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A new method for modeling dependence via extended common shock type model

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Abstract

Reinsurers cover a large variety of risks that come from different sources and can have hidden dependencies. It is therefore crucial for reinsurers to constantly fine-tune their internal risk dependency analysis models.
Quantitative risk management deals with a vector of one-period profit-and-loss random variables $X_1, \ldots, X_n$, which represent different risks within a portfolio $X = (X_1, \ldots, X_n)$, its aggregate position $\Psi(X) = X_1 + \ldots + X_n$ and a risk measure ρ which assigns to $\Psi(X)$ an amount $\rho(\Psi(X))$ that is the capital required to hold $\Psi(X)$ over the predetermined period of time.

The aim of this paper is to propose an original simple and flexible method for modeling dependence between the risks Xi which allows for an easy and efficient computation of the aggregate position $\Psi(X)$ and so as well of the risk measure ρ. Our idea is an extension of a common shock type model which applies to any kind of distribution contrary to standard shock model which may be applied only to infinitely divisible distributions.

Keywords
Risk aggregation, dependence structure, risk factors, common shock models

1. Introduction

The aim of this paper is to present an original method for modeling the joint distribution of a vector of random variables $\mathbf{X} := (X_1, \dots, X_n)$, which may represent a portfolio of risks X_i, given the marginal distributions of the X_i.

One of the most popular method used for this purpose consists in modeling the joint distribution of the vector of ranks of the X_i, that is, the copula (see [2] and references therein). More precisely, a copula is a multivariate distribution function $C(u_1, \dots, u_n)$ on $[0,1]^n$ which has uniform distributed marginals. Any multivariate distribution $F(x_1, \dots, x_n)$ can be written as the composition of $C(u_1, \dots, u_n)$ with the marginal distributions $F_i(x_i)$ of F. That is,

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)).$$

An alternative method for modeling dependence structures is presented in [5]. This method applies to Lévy processes, or equivalently to random variables with infinitely divisible distributions. Roughly speaking, the method consists in extending copulas to copulas between Lévy mesures.

The method we propose in this paper is an extension of a common shock type model and has been motivated by two main reasons. Firstly, the high rigidity of copula modeling. That is, the copula associated to a multivariate distribution is quite sensitive to modifications of the marginals. Secondly, many widely used families of copulas, as Clayton, Gumbel and Gauss (with one parameter), have the property that

$$\lim_{n\to\infty} P(X_2 \geq \mathrm{VaR}^p(X_2), \dots, X_n \geq \mathrm{VaR}^p(X_n) | X_1 \geq \mathrm{VaR}^p(X_1)) = 0,$$

and the convergence to 0 is exponentially fast in n. This means that those copulas are not appropriate to model a situation in which an extreme bad event is the common driver for severe losses of $X_1, \dots, X_n$.

The new method proposed in this paper solves the two issues described above without making use of copulas and can be applied to any type of distribution of marginal risks X_i. Our idea is based on what we call a risk-by-risk-factor decomposition applied to each risks X_i. Each piece represents the impact that an extreme bad event has on our risk. We then define the dependence structure of $\mathbf{X}$ in a simple way trough those pieces.

The method we propose yields a dependence structure which puts more dependence on one tail of the marginals than on the other when the marginal have skewed distributions, as for example losses and results distributions for (re)insurance portfolios. For marginals with symmetric distributions, our method yields a dependence structure which puts the same dependence on the two tails of the marginals.

We present our new method in the context of reinsurance risk. Nevertheless it is an abstract method which can be used in any context.

The paper is organized as follows. In section 2 we describe briefly some mathematical tools related to our method, as copulas, Lévy process and infinitely divisible distributions. In section 3 we present in details the motivations justifying the added value of our method. In section 4 we define the method, the risk-by-risk-factor decomposition and our modeling for different structures of dependence. In section 5 we compare it with Clayton, Gumbel and Gauss copula in both a flat and a hierarchical structure. In section 6 we underline the main advantages of our method.

2. Mathematical background

In this section we present concisely some mathematical concepts behind our methods. In section 2.1 we describe the dependence and risk measures used in the paper. In section 2.2 we recall the definition of copula functions toghether with the Clayton, Gumbel and Gauss copulas. In section 2.3 we present the notion of Lévy process with few important properties and in section 2.4 we introduce the concept of infinitely divisible distributions.

Section 2.3 is not important for the comprehension of the paper and it can be skipped. Nevertheless we decide to include it for sake of completeness. Sections 2.2 and 2.4 might be skipped by the reader already familiar with those concepts.

Throughout all the paper, we set by convention that losses have positive sign. Our attention is then focused on the right tail of the marginal distributions.

2.1 Dependency and risk measures

We refer to [2] for the definitions contained in this section.

To measure the dependence between two random variables X and Y, we use a rank dependence measure which assess the dependence between extreme events. The right tail dependence between X and Y at percentile p is defined as

$$\mathrm{RTD}^p(X,Y) := P(X \geq \mathrm{VaR}^p(X) | Y \geq \mathrm{VaR}^p(Y)).$$

Analogously, the left tail dependence between X and Y at percentile p is defined as

$$\mathrm{LTD}^p(X,Y) := P(X \leq \mathrm{VaR}^p(X) | Y \leq \mathrm{VaR}^p(Y)).$$

If the limit of $\mathrm{RTD}^p(X,Y)$ goes to 0, as p tends to 1, then we say that X and Y have asymptotically independent right tails, and analogously, if $\mathrm{LTD}^p(X,Y) \to 0$ goes to 0, as $p \to 0$, then we say that X and Y have asymptotically independent left tails.

To measure the risk of a r.v. X, we use different measures. The value-at-risk at percentile p of X is defined as

$$\mathrm{VaR}^p(X) := \inf\{x | P(X \geq x) \leq p\}.$$

Analogously, the tail value-at-risk TVaR and the XTvaR are given by

$$\mathrm{TVaR}^p(X) := E(X | X \geq \mathrm{VaR}^p(X))$$

and

$$\mathrm{XTVaR}^p(X) := \mathrm{TVaR}^p(X) - E(X).$$

We define also the diversification gain as the capital that can be saved when undertaking the risks jointly in a portfolio $\mathbf{X} := (X_1, \ldots, X_n)$ as consequence of diversification. If we look at the capital at percentile $p = 0.99$, we then define the diversification gain as

$$\mathrm{DG}(\mathbf{X}) := 1 - \frac{\mathrm{XTVaR}^{0.99}(X_1 + \cdots + X_n)}{\mathrm{XTVaR}^{0.99}(X_1) + \cdots + \mathrm{XTVaR}^{0.99}(X_n)}.$$

Notice that $\mathrm{XTVaR}^{0.99}(X_1) + \cdots + \mathrm{XTVaR}^{0.99}(X_n)$ is the capital which we would need to run $\mathbf{X}$ without benefit of diversification.

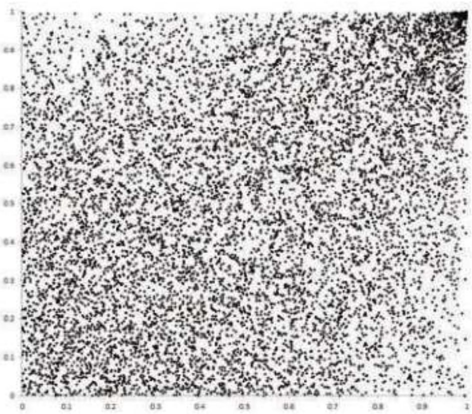

Figure 1: Scatter plot of Clayton copula with parameter 0.5

2.2 Copulas

One way of representing dependence between random variables are copulas [2].

A copula is defined as a multivariate distribution function $C(u_1, \ldots, u_n)$ on $[0,1]^n$ which has uniform distributed marginals.

By Sklar's theorem [4], any continuous multivariate distribution $F(x_1, \ldots, x_n)$, which represents the multivariate distribution of a portfolio $\mathbf{X} := (X_1, \ldots, X_n)$, is equivalent to the composition of a copula $C(u_1, \ldots, u_n)$ with the marginal distributions $F_i(x_i)$ of F, which represent the distributions of the risks X_i. That is,

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{1}$$

A nice property of copulas is that if we modify X_i in a comonotone way then the copula remains unchanged, that is, if g_i are increasing functions, then the copula of $(g_1(X_1), \ldots, g_n(X_n))$ is the same as the one of $(X_1, \ldots, X_n)$

There are a vast variety of copulas. We give belove an overview of the most common ones.

The Clayton copula is an Archimedean copula with the following expression, for $\theta > 0$,

$$C_\theta(u_1, \ldots, u_n) = [(1-u_1)^{-\theta} + \cdots + (1-u_n)^{-\theta} - n + 1]^{-1/\theta}.$$

Scatter plots of the bivariate Clayton can be found in [2] or see Figure 1. This copula puts more dependence on the right tail of the distribution of the marginals than on the left tail which is asymptotic independent. The higher the value of θ, the more the marginals depend on each other. The limit $\theta \to \infty$ corresponds to the comonotone dependence between the marginals and the limit $\theta \to 0$ corresponds to independent marginals.

The Gumbel copula is an Archimedean copula with the following expression, for $\theta \geq 1$,

$$C_\theta(u_1, \ldots, u_n) = \exp(-[(-\ln u_1)^\theta + \cdots + (-\ln u_n)^\theta]^{1/\theta}).$$

Scatter plots of the bivariate Gumbel can be found in [2]. This copula puts more dependence on the right tail of the distribution of the marginals than on the left tail which is asymptotic independent. The higher the value of θ, the more the marginals depend on each other. The limit $\theta \to \infty$ corresponds to the comonotone dependence between the marginals and the limit $\theta = 1$ corresponds to independent marginals.

The Gauss copula is an elliptic copula based on the multivariate normal distribution. Let $\Box_\Sigma$ be the multivariate normal distribution with mean **0** and correlation matrix Σ and let Φ be the standard normal distribution function, then

$$C_\Sigma(u_1,\ldots,u_n) = \Box_\Sigma(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n)).$$

Scatter plots of the bivariate Gauss copula can be found in [2]. This copula puts the same dependence on the right tail of the distribution of the marginals as on the left tail which are both asymptotic independent. The higher the values of Σ, the more the marginals depend on each other. The limit $\Sigma = 1$ corresponds to the comonotone dependence between the marginals and the limit $\Sigma = 0$ corresponds to independent marginals.

2.3 Lévy processes

By definition [1], a Lévy process is a stochastic process $\{X_t\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $X_0 = 0$, its increments are independent and stationary, that is, for every increasing sequence of times $t_0,\ldots,t_n$, the random variables X_{t_0}, $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent, and, for every $s > 0$, the distribution of $X_{t+s} - X_t$ does not depend on t, and right continuous in probability, i.e. for all $\epsilon > 0$ and $t \geq 0$,

$$\lim_{s\to t^+} P(|X_s - X_t| > \epsilon) = 0.$$

The distribution of a Lévy process is completely determined by its characteristic triplet (μ, σ^2, ν) where $\mu \in \mathbb{R}$ is the drift, $\sigma^2 \geq 0$ is the diffusion and ν is the Lévy measure, that is, a positive measure on $\mathbb{R} \setminus \{0\}$ which satisfy

$$\int_{\mathbb{R}\setminus\{0\}} \min\{x^2, 1\}\nu(dx) < \infty.$$

Roughly speaking, the Lévy measure describes the jumps of X_t. For every interval $[a, b] \subset \mathbb{R} \setminus \{0\}$, $\nu([a, b])$ is the average number (possibly infinite) of jumps of X_t in the time interval $[0, 1]$ whose fall in $[a, b]$. The integrability condition on ν above means that the average number of big jumps goes to 0, as the size of the jumps goes to ∞, and that the average number of small jumps might goes to ∞, as the size of the jumps goes to 0, but with a density prescribed by the condition.

By the Lévy-Khintchine formula, the characteristic function of a Lévy process with characteristic triplet $(\mu, \sigma^2, \nu$ has the following expression

$$E(e^{i\theta X_t}) = \exp(t[i\mu\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}}[1 - e^{i\theta x} - i\theta x \chi_{(-1,1)}(x)]\nu(dx)]).$$

A simple example of Lévy process is given by the jump-diffusion which is defined as the combination of a Brownian motion with drift and a compound Poisson process, that is

$$X_t := \mu t + \sigma B_t + \sum_{i=1}^{N_t} Y_i,$$

where N_t have Poisson distribution with parameter $t\lambda$, Y_i have distribution density f and B_t is a standard Brownian motion. In this case the characteristic triplet of the process is $(\mu, \sigma^2, \lambda f(dx))$ and the characteristic function has the following expression

$$E(e^{i\theta X_t}) = \exp(t[i\mu\theta - \frac{1}{2}\sigma^2\theta^2 + \lambda \int_{\mathbb{R}} (e^{i\theta x} - 1) f(dx)]).$$

The Lévy measure of a jump-diffusion process is finite, i.e. $\nu(\mathbb{R}) < \infty$, and hence it cannot have infinitely many jumps of a given size.

Any Lévy process can be obtained as the limit in distribution of a sequence of jump-diffusion processes.

2.4 infinitely divisible distributions

A random variable X is infinitely divisible [1] if, for any integer n, there exists independent and identically distributed random variables $X_1, \ldots, X_n$ such that $X_1 + \cdots + X_n$ has the same distribution as X. The distribution of X_i depends on X and on n.

Lévy processes and infinitely divisible distributions are strictly connected. Indeed, by the Lévy-Khintchine theorem, X is infinite divisible if and only if its characteristic function has the form

$$E(e^{i\theta X}) = \exp(i\mu\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} [1 - e^{i\theta x} - i\theta x \chi_{(-1,1)}(x)]\nu(dx)).$$

Hence, a Lévy process $\{X_t\}_{t\geq 0}$ is such that X_t is infinitely divisible, for every $t \geq 0$. Moreover, for any infinitely divisible random variable X there exists a Lévy process $\{X_t\}_{t\geq 0}$ such that X_1 coincides with X in distribution.

Examples of infinitely divisible distributions are Normal, LogNormal, Gamma, LogGamma, Exponential, Pareto, compound Poisson, compound Negative Binomial, compound Geometric, Fischer's F, Student's t, Cauchy, Gumbel, logistic.

3. Motivation

Copula is probably the most diffused tool used for modeling the dependence structure of a portfolio $\mathbf{X} = (X_1, \ldots, X_n)$, that is, for modeling the joint multivariate distribution $F(x_1, \ldots, x_n)$ of the vector $\mathbf{X}$ with marginals X_i.

One of the main reasons for this success is that one might think to be able to isolate in the copula function $C(u_1, \ldots, u_n)$ all the dependence structure of $\mathbf{X}$, independently on the distribution of the standalone risks X_i composing the portfolio. But this is not correct. Indeed, the equality (1) involves also the marginal distributions. In other words, if we keep the copula function C unchanged and we modify the distribution of one of the marginals X_i we then obtain a different joint distribution F, i.e. a different dependence structure[1].

An important feature of copulas to be aware of when modeling dependence is the following. Suppose we decide to use one of the copulas mentioned in Section 2.2 to model the dependence structure of $\mathbf{X}$ with the property that X_i have pair-wise the same dependence, non-comonotone nor independent, that is

$$P(X_j \geq \text{VaR}^{0.99}(X_j) | X_i \geq \text{VaR}^{0.99}(X_i))$$

is the same, for any $i \neq j$. Then, we have that

$$\lim_{n\to\infty} P(X_2 \geq \text{VaR}^{0.99}(X_2), \ldots, X_n \geq \text{VaR}^{0.99}(X_n) | X_1 \geq \text{VaR}^{0.99}(X_1)) = 0.$$

[1] See [3] for a critical review on the use of copulas.

That is, the probability that $X_2, \ldots, X_n$ incur severe losses greater than their corresponding 99%-quantiles, conditioned to X_1 has incurred the same kind of loss, converges to 0 as the number of risks growth to infinity. This means that Clayton, Gumbel and Gauss (with one parameter) copulas are not appropriate to model a situation in which an extreme bad event is the common driver for the severe losses of $X_1, \ldots, X_n$. Indeed, in that case we would expect that

$$P(X_j \geq \mathrm{VaR}^{0.99}(X_j)|X_i \geq \mathrm{VaR}^{0.99}(X_i))$$
$$\approx P(X_2 \geq \mathrm{VaR}^{0.99}(X_2), \ldots, X_n \geq \mathrm{VaR}^{0.99}(X_n)|X_1 \geq \mathrm{VaR}^{0.99}(X_1)), \quad \forall n \geq 2.$$

In our opinion copulas are more appropriate for the analysis of the dependence than for its modelization. In fact, copula is a complete source of information for rank dependence between the marginals. All the rank dependence measures, as Spearman's rank correlation and tail dependence, can be computed out of the copula. Nevertheless, copulas may be too rigid for modeling dependence.

In the following section we propose two real situations which show the limitations of using copulas for modeling the dependence structure.

3.1 Business case

Suppose that $\mathbf{X}_0$ represents a reinsurance portfolio composed by two buckets X_1, X_2 of Property treaties in Germany and in Poland, respectively, where the random variables X_1, X_2 represent the losses of our contracts. Suppose also we believe that X_1 and X_2 are distributed following two LogNormal distributions with parameters μ_1, σ_1 and μ_2, σ_2 and their dependence can be modeled with a Clayton copula function with parameter θ_0.

Consider now one of the following situations:

1. tomorrow we add a further bucket X_3 representing Property treaties in Czech Republic;
2. tomorrow we change the treaty conditions of X_1 and/or X_2;

Property treaties for Germany, Poland and Czech Republic have a catastrophic coverage component which might trigger common severe losses in our risks. Common catastrophic events behind those losses are windstorms.

In case (1), for modeling $\mathbf{X}_1 := (X_1, X_2, X_3)$, if we keep unchanged the parameter θ_0, setting

$$p := P(X_2 \geq \mathrm{VaR}^{0.99}(X_2)|X_1 \geq \mathrm{VaR}^{0.99}(X_1)),$$

we would have that

$$P(X_3 \geq \mathrm{VaR}^{0.99}(X_3), X_2 \geq \mathrm{VaR}^{0.99}(X_2)|X_1 \geq \mathrm{VaR}^{0.99}(X_1)) \approx 0.66^{1/\theta_0}p.$$

That is, for say $\theta_0 = 1$, the probability of X_3, X_2 have severe losses, conditioned to X_1 has a sever loss, is 66% of the probability of X_2 has a severe loss, conditioned to X_1 has a sever loss. In other words, out of 10 windstorms which hit Germany, in average $10p$ of them will hit also Poland and just $6.6p$ will hit Poland and Czech Republic. If we think to the geographical location of the three countries and the size of a windstorm, then we realize that the model cannot be quite correct.

Actually there is no way the modify the parameter θ_0 to overcome this issue. Moreover, if we increase the value of θ_0, we then increase the pair-wise dependence between the countries which would contradicts the dependence structure of $\mathbf{X}_0$. To be consistent with the model for $\mathbf{X}_0$, we could only modify the copula function.

In case (2), suppose that the treaty conditions in Germany and Poland change. We then obtain two effects. Firstly, the distributions of the losses of our risks X_1, X_2 change. Secondly, the dependence between X_1 and X_2 very likely change accordingly.

Indeed, suppose that, for instance, we increase the catastrophic component coverages in both X_1 and X_2. Hence, the dependence between X_1 and X_2 increases since the dependent part between X_1 and X_2 increases whereas the attritional part remains unchanged. A similar effects would happen also in case we change attachment point, or layers, or number of reinstatements, etc. for non-proportional treaties. The only case the dependence remains the same is the one for which X_1 and X_2 change in a comonotone way, i.e. $\tilde{X}_i = f_i(X_i)$ with f_i increasing. For example, if X_1 and X_2 are two quota-share contracts and shares change.

In both cases above, modeling dependence at the beginning with a specific copula function does not provide any advantage more than using the multivariate distribution.

4. A new method for modeling dependence

For the reasons mentioned in Section 3, we propose a new method to model the dependence structure of a portfolio which is flexible enough to deal in a simple way with modifications of the portfolio and which overcomes the issue of dimensionality for common risk factors.

We would like to underline that our method does not consist in defining a new family of copula functions. Our aim is to propose a way to define dependence directly on the marginals and which yields a copula function which depends on the marginal distributions.

Our method is described below.

4.1 Risk-by-risk-factor decomposition

In this section we describe our method for defining the dependence structure of a portfolio $\mathbf{X} = (X_1, \dots, X_n)$.

We start introducing what we call the risk-by-risk-factor decomposition which is the base of our new method. Let X be a random variable with probability density function f and let $\alpha \leq 1$ be a positive number[2]. Consider the function f_α defined by

$$f_\alpha(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\theta x} E(e^{i\theta X})^\alpha d\theta, \quad \text{for every } x \in \mathbb{R}.$$

Set Re$[\cdot]$ to be the real part of a complex number.

Definition. *We define the α piece of X as a random variable X^α with probability density function*

$$\tilde{f}_\alpha(x) := \max\{0, \mathrm{Re}[f_\alpha(x)]\} / \int_{\mathbb{R}} \max\{0, \mathrm{Re}[f_\alpha(x)]\} dx, \quad \textit{for every } x \in \mathbb{R},$$

[2]Assuming that the distribution function of X has a density is not relevant in our context. We can always approximate as good as we want X by a r.v. with absolutely continuos distribution function.

If X is an infinitely divisible (see Section 2.4), then f_α is a probability density function, for every α. Hence, $\tilde{f}_\alpha = f_\alpha$ and, if $\alpha_1, \ldots, \alpha_n$ are n positive real numbers such that $\alpha_1 + \cdots + \alpha_n = 1$, then, by defintion,

$$X^{\alpha_1} + \cdots + X^{\alpha_n} \stackrel{d}{=} X.$$

That is, the sum of the independent pieces $X^{\alpha_1}, \ldots, X^{\alpha_n}$ of X has the same distribution as X. If (μ, σ^2, ν) is the characteristic triplet of X (see Section 2.3), then the characteristic triplet of X^α is $(\alpha\mu, \alpha\sigma^2, \alpha\nu)$.

When X represents a risk in our portfolio, we can think to the α_k piece X^{α_k} of X as the part of the entire risk connected to an event, or risk factor, k. It is important to notice that X^{α_k} does not represent the event in itself but rather the impact that this event has on our risk.

Suppose that one event k has an impact on different risks X_i of our portfolio $\mathbf{X} = (X_1, \ldots, X_n)$, and that all the risks are infinitely divisible random variables (we remove this assumption below). We could describe this situation in the following way. For each risk i, we take an α_k^i piece of X_i which represent the impact of the event k on X_i, and we then treat, for sake of simplicity, those pieces as comonotone. By considering various events which affect independently our portfolio, we can apply again the same analysis for the remaining parts of the risks. In this way, for every i, the α_k^i pieces of X_i are independent and hence their sum gives us back X_i. This case in which all the risks are infinitely divisible random variables is a kind of common shock model.

In case a risk X_i is not infinitely divisible, we introduce the auxiliary random variable

$$\tilde{X}_i := X_i^{\alpha_1^i} + \cdots + X_i^{\alpha_n^i}.$$

The $\tilde{X}_i$ may be distributed differently than X_i. The difference between the distribution reduces to 0 as $\tilde{X}_i$ admits α_k^i pieces, $k = 1, \ldots, n$. We then treat X_i and $\tilde{X}_i$ as comonotone and define the dependence structure of $\mathbf{X}$ by using the copula which derives from our method applied to the portfolio $(X_1, \ldots, X_{i-1}, \tilde{X}_i, X_{i+1}, \ldots, X_n)$

This approach is justified in Section 5. Indeed the tests presented in that section show that the copulas which derive from our method depend on the marginal distributions in a weak way. That is, big changes in the marginals distributions have small impact on the implied copula.

In the following sections we describe in more details the parametrization of our method accordingly to the different type of realities we want to model. That allows to deal with the minimal number of parameters in each context.

4.2 Mono-dimensional dependence

In this section we describe our model in case of a mono-dimensional dependence structure. The reality we can model here corresponds to the case in which our portfolio $\mathbf{X} = (X_1, \ldots, X_n)$ has only one event which affects commonly, and with the same portion, all our risks. That is the typical situation in which Clayton or Gumbel copulas could be used.

Let $\alpha \leq 1$ be a positive number and let, for every i, X_i^α and $X_i^{1-\alpha}$ be the two pieces of X_i as defined in Section 4.1. Then, we define our dependence structure of $\mathbf{X}$ as follows

$$F(x_1, \ldots, x_n) := \int_{-\infty}^{\infty} F_{X_1^{1-\alpha}}(x_1 - x) \cdots F_{X_n^{1-\alpha}}(x_n - F_{X_n^\alpha}^{-1}(F_{X_1^\alpha}(x))) dF_{X_1^\alpha}(x).$$

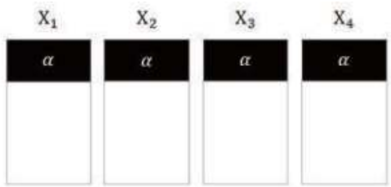

Figure 2: Mono-dimensional and symmetric dependence structure

That is, we look at the pieces X_i^α as comonotone and the other pieces $X_i^{1-\alpha}$ as independent. In Figure 2, we illustrate the decompositions of the risks in this case. The white parts in the figures are independent whereas the parts with the same color are comonotone.

The formula above provides an explicit analytical way to compute the multivariate distribution of **X**. Alternatively, we can easily simulate the multivariate distribution of **X** by generating $n+1$ independent uniformly distributed real numbers $u_0, \ldots, u_n$ in $[0,1]$ and obtain $X_1, \ldots, X_n$ by

$$X_i = F^{-1}_{X_i^\alpha}(u_0) + F^{-1}_{X_i^{1-\alpha}}(u_i).$$

See Figure 9 for examples of scatter plots of the copula derived by our method.

Differently suppose one event affects commonly, but with different portions, all our risks. In this situation neither Clayton nor Gumbel copulas could be used since they yield identical pair-wise dependence between risks.

Let $\alpha_1, \ldots, \alpha_n \leq 1$ be positive numbers and let, for every i, $X_i^{\alpha_i}$ and $X_i^{1-\alpha_i}$ be the two pieces of X_i as defined in Section 4.1. Then, we define our dependence structure as follows

$$F(x_1, \ldots, x_n) := \int_{-\infty}^{\infty} F_{X_1^{1-\alpha_1}}(x_1 - x) \cdots F_{X_n^{1-\alpha_n}}(x_n - F^{-1}_{X_n^{\alpha_n}}(F_{X_1^{\alpha_1}}(x))) dF_{X_1^{\alpha_1}}(x).$$

That is, we look at the pieces $X_i^{\alpha_i}$ as comonotone and the other pieces $X_i^{1-\alpha_i}$ as independent. In Figure 3, we illustrate the decompositions of the risks in this case. Again, the white parts in the figures are independent whereas the parts with the same color are comonotone. We can simulate the multivariate distribution of **X** by generating $n+1$ independent uniformly distributed real numbers $u_0, \ldots, u_n$ in $[0,1]$ and obtain $X_1, \ldots, X_n$ by

$$X_i = F^{-1}_{X_i^{\alpha_i}}(u_0) + F^{-1}_{X_i^{1-\alpha_i}}(u_i).$$

See Figure 8 for examples of scatter plots of the copula derived by our method.

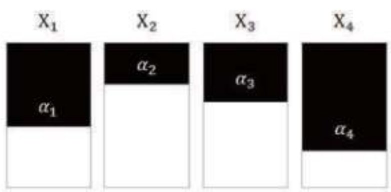

Figure 3: Mono-dimensional and non-symmetric dependence structure

4.3 Bi-dimensional dependence

In this section we describe our model in case of a bi-dimensional dependence structure. The reality we can model here corresponds to the case in which the risks X_i in our portfolio $\mathbf{X} = (X_1, \ldots, X_n)$ might have different pair-wise dependencies, but only with a gaussian-type structure. In this situation, neither Clayton nor Gumbel are appropriate copula function, while the Gauss copula could be used.

Set $m := n(n-1)/2$. Let $\alpha_1, \ldots, \alpha_m \leq 1$ be positive numbers such that

$$\alpha_1 + \cdots + \alpha_m \leq 1, \quad \alpha_{m+1} := 1 - (\alpha_1 + \cdots + \alpha_m),$$

and let $X_i^{\alpha_k}$ be the pieces of X_i as defined in Section 4.1. Then, we define our dependence structure of $\mathbf{X}$ as follows

$$F(x_1, \ldots, x_n) := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} F_{X_i^{\alpha_{m+1}}}(x_i - \sum_{k=1}^{i-1} F^{-1}_{X_i^{\alpha_i}}(F_{X_k^{\alpha_i}}(x_{k,i})) - \sum_{k=i}^{m} x_{i,k}) \prod_{k=i}^{m} dF_{X_i^{\alpha_k}}(x_{i,k}).$$

That is, for every k, we look at two pieces $X_i^{\alpha_k}$ and $X_j^{\alpha_k}$ as comonotone and the other pieces as independent. Observe that a α_k could be 0, meaning that the corresponding pair is an independent pair. In Figure 4, we illustrate the decompositions of the risks in this case. The white parts in the figures are independent whereas the parts with the same color are comonotone.

We can simulate the same dependence structure by generating n-dimensional vectors $\mathbf{u}^i$ of independent uniformly distributed real numbers in $[0,1]$ and obtain $X_1, \ldots, X_n$ by

$$\begin{aligned}
X_1 &= F^{-1}_{X_1^{\alpha_1}}((M_{1,1}\mathbf{u}^1)_1) + \cdots + F^{-1}_{X_1^{\alpha_m}}((M_{1,m}\mathbf{u}^m)_1) + F^{-1}_{X_1^{\alpha_{m+1}}}(\mathbf{u}_1^{m+1}) \\
X_2 &= F^{-1}_{X_2^{\alpha_1}}((M_{2,1}\mathbf{u}^2)_2) + \cdots + F^{-1}_{X_2^{\alpha_m}}((M_{2,m}\mathbf{u}^m)_2) + F^{-1}_{X_2^{\alpha_{m+1}}}(\mathbf{u}_2^{m+1}), \\
&\vdots \\
X_n &= F^{-1}_{X_n^{\alpha_1}}((M_{n,1}\mathbf{u}^n)_n) + \cdots + F^{-1}_{X_n^{\alpha_m}}((M_{n,m}\mathbf{u}^m)_n) + F^{-1}_{X_n^{\alpha_{m+1}}}(\mathbf{u}_n^{m+1}),
\end{aligned}$$

where $M_{i,k}$ are $n \times n$ lower triangular matrix with elements in $\{0,1\}$ such that $M_{i,k} = M_{k,i}$ and, for $k \leq i$, denoting by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ the standard base of $\mathbb{R}^n$,

$$M_{i,k} := \mathrm{Id} + \mathbf{e}_i \otimes (\mathbf{e}_k - \mathbf{e}_i).$$

That is, $M_{i,k}$ is the matrix such that, for every line $j \neq i$, all the elements in line j are equal to 0 but the element at column j which is equal to 1, and all the elements in line i are equal to 0 but the one at column k.

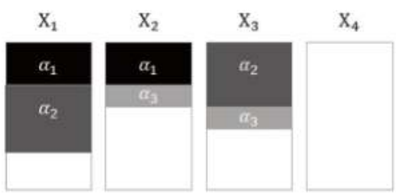

Figure 4: Bi-dimensional and symmetric dependence structure

Suppose the risks X_i in our portfolio $\mathbf{X} = (X_1, \ldots, X_n)$ might have different pair-wise dependencies, which might also be non-symmetric now. In this situation even the Gauss copula could not be used.

For every i, let $\alpha_1^i, \ldots, \alpha_m^i \leq 1$ be positive numbers such that

$$\alpha_1^i + \cdots + \alpha_m^i \leq 1, \quad \alpha_{m+1}^i := 1 - (\alpha_1^i + \cdots + \alpha_m^i),$$

and let $X_i^{\alpha_k^i}$ be the pieces of X_i as defined in Section 4.1. Then, we define our dependence structure of $\mathbf{X}$ as follows

$$F(x_1, \ldots, x_n) := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} F_{X_i^{\alpha_{m+1}^i}}\Big(x_i - \sum_{k=1}^{i} F^{-1}_{X_i^{\alpha_i^i}}\big(F_{X_k^{\alpha_i^k}}(x_{k,i})\big) - \sum_{k=i+1}^{m} x_{i,k}\Big) \prod_{k=i}^{m} dF_{X_i^{\alpha_k^i}}(x_{i,k}).$$

That is, for every k, we look at two pieces $X_i^{\alpha_k^i}$ and $X_j^{\alpha_k^j}$ as comonotonic and the other pieces as independent. We can simulate the same dependence structure by generating n-dimensional vectors $\mathbf{u}^i$ of independent uniformly distributed real numbers in $[0, 1]$ and obtain $X_1, \ldots, X_n$ by

$$\begin{aligned}
X_1 &= F^{-1}_{X_1^{\alpha_1^1}}((M_{1,1}\mathbf{u}^1)_1) + \cdots + F^{-1}_{X_1^{\alpha_m^1}}((M_{1,m}\mathbf{u}^m)_1) + F^{-1}_{X_1^{\alpha_{m+1}^1}}(\mathbf{u}_1^{m+1})\\
X_2 &= F^{-1}_{X_2^{\alpha_1^2}}((M_{2,1}\mathbf{u}^2)_2) + \cdots + F^{-1}_{X_2^{\alpha_m^2}}((M_{2,m}\mathbf{u}^m)_2) + F^{-1}_{X_2^{\alpha_{m+1}^2}}(\mathbf{u}_2^{m+1}),\\
&\vdots\\
X_n &= F^{-1}_{X_n^{\alpha_1^n}}((M_{n,1}\mathbf{u}^n)_n) + \cdots + F^{-1}_{X_n^{\alpha_m^n}}((M_{n,m}\mathbf{u}^m)_n) + F^{-1}_{X_n^{\alpha_{m+1}^n}}(\mathbf{u}_n^{m+1}).
\end{aligned}$$

4.4 Hierarchical dependence

In this section we describe our model in case of a hierarchical dependence structure. The reality we can model here corresponds to the case in which our portfolio $\mathbf{X} = (X_1, \ldots, X_n)$ has many events which can affects commonly, and with the same portion, various risks in a hierarchical way. In this situation, a hierarchical tree of Clayton or Gumbel copulas could be used.

To define this situation, suppose our hierarchical structure has m levels. For each level k, let $\mathcal{P}_k := \{P_1^k, \ldots, P_{m_k}^k\}$ be a partition of $\{1, \ldots, n\}$, that is, $P_1^k \cup \cdots \cup P_{m_k}^k = \{1, \ldots, n\}$ and P_i^k are pair-wise disjoints, such that $\mathcal{P}_1 := \{\{1, \ldots, n\}\}$ and $P_s^{k+1} \subset P_t^k$, $P_s^{k+1} \neq P_t^k$. That implies $m \leq n$.

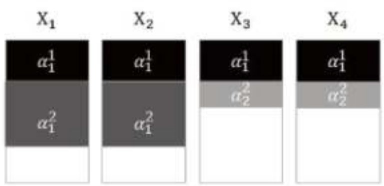

Figure 5: Hierarchical and symmetric dependence structure

For every $k = 1, \dots, m$, $s = 1, \dots, m_k$, let $\alpha_s^k \le 1$ be positive numbers such that, for every $i = 1, \dots, n$, the sum of α_s^k such that P_s^k contains i is less or equal than 1. Set

$$\beta_i := 1 - \sum_{\{s,k: i \in P_s^k\}} \alpha_s^k.$$

Let $X_i^{\alpha_s^k}$ and $X_i^{\beta_i}$ be the pieces of X_i as defined in Section 4.1. Then, we define our dependence structure of **X** as follows

$$F(x_1, \dots, x_n) := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} F_{X_i^{\beta_i}}\Big(x_i - \sum_{\{s,k: i \in P_s^k\}} F^{-1}_{X_i^{\alpha_s^k}}\big(F_{X_{j(s,k)}^{\alpha_s^k}}(x_{s,k})\big)\Big) \prod_{(s,k)\in I} dF_{X_{j(s,k)}^{\alpha_s^k}}(x_{s,k}),$$

where $j(s,k) := \min\{j : j \in P_s^k\}$ and $I := \{(s,k) : 1 \le k \le m, 1 \le s \le m_k\}$.

That is, we look at the pieces $X_i^{\alpha_s^k}$, for i in P_s^k, as comonotonic and the other pieces as independent. In Figure 5, we illustrate the decompositions of the risks in this case. The white parts in the figures are independent whereas the parts with the same color are comonotone.

We can simulate the same dependence structure of **X** by generating $m_1 + \cdots + m_m + n$ independent uniformly distributed real numbers $u_{s,k}$ in $[0,1]$, for every $k = 1, \dots, m$, $s = 1, \dots, m_k$, and with $s := 1$, for $k > m$, and obtain $X_1, \dots, X_n$ by

$$\begin{aligned}
X_1 &= F^{-1}_{X_1^{\alpha^1(1)}}(u_{s^1(1),1}) + \cdots + F^{-1}_{X_1^{\alpha^m(1)}}(u_{s^m(1),1}) + F^{-1}_{X_1^{\beta_1}}(u_{m_1+1,1})\\
X_2 &= F^{-1}_{X_2^{\alpha^1(2)}}(u_{s^1(2),2}) + \cdots + F^{-1}_{X_2^{\alpha^m(2)}}(u_{s^m(2),2}) + F^{-1}_{X_2^{\beta_2}}(u_{m_2+1,2}),\\
&\vdots\\
X_n &= F^{-1}_{X_n^{\alpha^1(n)}}(u_{s^1(n),n}) + \cdots + F^{-1}_{X_n^{\alpha^m(n)}}(u_{s^m(n),n}) + F^{-1}_{X_n^{\beta_n}}(u_{m_n+1,n}),
\end{aligned}$$

where $s^k(i) := s$, where s is such that $i \in P_s^k$, $\alpha^k(i) := \alpha^k_{s^k(i)}$ and $m_i := 0$, for $i > m$.

Suppose many events can affects commonly, but with different portions, various risks in a hierarchical way. In this situation, even a hierarchical tree of Clayton or Gumbel copulas can not be used.

For every $k = 1, \dots, m$, $s = 1, \dots, m_k$, $i = 1, \dots, n$, let $\alpha_{s,i}^k \le 1$ be positive numbers such that, for every $i = 1, \dots, n$, the sum of $\alpha_{s,i}^k$ such that P_s^k contains i is less or equal than 1. Set

$$\beta_i := 1 - \sum_{\{s,k: i \in P_s^k\}} \alpha_{s,i}^k.$$

Let $X_i^{\alpha_{s,i}^k}$ and $X_i^{\beta_i}$ be the pieces of X_i as defined in Section 4.1. Then, we define our dependence structure of **X** as follows

$$F(x_1, \dots, x_n) := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} F_{X_i^{\beta_i}}\Big(x_i - \sum_{\{s,k: i \in P_s^k\}} F^{-1}_{X_i^{\alpha_{s,i}^k}}\big(F_{X_{j(s,k)}^{\alpha_{s,j(s,k)}^k}}(x_{s,k})\big)\Big) \prod_{(s,k)\in I} dF_{X_{j(s,k)}^{\alpha_{s,j(s,k)}^k}}(x_{s,k}).$$

That is, we look at the pieces $X_i^{\alpha^k_{s,i}}$, for i in P_s^k, as comonotonic and the other pieces as independent. We can simulate the same dependence structure of **X** by generating $m_1 + \cdots + m_m + n$ independent uniformly distributed real numbers $u_{s,k}$ in $[0,1]$, for every $k = 1, \ldots, m$, $s = 1, \ldots, m_k$, and with $s := 1$, for $k > m$, and obtain $X_1, \ldots, X_n$ by

$$
\begin{aligned}
X_1 &= F^{-1}_{X_1^{\alpha^1(1)}}(u_{s^1(1),1}) + \cdots + F^{-1}_{X_1^{\alpha^m(1)}}(u_{s^m(1),1}) + F^{-1}_{X_1^{\beta_1}}(u_{m_1+1,1}) \\
X_2 &= F^{-1}_{X_2^{\alpha^1(2)}}(u_{s^1(2),2}) + \cdots + F^{-1}_{X_2^{\alpha^m(2)}}(u_{s^m(2),2}) + F^{-1}_{X_2^{\beta_2}}(u_{m_2+1,2}), \\
&\vdots \\
X_n &= F^{-1}_{X_n^{\alpha^1(n)}}(u_{s^1(n),n}) + \cdots + F^{-1}_{X_n^{\alpha^m(n)}}(u_{s^m(n),n}) + F^{-1}_{X_n^{\beta_n}}(u_{m_n+1,n}),
\end{aligned}
$$

where now $\alpha^k(i) := \alpha^k_{s^k(i),i}$.

4.5 Multi-dimensional dependence

In this section we describe our model in the most general case. That is the case in which there are many events which can affects commonly various risks in our portfolio **X** $= (X_1, \ldots, X_n)$ without any precise structure.

To define this situation, fix m and, for each level $k = 1, \ldots, m$, let $\mathcal{P}_k := \{P_1^k, \ldots, P_{m_k}^k\}$ be a partition of $\{1, \ldots, n\}$. Notice that, differently to the hierarchical dependence structure (Section 4.4), we do not require any property on the partitions. The definition of the dependence structure is formally equal to the one in Section 4.4.

5. Tests and comparisons

We compare our new method with Clayton, Gumbel and Gauss copulas in three different sets of tests.

For all the tests we consider marginals with two types of distributions: a LogNormal with $\mu = 10, \sigma = 1$ and a Pareto with $\alpha = 3, x_0 =$ 1m. To compute the Fourier transform and its inverse which are used for defining the dependendy structure by our method, we use the Fast Fourier Transform algorithm with 2.5k points equidistributed from 0 to 5m for the LogNormal and 5k points equidistributed from 0 to 80m for the Pareto marginals. For the aggregation and computation of statistics we use 100k simulations.

Figure 6: Hierarchical structure of Clayton copulas used for comparison with our method

Table 1: Conditional probability for more than two joint marginals

	Our method	Clayton
p_2 = RTD 99%	40%	40%
p_3	39%	24%
p_4	38%	18%
p_5	38%	15%
p_6	38%	14%
p_7	37%	12%
p_8	37%	10%
p_9	37%	9%
p_{10}	37%	6%
p_{11}	37%	4%
p_{12}	37%	4%
p_{13}	37%	4%
p_{14}	37%	4%
p_{15}	37%	4%
p_{16}	37%	3%
p_{17}	37%	3%
p_{18}	37%	3%
p_{19}	37%	3%
p_{20}	37%	3%

In the first set of tests, we aggregate two marginals with the four different methods and for three different values of dependeny, more precisely for RTD 99% equal to 13%, 28% and 61%. See results in Tables 2, 3. The main figures to be compared are the diversification gain absolute and relative.

For the second tests we build a small tree as defined in Figure 6. We compare our method described in Section 4.4 with a tree of Clayton copulas in which at each node we store the sum of its children. That is, the Clayton copula at the black node describes the dependence between $X_1 + X_2 + X_3$ and $X_4 + X_5$. See results in Tables 4 and 5. The main figures to be compared are the diversification gain absolute and relative.

In both families of tests above, we observe three main features of our method.

Firstly, from the values of RTD 99% and LTD 99%, we deduce that our method puts more dependence on the right tail of the marginals than on the left tails which are asymptotically independent. This is due to the skewness of the marginals. For symmetric marginals, the implied copula would have same tail dependence on the right and on the left.

Secondly, similar type of marginals have pretty identical implied copula.

Thirdly, the diversification gain at 99% with our method is higher with respect to the other copulas. We can explain this feature looking at the scatter plots of the implied copula (Figure 9) and, say, the Clayton copula (Figure 1). Indeed, for the Clayton copula, the points which fall in the 1% upper and right strips of the scatter plot, but in the upper-right $1\% \times 1\%$ square, do not fall far away from the square. While, with our method, those points might fall far away from the square. This feature yields a lower TVaR for the aggregated distribution under the same RTD 99%.

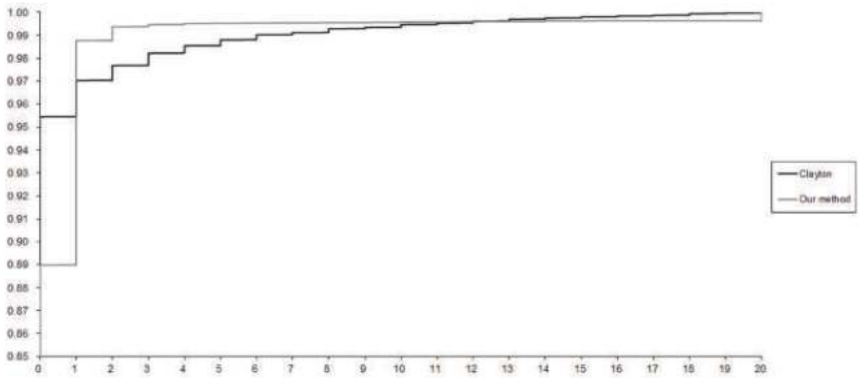

Figure 7: Distributions of number of risks from X1 toX20 which suffer bad losses simultanously with a Clayton copula and with our dependence structure

For the third test, we aggregate twenty marginals with a Clayton copula with parameter 0.8 and with our method. In both case we have a pairwise RTD 99% of 40%. Here we compare how the conditional dependence changes as we increase the dimension. That is, how the quantity

$$p_i := P(X_2 \geq \mathrm{VaR}^{0.99}(X_2), \ldots, X_i \geq \mathrm{VaR}^{0.99}(X_i) | X_1 \geq \mathrm{VaR}^{0.99}(X_1))$$

changes from $i = 2$ to 20. The results are shown in Figure 7 and Table 1. Gumbel and Gauss (with one parameter) manifest the same kind of behaviour as Clayton copula.

The third test shows the totally different behaviour of our method with respect to the other copulas. That is, increasing the number of marginals leaves essentially unchanged the conditional probability p_i with our method, while with the others copulas the value of p_i changes drastically as i increases.

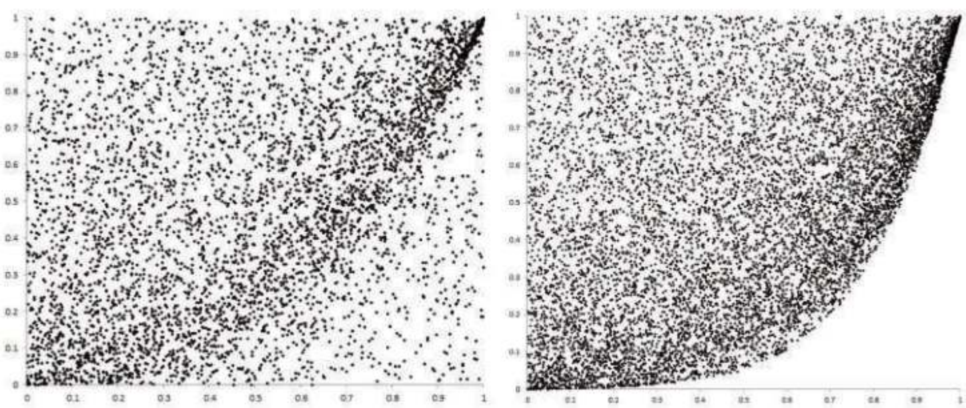

Figure 8: Scatter plots of our non-symmetric dependence structure applied to two LogNormal marginals μ = 10, σ = 1 with α1 = 0.67, α2 = 0.33 and α1 = 1, α2 = 0.33, respectively

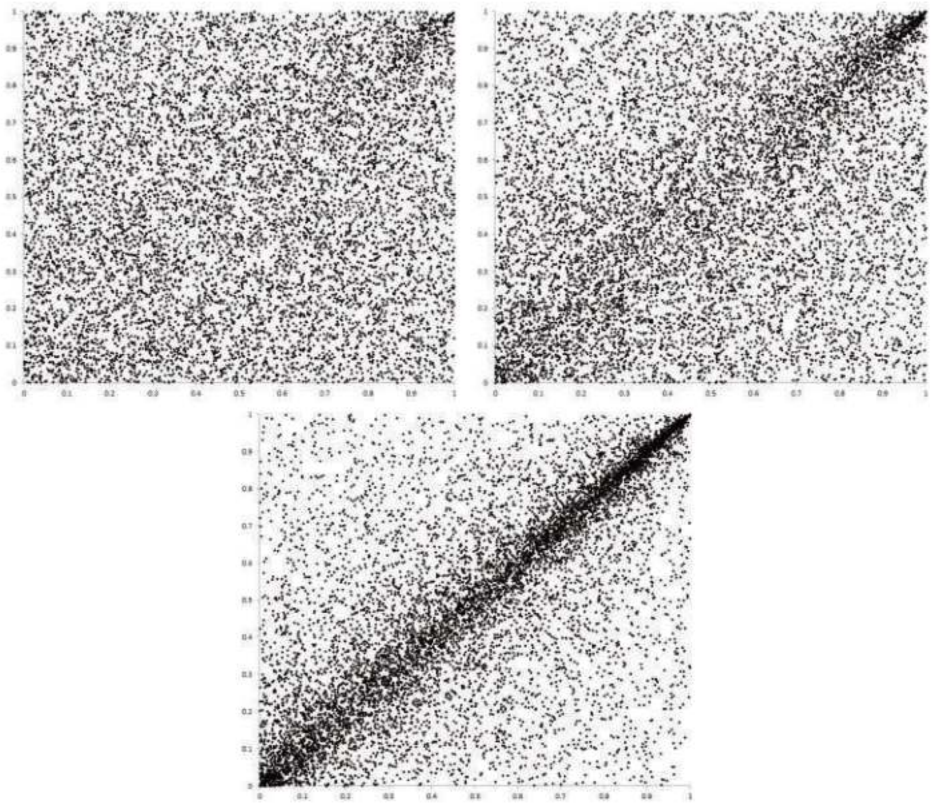

Figure 9: Scatter plots of our dependence structure applied to two LogNormal marginals $\mu = 10$, $\sigma = 1$ with α=0.145,0.33,0.675, respectively

6. Conclusions

The main advantages of our method are:

- flexibility to model the dependence structure under changes in the composition of the portfolio;
- flexibility to model many different situations with any kind of distribution of marginal risks (not only infinitely divisible distirbutions), and different risk structures, including non-symmetric and multi-dimensional structures, both which cannot be modeled with Clayton, Gumbel nor Gauss copulas;
- simplicity of the implementation and simulation;
- simplicity of the computation of the dependence and risk measures;
- fair treatment of common risk drivers, which is essentially independent on the number of risks, contrary to what happens with Clayton, Gumbel or Gauss (with one parameter) copulas;
- it yields dependence structures which put more dependence on one tail of the marginals than on the other when the marginal have skewed distributions, and dependence structures which put the same dependence on the two tails of the marginals, when the marginals have symmetric distributions.

Acknowledgement

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Appendices

Table 2: Pareto marginals, amounts in thousand

	Our method	Gumbel	Clayton	Gauss	Our method	Gumbel	Clayton	Gauss	Our method	Gumbel	Clayton	Gauss	Indep.	Comon.
Dependency	14%	1.10	0.25	0.48	32%	1.28	0.50	0.70	66%	2.10	1.40	0.92	0	1
Aggregate cdf:														
Mean	3,000	3,000	3,000	3,000	3,000	3,000	3,000	3,000	3,000	3,000	3,000	3,000	3,000	3,000
Std	1,284	1,320	1,341	1,388	1,362	1,455	1,461	1,507	1,532	1,630	1,558	1,643	1,241	1,711
VaR 80%	3,447	3,410	3,404	3,441	3,445	3,396	3,392	3,437	3,444	3,396	3,371	3,427	3,425	3,420
VaR 90%	4,145	4,098	4,114	4,222	4,177	4,133	4,129	4,264	4,260	4,232	4,202	4,299	4,079	4,301
VaR 95%	4,976	4,947	4,991	5,161	5,080	5,055	5,079	5,282	5,284	5,296	5,255	5,396	4,875	5,402
VaR 99%	7,751	7,783	7,917	8,242	8,098	8,303	8,386	8,681	8,736	9,001	8,855	9,116	7,326	9,240
Risk measures:														
VaR 99.5%	9,474	9,588	9,768	10,108	10,000	10,348	10,323	10,723	10,919	11,374	11,151	11,457	8,896	11,579
TVaR 99%	10,997	11,243	11,433	11,637	11,649	12,229	12,424	12,489	12,870	13,469	13,331	13,492	10,369	13,807
XTVaR 99%	7,973	8,243	8,433	8,638	8,626	9,229	9,420	9,488	9,846	10,469	10,331	10,491	7,368	10,810
DG	26%	24%	22%	20%	20%	15%	13%	12%	9%	3%	4%	3%	32%	0%
DG/DG for Indep.	82%	75%	69%	63%	63%	46%	40%	38%	28%	10%	14%	9%	100%	0%
Dep. measures:														
RTD 1%	13%	13%	13%	13%	28%	28%	28%	28%	61%	61%	61%	61%	1%	100%
LTD 1%	3%	2%	1%	13%	5%	4%	2%	28%	15%	16%	2%	61%	1%	100%

Table 3: LogNormal marginals, amounts in thousand

	Our method	Gumbel	Clayton	Gauss	Our method	Gumbel	Clayton	Gauss	Our method	Gumbel	Clayton	Gauss	Indep.	Comon.
Dependency	14.5%	1.10	0.25	0.48	33%	1.28	0.50	0.70	67.5%	2.10	1.40	0.92	0	1
Aggregate cdf:														
Mean	73	73	73	73	73	73	73	73	73	73	73	73	73	73
Std	72	74	75	79	78	80	81	85	87	91	90	92	67	95
VaR 80%	102	101	100	103	101	100	99	103	102	100	99	102	102	102
VaR 90%	145	144	145	152	147	146	146	155	153	153	152	158	143	159
VaR 95%	195	195	199	209	202	202	204	218	215	219	218	225	190	227
VaR 99%	351	355	366	381	373	385	391	410	416	432	434	439	331	448
Risk measures:														
VaR 99.5%	437	448	462	476	471	493	501	516	530	556	557	561	408	577
TVaR 99%	498	516	530	539	543	574	582	589	614	644	646	647	460	668
XTVaR 99%	425	444	457	467	470	501	510	517	542	572	573	576	387	595
DG	29%	26%	23%	22%	21%	16%	14%	13%	9%	4%	4%	4%	35%	0%
DG/DG for Indep.	82%	73%	67%	62%	60%	45%	41%	38%	26%	11%	11%	10%	100%	0%
Dep. measures:														
RTD 1%	13%	13%	13%	13%	28%	28%	28%	28%	61%	61%	61%	61%	1%	100%
LTD 1%	2%	2%	1%	13%	5%	4%	2%	28%	19%	16%	2%	61%	1%	100%

Table 4: Pareto marginals, amounts in thousand

	Our method	Clayton Tree	Indep.	Comon.
Mean	7,500	7,500	7,500	7,500
Std	2,838	3,133	1,881	4,288
VaR 80%	8,553	8,383	8,389	8,552
VaR 90%	10,024	10,010	9,470	10,765
VaR 95%	11,784	12,021	10,675	13,588
VaR 99%	17,700	18,925	14,282	23,283
Risk measures:				
VaR 99.5%	21,523	23,853	16,129	29,368
TVaR 99%	25,042	27,896	18,053	34,829
XTVaR 99%	17,441	20,380	10,558	27,328
DG	36%	25%	61%	0%
DG/DG for Indep.	96%	41%	100%	0%

Table 5: LogNormal marginals, amounts in thousand

	Our method	Clayton Tree	Indep.	Comon.
Mean	181	181	181	181
Std	156	172	106	240
VaR 80%	251	241	242	255
VaR 90%	341	342	305	396
VaR 95%	448	464	372	570
VaR 99%	779	857	563	1,130
Risk measures:				
VaR 99.5%	966	1,081	662	1,457
TVaR 99%	1,117	1,241	727	1,689
XTVaR 99%	930	1,059	546	1,507
DG	38%	30%	64%	0%
DG/DG for Indep.	94%	47%	100%	0%

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