(a) Dynamics of the VA sub-account

(b) Value of the guarantee offered by the GMWB option

Figure 3.2 (b) shows proportionally the point in time at which the guarantee offered by the GMWB option is activated. At the point in time at which the value of the account is reduced to zero, the option is activated, allowing the policyholder to continue to withdraw the amount of \$7 per annum until maturity of the policy. To calculate the probability of ruin of the policy defined in (2.7), having generated a family of ω trajectories (with $\omega \in \mathbb{N}$), a counter s_j was introduced which, once a specific value of μ is set, for each trajectory takes the value 1 if the V_T account balance at time T is zero, otherwise 0 if it is positive:

$$s_j := \begin{cases} 0 & \text{se } V_T > 0 \\ 1 & \text{se } V_T = 0 \end{cases} \quad \text{for } j = 1, \dots, \omega$$

The estimate of the probability of ruin $\hat{\xi}_T$ was obtained by considering the 10,000 trajectories generated and calculating the average of the sequence of 0s and 1s obtained. Formally, we have:

$$\hat{\xi}_T = \frac{\sum_{j=1}^{\omega} s_j}{\omega}$$

This procedure was repeated considering the different values of μ and σ . Table 3.1 summarizes the results obtained from the simulations. In this regard, we can see that, for a given volatility σ , there is a decreasing relationship between the estimated probability of ruin $\hat{\xi}_T$ and the expected rate of return μ of the assets underlying the VA policy, while for a given expected return μ , there is an increasing relationship between $\hat{\xi}_T$ and σ . The values obtained from the simulations carried out are relatively close to those calculated by Milevsky and Salisbury (see Table 3 in [11]).

Table 3.1: Estimate of the probability of ruin

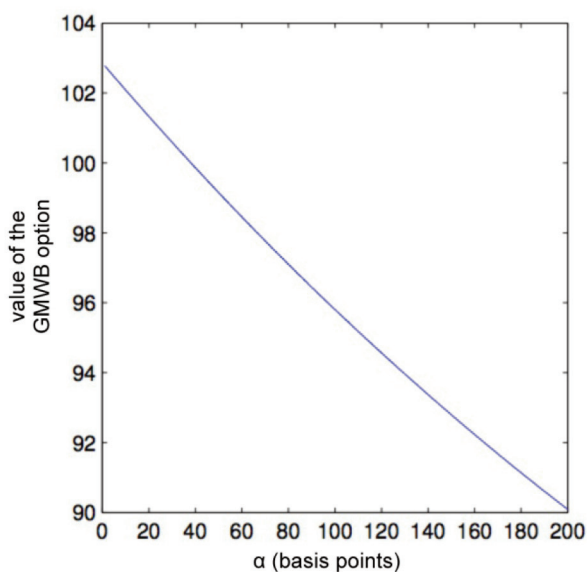
| | $\mu = 4\%$ | $\mu = 6\%$ | $\mu = 8\%$ | $\mu = 10\%$ | $\mu = 12\%$ |
|-----------------|-------------|-------------|-------------|--------------|--------------|
| $\sigma = 10\%$ | 17.08% | 5.44% | 1.21% | 0.18% | 0.04% |
| $\sigma = 15\%$ | 32.31% | 18.65% | 9.15% | 4.08% | 1.4% |
| $\sigma = 18\%$ | 39.33% | 26.36% | 16.0% | 8.87% | 4.44% |
| $\sigma = 25\%$ | 51.78% | 41.67% | 31.93% | 23.37% | 16.22% |

Taking the same values for the parameters used in [11], we then estimated the fair price of the option, namely the value of the fees α charged by the insurance company which satisfy the equivalence condition in (2.9) as the amount withdrawn G and the volatility of the investment σ change. The results obtained are comparable with those calculated in [11].

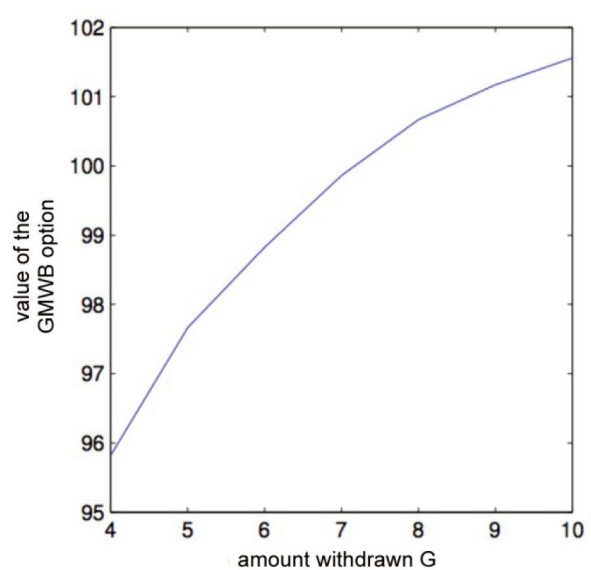
Before generalizing Milevsky and Salisbury's model, we wanted to analyse the impact of the various parameters of this model on the value of

the option. The results obtained are shown in Figure 3.3 where we can see how the value of the option changes according to variations in: the fees charged α (a), the amount withdrawn G (b), the risk-free interest rate r (c) and the volatility of the asset price σ (d). In all analyses, we considered an initial investment of \$100, an interest capitalization of 250 times per annum (or almost daily, if we consider only the days when the Exchange is open) and 7000 trajectories that describe the potential performance of the VA sub-account.

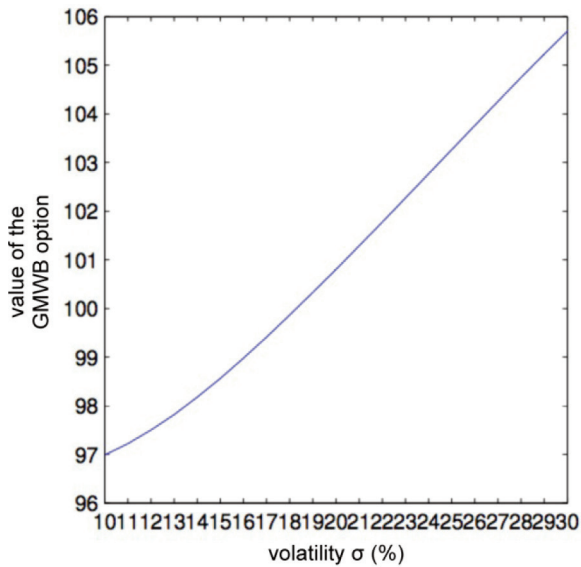
Figure 3.3: Influence of parameters α , G , r and σ on the value of the GMWB option



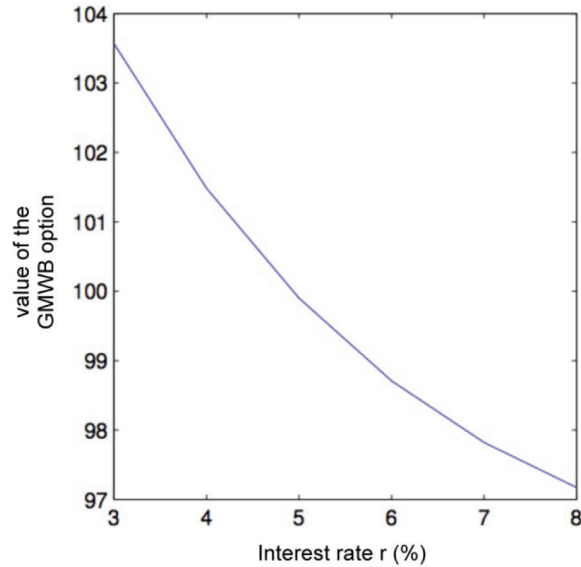
(a) $G = \$7$, $r = 5\%$, $\sigma = 18\%$



(b) $\alpha = 40$ basis points, $r = 5\%$, $\sigma = 18\%$



$\alpha = 40$ basis points, $G = \$7$, $r = 5\%$



$\alpha = 40$ basis points, $\sigma = 18\%$, $G = \$7$

However, the parameters that characterize the contract, the values of which are determined by the insurance company, are only the fees charged α and the amount withdrawn G . Volatility σ and interest rate r are determined by the market, therefore they should be assumed to be given by the insurance company. To analyze the relationship between the value of the option and parameters

α and G simultaneously, we generalized Figure 3.3 (a) considering different values for G and keeping all other parameters constant. For each value of G , we then identified the value for α that made the price of the option fair:

$$\alpha : V_T = 100$$

The results obtained are shown in Figures 3.4 and 3.5.

Figure 3.4: Value of the GMWB option in relation to changes in fees α and considering the different values of G

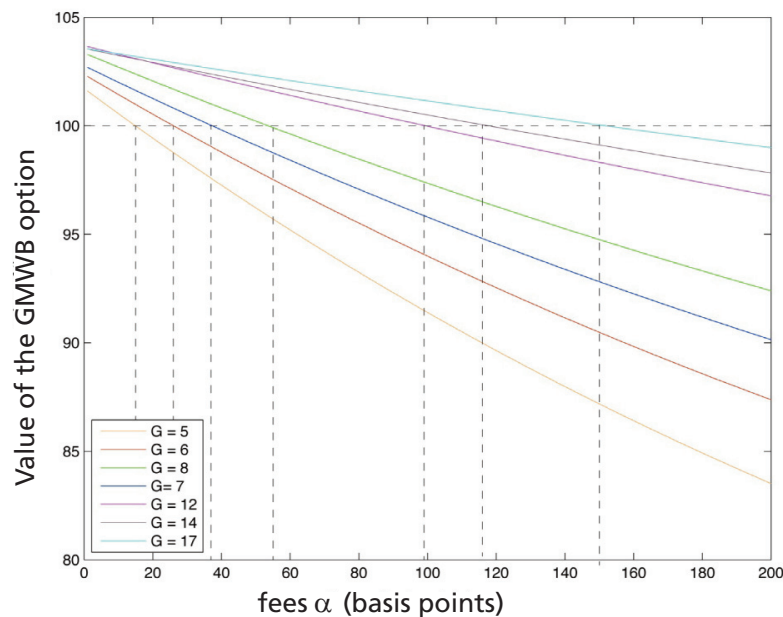
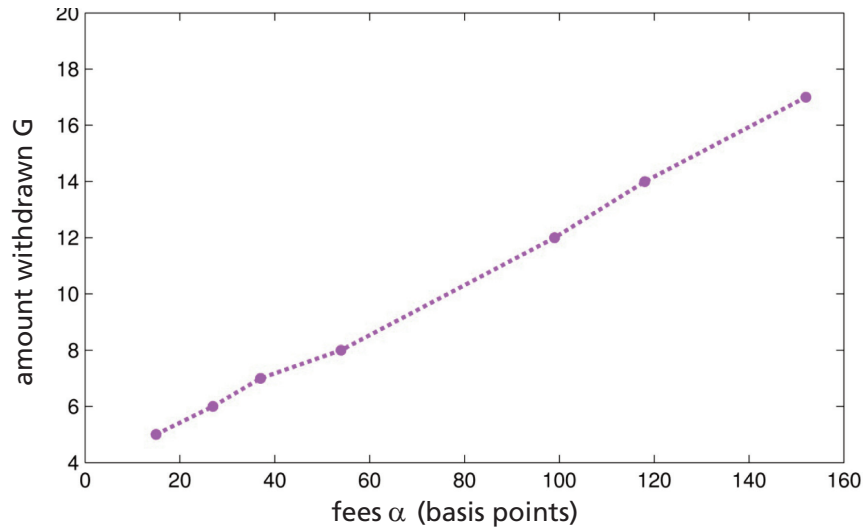


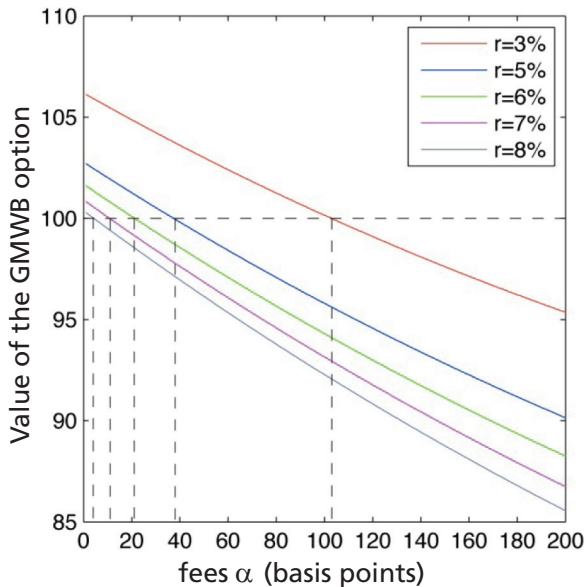
Figure 3.5: Value of α and G that make the price of the option fair where $r = 5\%$ and $\sigma = 18\%$



We then tried to understand the impact of volatility and interest rate on the pairs $(\alpha; G)$ identified.

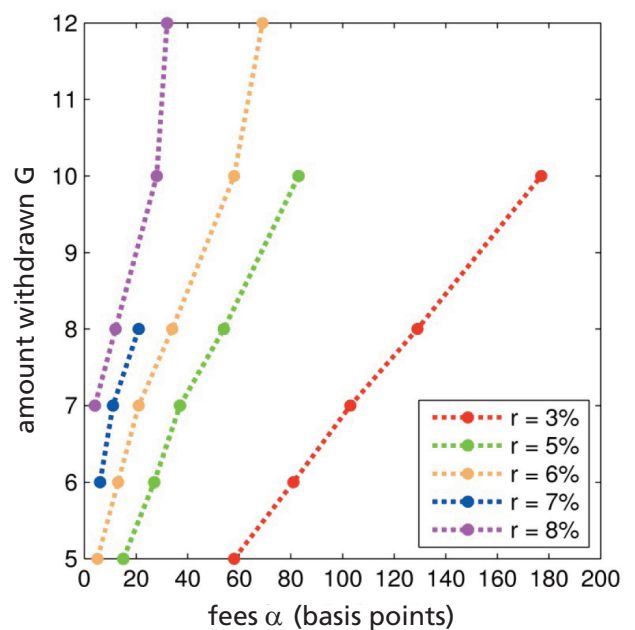
Figure 3.6 shows how, where $G = 7$, the trajectories that describe the performance of the value of the option as a function of α experience a downwards shift as the interest rate r increases. Thus, for a given value of G and σ , the higher the interest rate r , the lower the fees the insurance company has to charge to make the policy more attractive and therefore attract demand from investors who otherwise would turn to more profitable investments.

Figure 3.6: Value of the GMWB option in relation to changes in fees α for $G = 7$ and different values of r



By repeating the same procedure, considering different values of G and determining the corresponding values of α such that $V_T = 100$, we identified the pairs (α, G) that make the price of the option fair for different values of interest rate r , as shown in Figure 3.7.

Figure 3.7: Values of α and G that make the option price fair in relation to changes in interest rates



Similarly, we proceeded to establish the relationship between the fair price of the option and volatility. The results obtained are shown in Figures 3.8 and 3.9.

Figure 3.8: Value of the GMWB option in relation to changes in fees α for $G = 8$ and different values of σ

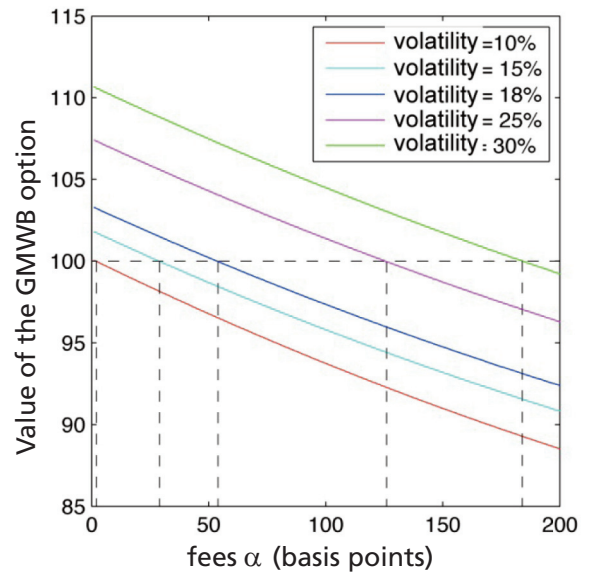
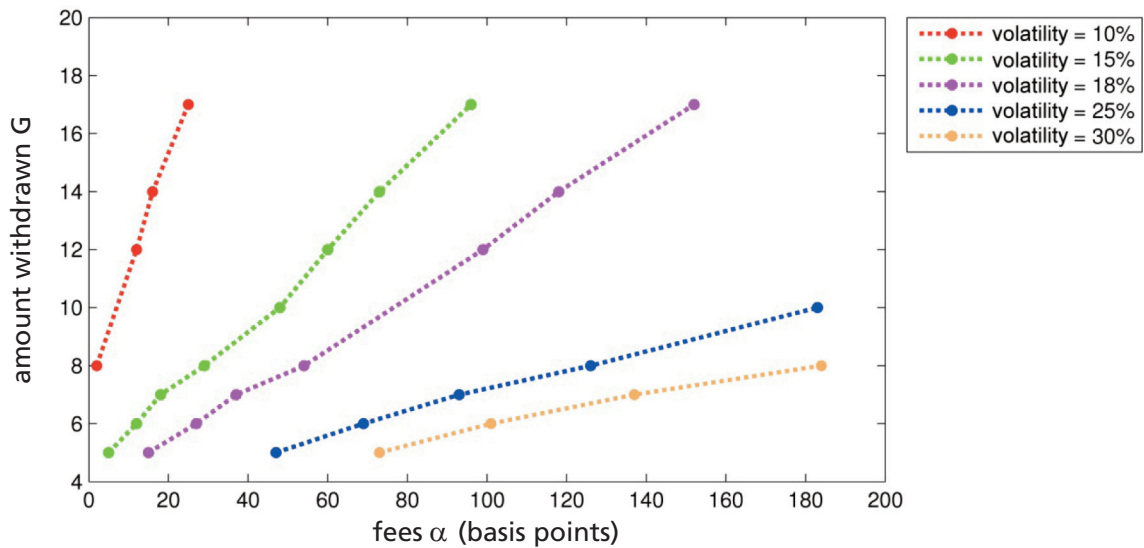


Figure 3.9: Values of α and G that make the option price fair in relation to changes in volatility



Note that, in Figure 3.9, a “small” variation in the amount withdrawn leads to a more than proportional increase in the fees charged by the insurance company. Moreover, as uncertainty in future market performance increases, the amount

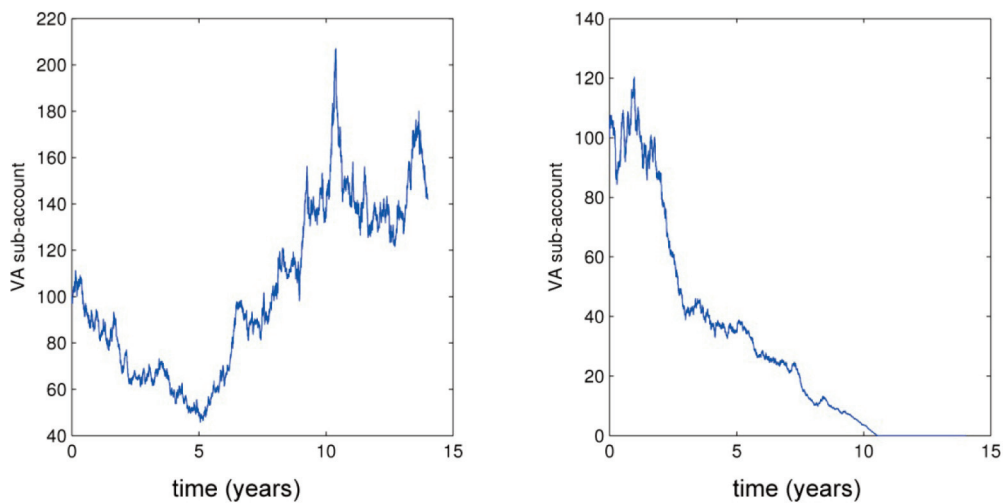
that individuals are willing to pay to ensure that they can make fixed, secure withdrawals, independently of the performance of the policy on the market, also increases.

3.1 Assumption of stochastic volatility

The analyses performed on the basis of the assumption of deterministic volatility were repeated, considering volatility as a stochastic process in itself.

In particular, by using the Scott model described in section 2.2 and discretizing the equations that describe the dynamics of the VA sub-account, different potential scenarios were generated, some of which are shown in figure (3.10).

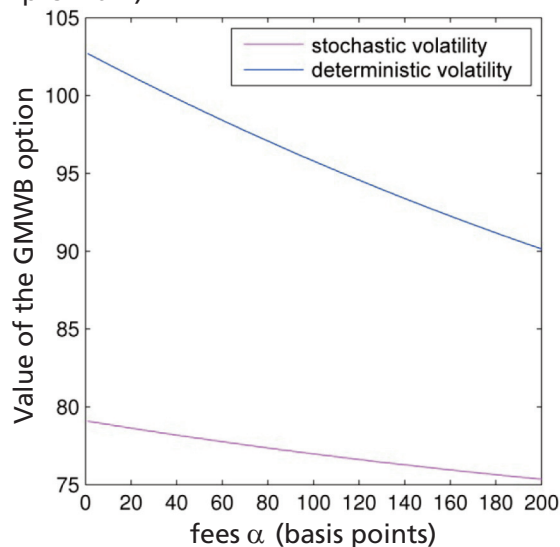
Figure 3.10: Examples of the dynamics of the VA sub-account ($a = 20$, $m = 18\%$, $b = 1.4$, $G = 7$, $\alpha = 40$ basis points)



Using the same values for the parameters considered in the deterministic volatility scenarios (also assuming that $a = 20$, $m = 18\%$ and $b = 1.4$) and also, in this case, using the assumption of the risk-neutrality of investors to consider the influence of the fees on the value of the option, we noted, however, that the contract never reaches a fair price. In fact, a zero risk premium ($\mu - r = 0$) means that the market does not compensate investors for the greater risk borne. Therefore,

under this assumption, all other factors being equal, the insurance company must charge lower fees in order to make the policy attractive to investors. The policy is therefore necessarily undervalued. See also Figure 3.11 in this respect, which compares the curve representing the influence of the fees on the value of the option obtained under the assumption of constant volatility with that of stochastic volatility: the first is positioned above the second.

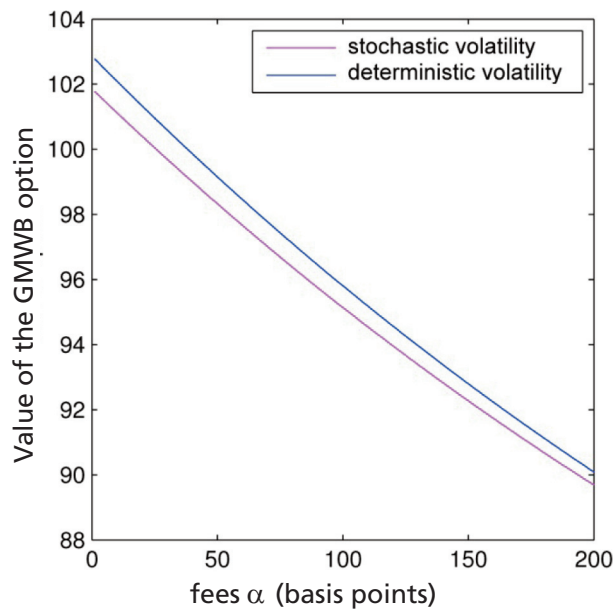
Figure 3.11: Value of the GMWB option in relation to changes in fees α under deterministic and stochastic volatility (zero risk premium)



The alternative is to apply a positive risk premium rather than a zero one. In this regard, let us consider figure 3.12, constructed assuming that $\mu = 10\%$ and $r = 5\%$: a value of α exists that makes the price of the option fair. This value, in particular, is lower than that obtained under the

assumption of deterministic volatility and lies at around 35 basis points. Therefore, the estimates obtained using this stochastic volatility model provide a more realistic view of market data than those obtained using Milevsky and Salisbury's deterministic volatility model.

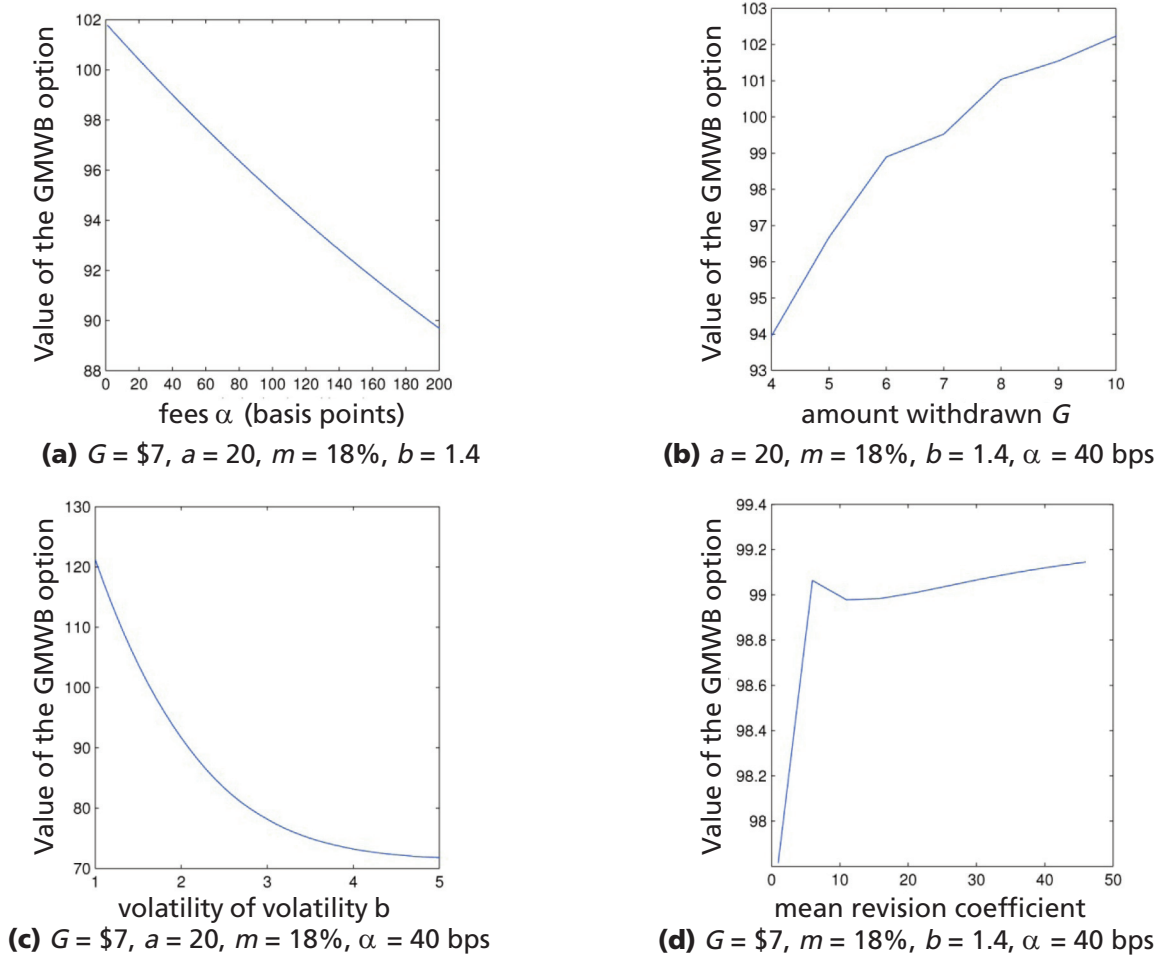
Figure 3.12: Value of the GMWB option in relation to changes in fees α under deterministic and stochastic volatility (5% positive risk premium)



In Figure 3.13, we consider the impact of other parameters on the value of the option. In scenarios (a) and (b), the trends followed by the option value are similar to those obtained under the assumption of deterministic volatility (decreasing in the first scenario and increasing in the second). Two parameters which instead appear only in the stochastic volatility model are the mean reversion

coefficient a and "volatility of volatility" b . Note in particular in scenarios (c) and (d) of Figure 3.13, the influence of these parameters on option value. The variance rate b of the volatility process has a negative impact on the value of the option, in the sense that by adding variability to the process followed by volatility, the value of the option decreases.

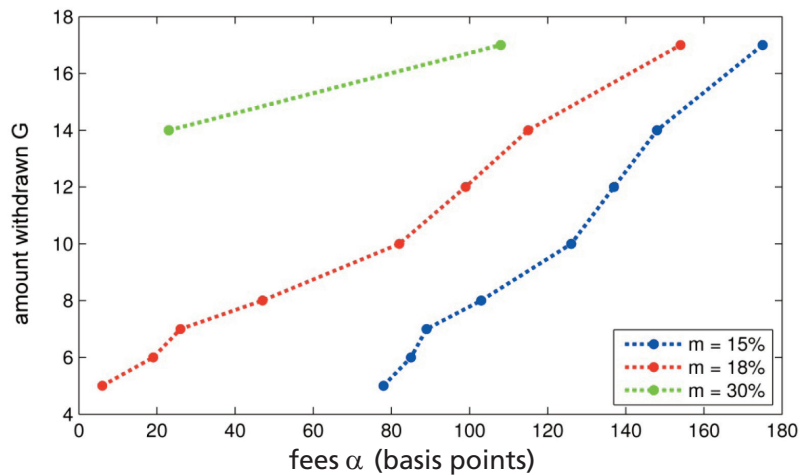
Figure 3.13: Influence of parameters α , G , b and a on the value of the GMWB option



Using the same procedure described for the deterministic volatility model, we generated the pairs $(\alpha; G)$ that make the option price fair. The

performance of these pairs in relation to changes in the long-run mean of Y is shown in Figure 3.14.

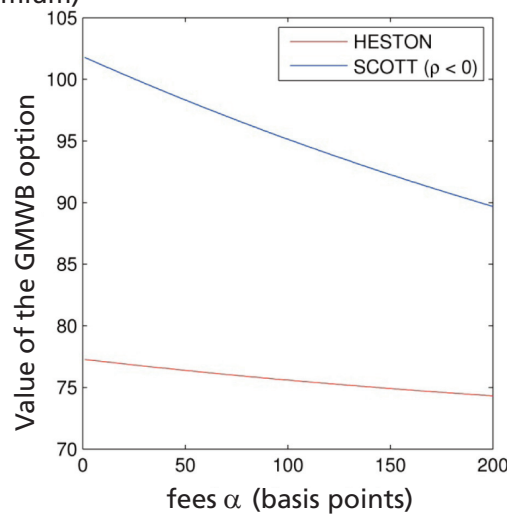
Figure 3.14: Values of α and G that make the price of the option fair in relation to changes in the long-run mean



The analyses performed considering the generalized Scott model were repeated with reference to the Heston model. The estimates obtained are consistent with those presented in the case of the generalized Scott model. For example, Figure 3.15 shows a comparison of the trajectories that describe

the performance of the option value in relation to changes in fees, considering both the generalized Scott model and the Heston model. Note how, from a qualitative point of view, the value of the option in the second scenario records a similar trend, although positioned lower on the graph.

Figure 3.15: Value of the GMWB option in relation to changes in fees α in the two stochastic volatility models (positive risk premium)



The simulations presented in this analysis have been obtained by selecting, from the generation of the Wiener process, very specific values for the number of trajectories used and the number of capitalizations considered in each period. We therefore decided to check whether the choices

made in the numerical experiments may in some way have influenced the results obtained. To achieve this, in Figures 3.16, 3.17, 3.18 and 3.19, we considered the influence of the two aforementioned parameters on the value of the option.

Figure 3.16: Influence of the number of trajectories generated on the value of the option considering the same "initial state" for the family of trajectories generated and dividing the time horizon into 250 time periods

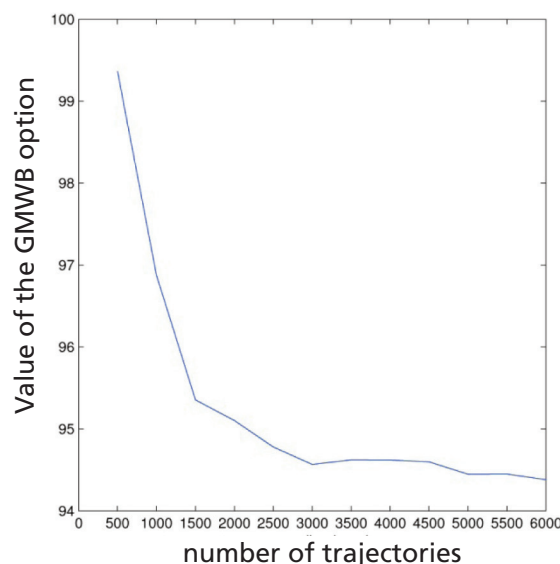


Figure 3.17: Influence of the number of trajectories generated on the value of the option

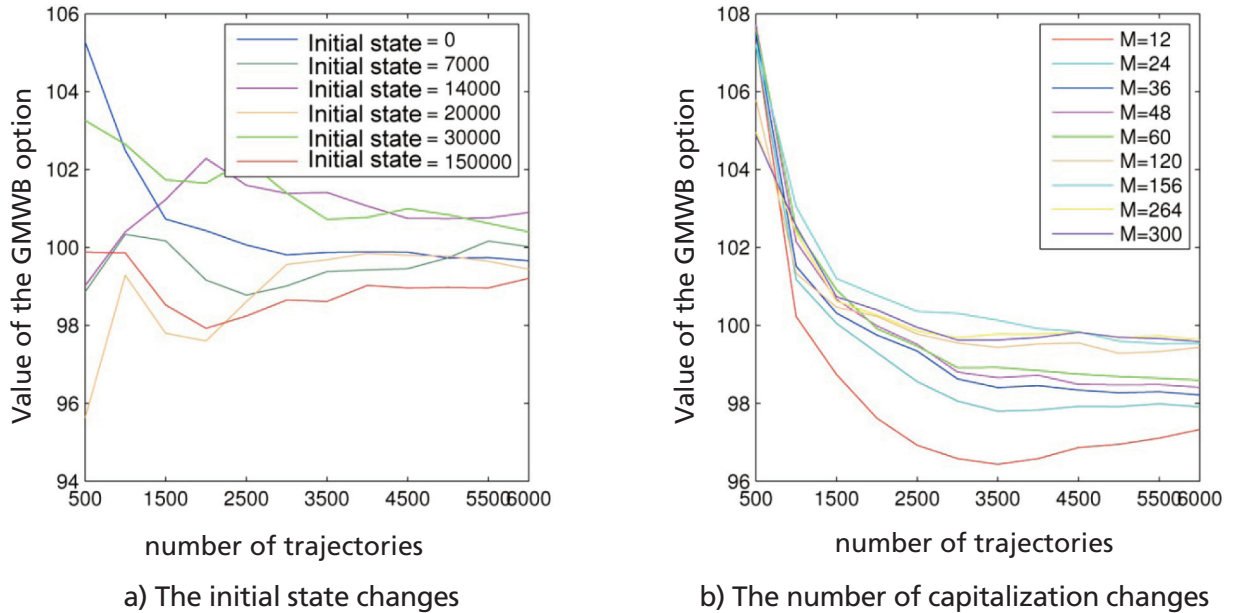


Figure 3.18: Influence of the number of capitalization periods on the value of the option, considering the same "initial state" for the 3000 trajectories generated

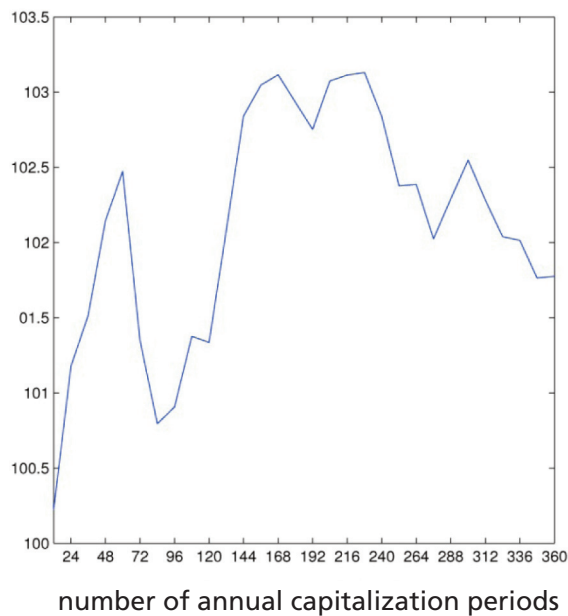
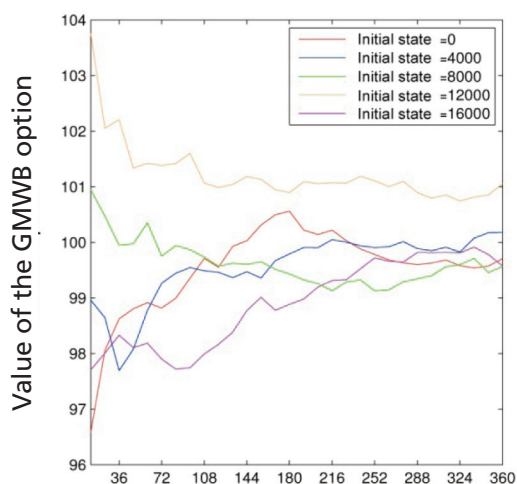
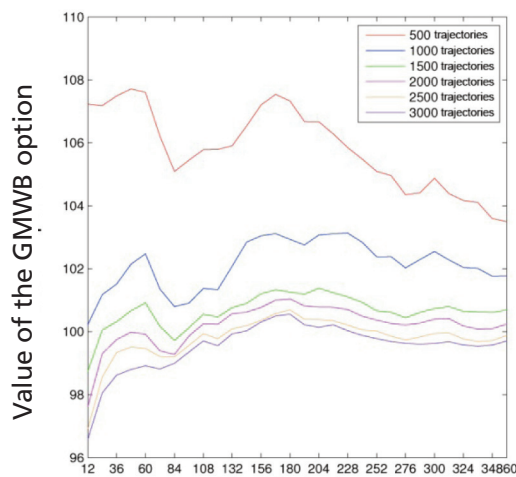


Figure 3.19: Influence of the number of capitalizations on the value of the option



number of annual capitalization periods

a) The initial state changes



number of annual capitalization periods

b) The number of trajectories generated changes

These figures demonstrate the validity of the model: all of the values considered in the simulations (3000 trajectories, initial state = 0 and 250

time periods annually) are those that best approximate the performance of the value of the GMWB option.

4. Conclusion

In this paper, the consideration of a stochastic volatility model and Monte Carlo simulation have made it possible to obtain very realistic estimates of insurance premiums observed on the markets for this type of option. In particular, it is interesting to observe how the proposed model is considerably more accurate than Milevsky and Salisbury's model. In fact, when actual premium values are between 30 and 50 basis points, Milevsky and Salisbury's model provides estimates of between 73 and 158 basis points, while the Scott and Heston models provide premium valuations of around 35 basis points. This improvement is extremely interesting from a practical point of view.

One potential future development of this analysis could to extend the proposed model to also consider the probability of death of the policyholder. Indeed, GMWB contracts can have very long time horizons (20-30 years), therefore it could be interesting to introduce the mortality rate to the explanatory variables of the model [4]. Another future development could also be to extend the valuation of the GMWB option to a dynamic scenario in which the amount of money payable by the policyholder is not a priori fixed but is determined by the policyholder himself over time. [5]

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