

The Influence of Risk Measures and Tail Dependencies on Capital Allocation

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Abstract

(Re)insurance companies need to model their liabilities' portfolio to compute the risk-based capital (RBC) needed. The RBC depends on both the distribution functions and the dependence functions that are applied. We investigate the impact of those assumptions on an important concept for (re)insurance industries: the diversification gain. Several copula models are considered in order to focus on the role of dependencies. To be consistent with the frameworks of both Solvency II and of the Swiss Solvency Test, we deal with two measures of risk: the value at risk and the expected shortfall. We also point out the behavior of different capital allocation principles according both to the assumptions on dependence and to the choice of the risk measure.

Key words: Capital Allocation, Copula, Dependence, Diversification Gain, Monte Carlo Methods, Risk Measure.

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1. Introduction

The new risk based solvency regulations require (re)insurance companies to model their liabilities to compute the risk-based capital (RBC) needed. This concept relies heavily on the portfolio model that is at the heart of this computation. Particularly, it depends on both the distribution functions (dfs) and the dependence functions that are applied. Besides the debate among regulators on the type of risk measures that should be applied for estimating the RBC, there are few studies that systematically explore the impact of those assumptions on the results of the model.

The aim of this paper is precisely to explore various models and show how they influence an important concept for the (re)insurance industry: the diversification gain. Considering a simple portfolio composed of two risks, namely X and Y , we define the diversification gain according to Bürgi et al. (2009). This definition requires the choice of a risk measure for computing the risk based capital of a single risk and of the portfolio, see SCOR (2008). In our analysis we deal with two well known measures of risk, i.e. the value at risk (VaR) and the expected shortfall (ES), in agreement with Embrechts et al. (2005). To be consistent with the frameworks of both Solvency II and of the Swiss Solvency Test, we consider those risk measures at the 99.5% and 99% threshold respectively.

The study is performed assuming identical marginal dfs for X and Y , in order to focus on dependence and tail assumptions. Two typical distributions have been chosen: the lognormal df, which is very popular for modelling insurance risks, and the Fréchet df, to explore extreme value distributions (see Embrechts et al. (1997)).

For computing the joint df, we use copulas to model dependencies (we refer to Nelsen (2006), Nešlehová (2008) and Frees and Valdez (1998), among others). In particular, two families of copulas are taken into account: Archimedean and elliptical. Within the former family the Gaussian and the Student's t copula are considered, while from the latter family, besides the Gumbel copula, we choose also a flipped Clayton copula. The aim of this choice is to check the behavior of the portfolio in cases characterized by strong upper tail dependencies.

In the first stage of the work, Monte Carlo methods are used to obtain estimations of expected value, VaR, ES and RBC. With those values we can estimate the diversification gain obtained by combining both risks. To ensure consistent comparison, different copulas are parameterized using the same value for the Kendall's tau rank correlation coefficient. This allows us to impute differences due to the structure of the dependence with the strength remains constant.

In the second stage, we consider the required capital, in terms of RBC, according to different allocation principles related to the choice of the risk measure (see Albrecht (2004) and Artzner et al. (1999)). The Euler principle and the haircut allocation principle are compared. We analyze both the change of the dependence strength, indicated by the value of Kendall's tau, and the change in the joint distribution, described by the choice of a specific copula.

The rest of this paper is organised as follows. Section 2 provides the definitions of risk measures. An overview on copulas and a description of the main families is given in Section 3. Section 4 introduce rank correlations and tail dependence coefficients. In Section 5, we focus on the evaluation of the RBC and of the diversification gain. The impact of dependence on capital allocation is discussed in Section 6. Section 7 gives an outlook on future research and we conclude in Section 8.

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2. Risk measures

Definition 2.1 (VaR) For a rv X with $E(|X|) < \infty$ and $df F_X$, the VaR at confidence level $\alpha \in (0, 1)$ is defined as:

$$\text{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : P(X > x) \leq 1 - \alpha\} = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}. \quad (2.1)$$

VaR is not a coherent measure of risk, according to the characterization given in Artzner et al. (1999). In their work, the authors proposed four axioms, derived using economic reasoning, that a measure of risk should satisfy to be defined coherent.

Definition 2.2 (coherence) A risk measure $\varrho: \mathcal{M} \rightarrow \mathbb{R}$ on a convex cone \mathcal{M} , is called coherent if it satisfy the following four axioms:

- **Translation invariance.** For all $X \in \mathcal{M}$ and every $x \in \mathbb{R}$ we have $\varrho(X + x) = \varrho(X) + x$.
- **Subadditivity.** For all $X, Y \in \mathcal{M}$ we have $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$.
- **Positive homogeneity.** For all $X \in \mathcal{M}$ and every $\lambda > 0$ we have $\varrho(\lambda X) = \lambda \varrho(X)$.
- **Monotonicity.** For $X, Y \in \mathcal{M}$ s.t. $X \leq Y$ a.s. we have $\varrho(X) \leq \varrho(Y)$.

In the same article is shown that VaR does not satisfy subadditivity, hence VaR is not coherent.

About risk measures, we remind that VaR is currently at the base of the Solvency II European project as the methodology for evaluating the risk, while the level selected for the calculation is $\alpha = 99.5\%$.

A measure of risk which is coherent according to Definition 2.2 is the expected shortfall (ES), also known as tail VaR (TVaR), worst conditional expectation (WCE) or conditional VaR (CVaR)¹.

3. Copulas

Copulas were originally introduced as mathematical functions and then recognized to be a useful tool to model dependence. The concept of copula can be traced back at least to the work of Wassily Hoeffding and Maurice Fréchet, thought the term itself was coined by Sklar. The term derive from the latin word “copula”, contraction of *co-apula, meaning connection, bond, tie (co means together and apere means to join). An interesting review of the develop of copula theory and its applications is found in Genest et al. (2009). A must-reads literature on copulas is given in Embrechts (2009).

Definition 2.3 (ES) For a rv X with $E(|X|) < \infty$ and $df F_X$, the expected shortfall at confidence level $\alpha \in (0, 1)$ is defined as:

$$\text{ES}_\alpha(X) = \frac{1}{1 - \alpha} \int_\alpha^1 q_u(F_X) du, \quad (2.2)$$

where $q_u(F_X)$ is the quantile function of F_X .

To better understand the link between VaR and ES we can write the latter simply as:

$$\text{ES}_\alpha(X) = \frac{1}{1 - \alpha} \int_\alpha^1 \text{VaR}_u(X) du. \quad (2.3)$$

Hence the expected shortfall, contrary to VaR, takes (the shape of) the tail into account. ES is always greater or equal to VaR for a chosen confidence level α , i.e. $\text{ES}_\alpha(X) \geq \text{VaR}_\alpha(X)$.

It is possible to argue that for continuous distribution a more intuitive expression can be derived which shows that ES can be interpreted as the expected value given that VaR is exceeded.

Proposition 2.4 For a rv X with $E(|X|) < \infty$ and continuous $df F_X$, the expected shortfall at confidence level $\alpha \in (0, 1)$ results to be:

$$\text{ES}_\alpha(X) = E[X | X \geq \text{VaR}_\alpha(X)]. \quad (2.4)$$

We will use the notation $\varrho_\alpha(X)$ to indicate the value of a generic risk measure, at confidence level α , for the risk X . Remarks A.2 and B.2 in the appendix provide analytical expressions for both VaR and ES in case X is lognormal or Fréchet distributed, respectively.

(1) In the literature some authors give different definitions for ES, TVaR, WCE and CVaR, most of which lead to the same result when applied to continuous (loss) distribution; we refer to Acerbi and Tasche (2002) for more on this topic. The nomenclature we adopt is in agreement with Embrechts et al. (2005).

Definition 3.1 (bivariate copula) A right-continuous function $C: [0, 1]^2 \rightarrow [0, 1]$ is a copula if and only if:

- (i) $C(u, 0) = C(0, v) = 0$ for all $u, v \in [0, 1]$;
- (ii) $C(u, 1) = u$ and $C(1, v) = v$ for all $u, v \in [0, 1]$;
- (iii) C is quasi-monotone, i.e., for any $0 \leq u_1 \leq u_2 \leq 1$ and any $0 \leq v_1 \leq v_2 \leq 1$,

$$C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0. \quad (3.1)$$

In other words, a bivariate copula is a cumulative distribution function (cdf) on $[0, 1]^2$ whose margins are standard uniform.

In this paper we deal with bivariate rvs. However, for most of the concepts used, the extension in higher dimension ($d > 2$) is straight forward.

The usefulness of copulas for describing dependencies is revealed by the fundamental theorem of Sklar.

Theorem 3.2 (Sklar) *Let H be a bivariate cdf with margins F and G . Then there always exist at least one copula C such that for all $x, y \in [-\infty, \infty]$:*

$$H(x, y) = C(F(X), G(Y)). \quad (3.2)$$

Furthermore, C is unique on $\text{Ran}F \times \text{Ran}G$ where $\text{Ran}F$ and $\text{Ran}G$ denote the ranges of the marginal df F and G . Conversely, if C is a copula and F, G are univariate dfs, then the function H defined as above is a joint distribution function with margins F, G . Proof: See Sklar (1959).

Hence, in the light of Sklar's result, copulas allow to separate the dependence structure from the behavior of the univariate margins. Theorem 3.3 shows another attractive feature of the copula representation inviting to interpret a copula, associated with a random vector, as being its dependence structure.

Theorem 3.3 (invariance) *Let (X, Y) be a random vector with continuous margins and copula C . If T_1, T_2 are strictly increasing functions, then $(T_1(X), T_2(Y))$ also has copula C . Proof: See Embrechts et al. (2003), p. 6.*

One of the basic copulas is the independence copula Π , given by $\Pi(u, v) = uv$. Two others fundamental dependence concepts play an important role in copula theory.

Definition 3.4 (countermonotonicity) *X, Y are countermonotonic if and only if exist a rv Z and functions f_x increasing and f_y decreasing (or viceversa) such that $(X, Y) \stackrel{d}{=} (f_x(Z), f_y(Z))$.*

Definition 3.5 (comonotonicity) *X, Y are comonotonic if and only if exist a rv Z and increasing functions f_x, f_y such that $(X, Y) \stackrel{d}{=} (f_x(Z), f_y(Z))$.*

Theorem 3.6 (Fréchet-Hoeffding bounds)

Let C be a bivariate copula, then

$$W(u, v) \leq C(u, v) \leq M(u, v), \quad u, v \in [0, 1], \quad (3.3)$$

where $W(u, v) = \max(u + v - 1, 0)$ and $M(u, v) = \min(u, v)$.

Proof: See Embrechts et al. (2005), p. 189.

$W(u, v)$ and $M(u, v)$ are labeled the countermonotonicity and the comonotonicity copula, respectively.

Definition 3.7 (comprehensive) *A family of copulas that includes M, Π and W is called comprehensive.*

Definition 3.8 (copula density) *If rvs X, Y have continuous marginal df F and G with probability density function (pdf) f_x and g_y , respectively, then the joint pdf of H can be written as:*

$$h_H(x, y) = f_x(x)g_y(y)c(F(x), G(y)), \quad (x, y) \in \mathbb{R}^2, \quad (3.4)$$

where the copula density c is given by:

$$c(u, v) = \frac{\partial^2}{\partial u \partial v} C(u, v), \quad u, v \in [0, 1]. \quad (3.5)$$

The second mixed derivative of a copula C , i.e. its density, can be interpreted as a local dependence measure. Getting back to equation 3.4, we linger on the split of the pdf between the pdf corresponding to independence and the actual dependence structure, i.e.:

$$h_H(x, y) = \underbrace{f_x(x)g_y(y)}_{\text{joint pdf corresponding to independence}} c(F(x), G(y)).$$

Consistently, if we consider the independence copula Π we got $c = 1$ and thus the joint pdf factors into the product of the marginals f_x and g_y solely, as requested by the definition of independent rvs.

We observe here that copulas do not always have joint densities; examples of copulas that are not absolutely continuous are the comonotonicity and the countermonotonicity copulas. We have substantially two types of copulas according to the way we can obtain them:

- **Implicit copulas.** We can extract an implicit copula from any distribution with continuous marginal dfs; examples of this type are the elliptical copulas, i. e. the copulas derived from elliptically contoured (or elliptical) distributions.
- **Explicit copulas.** There are many copulas which we can write down in a simple closed form; the Archimedean family is an example of this type.

Among elliptical copulas, in this paper we deal with the Gauss copula and the (Student's) t copula, extracted from the multivariate normal distribution and from the multivariate t distribution, respectively.

• **Gauss.**

$$C_\rho^{Ga}(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)), \quad \rho \in (-1, 1), \quad (3.6)$$

where Φ is the cdf of a standard univariate normal distribution and Φ_ρ denotes the cdf of a bivariate normal with standard margins and correlation coefficient ρ .

• **Student's t .**

$$C_{v,\rho}^t(u, v) = \mathbf{t}_{v,\rho}(t_v^{-1}(u), t_v^{-1}(v)), \quad \rho \in (-1, 1), v > 0, \quad (3.7)$$

where t_v is the cdf of a standard univariate t distribution with v degrees of freedom and $\mathbf{t}_{v,\rho}$ denotes the cdf of a bivariate Student's t with standard t_v margins and dispersion coefficient ρ .

The Student's t copula allows for joint heavy tails and an increased probability of joint extreme events compared with the Gauss copula. Moreover, with respect to the latter, the t copula introduces an additional parameter, namely the degrees of freedom ν . Since the Student's distribution tends to the Gaussian when $\nu \rightarrow \infty$, increasing the value of ν decreases the tendency to exhibit extreme co-movements.

For explaining how bivariate Archimedean copula are constructed, we need to introduce the pseudo-inverse function.

Definition 3.9 (pseudo-inverse) Let $\phi : [0,1] \rightarrow [0,\infty]$ be a continuous and strictly decreasing function with $\phi(1) = 0$ and $\phi(0) \leq \infty$. The pseudo-inverse of ϕ with domain $[0,\infty]$ is defined as:

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t), & 0 \leq t \leq \phi(0), \\ 0, & \phi(0) < t \leq \infty. \end{cases} \quad (3.8)$$

Theorem 3.10 (Archimedean copula) Let $\phi : [0,1] \rightarrow [0,\infty]$ be a continuous and strictly decreasing function with $\phi(1) = 0$, $\phi(0) \leq \infty$ and let $\phi^{[-1]}(t)$ be its pseudo-inverse. Then the function:

$$C(u, v) = \phi^{[-1]}(\phi(u) + \phi(v)), \quad (3.9)$$

is a copula if and only if ϕ is convex.

Proof. See Nelsen (2006), pp. 111, 112.

The function ϕ is called the generator of the copula, because it characterizes the specific copula within the Archimedean family.

Definition 3.11 (Archimedean generator)

A continuous, strictly decreasing, convex function $\phi : [0,1] \rightarrow [0,\infty]$ s.t. $\phi(1) = 0$ is called an Archimedean generator. It is called a strict generator if $\phi(0) = \infty$.

In Bürgi et al. (2009) the authors stress as Archimedean copulas are commutative, that is $C(u, v) = C(v, u)$ for all $u, v \in [0, 1]$, and associative, that is $C(C(u, v), z) = C(u, C(v, z))$ for all $u, v, z \in [0,1]$.

If we choose as generator $\phi(t) = -\ln(t)$ we obtain the independence copula.

Among Archimedean copulas, in this paper we deal with the Clayton and the Gumbel copula.

• **Clayton.** $\phi(t) = \frac{1}{\theta}(t^{-\theta} - 1)$, hence: (3.10)

$$C_{\theta}^{Cl}(u, v) = \left[\max(u^{-\theta} + v^{-\theta} - 1, 0) \right]^{-1/\theta}, \quad \theta \in [-1, \infty) \setminus \{0\}.$$

• **Gumbel.** $\phi(t) = (-\ln t)^{\theta}$, thus: (3.11)

$$C_{\theta}^{Gu}(u, v) = \exp\left(-\left[(-\ln u)^{\theta} + (-\ln v)^{\theta}\right]^{1/\theta}\right), \quad \theta \geq 1.$$

The Clayton copula, unlike the above mentioned elliptical copulas, allows for asymmetries. We have seen that the t copula allows for joint extreme events but not for asymmetries. The Clayton copula provide more flexibility, exhibiting greater dependence in the lower tail than in the upper tail. We refer to Section 4 for more on the shape of copulas and for the concept of tail dependence, in particular. However, we notice here something additionally about the Clayton copula.

In the literature C_{θ}^{Cl} is defined as in equation 3.10 so to have the asymmetry described above (see e.g. Nelsen (2006), p. 116). However in insurance, as confirmed in Bürgi et al. (2009), the dependence ought to be modelled for the upper tail, because the fewest number of large amount claims are more relevant for the company with respect to the largest number of small amount claims. To this aim, from next Section we will work with a flipped Clayton copula, obtained by the transformation $(u, v) \rightarrow (1 - u, 1 - v)$. We will refer to this copula as the Clayton-M, indicating it with C_{θ}^{Cl-M} . More flipped copulas are given in Venter (2002), pp. 90-91.

The Clayton copula is a comprehensive copula, in fact: if $\theta \rightarrow 0$, C_{θ}^{Cl} approaches the independence copula, Π ; if $\theta \rightarrow \infty$, C_{θ}^{Cl} approaches the comonotonicity copula, M ; if $\theta \rightarrow -1$, C_{θ}^{Cl} approaches the countermonotonicity copula, W . The same is valid for C_{θ}^{Cl-M} .

The Gumbel copula, originally proposed in Gumbel (1960), interpolates between independence and perfect dependence having θ to represent the strength of the dependence. In particular, if $\theta = 1$ we obtain the independence copula, while the limit as $\theta \rightarrow \infty$ is the comonotonicity copula.

A more comprehensive list of Archimedean copulas can be found in Nelsen (2006), pp. 116-119, where 22 families are listed. Chapter 4.3 of Denuit et al. (2005) gathers explicit formulas of copula densities for several families of bivariate copula (both elliptical and Archimedean).

4. Rank Correlation and Tail Dependence

We introduce two rank correlations that are both scalar measures of concordance according to the definition proposed in Scarsini (1984), pp. 205-206². Differently from the correlation coefficient ρ , named after the British statistician Karl Pearson, rank correlations are based on the copula of (X, Y) only and not on the margins. This feature represents a relevant advantage when it comes to measure the strength of the dependence, as shown in Section 5.

The first measure of concordance we propose was originally discussed by G. T. Fechner around 1900 and rediscovered by the British statistician Sir Maurice Kendall in Kendall (1938). For a complete historical review we refer to Kruskal (1958).

Definition 4.1 (Kendall's tau) For the random pair (X, Y) the Kendall's tau is defined as:

$$\rho_\tau(X, Y) = \mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) > 0\} - \mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) < 0\}, \quad (4.1)$$

where (\tilde{X}, \tilde{Y}) is an independent copy of (X, Y) .

As can be seen from its definition, Kendall's tau for (X, Y) is simply the probability of concordance minus the probability of discordance. For continuous marginal distributions, Kendall's tau depends only on the unique copula of the risks as the following proposition says.

Proposition 4.2 Let (X, Y) be a vector of continuous rvs with copula C . Then Kendall's tau for (X, Y) is given by

$$\rho_\tau(X, Y) = 4 \int \int_{[0,1]^2} C(u, v) dC(u, v) - 1, \quad (4.2)$$

that is equivalent to say:

$$\rho_\tau(X, Y) = 4\mathbb{E}(C(U, V)) - 1, \quad (4.3)$$

where U, V are standard uniform.

The second measure of concordance we introduce is named after the English psychologist Charles Spearman who proposed it in Spearman (1904).

Definition 4.3 (Spearman's rho) For rvs X and Y with marginal dfs F and G the Spearman's rho is defined as:

$$\rho_S(X, Y) = \rho(F(X), G(Y)). \quad (4.4)$$

Hence Spearman's rho is simply the linear correlation of the probability-transformed rvs, which for continuous rvs is the linear correlation of their copula.

Thus, in the case of continuous marginal distributions, Spearman's rho depends only on the unique copula of the risks as the following proposition says.

Proposition 4.4 Let (X, Y) be a vector of continuous rvs with copula C . Then Spearman's rho for (X, Y) is given by: (4.5)

$$\rho_S(X, Y) = 12 \int \int_{[0,1]^2} uv dC(u, v) - 3 = 12 \int \int_{[0,1]^2} C(u, v) dudv - 3,$$

that is equivalent to say:

$$\rho_S(X, Y) = 12\mathbb{E}(UV) - 3, \quad (4.6)$$

where, if $X \sim F$ and $Y \sim G$, $U = F(X)$ and $V = G(Y)$.

The fact that both ρ_τ and ρ_S are copula-based measures implies that they inherit the property of invariance under strictly increasing transformations, one of the main advantage of rank correlations. We collect some facts and useful considerations about rank correlations in theorem 4.5.

Theorem 4.5 (rank correlations) Let X and Y be rvs with continuous distributions F and G , joint distribution H and copula C . The following are true:

- $\rho_\tau(X, Y) = \rho_\tau(Y, X)$, $\rho_S(X, Y) = \rho_S(Y, X)$.
- If X and Y are independent then $\rho_\tau(X, Y) = \rho_S(X, Y) = 0$.
- $-1 \leq \rho_\tau(X, Y), \rho_S(X, Y) \leq 1$.
- For $T: \mathbb{R} \rightarrow \mathbb{R}$ strictly monotone on $\text{Ran}(X)$, both ρ_τ and ρ_S satisfy $\kappa(T(X), Y) = \kappa(X, Y)$ or $\kappa(T(X), Y) = -\kappa(X, Y)$ if T is increasing or decreasing, respectively.
- $\rho_\tau(X, Y) = \rho_S(X, Y) = 1 \Leftrightarrow C = M \Leftrightarrow Y = T(X)$ a.s. with T increasing.
- $\rho_\tau(X, Y) = \rho_S(X, Y) = -1 \Leftrightarrow C = W \Leftrightarrow Y = T(X)$ a.s. with T decreasing.

Proof. See Embrechts et al. (2002), p. 196, and Nešlehová (2008), p. 50.

Moreover, for particular types of copula is possible to establish simpler relations between the copula itself and the rank correlations.

Theorem 4.6 (rank correlations for Gauss copula) Consider the Gauss copula defined as in equation 3.6. Then the rank correlations are:

$$\rho_\tau(C_\rho^{Ga}) = \rho_\tau(X, Y) = \frac{2}{\pi} \arcsin \rho, \quad (4.7)$$

$$\rho_S(C_\rho^{Ga}) = \rho_S(X, Y) = \frac{6}{\pi} \arcsin \frac{1}{2} \rho. \quad (4.8)$$

Proof. See Embrechts et al. (2005), pp. 215, 216.

(2) A formal proof can be found in Nelsen (2006), p. 169.

Some remarks on this theorem:

- Note that the right-hand side of equation 4.8 may be approximated by the value of the linear correlation itself, i.e. ρ .
- The relationship 4.7 between Kendall's tau and the correlation parameter of the Gauss copula C_ρ^{Ga} holds more generally for all elliptical distribution, hence including the t copula $C_{v,\rho}^t$.
- The relationship 4.8 between Spearman's rho and the correlation parameter of the Gauss copula C_ρ^{Ga} does not hold for all elliptical distributions.

Concerning Archimedean family, it is possible to link Kendall's tau and the generator of the specific copula.

Theorem 4.7 (Kendall's tau for Archimedean copulas) Let X and Y be continuous rvs with unique Archimedean copula C and generator ϕ . Then:

$$\rho_\tau(X, Y) = 1 + 4 \int_0^1 \frac{\phi(t)}{\phi'(t)} dt, \tag{4.9}$$

$$= 1 - 4 \int_0^\infty t \left(\frac{d}{dt} \phi^{[-1]}(t) \right)^2 dt. \tag{4.10}$$

Proof. For 4.9 see Genest and MacKay (1986), pp. 282-283; for 4.10 see Joe (1997).

Thorough this result explicit relations between Kendall's tau and copulas parameter can be found, which are summarized in table 1.

The last copula-based measures we introduce are called coefficients of tail dependence.

Definition 4.8 (upper tail dependence) Let X and Y be rvs with dfs F and G , respectively. Then, the coefficient of upper tail dependence of X and Y is defined as:

$$\lambda_u = \lambda_u(X, Y) = \lim_{q \rightarrow 1^-} \mathbb{P}(Y > G^-(q) \mid X > F^-(q)), \tag{4.11}$$

provided a limit $\lambda_u \in [0, 1]$ exists. If $\lambda_u \in (0, 1]$ X and Y are said to be asymptotically dependent in the upper tail. If $\lambda_u = 0$ they are asymptotically independent in the upper tail.

Definition 4.9 (lower tail dependence) Let X and Y be rvs with dfs F and G , respectively. Then, the coefficient of lower tail dependence of X and Y is defined as:

$$\lambda_l = \lambda_l(X, Y) = \lim_{q \rightarrow 0^+} \mathbb{P}(Y \leq G^-(q) \mid X \leq F^-(q)), \tag{4.12}$$

provided a limit $\lambda_l \in [0, 1]$ exists. If $\lambda_l \in (0, 1]$ X and Y are said to be asymptotically dependent in the lower tail. If $\lambda_l = 0$ they are asymptotically independent in the lower tail.

As for rank correlation, these definitions make most sense in the case that F and G are continuous dfs.

Proposition 4.10 Let X and Y be rvs with continuous dfs F and G , respectively, then we get the following expressions for λ_u and λ_l in terms of the unique copula C :

$$\lambda_l = \lim_{q \rightarrow 0^+} \frac{C(q, q)}{q}, \tag{4.13}$$

$$\lambda_u = \lim_{q \rightarrow 0^+} \frac{\hat{C}(q, q)}{q}, \tag{4.14}$$

where \hat{C} denotes the survival copula, defined as $\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$.

In Table 1 we give the available analytical expressions for rank correlations and tail dependencies referred to the four copulas used in this paper.

Copula	ρ_τ	ρ_S	λ_l	λ_u
C_ρ^{Ga}	$\frac{2}{\pi} \arcsin \rho$	$\frac{6}{\pi} \arcsin \frac{1}{2} \rho$	$\begin{cases} 0, & \rho \neq 1, \\ 1, & \rho = 1, \end{cases}$	$\begin{cases} 0, & \rho \neq 1, \\ 1, & \rho = 1, \end{cases}$
$C_{v,\rho}^t$	$\frac{2}{\pi} \arcsin \rho$	-	$\begin{cases} 2t_{v+1} \left(-\frac{\sqrt{v+1}\sqrt{1-\rho}}{\sqrt{1+\rho}} \right), & \rho \neq 1, \\ 1, & \rho = 1, \end{cases}$	$\begin{cases} 2t_{v+1} \left(-\frac{\sqrt{v+1}\sqrt{1-\rho}}{\sqrt{1+\rho}} \right), & \rho \neq 1, \\ \rho = 1, & \rho = 1, \end{cases}$
C_θ^{Cl-M}	$\frac{\theta}{(\theta+2)}$	-	0	$\begin{cases} 2^{-1/\theta}, & \theta > 0, \\ 0, & \theta \leq 0, \end{cases}$
C_θ^{Gu}	$1 - \frac{1}{\theta}$	-	0	$2 - 2^{1/\theta}$

Table 1
Rank correlations and tail dependencies for four copula classes.

Remark 4.11 Concerning the tail dependence, we observe that:

- While independence of X and Y implies $\lambda_u = \lambda_l = 0$, the converse is not true in general.
- For elliptical copulas, since $C = \hat{C}$, we have $\lambda_u = \lambda_l$.
- For the Gauss copula, both λ_u and λ_l are equal to 0 (provided $\rho \neq 1$) meaning that, regardless of high correlation ρ we choose, if we go far enough into the tail, extreme events appear to occur independently in X and Y .
- For the t copula the tail dependence coefficient increase with ρ increasing and decrease with v increasing; the t copula gives asymptotic dependence in the tail even when ρ is negative or zero.
- The Clayton copula is lower tail dependent, hence the Clayton-M is upper tail dependent.
- The Gumbel copula is upper tail dependent.

5. Diversification Gain

In this section, we investigate through Monte Carlo methods the role of dependence on the risk based capital. We define RBC as the surplus of the risk measure on the risk premium.

Definition 5.1 (RBC)

$$RBC_{\varrho_\alpha}(X) = \varrho_\alpha(X) - E(X). \tag{5.1}$$

According to the Swiss Solvency Test (SST) guidelines, Swiss-based insurances have to adopt the ES at 99% as the risk measure. In order to meet the solvency requirements under the Solvency 2 guidelines, European insurances will have to utilize as the risk measure, say, the VaR calibrated to a confidence level of 99.5%. In our analysis we explore both cases. The main goal of this section is to provide a quantitative judgment about the diversification gain, as defined in Bürgi et al. (2009). The diversification gain represents the percentage of the RBC that a (re)insurance company can save in the management of its portfolio, on account of the positive aggregation of more risks.

Definition 5.2 (Diversification Gain) *The diversification gain for a portfolio Z, aggregating the risks X₁, ..., X_d is given by:*

$$D_{\varrho_\alpha}(Z) = 100\% - \frac{RBC_{\varrho_\alpha}(\sum_{i=1}^d X_i)}{\sum_{i=1}^d RBC_{\varrho_\alpha}(X_i)}. \tag{5.2}$$

We study a simple portfolio, Z, composed by two risks, X and Y. In a first stage both risks are assumed to have a lognormal distribution, namely $X, Y \sim \text{logN}(9.58, 0.83)$. The lognormal df is very popular for modelling insurance risks and we select μ and σ such that the coefficient of variation, that is the ratio of the standard deviation to the mean, is equal to 1. For computing the joint df, we use copulas to model the structure of the dependence. According to the families introduced in Section 3, we examine two elliptical copulas, the Gauss and the Student's t, together with two Archimedean copulas, the Clayton-M and the Gumbel. About the t copula, if not specified, we assume $\nu = 1$.

Concerning the strength of the dependence and in order to ensure consistent comparison, different copulas are parameterized using the same value for the Kendall's tau, through the equations given in Table 1.

Figure 1 illustrates how different copulas implies different structures of dependence. A comparison with Figures 2 and 3 shows how the strength of the dependence acts on these structures.

Through the programming language MATLAB, we implement ad-hoc procedures to compute all the values necessary to quantify the diversification gain. We simulate a realization (x, y) of the bivariate random vector (X, Y), according to the specific copula, to its parametrization and to the marginal distributions. Marshall and Olkin (1988) provide an algorithm for the simulation of Archimedean copulas. Algorithms for simulating from both the Gauss and the t copula can be found in Embrechts et al. (2005). We repeat the simulation process 2,000,000 times for each characterization of the bivariate distribution. This allow us to derive Monte Carlo estimates for the portfolio of the expected value, E(Z), the value at risk at 99.5%, VaR_{99.5%}(Z), the expected shortfall at 99%, ES_{99%}(Z). For the sake of simplicity and without loss of generality, we drop percentiles from notation, keeping them fixed at the above mentioned levels.

From these results and from Equation 5.1, we calculate the RBC_{VaR}(Z) and the RBC_{ES}(Z) according to the measure of risk adopted. Finally, in accordance with Equation 5.2, we determine the D_{VaR}(Z) and the D_{ES}(Z).

Table 2 provides the results for three different values of Kendall's tau, namely 0.05, 0.35 and 0.7. For this specific portfolio 10,000,000 simulations are considered. For each determination of the Kendall's tau, the corresponding values of the copula parameters are given. The Student's t copula require an additional parameter, the degrees of freedom, that is not implied by the rank correlation. We select three values for ν , namely 1, 3 and 7, to have a more complete picture about the t copula.

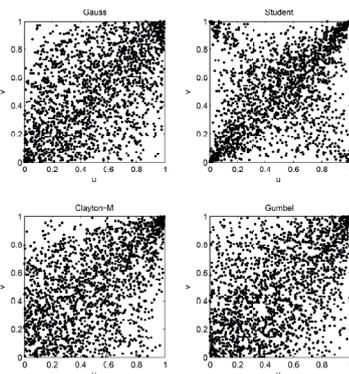


Fig. 1. Rank scatter plots for copulas parameterized with $\rho_\tau = 0.35$.

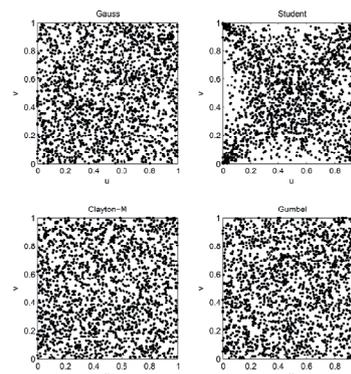


Fig. 2. Rank scatter plots for $\rho_\tau = 0.05$

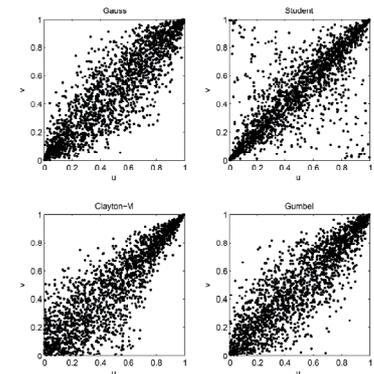


Fig.3. Rank scatter plots for $\rho_\tau = 0.70$

	Clayton-M	Gumbel	Student's t			Gauss	
$\rho_\tau = 0.05$	$\theta = 0.1053$	$\theta = 1.0526$	$\rho = 0.0785$			$\rho = 0.0785$	
			$\nu = 1$	$\nu = 3$	$\nu = 7$		
	E(Z)	40,851	40,833	40,857	40,843	40,844	40,858
	VaR _{99.5%} (Z)	183,865	184,026	205,621	191,460	184,264	177,947
	ES _{99%} (Z)	202,591	204,287	231,094	214,912	204,157	194,966
	RBC _{VaR} (Z)	143,013	143,193	164,764	150,617	143,420	137,090
	RBC _{ES} (Z)	161,740	163,454	190,237	174,069	163,313	154,108
D _{VaR} (Z)	30.14%	29.95%	19.64%	26.52%	29.95%	33.12%	
D _{ES} (Z)	30.83%	29.98%	18.74%	25.68%	30.26%	34.33%	
$\rho_\tau = 0.35$	$\theta = 1.0769$	$\theta = 1.5385$	$\rho = 0.5225$			$\rho = 0.5225$	
			$\nu = 1$	$\nu = 3$	$\nu = 7$		
	E(Z)	40,859	40,848	40,846	40,848	40,836	40,860
	VaR _{99.5%} (Z)	233,680	227,299	223,841	217,209	211,411	206,581
	ES _{99%} (Z)	262,338	254,897	251,220	244,014	235,852	227,589
	RBC _{VaR} (Z)	192,821	186,451	182,995	176,361	170,576	165,722
	RBC _{ES} (Z)	221,479	214,048	210,374	203,167	195,016	186,729
D _{VaR} (Z)	5.81%	9.15%	10.57%	13.85%	16.45%	19.07%	
D _{ES} (Z)	5.47%	8.64%	9.96%	13.31%	16.50%	20.23%	
$\rho_\tau = 0.70$	$\theta = 4.6667$	$\theta = 3.3333$	$\rho = 0.8910$			$\rho = 0.8910$	
			$\nu = 1$	$\nu = 3$	$\nu = 7$		
	E(Z)	40,845	40,826	40,863	40,860	40,840	40,860
	VaR _{99.5%} (Z)	244,219	242,780	240,094	238,924	237,395	236,003
	ES _{99%} (Z)	273,178	271,486	268,911	267,953	265,350	263,269
	RBC _{VaR} (Z)	203,374	201,954	199,231	198,064	196,555	195,143
	RBC _{ES} (Z)	232,333	230,660	228,048	227,093	224,510	222,409
D _{VaR} (Z)	0.44%	1.30%	2.84%	3.19%	3.77%	4.63%	
D _{ES} (Z)	0.43%	1.25%	2.62%	3.00%	3.83%	5.03%	

Table 2
Results for a portfolio composed by $X, Y \sim \log N(9.58, 0.83)$.

Considering figures available in Table 2, a first consideration can be done about risk measure. For this type of df, characterized by a (moderately) heavy tail, the difference between $VaR_{99.5\%}(Z)$ and $ES_{99\%}(Z)$ is relatively small. Nonetheless, due to this difference, the RBC increase as we move from one risk measure, VaR, to the other, ES. Regarding RBC, empirical results confirm common intuition. The risk based capital increase as the strength of the dependence increase. For the Gauss copula, we have $RBC_{ES}(Z) = 154,108$ if $\rho_\tau = 0.05$, $RBC_{ES}(Z) = 186,729$ if $\rho_\tau = 0.35$ and $RBC_{ES}(Z) = 222,409$ if $\rho_\tau = 0.70$. Consequently, the diversification gain reflects these movements. Looking still at the Gauss copula, we obtain $D_{ES}(Z) = 34.33\%$ if $\rho_\tau = 0.05$, $D_{ES}(Z) = 20.23\%$ if $\rho_\tau = 0.35$ and $D_{ES}(Z) = 5.03\%$ if $\rho_\tau = 0.70$. A similar behavior is obtained in case we take VaR instead of ES. Focusing on the structure of the dependence, it is interesting to observe an order in terms of conservativeness, among the copulas analyzed. Both when $\rho_\tau = 0.35$ and when $\rho_\tau = 0.70$, we have that the Gauss copula provide the highest diversification gain, followed by the Gumbel, the t and the Clayton-M copula. Further analysis have taken into account more values for rank correlations³.

(3) All the numerical results mentioned in the paper are available to the interested readers.

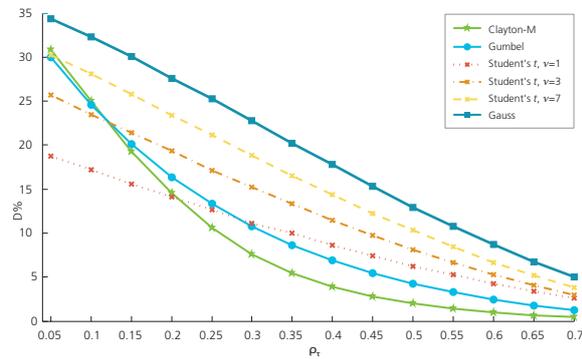


Fig. 4. Diversification gain as a function of both the strength and the structure of the dependence. The risk measure is the expected shortfall. X, Y are both lognormal distributed.

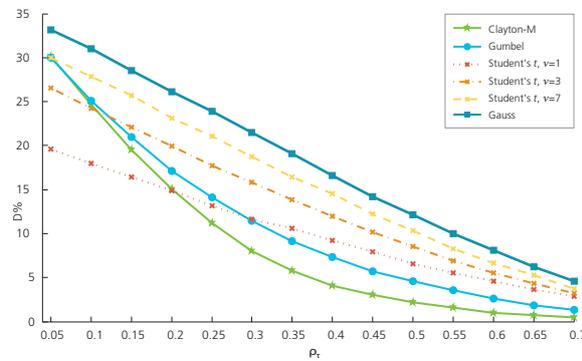


Fig. 5. Diversification gain as a function of both the strength and the structure of the dependence. The risk measure is the value at risk. X, Y are both lognormal distributed.

The trend of the diversification gain is represented, in case ES or VaR is used, in Figure 4 and Figure 5, respectively. Figure 4 depicts clearly the behavior above mentioned. Excepts for very low levels of dependence, the Clayton-M copula results to be the more conservative in terms of the diversification gain. This means that when a (re)insurance company has to assume a copula to model the dependence among the risks of its (bivariate) portfolio, the C^{Cl-M} would guarantee a prudential choice. For instance, when $\rho_\tau = 0.35$, if the model is based on a Gauss copula, then the (re)insurance company can state $D_{ES}(Z)$ equal to 20.23% while if the real model would follow a Clayton-M copula, then the $D_{ES}(Z)$ would be only 5.47%. Thus, a warning has to be sent with regard to the prudentiality of certain assumptions. Only for very low dependencies, the t copula with $\nu = 1$ supply the lower diversification gain among the set of copulas considered. Another interesting features illustrated in Figure 4, is the tendency of the results provided by the t copula to those given by the Gauss copula, as the degrees of freedom increase. When $\nu = 7$ the divergence is already significantly reduced. Figure 5, based on VaR as risk measure, confirms all the remarks valid for Figure 4.

Hereafter, we will use the ES to illustrate the results, pointing out differences with VaR, if any. Some analysis has been done to check the behavior under others parametrization of the lognormal dfs. No particular discrepancy has emerged.

Figure 6 highlights the influence of the dependence on the RBC. For each copula model, the darker column quantifies the $RBC_{VaR}(Z)$, and the lighter column represents the $RBC_{ES}(Z)$. The lines refer to the diversification gain. As noticed above, the difference among the two risk measures used is modest within this portfolio.

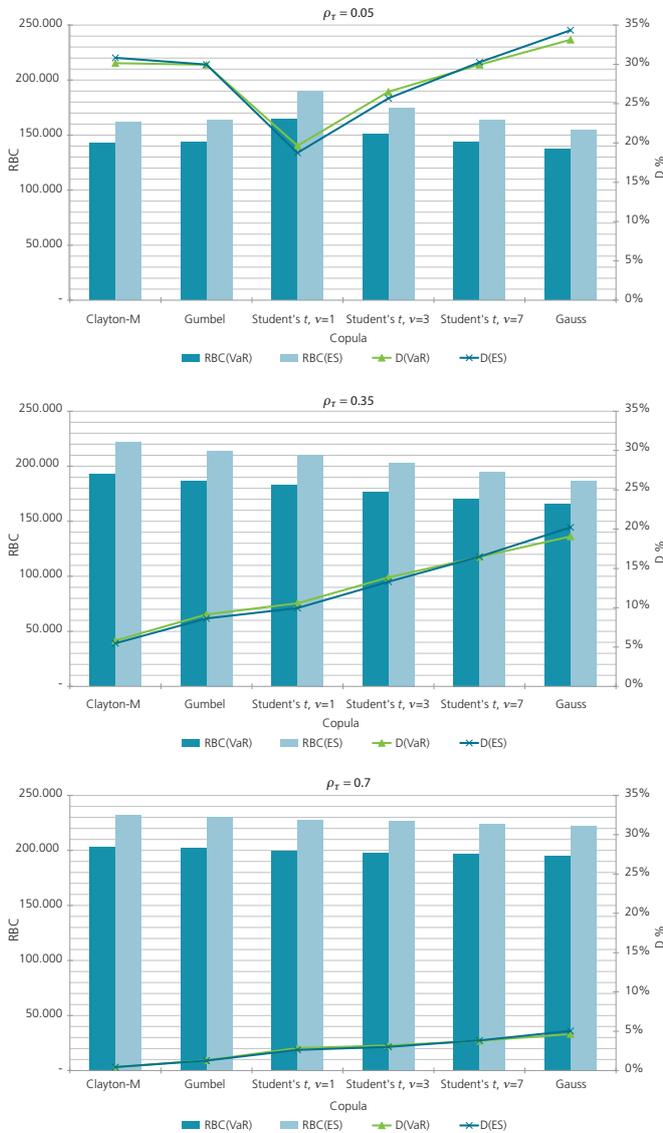


Fig. 6. RBC and D as a function of copulas, for $\rho_{\tau} = 0.05$ (top), $\rho_{\tau} = 0.35$ (middle) and $\rho_{\tau} = 0.70$ (bottom). X, Y are both log-normal distributed.

	Clayton-M	Gumbel	Student's t			Gauss
$\rho_{\tau} = 0.35$	$\theta = 1.0769$	$\theta = 1.5385$	$\rho = 0.5225$			$\rho = 0.5225$
			$v = 1$	$v = 3$	$v = 7$	
$E(Z)$	24,973	25,049	24,956	24,909	24,872	24,672
$VaR_{99.5\%}(Z)$	307,964	298,570	297,173	293,741	290,529	288,033
$ES_{99\%}(Z)$	579,554	573,396	563,277	543,112	529,311	499,621
$RBC_{VaR}(Z)$	282,991	273,520	272,217	268,833	265,657	263,361
$RBC_{ES}(Z)$	554,582	548,346	538,321	518,204	504,439	474,949
$D_{VaR}(Z)$	4.65%	6.50%	7.39%	8.96%	9.68%	9.80%
$D_{ES}(Z)$	4.38%	6.41%	6.68%	9.21%	11.41%	14.01%

Table 3 Results for a portfolio composed by X, Y both Fréchet distributed with $\alpha = 1.5$ and $s = 4,657.15$

To explore extreme value distributions, we repeat all the previous analysis with a new portfolio Z . Z is now composed from two risks, X, Y , both Fréchet distributed with $\alpha = 1.5$ and $s = 4,657.15$. The shape parameter, α , is chosen to grant both an important tail and sufficient stability in the simulation process. Table 3 provides results for $\rho_{\tau} = 0.35$.

Focusing on $D_{ES}(Z)$, we observe an estimated value ranging from 4.38%, in case we assume a Clayton-M copula, to 14.01%, in case our model is based on a Gauss copula. The main difference, that has to be stressed with respect to the previous portfolio, is the relevance of the choice of the risk measure. For instance, when the copula is the Clayton-M, we have $RBC_{VaR}(Z) = 282,991$ and $RBC_{ES}(Z) = 554,582$. A similar discrepancy is present indifferently from the structure of the dependence used. The divergence between the capital requirements, according to the risk measure applied, is due to the marginal distributions. The tail of the Fréchet df, differently from the tail of the lognormal df, emphasizes the diversity between the two risk measures. As we noticed in Section 2, the ES is able, contrary to the VaR, to take into account the shape of the tail.

Figure 7 illustrates the trend of the diversification gain as a function of the dependence. Remarks provided for Figure 4 are valid here as well.

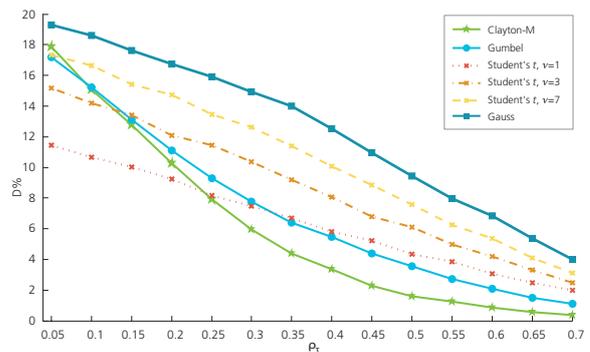


Fig. 7. Diversification gain as a function of both the strength and the structure of the dependence. The risk measure is the expected shortfall. X, Y are both Fréchet distributed.

Figure 8 highlights the influence of the dependence on the RBC. The importance of the choice of the risk measure is eye-catching and other analysis confirm that its preserved also varying the strength of the dependence.

Some analysis has been done to check the behavior under others parametrization of the Fréchet dfs. No particular discrepancy has emerged, but the stability of the Monte Carlo process, for $\alpha \rightarrow 1$, has to be supported by an increasing number of simulations.

As a further analysis, we mixed in a new portfolio $Z = X + Y$ a (moderately) heavy tail df with an extreme value df.

In particular, X is lognormal distributed with $\mu = 6.52$ and $\sigma = 2.15$, and Y is Fréchet distributed with $\alpha = 1.5$ and $s = 4$, 657.15.

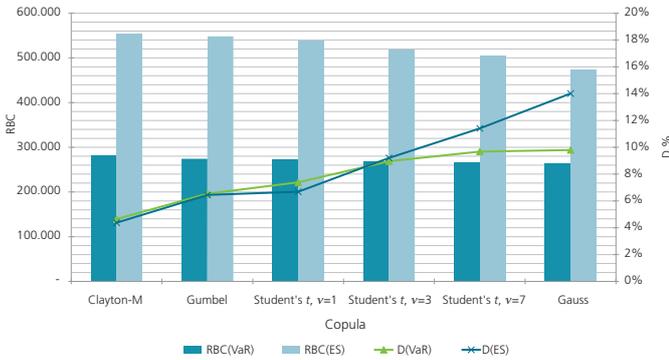


Fig. 8. RBC and D as a function of copulas, for $\rho_\tau = 0.35$. X, Y are both Fréchet distributed.

The new parametrization of the lognormal df has been chosen such that the weight of the two risks is almost equivalent in terms of capital allocation (see Section 6). Table 4 provides results for $\rho_\tau = 0.35$.

	Clayton-M	Gumbel	Student's t			Gauss
	$\theta = 1.0769$	$\theta = 1.5385$	$\rho = 0.5225$			$\rho = 0.5225$
$\rho_\tau = 0.35$			$\nu = 1$	$\nu = 3$	$\nu = 7$	
$E(Z)$	19,254	19,286	19,162	19,224	19,323	19,414
$VaR_{99.5\%}(Z)$	315,345	310,762	308,896	303,959	301,702	301,297
$ES_{99\%}(Z)$	562,579	552,309	540,872	530,119	526,597	521,544
$RBC_{VaR}(Z)$	296,092	291,475	289,734	284,736	282,379	281,883
$RBC_{ES}(Z)$	543,325	533,023	521,709	510,895	507,275	502,131
$D_{VaR}(Z)$	4.56%	6.19%	6.44%	8.69%	9.32%	9.95%
$D_{ES}(Z)$	4.78%	7.07%	7.15%	9.86%	12.02%	14.25%

Table 4
Results for a portfolio composed by X, Y , where X is lognormal distributed with $\mu = 6.52$ and $\sigma = 2.15$, and Y is Fréchet distributed with $\alpha = 1.5$ and $s = 4,657.15$.

We observe once more the order among dependence structures in terms of conservativeness. Figure 9 illustrates the trend of the diversification gain as a function of the dependence. No contradiction arose with respect to the comments made about Figure 4.

Figure 10 highlights the influence of the dependence on the RBC. The presence in the portfolio of one extreme value df is sufficient to emphasize the difference between the $RBC_{VaR}(Z)$ and the $RBC_{ES}(Z)$.

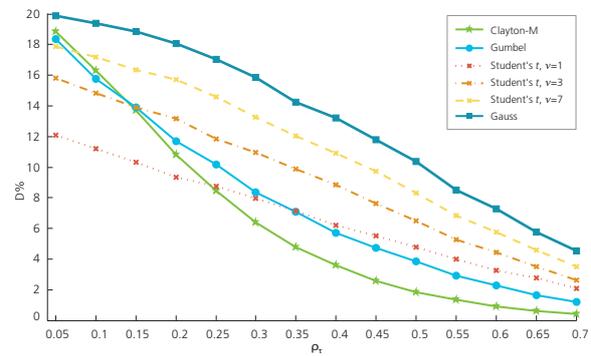


Fig. 9. Diversification gain as a function of both the strength and the structure of the dependence. The risk measure is the expected shortfall. X is lognormal distributed and Y is Fréchet distributed.

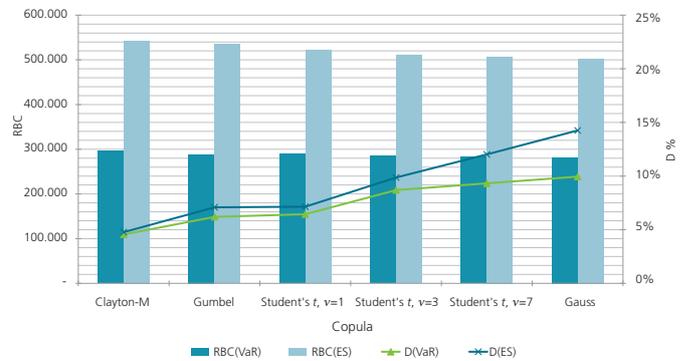


Fig. 10. RBC and D as a function of copulas, for $\rho_\tau = 0.35$. X is lognormal distributed and Y is Fréchet distributed.

6. Capital Allocation

In this section, we investigate through Monte Carlo methods the role of dependence on capital allocation. We refer to Goovaerts et al. (2003) for a broad discussion on allocation principles. A capital allocation principle is a method to split the overall risk capital of a portfolio among its components. To the purpose of this analysis, two allocation principles are described. As in previous Section, we drop the α indicating the percentile from the notation.

- **Euler principle.** According to the Euler principle, the expected shortfall contribution of risk X_i to the portfolio $Z = X_1 + \dots + X_d$, is given by:

$$ES(X_i, Z) = E \left[X_i \mid \sum_{i=1}^d X_i \geq \text{VaR} \left(\sum_{i=1}^d X_i \right) \right]. \quad (6.1)$$

Thus, the RBC allocated to risk X_i is equal to:

$$\text{RBC}_{ES}(X_i, Z) = ES(X_i, Z) - E(X_i). \quad (6.2)$$

We denote with $\text{RBC}_{ES}(X_i|Z)$ the percentage of RBC allocated to risk X_i , i. e.:

$$\text{RBC}_{ES}(X_i|Z) = \frac{\text{RBC}_{ES}(X_i, Z)}{\text{RBC}_{ES}(Z, Z)} = \frac{\text{RBC}_{ES}(X_i, Z)}{\text{RBC}_{ES}(Z)}. \quad (6.3)$$

For more on the Euler principle, we refer to Tasche (2008).

- **Haircut principle.** According to the haircut principle and in agreement with the above notation, the contribution of risk X_i to the portfolio Z is equal to:

$$\text{VaR}(X_i, Z) = \frac{\text{VaR}(X_i)}{\sum_{i=1}^d \text{VaR}(X_i)} \text{VaR}(Z). \quad (6.4)$$

Hence, the RBC allocated to risk X_i is given by:

$$\text{RBC}_{\text{VaR}}(X_i, Z) = \text{VaR}(X_i, Z) - E(X_i). \quad (6.5)$$

We denote with $\text{RBC}_{\text{VaR}}(X_i|Z)$ the percentage of RBC allocated to risk X_i , that corresponds to:

$$\text{RBC}_{\text{VaR}}(X_i|Z) = \frac{\text{VaR}(X_i)}{\sum_{i=1}^d \text{VaR}(X_i)}. \quad (6.6)$$

A description of the haircut principle is offered in Dhaene et al. (2009), Section 2.

Both principles lead to a full allocation of the capital requirement, i. e. $\sum_{i=1}^d \text{RBC}_{ES}(X_i|Z) = \text{RBC}_{ES}(Z)$ and $\sum_{i=1}^d \text{RBC}_{\text{VaR}}(X_i|Z) = \text{RBC}_{\text{VaR}}(Z)$. Concerning the haircut principle, the full allocation criterion is not satisfied if we substitute the right hand side of Equation 6.6 with:

$$\frac{\text{VaR}(X_i)}{\text{VaR}(\sum_{i=1}^d X_i)}. \quad (6.7)$$

This fail is a direct consequence of the non coherence of VaR mentioned in Section 2. We will return later on this alternative definition of the haircut principle.

Among capital allocation principles, Theorem 4.4 in Tasche (1999) proves that only the Euler principle is suitable for performance measurement. This feature is very important in steering the portfolio towards profitability through RORAC (return on risk adjusted capital) optimization. The RORAC of a risk (portfolio) represents the ratio between the expected profit and the risk capital contribution necessary to run that risk (portfolio). Roughly speaking, Tasche's theorem guarantee that if, according to the Euler principle, the RORAC of risk X_i is higher than the RORAC of the portfolio containing that risk, then an increase of the weight of risk X_i will improve the RORAC of the entire portfolio.

We study a simple portfolio, Z' , composed by two risks, X and Y' . In agreement with Section 5, both risks are assumed to have a lognormal distribution, namely $X, Y' \sim \log N(9.58, 0.83)$. Value at risk is computed at 99.5% and expected shortfall at 99%. Our goal is to study the Euler and the haircut capital allocation principles under several dependence assumptions. We also investigate how these principles react to changes in the riskiness of the portfolio components. Thus, we define two other portfolios, Z'' and Z''' , where the first component is still X but the second component is substituted by $Y'' \sim \log N(9.58, 0.70)$ and by $Y''' \sim \log N(9.58, 0.40)$, respectively. Hence, the riskiness of the second component is reduced, with respect to the original portfolio, varying its parameter σ .

The relation between σ and the variance of a lognormal rv is explicated in Appendix A. Through the programming language MATLAB, we implement ad-hoc procedures to quantify the allocated capital per risk. All the results are obtained with Monte Carlo methods that benefits of a minimum number of 2,000,000 iterations. Table 5 provides the results for portfolios where the dependence structure assumed is a Clayton-M copula or a Gauss copula. About the strength of the dependence, which is characterized via the Kendall's tau, results are given for $\rho_\tau = 0.20$ and $\rho_\tau = 0.50$.

		Clayton-M			Gauss		
		Z'	Z''	Z'''	Z'	Z''	Z'''
$\rho_\tau = 0.20$							
	RBC _{VaR} (Z)	174,239	146,924	116,991	151,383	128,099	109,411
	RBC _{ES} (Z)	200,040	168,377	133,076	169,847	143,639	124,131
Euler	RBC _{ES} (X Z)	49.97%	63.88%	86.89%	50.21%	69.83%	93.25%
	RBC _{ES} (Y Z)	50.03%	36.12%	13.11%	49.79%	30.17%	6.75%
	RBC _{ES} (X, Z)	99,956	107,565	115,629	85,283	100,302	115,746
	RBC _{ES} (Y, Z)	100,084	60,812	17,447	84,564	43,337	8,385
Haircut	RBC _{VaR} (X Z)	50.00%	58.25%	75.17%	50.15%	58.26%	75.25%
	RBC _{VaR} (Y Z)	50.00%	41.75%	24.83%	49.85%	41.74%	24.75%
	RBC _{VaR} (X, Z)	87,122	85,584	87,939	75,919	74,627	82,332
	RBC _{VaR} (Y, Z)	87,117	61,340	29,052	75,464	53,472	27,078
$\rho_\tau = 0.50$							
	RBC _{VaR} (Z)	201,787	168,429	125,686	180,858	151,972	118,774
	RBC _{ES} (Z)	231,055	190,749	141,822	204,212	171,353	134,091
Euler	RBC _{ES} (X Z)	50.04%	60.53%	82.07%	49.88%	63.35%	86.20%
	RBC _{ES} (Y Z)	49.96%	39.47%	17.93%	50.12%	36.65%	13.80%
	RBC _{ES} (X, Z)	115,617	115,453	116,389	101,871	108,551	115,582
	RBC _{ES} (Y, Z)	115,439	75,297	25,433	102,341	62,801	18,509
Haircut	RBC _{VaR} (X Z)	50.13%	58.31%	75.15%	49.99%	58.27%	75.14%
	RBC _{VaR} (Y Z)	49.87%	41.69%	24.85%	50.01%	41.73%	24.86%
	RBC _{VaR} (X, Z)	101,149	98,218	94,456	90,408	88,553	89,245
	RBC _{VaR} (Y, Z)	100,637	70,211	31,229	90,450	63,419	29,528

Table 5
Numerical results for the capital allocation. X and Y indicate the first and the second component of the general portfolio, denoted with Z.

Harking back to Equation 6.6, we notice that the denominator, $\sum_{i=1}^d \text{VaR}(X_i)$, does not take into account the dependence among risks. Thus, as our empirical data emphasize, the haircut allocation principle does not react neither to changes in the dependence structure nor to changes in the strength of the dependence within the portfolio. For instance, both when we have a Clayton-M copula and $\rho_\tau = 0.20$, or a Clayton-M copula and $\rho_\tau = 0.50$, as well as when we have a Gauss copula and $\rho_\tau = 0.20$, or a Gauss copula and $\rho_\tau = 0.50$, the RBC_{VaR}(Y|Z) \cong 24.8% (considering Z'''). Naturally, the capital requirement vary in terms of absolute amounts, but this is a consequence of the varied RBC_{VaR}(Z) only. Hence, continuing the previous example, the RBC_{VaR}(Y, Z) fluctuate from 29,052 to 31,229, and from 27,078 to 29,528. The alternative formulation of the haircut principle, using Equation 6.7, would grant a sensibility to the dependence assumptions, because of the denominator VaR ($\sum_{i=1}^d X_i$). Unfortunately, as above noticed, the incoherence of VaR would not grant a full allocation of the capital.

The Euler principle, instead, is able to catch different dependence conditions directly in terms of the weights assigned to the portfolio components. For instance, considering again Z''', RBC_{ES}(Y|Z) with a Gauss copula equals 6.75% when $\rho_\tau = 0.20$ and 13.80% when $\rho_\tau = 0.50$. Thus, to an increased strength of dependence, the Euler principle reacts assigning more weight to the less volatile risk. This behavior espouse the intuition that if risks are linked by a stronger dependence, we have to pay higher attention to the less volatile risks as well, because the probability that something go wrong for them too is increased by the stronger dependence with the more volatile risks. The same remark is valid considering a change in the structure of the dependence. In particular, if we move from the Gauss to the Clayton-M assumption, we are heightening tail dependence and increasing conservativeness, as described in Section 5. Hence, continuing the previous example, we register an increase of the weight of the less volatile risk that moves from 6.75% to 13.11%, when $\rho_\tau = 0.20$, and from 13.80% to 17.93%, when $\rho_\tau = 0.50$.

An extensive analysis has been conducted for all copulas discussed in Table 1. The order of conservativeness above mentioned leads, in the Euler principle, to an equivalent order in terms of the importance of the weights assigned to less volatile risks. Hence the values of, say, $RBC_{ES}(Y|Z)$, will range from the minimum weight assigned by the Gauss copula, followed by the Student's t copula and from the Gumbel copula, to the maximum weight assigned by the Clayton-M copula. We repeat all the analysis about capital allocation for two other portfolios.

We consider a portfolio composed by two Fréchet distributed rvs, both originally with $\alpha = 1.5$ and $s = 4,657.15$, and then we modify the second component. Similarly, we study a portfolio that mixes a Fréchet distributed rv with $\alpha = 1.5$ and $s = 4,657.15$ together with a lognormal distributed rv with $\mu = 6.52$ and $\sigma = 2.15$, such that the weight of the two risks is almost equivalent in the original composition, and then we reduce the riskiness of the Fréchet rv increasing its shape parameter, α (see Appendix B). All these further analysis confirm the above remarks about the link between (structure and strength of) dependence and allocation of capital.

7. Proposals for Future Research

We dealt with bivariate copulas but most of the concept illustrated can be easily extended in higher dimension. The fair comparison among structures of dependence, granted by the link between copulas parameter and rank correlation coefficients, could be maintained without loss of generality in case of one-parameter copula families, like the Clayton-M and the Gumbel. In case of elliptical copulas, instead, this fairness would require the comparison to deal with a subset of the specific copula family. For instance, if $d = 3$, we should define the Gauss copula to imply the same dependence among all the three portfolio components.

In Bürgi et al. (2009) the authors investigate on the diversification gain using a reference model based on a Clayton-M copula. The Monte Carlo techniques employed in our analysis could be used to extend their study varying the reference model (both in terms of copula and parameter) and reviewing the diversification gain. This procedure would allow to judge the conservativeness of assumptions on dependence with respect to each possible reference model.

However, the assumptions on copulas and their parameters are fundamental. Hence, further developments will involve the estimation and calibration of copulas from real data in order to obtain the right dependence and therefore to be able to give reliable assessment of diversification.

8. Conclusion

We pointed out the importance of dependence in the assessment of the capital requirements. The appraisal of RBC is heavily influenced from both the strength and the structure of dependence that is assumed to model the portfolio. Through the investigation of several copula models we identified an order among them in terms of conservativeness, with regard to the stated diversification gain. The same order is preserved varying both parametric and non parametric assumptions about marginal distributions. The results of our analysis send a warning concerning the prudence of certain assumptions. The main risk is to underestimate the RBC of the portfolio due to an improper choice of the copula model. Regarding risk

measures, we noted the sizeable differences in terms of RBC based on VaR and on ES when the distribution is fat tailed.

Finally, we discussed capital allocation principles. We observed that the split of the overall risk capital of a portfolio among its component can whether take dependence into account or not. The Euler principle reacts to changes in dependence assumptions. In particular, we showed that as dependence increase in strength or it is heightening with respect to tail dependence, the less volatile risk gains more weight coherently with the increased probability of a joint severe event.

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Appendix A. Lognormal distribution

Definition A.1 (Lognormal distribution) A continuous rv X is said to have a lognormal distribution, written $X \sim \text{logN}(\mu, \sigma^2)$, if its density is:

$$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right), & x > 0, \\ 0, & x \leq 0. \end{cases} \quad (\text{A.1})$$

It is possible to show that the mean and the variance of the lognormal distribution are respectively given by $E(X) = \exp(\mu + \frac{1}{2}\sigma^2)$ and $\text{VAR}(X) = E(X)^2(\exp(\sigma^2) - 1)$.

Considering a lognormal random variable, we can state the following remark.

Remark A.2 For a lognormal random variable $X \sim \text{logN}(\mu, \sigma^2)$:

$$\text{VaR}_\alpha(X) = \exp(\mu + \sigma\Phi^{-1}(\alpha)), \quad (\text{A.2})$$

$$\text{ES}_\alpha(X) = \exp(\mu + \sigma^2/2) \left(\frac{1 - \Phi(\Phi^{-1}(\alpha) - \sigma)}{1 - \alpha} \right), \quad (\text{A.3})$$

where Φ is the standard normal df.

For a confirmation of this remark, one can refer to Denuit et al. (2005), p. 98⁴.

Appendix B. Fréchet distribution

Definition B.1 (Fréchet distribution) A continuous rv X is said to have a Fréchet distribution if its cumulative density function is:

$$F_X(x) = \begin{cases} \exp\left(-\left(\frac{x}{s}\right)^{-\alpha}\right), & x > 0, \\ 0, & x \leq 0, \end{cases} \quad (\text{B.1})$$

where $\alpha > 0$ is the shape parameter and s is the scale parameter.

It is possible to show that the mean of the Fréchet distribution is given by $E(X) = \Gamma\left(1 - \frac{1}{\alpha}\right)s$ where $\Gamma(\gamma) = \int_0^\infty x^{\gamma-1} \exp(-x) dx$ is the Gamma function.

In the case of a Fréchet distribution also is possible to derive explicit formulas for VaR and ES as the following remark shows.

Remark B.2 For a Fréchet random variable⁵ X with shape parameter α and scale parameter s :

$$\text{VaR}_{\hat{\alpha}}(X) = s(-\ln(\hat{\alpha}))^{(-1/\alpha)}, \quad (\text{B.2})$$

$$\text{ES}_{\hat{\alpha}}(X) = s \frac{1}{1 - \hat{\alpha}} \Gamma\left(1 - \frac{1}{\alpha}, -\ln(\hat{\alpha})\right) \Gamma\left(1 - \frac{1}{\alpha}\right), \quad (\text{B.3})$$

where $\Gamma(\gamma)$ is the Gamma function and $\Gamma(\gamma, x)$ is the incomplete Gamma function, i.e.

$$\Gamma(\gamma, x) = \frac{1}{\Gamma(\gamma)} \int_0^x t^{\gamma-1} \exp(-t) dt.$$

(4) The authors distinguish between TVaR and ES and in particular they defined as TVaR what we defined as ES.

(5) We indicate here with $\hat{\alpha}$ the percentile to discern it from the shape parameter of the Fréchet distribution, α .

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